Econometricks: Short guides to econometrics

Trick 03: Review of Distribution Theory

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Content

Bivariate distributions

For observations of two discrete variables $y \in \{1, 2\}$ and $x \in \{1, 2, 3\}$, we can calculate

- the frequencies $n_{x,y}$,
-
-
-

freq. $n_{x,y}$	$y = 1$	$y = 2$	$f(x) = n_x/N$
$x = 1$	1	2	3/10
$x = 2$	1	2	3/10
$x = 3$	0	4	4/10
$f(y) = n_y/N$	2/10	8/10	1

Bivariate distributions

For observations of two discrete variables $y \in \{1, 2\}$ and $x \in \{1, 2, 3\}$, we can calculate

- the frequencies $n_{x,y}$,
- conditional distributions $f(y|x)$ and $f(x|y)$,
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$f(y) = n_y/N$	2/10	8/10	1

cond. distr. $f(y|x)$	$y = 1$	$y = 2$	$\sum_y$
$f(y|x = 1)$	1/3	2/3	1
$f(y|x = 2)$	1/3	2/3	1
$f(y|x = 3)$	0	1	1
$f(y|x = 1, x = 2, x = 3)$	1/5	4/5	1

cond. distr. $f(x|y)$	$f(x|y = 1)$	$f(x|y = 2)$	$f(x|y = 1, y = 2)$
$x = 1$	1/2	1/4	3/10
$x = 2$	1/2	1/4	3/10
$x = 3$	0	1/2	4/10
$\sum_x$	1	1	1

Bivariate distributions

For observations of two discrete variables $y \in \{1, 2\}$ and $x \in \{1, 2, 3\}$, we can calculate

- the frequencies $n_{x,y}$,
- conditional distributions $f(y|x)$ and $f(x|y)$,
- joint distributions $f(x, y)$, and
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cond. distr. $f(x|y)$	$f(x|y = 1)$	$f(x|y = 2)$	$f(x|y = 1, y = 2)$
$x = 1$	1/2	1/4	3/10
$x = 2$	1/2	1/4	3/10
$x = 3$	0	1/2	4/10
$\sum_x$	1	1	1

joint distr. $f(x, y)$	$f(x, y = 1)$	$f(x, y = 2)$
$f(x = 1, y)$	1/10	2/10
$f(x = 2, y)$	1/10	2/10
$f(x = 3, y)$	0	4/10

Bivariate distributions

For observations of two discrete variables $y \in \{1, 2\}$ and $x \in \{1, 2, 3\}$, we can calculate

- the frequencies $n_{x,y}$,
- conditional distributions $f(y|x)$ and $f(x|y)$,
- joint distributions $f(x, y)$, and
- marginal distributions $f_y(y)$ and $f_x(x)$.

freq. $n_{x,y}$	$y=1$	$y=2$	$f(x) = n_x/N$
$x=1$	1	2	3/10
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$f(y|x=1, x=2, x=3)$	1/5	4/5	1

cond. distr. $f(x|y)$	$f(x|y=1)$	$f(x|y=2)$	$f(x|y=1, y=2)$
$x=1$	1/2	1/4	3/10
$x=2$	1/2	1/4	3/10
$x=3$	0	1/2	4/10
$\sum_x$	1	1	1

joint distr. $f(x, y)$	$f(x, y=1)$	$f(x, y=2)$	marginal pr. $f_x(x)$
$f(x=1, y)$	1/10	2/10	3/10
$f(x=2, y)$	1/10	2/10	3/10
$f(x=3, y)$	0	4/10	4/10
marginal pr. $f_y(y)$	2/10	8/10	1

The joint density function

Two random variables X and Y have **joint density function**

- if x and y are discrete

$$f(x,y) = Prob(a \le x \le b, c \le y \le d) = \sum_{a \le x \le b} \sum_{c \le y \le d} f(x,y)$$

- if x and y are continuous

$$f(x,y) = Prob(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x,y)dxdy$$

Example

With $a = 1, b = 2, c = 2, d = 2$ and the following $f(x,y)$

joint distr. $f(x,y)$	$f(x, y=1)$	$f(x, y=2)$
$f(x=1,y)$	1/10	2/10
$f(x=2,y)$	1/10	2/10
$f(x=3,y)$	0	4/10

$Prob(1 \le x \le 2, 2 \le y \le 2) = f(y=2, x=1) + f(y=2, x=2) = 2/5.$

Bivariate probabilities

For values x and y of two discrete random variable X and Y, the **probability distribution**

$$f(x, y) = Prob(X = x, Y = y).$$

The axioms of probability require

$$f(x, y) \geq 0,$$

$$\sum_x \sum_y f(x, y) = 1.$$

If X and Y are continuous,

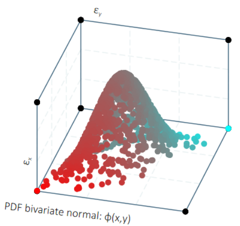
$$\int_x \int_y f(x, y) dxdy = 1.$$

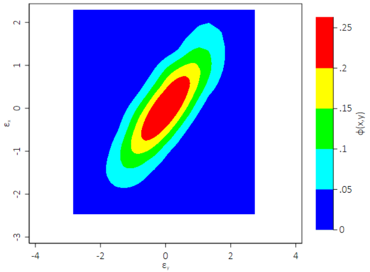
The bivariate normal distribution

The bivariate normal distribution is the joint distribution of two normally distributed variables. The density is

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-1/2[(\epsilon_x^2+\epsilon_y^2-2\rho\epsilon_x\epsilon_y)/(1-\rho^2)]}, \tag{1}$$

where $\epsilon_x = \frac{x-\mu_x}{\sigma_x}$, and $\epsilon_y = \frac{y-\mu_y}{\sigma_y}$.





The joint cumulative density function

The probability of a joint event of X and Y have **joint cumulative density function**

- if x and y are discrete

$$F(x, y) = Prob(X \leq x, Y \leq y) = \sum_{X \leq x} \sum_{Y \leq y} f(x, y)$$

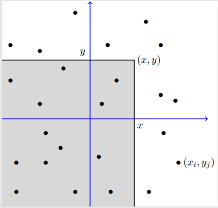
- if x and y are continuous

$$F(x, y) = Prob(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(t, s) ds dt$$

Example

With $x = 2, y = 2$ and the following $f(x, y)$

$f(x,y)$	$f(x, y = 1)$	$f(x, y = 2)$
$f(x = 1, y)$	1/10	2/10
$f(x = 2, y)$	1/10	2/10
$f(x = 3, y)$	0	4/10



$Prob(X \leq 2, Y \leq 2) = f(x = 1, y = 1) +$
$f(x = 2, y = 1) + f(x = 1, y = 2) + f(x = 2, y = 2) = 3/5.$

Bivariate probabilities

For values x and y of two discrete random variable X and Y, the **cumulative probability distribution**

$$F(x, y) = Prob(X \leq x, Y \leq y).$$

The axioms of probability require

$$0 \leq F(x, y) \leq 1,$$

$$F(\infty, \infty) = 1,$$

$$F(-\infty, y) = 0,$$

$$F(x, -\infty) = 0.$$

The marginal probabilities can be found from the joint cdf

$$f_x(x) = P(X \leq x) = Prob(X \leq x, Y \leq \infty) = F(x, \infty).$$

The marginal probability density

To obtain the marginal distributions $f_x(x)$ and $f_y(y)$ from the joint density $f(x,y)$, it is necessary to sum or integrate out the other variable. For example,

- if x and y are discrete
$$f_x(x) = \sum_y f(x,y),$$
- if x and y are continuous
$$f_x(x) = \int_y f(x,s)ds.$$

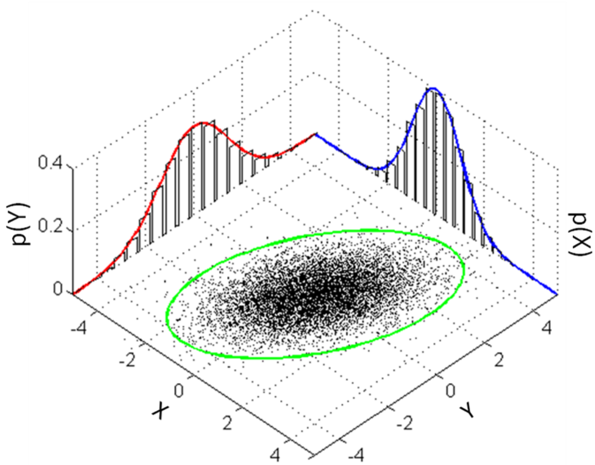
Example

$f(x,y)$	$f(x,y=1)$	$f(x,y=2)$	$f_x(x)$
$f(x=1,y)$	1/10	2/10	3/10
$f(x=2,y)$	1/10	2/10	3/10
$f(x=3,y)$	0	4/10	4/10
$f_y(y)$	2/10	8/10	1

$$f_x(x=1) = f(x=1,y=1) + f(x=1,y=2) = 3/10.$$

$$f_y(y=2) = f(x=1,y=2) + f(x=2,y=2) + f(x=3,y=2) = 4/5.$$

The bivariate normal distribution



Why do we care about marginal distributions?

Means, variances, and higher moments of the variables in a joint distribution are defined with respect to the marginal distributions.

- **Expectations**

  If x and y are discrete

  $$E[x] = \sum_x x f_x(x) = \sum_x x \left[\sum_y f(x,y) \right] = \sum_x \sum_y x f(x,y).$$

  If x and y are continuous

  $$E[x] = \int_x x f_x(x) = \int_x \int_y x f(x,y) dy dx.$$

- **Variances**

  $$Var[x] = \sum_x (x - E[x])^2 f_x(x) = \sum_x \sum_y (x - E[x])^2 f(x,y).$$

Covariance and correlation

For any function $g(x, y)$,

$$E[g(x,y)] = \begin{cases} \sum_x \sum_y g(x,y)f(x,y) & \textit{in the discrete case}, \\ \\ \int_x \int_y g(x,y)f(x,x)dydx & \textit{in the continuous case}. \end{cases} \tag{2}$$

The covariance of x and y is a special case:

$$\begin{aligned} Cov[x,y] &= E[(x-\mu_x)(y-\mu_y)] \\ &= E[xy] - \mu_x\mu_y = \sigma_{xy} \end{aligned}$$

If x and y are independent, then $f(x,y) = f_x(x)f_y(y)$ and

$$\begin{aligned} \sigma_{xy} &= \sum_x \sum_y f_x(x)f_y(y)(x-\mu_x)(y-\mu_y) \\ &= \sum_x (x-\mu_x)f_x(x) \sum_y (y-\mu_y)f_y(y) = E[x-\mu_x]E[y-\mu_y] = 0. \end{aligned}$$

- correlation $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$
- $\sigma_{xy} = 0$ does not imply independence (except for bivariate normal).

Independence: Pdf and cdf from marginal densities

- Two random variables are statistically independent if and only if their joint density is the product of the marginal densities:

$$f(x,y) = f_x(x)f_y(y) \Leftrightarrow x \textit{ and } y \textit{ are independent}.$$

- If (and only if) x and y are independent, then the marginal cdfs factors the cdf as well:

$$F(x,y) = F_x(x)F_y(y) = Prob(X \leq x, Y \leq y) = Prob(X \leq x)Prob(Y \leq y).$$

Example

$f(x,y)$	$f(x,y=1)$	$f(x,y=2)$	$f_x(x)$
$f(x=1,y)$	1/6	1/6	1/3
$f(x=2,y)$	1/6	1/6	1/3
$f(x=3,y)$	1/6	1/6	1/3
$f_y(y)$	1/2	1/2	1

$$f_x(x=3) \times f_y(y=2) = 1/3 \times 1/2 = 1/6.$$

$F(x,y)$	$F(x,y=1)$	$F(x,y=2)$
$F(x=1,y)$	1/6	2/6
$F(x=2,y)$	2/6	4/6
$F(x=3,y)$	3/6	1

$$P(x \leq 2)P(y \leq 2) = [f(x=2,y=1)+f(x=2,y=2)]\times$$

$$[f(x=1,y=2)+f(x=2,y=2)]$$

$$= [1/6+1/6][1/6+1/6] = 4/36 = 2/18.$$

The conditional density function

The **conditional distribution** over y for each value of x (and vice versa) has conditional densities

$$f(y|x) = \frac{f(x,y)}{f_x(x)} \quad f(x|y) = \frac{f(x,y)}{f_y(y)}.$$

The marginal distribution of x averages the probability of x given y over the distribution of all values of y $f_x(x) = E[f(x|y)f(y)]$. If x and y are independent, knowing the value of y does not provide any information about x, so $f_x(x) = f(x|y)$.

Example

cond. distr. $f(x\|y)$	$f(x\|y=1)$	$f(x\|y=2)$	$f(x\|y=1, y=2)$
$x=1$	1/2	1/4	3/10
$x=2$	1/2	1/4	3/10
$x=3$	0	1/2	4/10
$\sum_x$	1	1	1

joint distr. $f(x,y)$	$f(x, y=1)$	$f(x, y=2)$	marginal pr. $f_x(x)$
$f(x=1,y)$	1/10	2/10	3/10
$f(x=2,y)$	1/10	2/10	3/10
$f(x=3,y)$	0	4/10	4/10
marginal pr. $f_y(y)$	2/10	8/10	1

$$f(x=3|y=2) = \frac{f(x=3, y=2)}{f_y(y=2)} = 4/10 \times 10/8 = 1/2.$$

$$f_x(x=2) = E_y[f(x=2|y)f(y)] = f(x=2|y=1)f(y=1) + f(x=2|y=2)f(y=2)$$

$$= 1/2 \times 2/10 + 1/4 \times 8/10 = 1/10 + 2/10 = 3/10.$$

Conditional mean aka regression

A random variable may always be written as

$$\begin{aligned} y &= E[y|x] + (y - E[y|x]) \\ &= E[y|x] + \epsilon. \end{aligned}$$

Definition

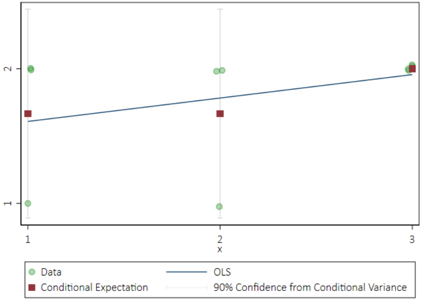
The regression of y on x is obtained from the **conditional mean**

$$E[y|x] = \begin{cases} \sum_y yf(y|x) & \textit{if } y \textit{ is discrete}, \\ \\ \int_y yf(y|x)dy & \textit{if } y \textit{ is continuous}. \end{cases} \tag{3}$$

Conditional mean aka regression

Predict y at values of x:

$$\sum_y yf(y|x=1) = 1 \times 1/3 + 2 \times 2/3 = 5/3.$$



Conditional variance

A **conditional variance** is the variance of the conditional distribution:

$$Var[y|x] = \begin{cases} \sum_y (y - E[y|x])^2 f(y|x) & \textit{if } y \textit{ is discrete}, \\ \\ \int_y (y - E[y|x])^2 f(y|x) dy, & \textit{if } y \textit{ is continuous}. \end{cases} \tag{4}$$

The computation can be simplified by using

$$Var[y|x] = E[y^2|x] - (E[y|x])^2 \geq 0. \tag{5}$$

Decomposition of variance $Var[y] = E_x[Var[y|x]] + Var_x[E[y|x]]$

- When we condition on x, the variance of y reduces on average.

  $Var[y] \geq E_x[Var[y|x]]$
- $E_x[Var[y|x]]$ is the average of variances **within** each x
- $Var_x[E[y|x]]$ is variance **between** y averages in each x.

Conditional expectations and variances

- $E[y|x=1] = 1.67$, $E[y|x=2] = 1.67$, and $E[y|x=3] = 2$
- $V[y|x=1] = 0.22$, $V[y|x=2] = 0.22$, and $V[y|x=3] = 0$

Example

$f(y\|x)$	$y=1$	$y=2$	
$f(y\|x=1)$	1/3	2/3	1
$f(y\|x=2)$	1/3	2/3	1
$f(y\|x=3)$	0	1	1

$$E[y|x=1] = 1/3 \times 1 + 2/3 \times 2 = 5/3$$

$$E[y|x=2] = 1/3 \times 1 + 2/3 \times 2 = 5/3$$

$$E[y|x=3] = 0 \times 1 + 1 \times 2 = 2$$

$f(x,y)$	$f(x, y=1)$	$f(x, y=2)$	$f_x(x)$
$f(x=1,y)$	1/10	2/10	3/10
$f(x=2,y)$	1/10	2/10	3/10
$f(x=3,y)$	0	4/10	4/10
$f_y(y)$	2/10	8/10	1

$$V[y|x=1] = 1^2 \times 1/3 + 2^2 \times 2/3 - (5/3)^2 = 2/9$$

$$V[y|x=2] = 1^2 \times 1/3 + 2^2 \times 2/3 - (5/3)^2 = 2/9$$

$$V[y|x=3] = 1^2 \times 0 + 2^2 \times 1 - 2^2 = 0$$

alternatively (requiring more differences)

$$V[y|x=1] = (1-5/3)^2 \times 1/3 + (2-5/3)^2 \times 2/3 = 2/9$$

Conditional expectations and variances

Average of variances **within** each x, $E[V[y|x]]$ is less or equal total variance $V[y]$.

Example

- Use the conditional mean to calculate $E[y]$:

$$E[y] = E_x[E[y|x]] = E[y|x=1]f(x=1) + E[y|x=2]f(x=2) + E[y|x=3]f(x=3)$$

$$= 5/3 \times 3/10 + 5/3 \times 3/10 + 2 \times 4/10 = 9/5.$$

$$E[y] = \sum_y f_y(y) = 1 \times 2/10 + 2 \times 8/10 = 9/5.$$

- Variation in y, $V[y|x=1] = 0.22$, $V[y|x=2] = 0.22$, and $V[y|x=3] = 0$ due to variation in x, is on average $E[V[y|x]] = 3/10 \times 2/9 + 3/10 \times 2/9 + 4/10 \times 0 = 2/15$.
- For each conditional mean $E[y|x=1] = 5/3$, $E[y|x=2] = 5/3$, and $E[y|x=3] = 2$, y varies with $V[E[y|x]] = E[(E[y|x])^2] - (E[y|x])^2 = 3/10 \times (5/3)^2 + 3/10 \times (5/3)^2 + 4/10 \times (2)^2 - (9/5)^2 = 2/75$.
- $E[V[y|x]] + V[E[y|x]] = V[y] = 2/75 + 2/15 = 4/25$.
  With degree of freedom correction $(n-1)$ (as reported in software):
  $E[V[y|x]] + V[E[y|x]] = V[y] = 2/75/(10-1) \times 10 + 2/15/(10-1) \times 10 = 8/45$.

Recall bivariate normal distribution is the joint distribution of two normally distributed variables. The density is

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-1/2[(\epsilon_x^2+\epsilon_y^2-2\rho\epsilon_x\epsilon_y)/(1-\rho^2)]}, \tag{6}$$

where $\epsilon_x = \frac{x-\mu_x}{\sigma_x}$, and $\epsilon_y = \frac{y-\mu_y}{\sigma_y}$.

The covariance is $\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$, where

- $-1 < \rho_{xy} < 1$ is the correlation between x and y
- $\mu_x, \sigma_x, \mu_y, \sigma_y$ are means and standard deviations of the marginal distributions of x or y

Properties of the bivariate normal

If x and y are bivariately normally distributed

$(x, y) \sim N_2[\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho_{xy}]$

- the marginal distributions are normal

$$f_x(x) = N[\mu_x, \sigma_x^2]$$

$$f_y(y) = N[\mu_y, \sigma_y^2]$$

- the conditional distributions are normal

$$f(y|x) = N[\alpha + \beta x, \sigma_y^2(1 - \rho^2)]$$

$$\alpha = \mu_y - \beta\mu_x; \beta = \frac{\sigma_{xy}}{\sigma_x^2}$$

- $f(x, y) = f_x(x)f_x(x)$ if $\rho_{xy} = 0$: x and y are independent if and only if they are uncorrelated

Useful rules

- $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$
- $E[ax + by + c] = aE[x] + bE[y] + c$
- $Var[ax+by+c] = a^2 Var[x]+b^2 Var[y]+2abCov[x, y] = Var[ax+by]$
- $Cov[ax + by, cx + dy] = acVar[x] + bdVar[y] + (ad + bc)Cov[x, y]$
- If X and Y are uncorrelated, then

  $Var[x + y] = Var[x - y] = Var[x] + Var[y].$

Useful rules

- Linearity

$$E[ax + by|z] = aE[x|z] + bE[y|z].$$

- Adam's Law / Law of Iterated Expectation

$$E[y] = E_x[E[y|x]]$$

- Adam's general Law / Law of Iterated Expectation

$$E[y|g_2(g_1(x))] = E[E[y|g_1(x)]|g_2(g_1(x))]$$

- Independence

  If x and y are independent, then

$$E[y] = E[y|x],$$

$$E[g_1(x)g_2(y)] = E[g_1(x)]E[g_2(y)].$$

Useful rules

- Taking out what is known

$$E[g_1(x)g_2(y)|x] = g_1(x)E[g_2(y)|x].$$

- Projection of y by $E[y|x]$, such that orthogonal to $h(x)$

$$E[(y - E[y|x])h(x)] = 0.$$

- Keeping just what is needed (y predictable from x needed, not residual)

$$E[xy] = E[xE[y|x]].$$

- Eve's Law (EVVE) / Law of Total Variance

$$Var[y] = E_x[Var[y|x]] + Var_x[E[y|x]]$$

- ECCE law / Law of Total Covariance

$$Cov[x, y] = E_z[Cov[y, x|z]] + Cov_z[E[x|z], E[y|z]]$$

Useful rules

- $Cov[x, y] = Cov_x[x, E[y|x]] = \int_x (x - E[x])\, E[y|x] f_x(x) dx.$
- If $E[y|x] = \alpha + \beta x$, then $\alpha = E[y] - \beta E[x]$ and $\beta = \frac{Cov[x,y]}{Var[x]}$
- Regression variance $Var_x[E[y|x]]$, because $E[y|x]$ varies with x
- Residual variance $E_x[Var[y|x]] = Var[y] - Var_x[E[y|x]]$, because y varies around the conditional mean
- Decomposition of variance $Var[y] = Var_x[E[y|x]] + E_x[Var[y|x]]$
- Coefficient of determination $= \frac{\text{regression variance}}{\text{total variance}}$
- If $E[y|x] = \alpha + \beta x$ and if $Var[y|x]$ is a constant, then

$$Var[y|x] = Var[y]\left(1 - Corr^2[y, x]\right) = \sigma_y^2 \left(1 - \sigma_{xy}^2\right)$$

References

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