

# Economet**tricks**: Short guides to econometrics

Trick 05: Simplifying Linear Regressions using Frisch-Waugh-Lovell

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# Content

1. Frisch-Waugh-Lovell theorem in equation algebra
2. Projection and residual maker matrices
3. Frisch-Waugh-Lovell theorem in matrix algebra

From the multivariate to the bivariate regression

Regress  $y_i$  on two explanatory variables, where  $x_i^2$  is the variable of interest and  $x_i^1$  (or further variables) are not of interest.

$$y_i = \beta_0 + \beta_2 x_i^2 + \beta_1 x_i^1 + \varepsilon_i.$$

Surprising and useful result:

- ▶ We can obtain **exactly the same** coefficients and residuals from a regression of two **demeaned** variables

$$\tilde{y}_i = \beta_0 + \beta_2 \tilde{x}_i^2 + \varepsilon_i.$$

- ▶ We can obtain **exactly the same** coefficient and residuals from a regression of two **residualized** variables

$$\varepsilon_i^y = \beta_2 \varepsilon_i^2 + \varepsilon_i.$$

## Why is the decomposition useful?

Allows breaking a multivariate model with  $K$  independent variables into  $K$  bivariate models.

- ▶ Relationship between two variables from a multivariate model can be shown in a two-dimensional scatter plot
- ▶ Absorbs fixed effects to reduce computation time (see `reghdfe` for Stata)
- ▶ Allows to separate variability between the regressors (multicollinearity) and between the residualized variable  $\tilde{x}_i^2$  and the dependent variable  $y_i$ .
- ▶ Understand biases in multivariate models tractably.

How to decompose  $y_i$  and  $x_i^2$ ?

Partial out  $x_i^1$  from  $y_i$  and from  $x_i^2$ .

- ▶ Regress  $x_i^2$  on all  $x_i^1$  and get residuals  $\varepsilon_i^2$ :

$$x_i^2 = \gamma_0 + \gamma_1 x_i^1 + \varepsilon_i^2,$$

this implies  $\text{Cov}(x_i^1, \varepsilon_i^2) = 0$ ,

- ▶ Regress  $y_i$  on all  $x_i^1$  and get residuals  $\varepsilon_i^y$ :

$$y_i = \delta_0 + \delta_1 x_i^1 + \varepsilon_i^y.$$

This implies  $\text{Cov}(x_i^1, \varepsilon_i^y) = 0$ .

From the residuals and the constants  $\gamma_0$  and  $\delta_0$  generate

- ▶  $\tilde{x}_i^2 = \gamma_0 + \varepsilon_i^2$ ,
- ▶  $\tilde{y}_i = \delta_0 + \varepsilon_i^y$ .

Finally,

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i^2 + \tilde{\varepsilon}_i = \beta_0 + \beta_2 \tilde{x}_i^2 + \varepsilon_i.$$

## Decomposition theorem

### Theorem

Decomposition theorem. *For multivariate regressions and detrended regressions, e.g.,*

$$y_i = \beta_0 + \beta_2 x_i^2 + \beta_1 x_i^1 + \varepsilon_i,$$

$$\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i^2 + \tilde{\varepsilon}_i,$$

*the same regression coefficients will be obtained with any non-empty subset of the explanatory variables, such that*

$$\tilde{\beta}_1 = \beta_2 \text{ and also } \tilde{\varepsilon}_i = \varepsilon_i.$$

Examining either set of residuals will convey precisely the same information about the properties of the unobservable stochastic disturbances.

## Detrended variables

Show that

$$y_i = \beta_0 + \beta_2 x_i^2 + \beta_1 x_i^1 + \varepsilon_i \quad (1)$$

$$= \tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_i^2 + \tilde{\varepsilon}_i. \quad (2)$$

Plug in the variables  $y_i = \delta_0 + \delta_1 x_i^1 + \varepsilon_i^y$  and  $x_i^2 = \gamma_0 + \gamma_1 x_i^1 + \varepsilon_i^2$  in the equation (1)

$$y_i = \delta_0 + \delta_1 x_i^1 + \varepsilon_i^y = \beta_0 + \beta_2(\gamma_0 + \gamma_1 x_i^1 + \varepsilon_i^2) + \beta_1 x_i^1 + \varepsilon_i$$

$$\tilde{y}_i = \delta_0 + \varepsilon_i^y = \beta_0 + \beta_2(\gamma_0 + \varepsilon_i^2) + (\beta_2 \gamma_1 - \delta_1 + \beta_1) x_i^1 + \varepsilon_i.$$

Because we partialled out  $x_i^1$  using OLS,  $x_i^1$  is mechanically uncorrelated to  $\varepsilon_i^2$  and to  $\varepsilon_i^y$ . Therefore, the regression coefficient  $(\beta_2 \gamma_1 - \delta_1 + \beta_1)$  of the partialled out variable  $x_i^1$  is zero. The equation simplifies with  $\tilde{x}_i^2 = \gamma_0 + \varepsilon_i^2$  to

$$\tilde{y}_i = \delta_0 + \varepsilon_i^y = \beta_0 + \beta_2(\gamma_0 + \varepsilon_i^2) + \varepsilon_i.$$

## Detrended variables

Regression anatomy: Only detrending  $x_i^2$  and not  $y_i$ . The regression constant, residuals, and the standard errors change but  $\beta_2$  remains

$$\begin{aligned} y_i = \delta_0 + \delta_1 x_i^1 + \varepsilon_i^y &= (\beta_0 + \delta_1 \bar{x}^1) + \beta_2 (\gamma_0 + \varepsilon_i^2) + (\varepsilon_i + \delta_1 x_i^1) \\ y_i &= \kappa + \beta_2 \tilde{x}^2 + \epsilon_i. \end{aligned} \tag{3}$$



## Residualized variables

$$\begin{aligned}\tilde{y}_i = \delta_0 + \varepsilon_i^y &= \beta_0 + \beta_2(\gamma_0 + \varepsilon_i^2) + \varepsilon_i \\ \varepsilon_i^y &= \beta_0 - \delta_0 + \beta_2\gamma_0 + \beta_2\varepsilon_i^2 + \varepsilon_i.\end{aligned}$$

The same result of the FWL Theorem holds as well for a regression of the residualized variables because  $\beta_0 = \delta_0 - \beta_2\gamma_0$ :

$$\varepsilon_i^y = \beta_2\varepsilon_i^2 + \varepsilon_i.$$

## Partition of $y$

Least squares partitions the vector  $y$  into two orthogonal parts

$$y = \hat{y} + e = Xb + e = Py + My.$$

- ▶  $n \times 1$  vector of data  $y$
- ▶  $n \times n$  projection matrix  $P$
- ▶  $n \times n$  residual maker matrix  $M$
- ▶  $n \times 1$  vector of residuals  $e$

## Projection matrix

$$Py = Xb = X(X'X)^{-1}X'y$$

$$\rightarrow P = X(X'X)^{-1}X'.$$

### Definition

Properties.

- ▶ *symmetric such that  $P = P'$ , thus orthogonal*
- ▶ *idempotent such that  $P = P^2$ , thus indeed a projection*
- ▶ *annihilator matrix  $PX = X$*

## Example for projection matrix

### Example

Show  $\mathbf{P}\mathbf{X} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X} = \mathbf{X}$ .

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}; \mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}; \mathbf{X}'\mathbf{X}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1.5 \end{bmatrix};$$

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{P}\mathbf{X} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (4)$$

Project  $\mathbf{y}$  on the column space of  $\mathbf{X}$ , i.e. regress  $\mathbf{y}$  on  $\mathbf{x}$  and predict  $E[\mathbf{y}] = \hat{\mathbf{y}}$ .

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \mathbf{P}\mathbf{y} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \quad (5)$$

## Residual maker matrix

$$My = e = y - Xb = y - X(X'X)^{-1}X'y$$

$$My = (I - X(X'X)^{-1}X')y$$

$$\rightarrow M = I - X(X'X)^{-1}X' = (I - P).$$

### Definition

Properties.

- ▶ *symmetric such that  $M = M'$*
- ▶ *idempotent such that  $M = M^2$*
- ▶ *annihilator matrix  $MX = 0$*
- ▶ *orthogonal to  $P$ :  $PM = MP = 0$ .*

## Example for residual maker matrix

### Example

Show  $\mathbf{MX} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{X} = (\mathbf{I} - \mathbf{P})\mathbf{X} = \mathbf{X} - \mathbf{X} = \mathbf{0}$ .

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix};$$

$$\mathbf{M} = (\mathbf{I} - \mathbf{P}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{MX} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (6)$$

Obtain residuals from a projection of  $\mathbf{y}$  on the column space of  $\mathbf{X}$ , i.e. regress  $\mathbf{y}$  on  $\mathbf{x}$  and predict  $\mathbf{y} - E[\mathbf{y}] = \mathbf{y} - \hat{\mathbf{y}}$ .

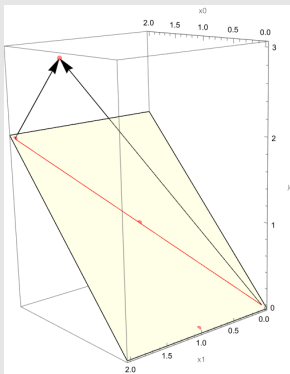
$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \mathbf{My} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \quad (7)$$

## Example for residual maker matrix

### Example

Column space of  $\mathbf{X}$  is  $\mathbf{x}_0$  and  $\mathbf{x}_1$ .

$$\begin{bmatrix} x_0^1 = 1 & x_1^1 = 0 & y^1 = 1 \\ x_0^2 = 1 & x_1^2 = 1 & y^2 = 2 \\ x_0^3 = 1 & x_1^3 = 0 & y^3 = 3 \end{bmatrix}; \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}; \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \quad (8)$$



The closest point from the vector  $\mathbf{y}' = [1, 2, 3]$  onto the column space of  $\mathbf{X}$ , is  $\hat{\mathbf{y}} = \mathbf{X}\mathbf{b}$ , here  $\hat{\mathbf{y}}' = [2, 2, 2]$ . At this point, we can draw a line orthogonal to the column space of  $\mathbf{X}$ .

## Decomposing the normal equations

The normal equations<sup>1</sup> in matrix form are  $\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}$ . If  $\mathbf{X}$  is partitioned into an interesting segment  $\mathbf{X}_2$  and an uninteresting  $\mathbf{X}_1$ , normal equations are

$$\begin{bmatrix} \mathbf{X}'_1\mathbf{X}_1 & \mathbf{X}'_1\mathbf{X}_2 \\ \mathbf{X}'_2\mathbf{X}_1 & \mathbf{X}'_2\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}'_1\mathbf{y} \\ \mathbf{X}'_2\mathbf{y} \end{bmatrix}.$$

The multiplication of the two equations can be done separately

$$\begin{bmatrix} \mathbf{X}'_1\mathbf{X}_1 & \mathbf{X}'_1\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}'_1\mathbf{y} \end{bmatrix} \quad (9)$$

$$\begin{bmatrix} \mathbf{X}'_2\mathbf{X}_1 & \mathbf{X}'_2\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}'_2\mathbf{y} \end{bmatrix}. \quad (10)$$

How can we find an expression for  $\mathbf{b}_2$  that does not involve  $\mathbf{b}_1$ ?

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<sup>1</sup>It is called a normal equation because  $\mathbf{y} - \mathbf{X}\mathbf{b}$  is normal to the range of  $\mathbf{X}$ .



## Solving for $\mathbf{b}_2$

Idea: Solve equation (9) for  $\mathbf{b}_1$  in terms of  $\mathbf{b}_2$ , then substituting that solution into the equation (10).

$$\begin{aligned} \begin{bmatrix} \mathbf{X}'_1\mathbf{X}_1 & \mathbf{X}'_1\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} &= [\mathbf{X}'_1\mathbf{y}] \\ \mathbf{X}'_1\mathbf{X}_1\mathbf{b}_1 + \mathbf{X}'_1\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}'_1\mathbf{y} \\ \mathbf{X}'_1\mathbf{X}_1\mathbf{b}_1 &= \mathbf{X}'_1\mathbf{y} - \mathbf{X}'_1\mathbf{X}_2\mathbf{b}_2 \\ \mathbf{b}_1 &= (\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1\mathbf{y} - (\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1\mathbf{X}_2\mathbf{b}_2 \\ &= (\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1(\mathbf{y} - \mathbf{X}_2\mathbf{b}_2) \end{aligned}$$

Multiplying out equation (10) gives

$$\begin{aligned} \begin{bmatrix} \mathbf{X}'_2\mathbf{X}_1 & \mathbf{X}'_2\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} &= [\mathbf{X}'_2\mathbf{y}] \\ \mathbf{X}'_2\mathbf{X}_1\mathbf{b}_1 + \mathbf{X}'_2\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}'_2\mathbf{y} \end{aligned}$$

Plugging in the solution for  $\mathbf{b}_1$  gives

$$\mathbf{X}'_2\mathbf{X}_1 \left( (\mathbf{X}'_1\mathbf{X}_1)^{-1}\mathbf{X}'_1(\mathbf{y} - \mathbf{X}_2\mathbf{b}_2) \right) + \mathbf{X}'_2\mathbf{X}_2\mathbf{b}_2 = \mathbf{X}'_2\mathbf{y}. \quad (11)$$

## Solving for $\mathbf{b}_2$

$$\mathbf{X}_2' \mathbf{X}_1 (\mathbf{X}_1' \mathbf{X}_1)^{-1} \mathbf{X}_1' (\mathbf{y} - \mathbf{X}_2 \mathbf{b}_2) + \mathbf{X}_2' \mathbf{X}_2 \mathbf{b}_2 = \mathbf{X}_2' \mathbf{y}.$$

The middle part of the first term is  $\mathbf{X}_1 (\mathbf{X}_1' \mathbf{X}_1)^{-1} \mathbf{X}_1'$ . This is the projection matrix  $\mathbf{P}_{X_1}$  from a regression of  $\mathbf{y}$  on  $\mathbf{X}_1$ .

$$\mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{y} - \mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{X}_2 \mathbf{b}_2 + \mathbf{X}_2' \mathbf{X}_2 \mathbf{b}_2 = \mathbf{X}_2' \mathbf{y}.$$

We can multiply by an identity matrix  $\mathbf{I}$  without changing anything

$$\mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{y} - \mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{X}_2 \mathbf{b}_2 + \mathbf{X}_2' \mathbf{I} \mathbf{X}_2 \mathbf{b}_2 = \mathbf{X}_2' \mathbf{I} \mathbf{y}.$$

$$\mathbf{X}_2' \mathbf{I} \mathbf{y} - \mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{y} = \mathbf{X}_2' \mathbf{I} \mathbf{X}_2 \mathbf{b}_2 - \mathbf{X}_2' \mathbf{P}_{X_1} \mathbf{X}_2 \mathbf{b}_2.$$

$$\mathbf{X}_2' (\mathbf{I} - \mathbf{P}_{X_1}) \mathbf{y} = \mathbf{X}_2' (\mathbf{I} - \mathbf{P}_{X_1}) \mathbf{X}_2 \mathbf{b}_2.$$

(12)

Now  $(\mathbf{I} - \mathbf{P}_{X_1})$  is the residual maker matrix  $\mathbf{M}_{X_1}$

$$\mathbf{X}_2' \mathbf{M}_{X_1} \mathbf{y} = \mathbf{X}_2' \mathbf{M}_{X_1} \mathbf{X}_2 \mathbf{b}_2.$$

Solving for  $\mathbf{b}_2$  gives

$$\mathbf{b}_2 = (\mathbf{X}_2' \mathbf{M}_{X_1} \mathbf{X}_2)^{-1} \mathbf{X}_2' \mathbf{M}_{X_1} \mathbf{y}.$$

## Solving for $\mathbf{b}_2$

$$\mathbf{b}_2 = (\mathbf{X}_2' \mathbf{M}_{\mathbf{X}_1} \mathbf{X}_2)^{-1} \mathbf{X}_2' \mathbf{M}_{\mathbf{X}_1} \mathbf{y}.$$

The residualizer matrix is symmetric and idempotent, such that  $\mathbf{M}_{\mathbf{X}_1} = \mathbf{M}_{\mathbf{X}_1} \mathbf{M}_{\mathbf{X}_1} = \mathbf{M}_{\mathbf{X}_1}' \mathbf{M}_{\mathbf{X}_1}$ .

$$\begin{aligned} \mathbf{b}_2 &= (\mathbf{X}_2' \mathbf{M}_{\mathbf{X}_1}' \mathbf{M}_{\mathbf{X}_1} \mathbf{X}_2)^{-1} \mathbf{X}_2' \mathbf{M}_{\mathbf{X}_1}' \mathbf{M}_{\mathbf{X}_1} \mathbf{y} \\ &= \left( (\mathbf{M}_{\mathbf{X}_1} \mathbf{X}_2)' (\mathbf{M}_{\mathbf{X}_1} \mathbf{X}_2) \right)^{-1} (\mathbf{M}_{\mathbf{X}_1} \mathbf{X}_2)' (\mathbf{M}_{\mathbf{X}_1} \mathbf{y}) \\ &= (\tilde{\mathbf{X}}_2' \tilde{\mathbf{X}}_2)^{-1} \tilde{\mathbf{X}}_2' \tilde{\mathbf{y}}. \end{aligned} \tag{13}$$

(14)

This is the OLS solution for  $\mathbf{b}_2$ , with  $\tilde{\mathbf{X}}_2$  instead of  $\mathbf{X}$  and  $\tilde{\mathbf{y}}$  instead of  $\mathbf{y}$ .

- ▶  $\tilde{\mathbf{X}}_2$  are residuals from a regression of  $\mathbf{X}_2$  on  $\mathbf{X}_1$
- ▶  $\tilde{\mathbf{y}}$  are residuals from a regression of  $\mathbf{y}$  on  $\mathbf{X}_1$

The solution of the regression coefficients  $\mathbf{b}_2$  in a regression that includes other regressors  $\mathbf{X}_1$  is the same as first regressing all of  $\mathbf{X}_2$  and  $\mathbf{y}$  on  $\mathbf{X}_1$ , then regressing the residuals from the  $\mathbf{y}$  regression on the residuals from the  $\mathbf{X}_2$  regression.

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