Econometricks: Short guides to econometrics

Trick 07: The Generalized Method of Moments

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Asymptotic properties of the GMM estimator

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Minimize the quadratic form

The overidentified GMM estimator $\hat{\theta}_{GMM}(W_n)$ for K parameters in θ identified by $L > K$ moment conditions is a function of the weighting matrix W_n for a sample of $i = 1, ..., n$ observations:

$$\hat{\theta}_{GMM}(W_n) = \underset{\theta}{\text{argmin}} q_n(\theta),$$

where the quadratic form $q_n(\theta)$ is the criterion function and is given as a function of the sample moments $\bar{m}_n(\theta)$

$$q_n(\theta) = \bar{m}_n(\theta)' W \bar{m}_n(\theta).$$

The sample moments are a function

$$\bar{m}_n(\theta) = 1/n \sum_{i=1}^{N} m(X_i, Z_i, \theta_0)$$

of the model variables X_i, the instruments Z_i, and the true parameters θ_0.

$$\underset{1\times 1}{q_n(\theta)} = \underset{1\times L}{\bar{m}_n(\theta)'} \underset{L\times L}{W} \underset{L\times 1}{\bar{m}_n(\theta)}.$$

Quadratic form criterion function $q_n(\theta) \geq 0$ is a scalar!

Weighting matrix W is symmetric (and positive definite that is $x'Wx > 0$ for all non-zero x)!

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First order Taylor expansion of sample moments $\bar{m}_n(\hat{\theta}_{GMM})$ around $\bar{m}_n(\theta_0)$ at true parameters gives:

$$\bar{m}_n(\hat{\theta}_{GMM}) \approx \bar{m}_n(\theta_0) + \bar{G}_n(\bar{\theta})(\hat{\theta}_{GMM} - \theta_0),$$

where $\bar{G}_n(\bar{\theta}) = \frac{\partial \bar{m}_n(\bar{\theta})}{\partial \bar{\theta}'}$ and $\bar{\theta}$ is a point between $\hat{\theta}_{GMM}$ and θ_0.

Check the dimensions

First order Taylor expansion of sample moments $\bar{m}_n(\hat{\theta}_{GMM})$ around $\bar{m}_n(\theta_0)$ at true parameters gives:

$$\underset{L\times 1}{\bar{m}_n(\hat{\theta}_{GMM})} \approx \underset{L\times 1}{\bar{m}_n(\theta_0)} + \underset{L\times K}{\bar{G}_n(\bar{\theta})}\underset{K\times 1}{(\hat{\theta}_{GMM} - \theta_0)},$$

where $\bar{G}_n(\bar{\theta}) = \frac{\partial \underset{L\times 1}{\bar{m}_n(\bar{\theta})}}{\partial \underset{1\times K}{\bar{\theta}'}}$ and $\bar{\theta}$ is a point between $\hat{\theta}_{GMM}$ and θ_0, because of the

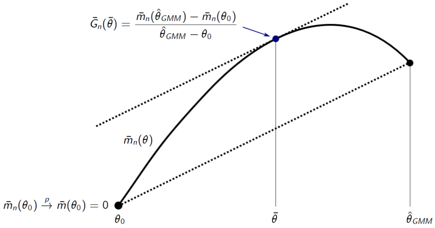
Mean value theorem...

Approximation introduced $\bar{\theta}$

...where $\bar{G}_n(\bar{\theta}) = \frac{\partial \bar{m}_n(\bar{\theta})}{\partial \bar{\theta}'}$ and $\bar{\theta}$ is a point between $\hat{\theta}_{GMM}$ and θ_0.

Mean value theorem:

$$\bar{G}_n(\bar{\theta}) = \frac{\bar{m}_n(\hat{\theta}_{GMM}) - \bar{m}_n(\theta_0)}{\hat{\theta}_{GMM} - \theta_0} \text{ for } \theta_0 < \bar{\theta} < \hat{\theta}_{GMM}.$$



Do the minimization

To minimize the quadratic form criterion, we take the first derivative of

$$q_n(\theta) = \bar{m}_n(\theta)' W \bar{m}_n(\theta)$$

$$\frac{\partial q_n(\hat{\theta}_{GMM})}{\partial \hat{\theta}_{GMM}} = 2\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\hat{\theta}_{GMM}) = 0.$$

Express as much as possible asymptotically

$$\frac{\partial q_n(\hat{\theta}_{GMM})}{\partial \hat{\theta}_{GMM}} = 2\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\hat{\theta}_{GMM}) = 0,$$

Plug in the approximation from before

$$\bar{m}_n(\hat{\theta}_{GMM}) \approx \bar{m}_n(\theta_0) + \bar{G}_n(\bar{\theta})(\hat{\theta}_{GMM} - \theta_0)$$

to obtain

$$\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\theta_0) + \bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta})(\hat{\theta}_{GMM} - \theta_0) \approx 0$$

which we rearrange to get the very useful

$$\hat{\theta}_{GMM} \approx \theta_0 - (\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta}))^{-1} \bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\theta_0).$$

So the estimate $\hat{\theta}_{GMM}$ is approximately the true parameter θ_0 plus a sampling error that depends on the sample moment $\bar{m}_n(\theta_0)$.

Quickly check dimensions

Useful approximation

$$\underset{K\times 1}{\hat{\theta}_{GMM}} \approx \underset{K\times 1}{\theta_0} - (\underset{K\times L}{\bar{G}_n(\hat{\theta}_{GMM})'}\underset{L\times L}{W_n}\underset{L\times K}{\bar{G}_n(\bar{\theta})})^{-1}\underset{K\times L}{\bar{G}_n(\hat{\theta}_{GMM})'}\underset{L\times L}{W_n}\underset{L\times 1}{\bar{m}_n(\theta_0)}.$$

So the estimate $\hat{\theta}_{GMM}$ is approximately the true parameter θ_0 plus a sampling error that depends on the sample moment $\bar{m}_n(\theta_0)$.

Asymptotic properties of the GMM estimator

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Three assumptions: moment conditions

Definition

GMM1: Moment Conditions and Identification.

$$\bar{m}(\theta_a) \neq \bar{m}(\theta_0) = E[m(X_i, Z_i, \theta_0)] = 0.$$

Identification implies that the probability limit of the GMM criterion function is uniquely minimized at the true parameters.

Three assumptions: law of large numbers

Definition

GMM2: Law of Large Numbers Applies.

$$\bar{m}_n(\theta) = 1/n \sum_{i=1}^{N} m(X_i, Z_i, \theta_0) \xrightarrow{p} E[m(X_i, Z_i, \theta_0)].$$

The data meets the conditions for a law of large numbers to apply, so that we may assume that the empirical moments converge in probability to their expectation.

Three assumptions: central limit theorem

Definition

GMM3: Central Limit Theorem Applies.

$$\sqrt{n}\bar{m}_n(\theta) = \sqrt{n}/n \sum_{i=1}^{N} m(X_i, Z_i, \theta_0) \xrightarrow{d} N[0, \Phi].$$

The empirical moments obey a central limit theorem. This assumes that the moments have a finite asymptotic covariance matrix $E[m(X_i, Z_i, \theta_0) m(X_i, Z_i, \theta_0)'] = \Phi$.

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Consistency

Recall the useful approximation of the estimator:

$$\hat{\theta}_{GMM} \approx \theta_0 - (\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta}))^{-1} \bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\theta_0).$$

Assumption GMM2 implies that

$$\bar{m}_n(\theta_0) = 1/n \sum_{i=1}^{N} m(X_i, Z_i, \theta_0) \xrightarrow{p} E[m(X_i, Z_i, \theta_0)] = \bar{m}(\theta_0).$$

That is, the sample moment equals the population moment in probability. Assumption GMM1 implies that

$$\bar{m}(\theta_0) = 0.$$

Then

$$\bar{m}_n(\theta_0) \xrightarrow{p} \bar{m}(\theta_0) = 0,$$

such that

$$\hat{\theta}_{GMM} \xrightarrow{p} \theta_0 \text{ for } N \to \infty$$

That is, by GMM1 and GMM2 the GMM estimator is consistent.

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Asymptotic normality

Recall the useful approximation of the estimator:

$$\hat{\theta}_{GMM} \approx \theta_0 - (\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta}))^{-1} \bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{m}_n(\theta_0).$$

Rewrite to obtain

$$\sqrt{n}(\hat{\theta}_{GMM} - \theta_0) \approx -(\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta}))^{-1} \bar{G}_n(\hat{\theta}_{GMM})' W_n \sqrt{n}\bar{m}_n(\theta_0),$$

The right hand side has several parts for which we made assumptions on what happens when $N \to \infty$. Under the central limit theorem (GMM3)

$$\sqrt{n}\bar{m}_n(\theta_0) \stackrel{d}{\to} N[0, \Phi]$$

$$plim W_n = W$$

$$plim \bar{G}_n(\hat{\theta}_{GMM}) = plim \bar{G}_n(\bar{\theta}) = plim \frac{\partial m(X_i, Z_i, \theta_0)}{\partial \theta_0'} = E\left[\frac{\partial \bar{m}(\theta_0)}{\partial \theta_0'}\right] = \Gamma(\theta_0)$$

Asymptotic normality

With $plim W_n = W$ and

$$plim \bar{G}_n(\hat{\theta}_{GMM}) = plim \bar{G}_n(\bar{\theta}) = \Gamma(\theta_0)$$

the expression

$$\sqrt{n}(\hat{\theta}_{GMM} - \theta_0) \approx -(\bar{G}_n(\hat{\theta}_{GMM})' W_n \bar{G}_n(\bar{\theta}))^{-1} \bar{G}_n(\hat{\theta}_{GMM})' W_n \sqrt{n} \bar{m}_n(\theta_0)$$

becomes

$$\sqrt{n}(\hat{\theta}_{GMM} - \theta_0) \approx -(\Gamma(\theta_0)' W \Gamma(\theta_0))^{-1} \Gamma(\theta_0)' W \sqrt{n} \bar{m}_n(\theta_0)$$

from which we get the variance V. So

$$\sqrt{n}(\hat{\theta}_{GMM} - \theta_0) \xrightarrow{d} N[0, V]$$

with

$$\underset{K \times K}{V} = 1/n[\Gamma(\theta_0)' W \Gamma(\theta_0)]^{-1} [\Gamma(\theta_0)' W \Phi W' \Gamma(\theta_0)] [\Gamma(\theta_0)' W \Gamma(\theta_0)]^{-1}$$

That is by GMM1, GMM2, and GMM3 the GMM estimator is asymptotic normal.

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Asymptotic efficiency

Which weighting matrix W gives us the smallest possible asymptotic variance of the GMM estimator $\hat{\theta}_{GMM}$.
The variance of the GMM estimator V depends on the choice of W

$$V = 1/n[\Gamma(\theta_0)' W \Gamma(\theta_0)]^{-1}[\Gamma(\theta_0)' W \Phi W' \Gamma(\theta_0)][\Gamma(\theta_0)' W \Gamma(\theta_0)]^{-1}$$

So let us minimize V to get the *optimal* weight matrix. Try from GMM3

$$\underset{n\to\infty}{plim} W_n = W = \Phi^{-1}$$

$$V_{GMM,optimal} = 1/n[\Gamma(\theta_0)'\Phi^{-1}\Gamma(\theta_0)]^{-1}[\Gamma(\theta_0)'\Phi^{-1}\Phi\Phi^{-1'}\Gamma(\theta_0)][\Gamma(\theta_0)'\Phi^{-1}\Gamma(\theta_0)]^{-1}$$

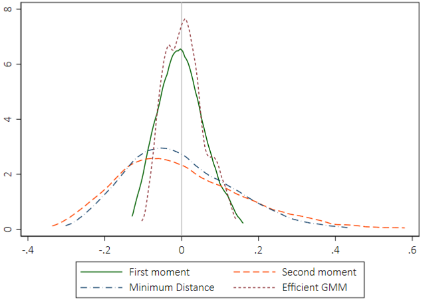
Which can be simplified to

$$V_{GMM,optimal} = 1/n[\Gamma(\theta_0)'\Phi^{-1}\Gamma(\theta_0)]^{-1}$$

Asymptotic efficiency

$$V_{GMM,optimal} = 1/n[\Gamma(\theta_0)'\Phi^{-1}\Gamma(\theta_0)]^{-1}$$

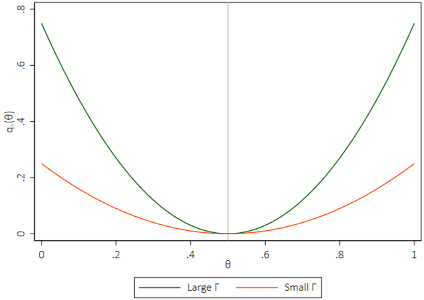
If Φ is small, there is little variation of this specific sample moment around zero and the moment condition is very informative about θ_0. So it is best to assign a high weight to it.



Asymptotic efficiency

$$V_{GMM,optimal} = 1/n[\Gamma(\theta_0)'\Phi^{-1}\Gamma(\theta_0)]^{-1}$$

If Γ is large, there is a large penalty from violating the moment condition by evaluating at $\theta \neq \theta_0$. Then the moment condition is very informative about θ_0. V is inversely related to Γ.



Estimate the variance in practice

$$\hat{V}_{GMM,optimal} = 1/n[\bar{G}_n(\hat{\theta})'\Phi_n^{-1}\bar{G}_n(\hat{\theta})]^{-1}$$

Consistent estimator

$$\Phi_n = NV(\bar{m}_n(\hat{\theta}))$$

$$\bar{G}_n(\hat{\theta}) = \frac{\partial m(X_i, Z_i, \hat{\theta})}{\partial \hat{\theta}'}$$

References I

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