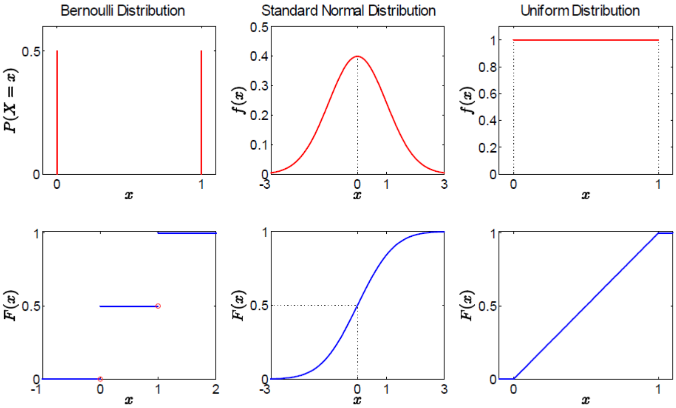
Econometricks: Short guides to econometrics

Trick 02: Specific Distributions

Davud Rostam-Afschar (Uni Mannheim)

Content

Specific Distributions



Thanks to Ping Yu

The **Bernoulli distribution** for a single binomial outcome (trial) is

$$\begin{aligned} Prob(x = 1) &= p, \\ Prob(x = 0) &= 1 - p, \end{aligned}$$

where $0 \leq p \leq 1$ is the probability of success.

- $E[x] = p$ and
- $V[x] = E[x^2] - E[x]^2 = p - p^2 = p(1 - p)$.

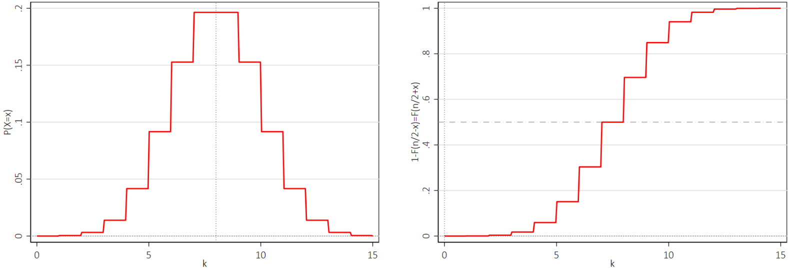
The distribution for x successes in n trials is the **binomial distribution**,

$$Prob(X = x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \quad x = 0, 1, \dots, n.$$

The mean and variance of x are

- $E[x] = np$ and
- $V[x] = np(1-p)$.

Example of a binomial $[n = 15, p = 0.5]$ distribution:



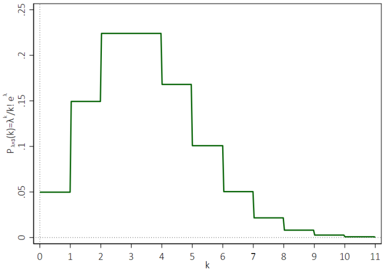
The limiting form of the binomial distribution, $n \to \infty$, is the **Poisson distribution**,

$$Prob(X = x) = \frac{e^{\lambda}\lambda^{x}}{x!}.$$

The mean and variance of x are

- $E[x] = \lambda$ and
- $V[x] = \lambda$.

Example of a Poisson [3] distribution:

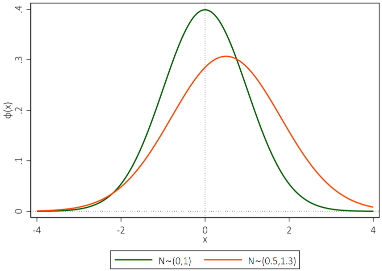


The normal distribution

Random variable $x \sim N[\mu, \sigma^2]$ is distributed according to the **normal distribution** with mean μ and standard deviation σ obtained as

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}. \tag{1}$$

The density is denoted $\phi(x)$ and the cumulative distribution function is denoted $\Phi(x)$ for the standard normal. Example of a standard normal, ($x \sim N[0,1]$), and a normal with mean 0.5 and standard deviation 1.3:



Transformation of random variables

Continuous variable x may be transformed to a discrete variable y. Calculate the mean of variable x in the respective interval:

$$
\begin{aligned}
Prob(Y = \mu_1) &= P(-\infty < X \leq a), \\
Prob(Y = \mu_2) &= P(a < X \leq b), \\
Prob(Y = \mu_3) &= P(b < X \leq \infty).
\end{aligned}
$$

Method of transformations

If x is a continuous random variable with pdf $f_x(x)$ and if $y = g(x)$ is a continuous monotonic function of x, then the density of y is obtained by

$$Prob(y \leq b) = \int_{-\infty}^{b} f_x(g^{-1}(y))|g^{-1\prime}(y)|dy.$$

With $f_y(y) = f_x(g^{-1}(y))|g^{-1\prime}](y)|dy$, this equation can be written as

$$Prob(y \leq b) = \int_{-\infty}^{b} f_y(y)dy.$$

Example

If $x \sim N[\mu, \sigma^2]$, then the distribution of $y = g(x) = \frac{x-\mu}{\sigma}$ is found as follows:

$$g^{-1}(y) = x = \sigma y + \mu$$

$$g^{-1\prime}(y) = \frac{dx}{dy} = \sigma$$

Therefore with $f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}[(g^{-1}(y)-\mu)^2/\sigma^2]}|g^{-1\prime}(y)|$

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-[(\sigma y+\mu)-\mu]^2/2\sigma^2}|\sigma| = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

- Preservation under linear transformation:

  If $x \sim N[\mu, \sigma^2]$, then $(a + bx) \sim N[a + b\mu, b^2\sigma^2]$.

- Convenient transformation $a = -\mu/\sigma$ and $b = 1/\sigma$:

  The resulting variable $z = \frac{(x-\mu)}{\sigma}$ has the standard normal distribution with density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

- If $x \sim N[\mu, \sigma^2]$, then $f(x) = \frac{1}{\sigma}\phi[\frac{x-\mu}{\sigma}]$
- $Prob(a \leq x \leq b) = Prob\left(\frac{a-\mu}{\sigma} \leq \frac{x-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$
- $\phi(-z) = 1 - \phi(z)$ and $\Phi(-x) = 1 - \Phi(x)$ because of symmetry

Method of transformations

If $z \sim N[0,1]$, then $z^2 \sim \chi^2[1]$ with pdf $\frac{1}{\sqrt{2\pi y}} e^{-y/2}$.

Example

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$y = g(x) = x^2$$

$$g^{-1}(y) = x = \pm\sqrt{y} \text{ there are two solutions to } g_1, g_2.$$

$$g^{-1\prime}(y) = \frac{dx}{dy} = \pm 1/2y^{-1/2}$$

$$f_y(y) = f_x(g_1^{-1}(y))|g_1^{-1\prime}(y)| + f_x(g_2^{-1}(y))|g_2^{-1\prime}(y)|$$

$$f_y(y) = f_x(\sqrt{y})|1/2y^{-1/2}| + f_x(-\sqrt{y})| - 1/2y^{-1/2}|$$

$$f_y(y) = \frac{1}{2\sqrt{2\pi y}} e^{-\frac{y}{2}} + \frac{1}{2\sqrt{2\pi y}} e^{-\frac{y}{2}} = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$$

Distributions derived from the normal

- If $z \sim N[0,1]$, then $z^2 \sim \chi^2[1]$ with $E[z^2] = 1$ and $V[z^2] = 2$.
- If $x_1, ..., x_n$ are n independent $\chi^2[1]$ variables, then

$$\sum_{i=1}^{n} x_i \sim \chi^2[n].$$

- If z_i, $i = 1, ..., n$, are independent $N[0,1]$ variables, then

$$\sum_{i=1}^{n} z_i^2 \sim \chi^2[n].$$

- If z_i, $i = 1, ..., n$, are independent $N[0,\sigma^2]$ variables, then

$$\sum_{i=1}^{n} \left(\frac{z_i}{\sigma}\right)^2 \sim \chi^2[n].$$

- If x_1 and x_2 are independent χ^2 variables with n_1 and n_2 degrees of freedom, then

$$x_1 + x_2 \sim \chi^2[n_1 + n_2].$$

The χ^2 distribution

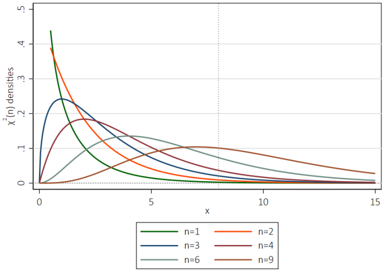
Random variable $x \sim \chi^2[n]$ is distributed according to the **chi-squared distribution** with n degrees of freedom

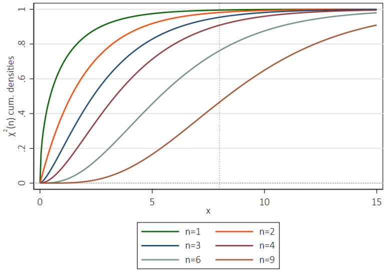
$$f(x|n) = \frac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}, \tag{2}$$

where Γ is the Gamma-distribution (more below).

- $E[x] = n$
- $V[x] = 2n$

Example of a $\chi^2[3]$ distribution:



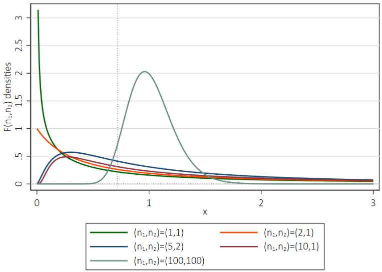


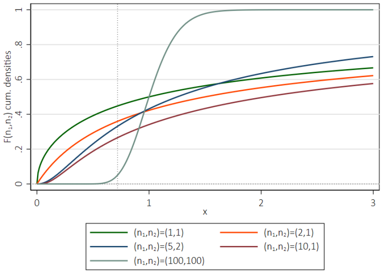
The F-distribution

If x_1 and x_2 are two independent chi-squared variables with degrees of freedom parameters n_1 and n_2, respectively, then the ratio

$$F[n_1, n_2] = \frac{x_1/n_1}{x_2/n_2} \tag{3}$$

has the **F distribution** with n_1 and n_2 degrees of freedom.



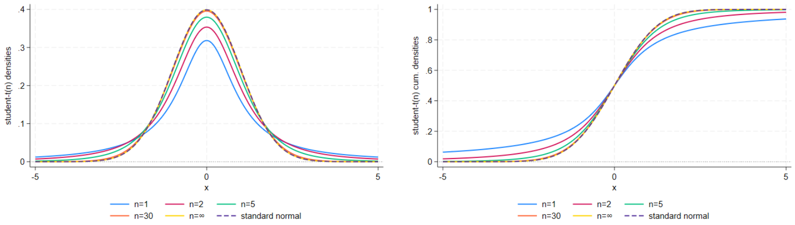


The student t-distribution

If x_1 is an $N[0,1]$ variable, often denoted by z, and x_2 is $\chi^2[n_2]$ and is independent of x_1, then the ratio

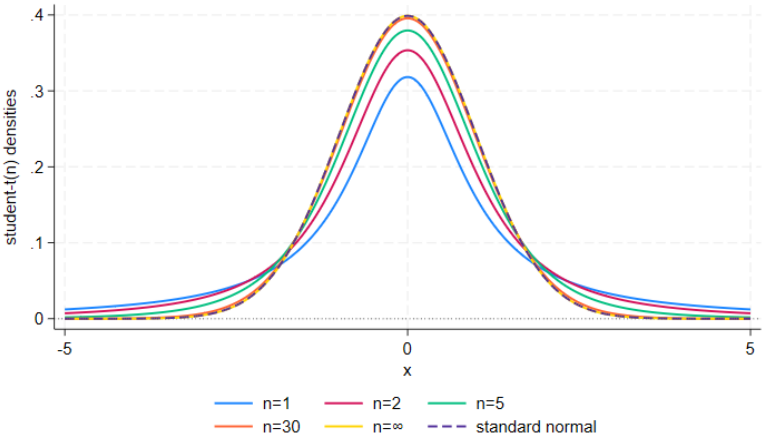
$$t[n_2] = \frac{x_1}{\sqrt{x_2/n_2}}. \tag{4}$$

has the **t distribution** with n_2 degrees of freedom.
Example for the t distributions with 3 and 10 degrees of freedom with the standard normal distribution.



Comparing (3) with $n_1 = 1$ and (4), if $t \sim t[n]$, then $t^2 \sim F[1, n]$.

The $t[30]$ approx. the standard normal



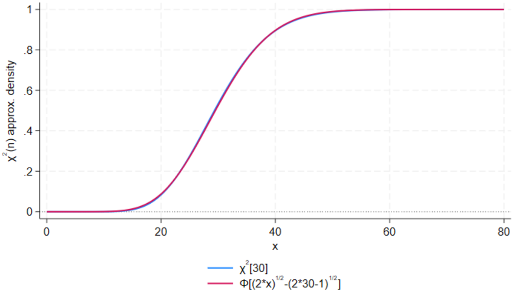
Approximating a χ^2

For degrees of freedom greater than 30 the distribution of the chi-squared variable x is approx.

$$z = (2x)^{1/2} - (2n-1)^{1/2}, \tag{5}$$

which is approximately standard normally distributed. Thus,

$$Prob(\chi^2[n] \leq a) \approx \Phi[(2a)^{1/2} - (2n-1)^{1/2}].$$



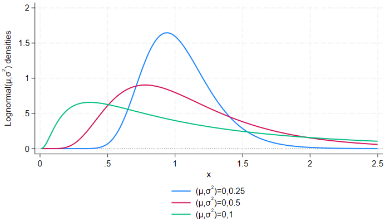
The **lognormal distribution**, denoted $LN[\mu, \sigma^2]$, has been particularly useful in modeling the size distributions.

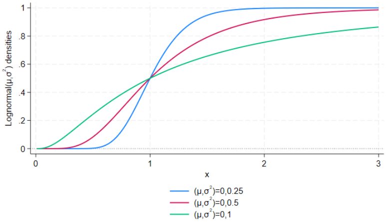
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}[(\ln x - \mu)/\sigma]^2}, \qquad x > 0$$

A lognormal variable x has

- $E[x] = e^{\mu + \sigma^2/2}$, and
- $Var[x] = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.

If $y \sim LN[\mu, \sigma^2]$, then $\ln y \sim N[\mu, \sigma^2]$.

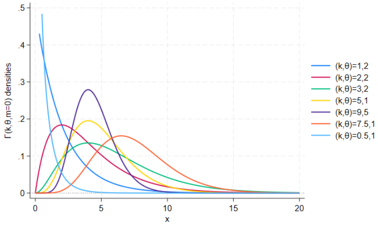


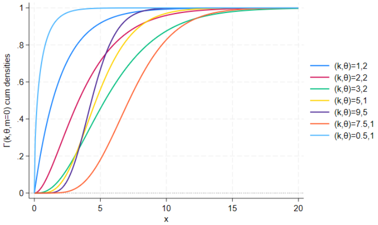


The general form of the **gamma distribution** is

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}, \qquad x \geq 0, \beta = 1/\theta > 0, \alpha = k > 0. \tag{6}$$

Many familiar distributions are special cases, including the **exponential distribution**($\alpha = 1$) and **chi-squared**($\beta = 1/2, \alpha = n/2$). The **Erlang distribution** results if α is a positive integer. The mean is α/β, and the variance is α/β^2. The **inverse gamma distribution** is the distribution of $1/x$, where x has the gamma distribution.



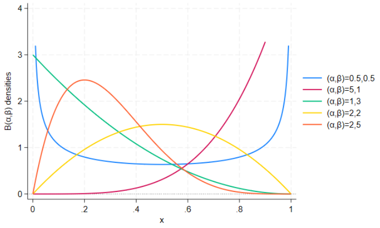


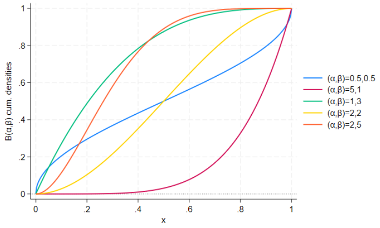
The beta distribution

For a variable constrained between 0 and $c > 0$, the **beta distribution** has proved useful. Its density is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{x}{c}\right)^{\alpha-1} \left(1 - \frac{x}{c}\right)^{\beta-1} \frac{1}{c}, \qquad 0 \le x \le 1.$$

It is symmetric if $\alpha = \beta$, asymmetric otherwise. The mean is $c\alpha/(\alpha+\beta)$, and the variance is $c^2\alpha\beta/[(\alpha+\beta+1)(\alpha+\beta)^2]$.



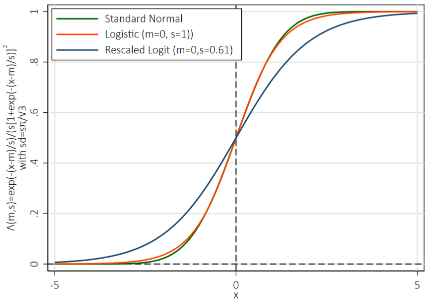


The logistic distribution

The **logistic distribution** is an alternative if the normal cannot model the mass in the tails; the cdf for a logistic random variable with $\mu = 0, s = 1$ is

$$F(x) = \Lambda(x) = \frac{1}{1 + e^{-x}}.$$

The density is $f(x) = \Lambda(x)[1 - \Lambda(x)]$. The mean and variance of this random variable are zero and $\sigma^2 = \pi^2/3$.



The Wishart distribution

The **Wishart distribution** describes the distribution of a random matrix obtained as

$$f(\boldsymbol{W}) = \sum_{i=1}^{n} (x_i - \mu)(x_i - \mu)'.$$

where x_i is the ith of nK element random vectors from the multivariate normal distribution with mean vector, μ, and covariance matrix, Σ. The density of the Wishart random matrix is

$$f(\boldsymbol{W}) = \frac{\exp\left[-\frac{1}{2} trace(\Sigma^{-1}\boldsymbol{W})\right] |\boldsymbol{W}|^{-\frac{1}{2}(n-K-1)}}{2^{nK/2}|\Sigma|^{K/2}\pi^{K(K-1)/4}\prod_{j=1}^{K}\Gamma\left(\frac{n+1-j}{2}\right)}.$$

The mean matrix is $n\Sigma$. For the individual pairs of elements in $\boldsymbol{W}$,

$$Cov[w_{ij}, w_{rs}] = n(\sigma_{ir}\sigma_{js} + \sigma_{is}\sigma_{jr}).$$

The Wishart distribution is a multivariate extension of χ^2 distribution. If $\boldsymbol{W} \sim W(n, \sigma^2)$, then $\boldsymbol{W}/\sigma^2 \sim \chi^2[n]$.

Common distributions and their properties

	Normal	Logistic
Parameters	$\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_{>0}$	$\mu \in \mathbb{R}$, $s \in \mathbb{R}_{>0}$
Support	$x \in \mathbb{R}$	$x \in \mathbb{R}$
PDF	$\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\lambda\left(\frac{x-\mu}{s}\right) = \frac{e^{-(x-\mu)/s}}{s\left(1+e^{-(x-\mu)/s}\right)^2}$
CDF	$\Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$	$\Lambda\left(\frac{x-\mu}{s}\right) = \frac{1}{1+e^{-(x-\mu)/s}}$
Mean	μ	μ
Median	μ	μ
Mode	μ	μ
Variance	σ^2	$\frac{s^2\pi^2}{3}$
Skewness	0	0
Ex. Kurtosis	0	$6/5$
MGF	$\exp(\mu t + \sigma^2 t^2/2)$	$e^{\mu t} B(1 - st, 1 + st)$ for $t \in (-1/s, 1/s)$

- PDF denotes probability density function, CDF cumulative distribution function, MGF moment-generating function.
- μ mean (location), σ, s (scale).
- $B(z_1, z_2)$ is beta function $\int_0^1 t^{z_1 - 1}(1-t)^{z_2-1}\, dt$ for complex number inputs z_1, z_2 with $\Re(z_1), \Re(z_2) > 0$.
- Excess Kurtosis is defined as Kurtosis minus 3.

Common distributions and their properties

	t	Log-normal
Parameters	$n \in \mathbb{R}_{>0}$	$\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_{>0}$
Support	$x \in \mathbb{R}$	$x \in \mathbb{R}_{>0}$
PDF	$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\ \Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$	$\frac{1}{x\sigma\sqrt{2\pi}}\ \exp\left(-\frac{(\ln x-\mu\ )^2}{2\sigma^2}\right)$
CDF	$\frac{1}{2}+x\ \Gamma\left(\frac{n+1}{2}\right)\times$ $\frac{{}_2F_1\left(\frac{1}{2},\ \frac{n+1}{2};\ \frac{3}{2};\ -\frac{x^2}{n}\right)}{\sqrt{\pi n}\ \Gamma\left(\frac{n}{2}\right)}$	$\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\ln x-\mu}{\sigma\sqrt{2}}\right)\right]$ $=\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)$
Mean	0 for $n>1$	$\exp\left(\mu+\frac{\sigma^2}{2}\right)$
Median	0	$\exp(\mu)$
Mode	0	$\exp\left(\mu-\sigma^2\right)$
Variance	$\frac{n}{n-2}$ for $n>2$, ∞ for $1<n\leq 2$	$\left[\exp(\sigma^2)-1\right]\exp\left(2\mu+\sigma^2\right)$
Skewness	0 for $n>3$	$\left[\exp\left(\sigma^2\right)+2\right]\sqrt{\exp(\sigma^2)-1}$
Ex. Kurtosis	$\frac{6}{n-4}$ for $n>4$, ∞ for $2<n\leq 4$	$1\exp\left(4\sigma^2\right)+2\exp\left(3\sigma^2\right)+3\exp\left(2\sigma^2\right)-6$
MGF	does not exist	not determined by its moments

- n denote degrees of freedom.
- ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is a particular instance of the hypergeometric function.

Common distributions and their properties

	Γ	Γ
Parameters	$k > 0 \in \mathbb{R}$ (shape), $\theta > 0 \in \mathbb{R}$ scale	$\alpha > 0 \in \mathbb{R}$ (shape), $\beta > 0 \in \mathbb{R}$ (rate)
Support	$x \in \mathbb{R}(0, \infty)$	$x \in \mathbb{R}(0, \infty)$
PDF	$f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$
CDF	$F(x) = \frac{1}{\Gamma(k)} \gamma\left(k, \frac{x}{\theta}\right)$	$F(x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x)$
Mean	$k\theta$	$\frac{\alpha}{\beta}$
Median	No simple closed form	No simple closed form
Mode	$(k-1)\theta$ for $k \geq 1$, 0 for $k < 1$	$\frac{\alpha-1}{\beta}$ for $\alpha \geq 1$, 0 for $\alpha < 1$
Variance	$k\theta^2$	$\frac{\alpha}{\beta^2}$
Skewness	$\frac{2}{\sqrt{k}}$	$\frac{2}{\sqrt{\alpha}}$
Ex. Kurtosis	$\frac{6}{k}$	$\frac{6}{\alpha}$
MGF	$(1-\theta t)^{-k}$ for $t < \frac{1}{\theta}$	$\left(1 - \frac{t}{\beta}\right)^{-\alpha}$ for $t < \beta$

- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t, \qquad \Re(z) > 0$, for complex numbers with a positive real part.
- lower incomplete gamma function is $\gamma(s, x) = \int_0^x t^{s-1}\, e^{-t}\, dt$, for complex numbers with a positive real part.

Common distributions and their properties

	χ^2	F
Parameters	$n \in \mathbb{N}_{>0}$	$n_1,\ n_2 \in \mathbb{N}_{>0}$
Support	$x \in \mathbb{R}_{>0}$ if $n = 1$, else $x \in \mathbb{R}_{\geq 0}$	$x \in \mathbb{R}_{>0}$ if $n_1 = 1$, else $x \in \mathbb{R}_{\geq 0}$
PDF	$\frac{1}{2^{n/2}\Gamma(n/2)}\ x^{n/2-1}e^{-x/2}$	$n_1^{\frac{n_1}{2}}\ n_2^{\frac{n_2}{2}}\ \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}\ \frac{x^{\frac{n_1}{2}-1}}{(n_1x+n_2)^{\frac{n_1+n_2}{2}}}$
CDF	$\frac{1}{\Gamma(n/2)}\ \gamma\left(\frac{n}{2},\ \frac{x}{2}\right)$	$I\left(\frac{n_1x}{n_1x+n_2},\ \frac{n_1}{2},\ \frac{n_2}{2}\right)$
Mean	n	$\frac{n_2}{n_2-2}$ for $n_2 > 2$
Median	No simple closed form	No simple closed form
Mode	$\max(n-2, 0)$	$\frac{n_1-2}{n_1}\ \frac{n_2}{n_2+2}$ for $n_1 > 2$
Variance	$2n$	$\frac{2\,n_2^2\,(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$ for $n_2 > 4$
Skewness	$\sqrt{8/n}$	$\frac{(2n_1+n_2-2)\sqrt{8(n_2-4)}}{(n_2-6)\sqrt{n_1(n_1+n_2-2)}}$ for $n_2 > 6$
Ex. Kurtosis	$\frac{12}{n}$	$12\frac{n_1(5n_2-22)(n_1+n_2-2)+(n_2-4)(n_2-2)^2}{n_1(n_2-6)(n_2-8)(n_1+n_2-2)}$ for $n_2 > 8$
MGF	$(1-2t)^{-n/2}$ for $t < \frac{1}{2}$	does not exist

- n, n_1, n_2 known as degrees of freedom.
- Regularized incomplete beta function $I(x, a, b) = \frac{B(x,a,b)}{B(a,b)}$ with $B(x,\ a,\ b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt$.

Common distributions and their properties

	B
Parameters	$\alpha, \beta \in \mathbb{R}_{>0}$
Support	$x \in [0,1]$ or $x \in (0,1)$
PDF	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$
CDF	$I(x,\ \alpha, \beta)$
Mean	$\frac{\alpha}{\alpha+\beta}$
Median	$I_{\frac{1}{2}}^{[-1]}(\alpha,\beta) \approx \frac{\alpha-\frac{1}{3}}{\alpha+\beta-\frac{2}{3}}$ for $\alpha,\beta > 1$
Mode	*
Variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Skewness	$\frac{2\,(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$
Ex. Kurtosis	$\frac{6[(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$
MGF	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$

- $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and Γ is the Gamma function.
- $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t,\qquad \Re(z) > 0$, for complex numbers with a positive real part.
- Regularized incomplete beta function $I(x,\ a, b) = \frac{B(x,a,b)}{B(a,b)}$ with $B(x,\ a,\ b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt$.
- * $\frac{\alpha-1}{\alpha+\beta-2}$ for $\alpha,\beta > 1$; any value in $(0,1)$ for $\alpha,\beta = 1$; $\{0,1\}$ (bimodal) for $\alpha,\beta < 1$; 0 for $\alpha \leq 1, \beta > 1$; 1 for $\alpha > 1, \beta \leq 1$.

References I

GREENE, W. H. (2011): *Econometric Analysis*. Prentice Hall, 5 edn.