Econometricks: Short guides to econometrics

Trick 04: The Least Squares Estimator

D. Rostam-Afschar

Content

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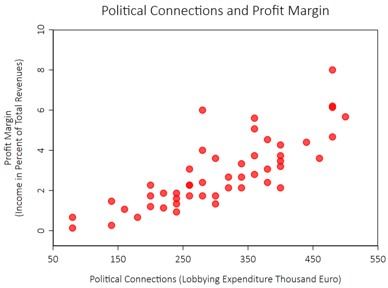
1. What is the Relationship between Two Variables?
2. The Econometric Model
3. Estimation with OLS
4. Properties of the OLS Estimator in the Small and in the Large
5. Politically Connected Firms: Causality or Correlation?

Political Connections and Firms

Firm profits increase with the degree of political connections



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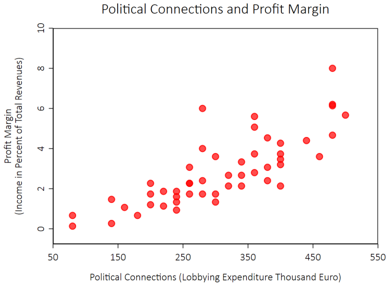

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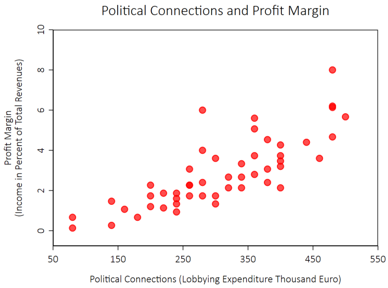

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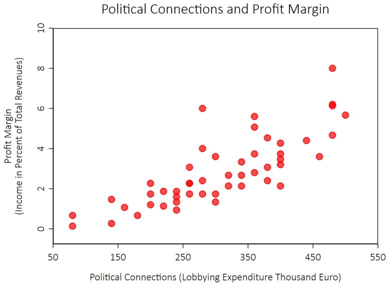

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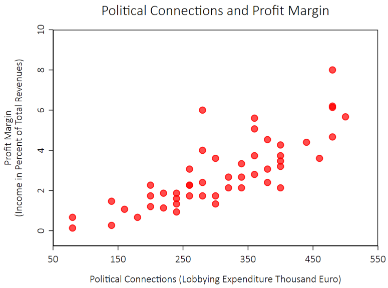

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- Show properties of the OLS estimator in small & large samples
- Apply Monte Carlo Simulations to assess properties of OLS

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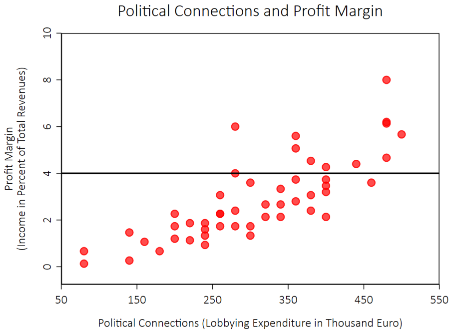
Specification of a Linear Regression

- dependent variable
  y_i = profits of firm i
- explanatory variables
  $x_{i1}, \dots, x_{iK}$ $k = 1, \dots K$
  political connections, other firm characteristics
- $x_{i0} = 1$ is a constant
- parameters to be estimated
  $\beta_0, \beta_1, \dots, \beta_K$ are $K + 1$
- u_i is called the error term



$$y_i = (\beta_0 = 4) + (\beta_1 = 0)x_{i1} + u_i.$$

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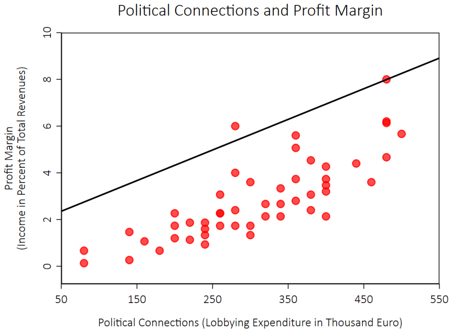

$$y_i = (\beta_0 = 2.36) + (\beta_1 = 0.01)x_{i1} + u_i.$$

How Were the Data Generated?

The *data generating process* is fully described by a set of assumptions.

The Five Assumptions of the Econometric Model

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- LRM5: Identifiability

Data Generating Process: Linearity

Definition

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- parameters β_k are constant across individual firms i and $j \neq i$.

Anscombe's Quartet

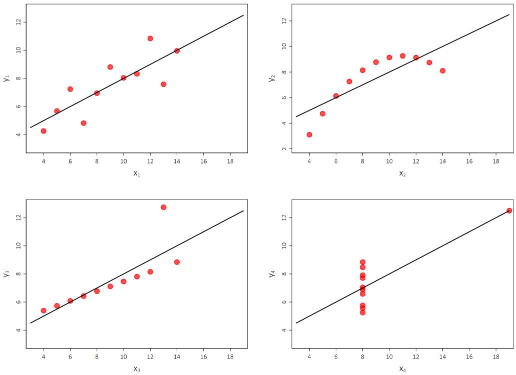

Figure 1: All four sets are identical when examined using linear statistics, but very different when graphed. Correlation between x and y is 0.816. Linear Regression y = 3.00 + 0.50x.

Data Generating Process: Random Sampling

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LRM2: Simple Random Sampling.

$$\{x_{i1}, \ldots, x_{iK}, y_i\}_{i=1}^{N} \quad \textit{i.i.d. (independent and identically distributed)}$$

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This assumption is guaranteed by simple random sampling provided there is no systematic non-response or truncation.

Density of Population and Truncated Sample

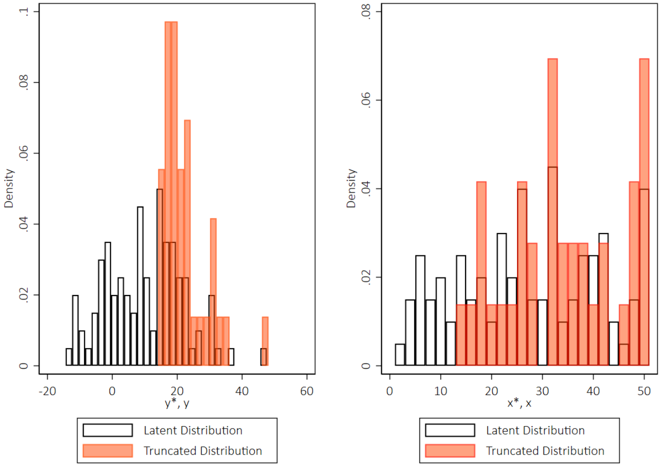

Figure 2: Distribution of a dependent variable and an independent variable truncated at $y^* = 15$.

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LRM3a or LRM3b imply LRM3c and LRM3d. LRM3c implies LRM3d.

(Conditional) Mean Independence

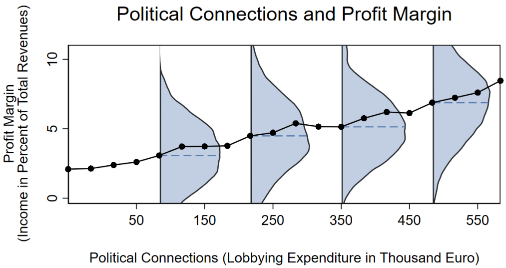

Figure 3: Distributions of the dependent variable conditional on values of an independent variable.

Weaker exogeneity assumption if interest only in, say, x_{i1}:

Conditional Mean Independence

$E(u_i | x_{i1}, x_{i2}, \ldots, x_{iK}) = E(u_i | x_{i2}, \ldots, x_{iK})$

Given the control variables $x_{i2}, \ldots, x_{iK}$, the mean of u_i does not depend on the variable of interest x_{i1}.

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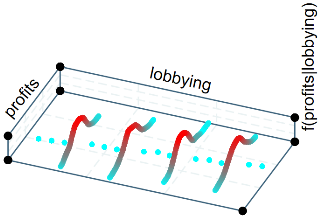
LRM4b allows the variance of the error term to depend on a function g of the explanatory variables.

Heteroskedasticity



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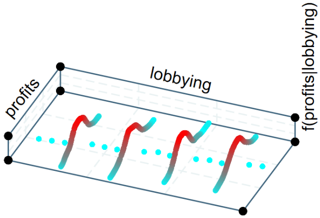

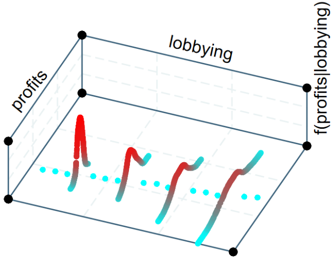

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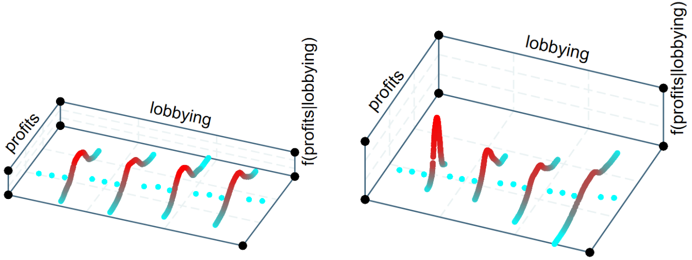

Figure 4: The simple regression model under homo- and heteroskedasticity. $Var(profits|lobbying, employees)$ increasing with *lobbying*.

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LRM5 means that every explanatory variable adds additional information.

The Identifying Variation from x_{ik}

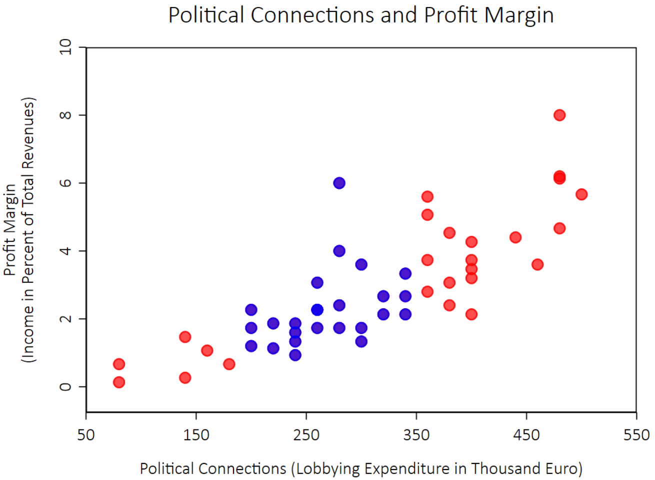

Figure 5: The number of red and blue dots is the same. Using which would you get a more accurate regression line?

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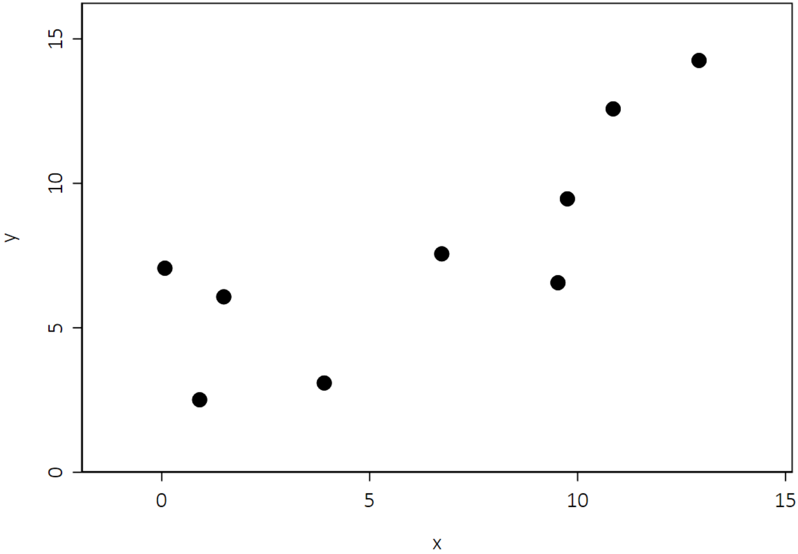
How to Describe the Relationship Best?



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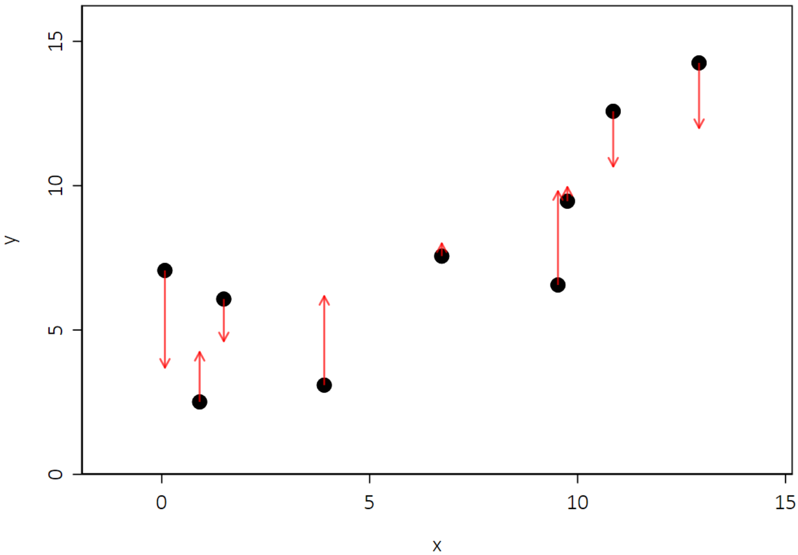

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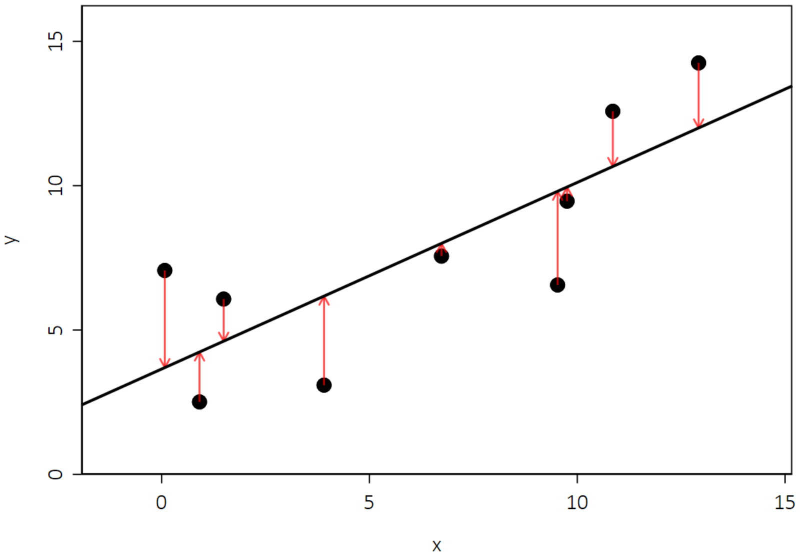

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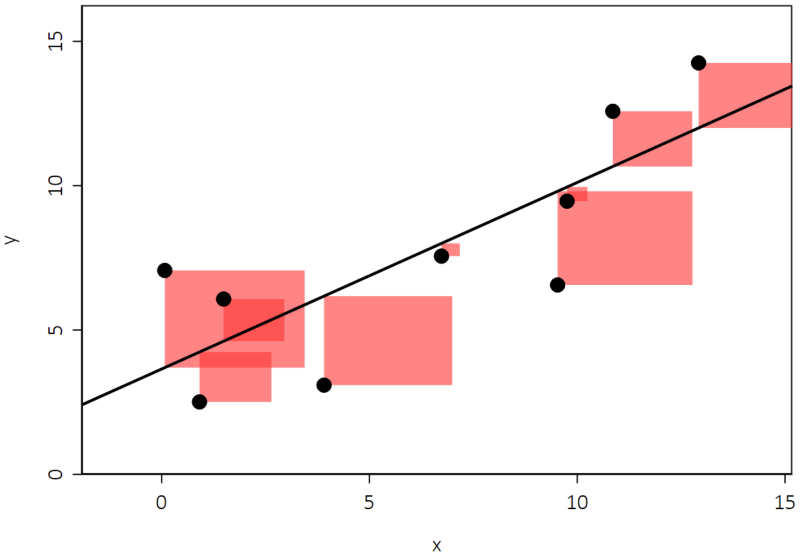

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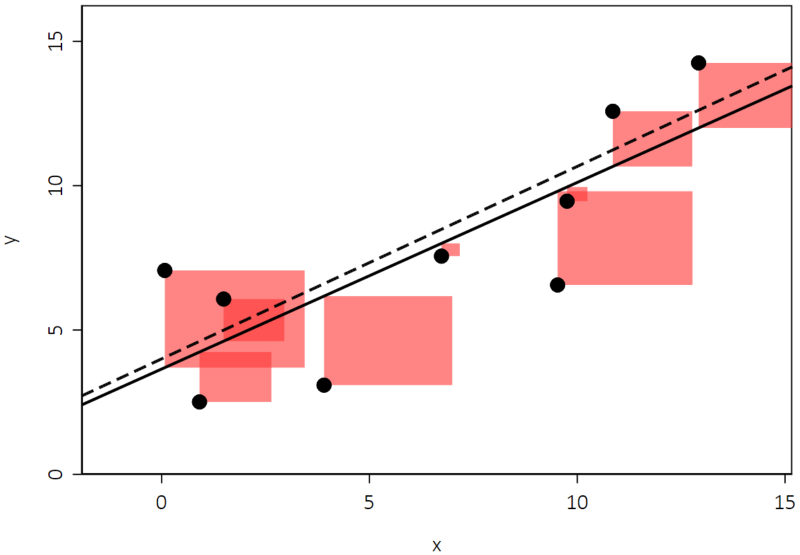

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Figure 6: Watercolor caricature of Legendre by Boilly (1820), the only existing portrait known.

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Figure 7: Portrait of Gauss by Jensen (1840).

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$$\hat{\beta}_1 = cov(x, y)/(s_x s_x) = R s_y / s_x,$$

where $R \equiv cov(x, y)/(s_x s_y)$ is **Pearson's correlation coefficient** with s_z denoting the standard deviation of z.

OLS estimator Measures Linear Correlation

Equivalently,

$$R = s_x/s_y\hat{\beta}_1 = \frac{\hat{\beta}_1\sum_{i=1}^{N}(x_{i1} - \bar{x})}{\sum_{i=1}^{N}(y_i - \bar{y})} = \frac{\sum_{i=1}^{N}(\hat{\beta}_1 x_{i1} - \hat{\beta}_1\bar{x})}{\sum_{i=1}^{N}(y_i - \bar{y})}.$$

Squaring gives

$$R^2 = \frac{\sum_{i=1}^{N}(\hat{y}_i - \bar{y})^2}{\sum_{i=1}^{N}(y_i - \bar{y})^2} = 1 - \frac{\sum_{i=1}^{N}\hat{u}_i^2}{\sum_{i=1}^{N}(y_i - \bar{y})^2}.$$

R^2 as measure of the **goodness of fit**:
The fit improves with the fraction of the sample variation in y that is explained by the x.

The Case with K Explanatory Variables

The more general case with K explanatory variables is

$$\underset{(K+1)\times 1}{\hat{\beta}} = \underset{(K+1)\times(K+1)}{(X'X)^{-1}} \underset{(K+1)\times N}{X'} \underset{N\times 1}{y}$$

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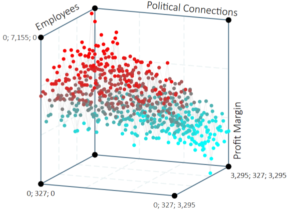

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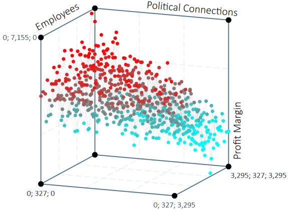

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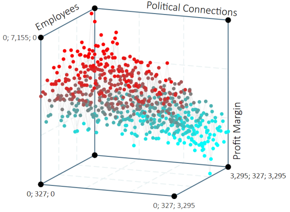

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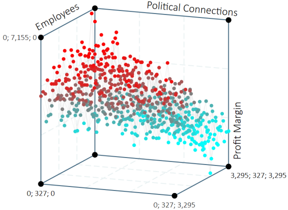

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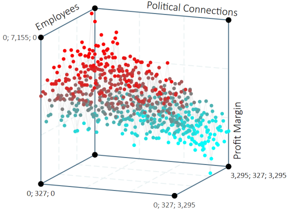

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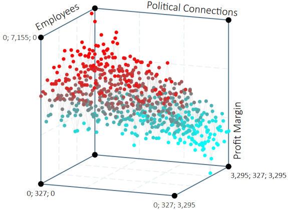

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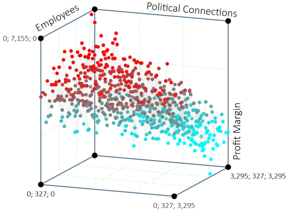

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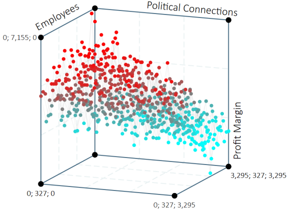

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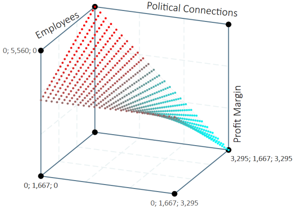

Figure 9: OLS surface visualized with GRAPH3D for Stata.

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Content

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Small Sample Properties

Figure 10: What is a small sample?
Source: Familien-Duell
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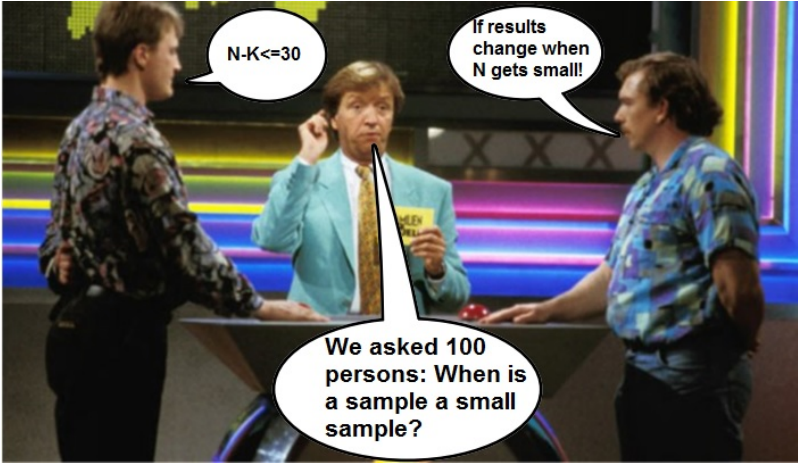

Figure 11: What is a small sample? (Wooldridge, 2009, p. 755): "But large sample approximations have been known to work well for sample sizes as small as $N = 20$." *Source:* Familien-Duell Grundy Light Entertainment.

Unbiasedness and Normality of $\hat{\beta}_k$

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  - **multicollinearity** (high (but not perfect) correlation between two or more of the independent variables).

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  where $E[u|X] = 0$ by assumptions of the model. $\square$

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$$= \sigma^2 (X'X)^{-1},$$

where we used the fact that $\hat{\beta} - \beta$ is just an affine transformation of u by the matrix $(X'X)^{-1}X'$. $\square$

Estimator for Variance

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Suppose the data generating process is

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Try it yourself...

How to Establish Asymptotic Properties of $\hat{\beta}_k$?

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Law of Large Numbers

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As N increases, the distribution of $\hat{\beta}_k$ becomes more tightly centered around β_k.

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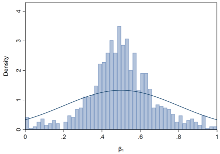

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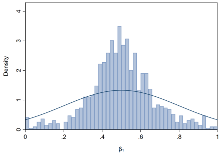

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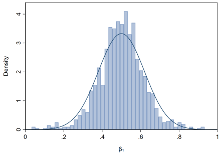

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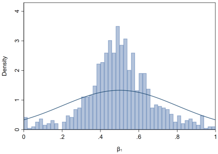

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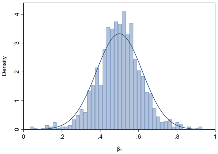

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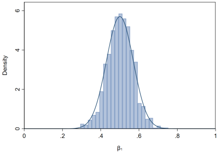

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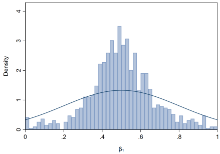

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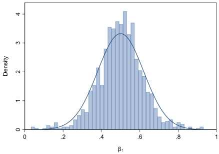

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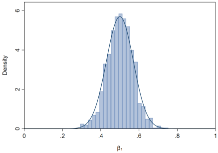

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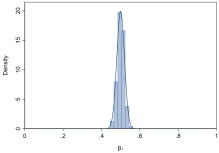

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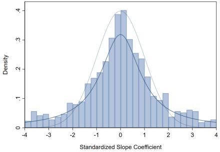

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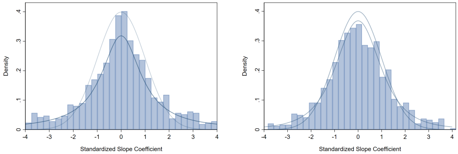

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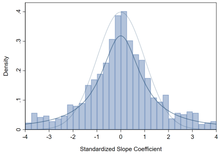

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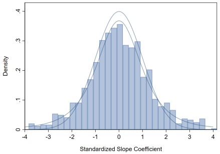

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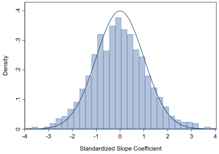

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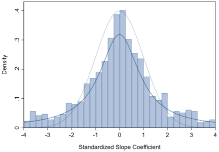

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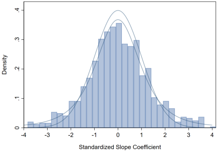

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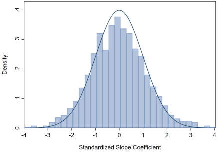

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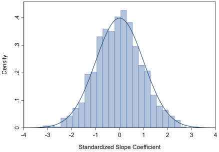

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Consistency, Asymptotically Normality

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Consistency, Asymptotically Normality

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Consistency, Asymptotically Normality

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Note: In practice we can almost never be sure that the errors are homoskedastic and should therefore always use robust standard errors.

Sketch of Proof for Asymptotic Properties

- The OLS estimator of $\hat{\beta}$ is consistent and asymptotic normal
  Estimator $\hat{\beta}$ can be written as: $\hat{\beta} = \left(\frac{1}{N}X'X\right)^{-1}\frac{1}{N}X'y =$
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  We can use the law of large numbers to establish that :
  $\frac{1}{N}\sum_{i=1}^{N} x_i x_i' \ \xrightarrow{p}\ \mathrm{E}[x_i x_i'] = \frac{Q_{xx}}{N}, \qquad \frac{1}{N}\sum_{i=1}^{N} x_i u_i \ \xrightarrow{p}\ \mathrm{E}[x_i u_i] = 0$

Sketch of Proof for Asymptotic Properties

- The OLS estimator of $\hat{\beta}$ is consistent and asymptotic normal
  Estimator $\hat{\beta}$ can be written as: $\hat{\beta} = \left(\frac{1}{N}X'X\right)^{-1}\frac{1}{N}X'y =$
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  Applying Slutsky's theorem again we'll have:
  $\sqrt{N}(\hat{\beta} - \beta) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i x_i'\right)^{-1}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i u_i\right) \ \xrightarrow{d}$
  $Q_{xx}^{-1} N \cdot \mathcal{N}(0, \sigma^2 \frac{Q_{xx}}{N}) = \mathcal{N}(0, \sigma^2 Q_{xx}^{-1} N)$ $\square$

OLS Properties in the Small and in the Large

Set of assumptions	(1)	(2)	(3)	(4)	(5)	(6)
LRM1: linearity		f u	l f	i l	l e	d
LRM2: simple random sampling		f u	l f	i l	l e	d
LRM5: identifiability		f u	l f	i l	l e	d
LRM4: error variance						
- LRM4a: homoskedastic	✓	✓	✓	×	×	×
- LRM4b: heteroskedastic	×	×	×	✓	✓	✓
LRM3: exogeneity						
- LRM3a: normality	✓	×	×	✓	×	×
- LRM3b: independent	✓	✓	×	×	×	×
- LRM3c: mean indep.	✓	✓	✓	✓	✓	×
- LRM3d: uncorrelated	✓	✓	✓	✓	✓	✓
Small sample properties of $\hat{\beta}$						
- unbiased	✓	✓	✓	✓	✓	×
- normally distributed	✓	×	×	✓	×	×
- efficient	✓	✓	✓	×	×	×
Large sample properties of $\hat{\beta}$						
- consistent	✓	✓	✓	✓	✓	✓
- approx. normal	✓	✓	✓	✓	✓	✓
- asymptotically efficient	✓	✓	✓	×	×	×

- *Notes:* ✓ = fulfilled, × = violated

Tests in Small Samples I

Assume LRM1,

Tests in Small Samples I

Assume LRM1, LRM2,

Assume LRM1, LRM2, LRM3a,

Tests in Small Samples I

Assume LRM1, LRM2, LRM3a, LRM4a,

Tests in Small Samples I

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For example, to perform a two-sided test of H_0 against the alternative hypotheses $H_A : \beta_k \neq q$ on the 5% significance level, we calculate the t-statistic and compare its absolute value to the 0.975-quantile of the t-distribution. With $N = 30$ and $K = 2$, H_0 is rejected if $|t| > 2.052$.

Tests in Small Samples II

A null hypotheses of the form $H_0 : r_{j1}\beta_1 + \ldots + r_{jK}\beta_K = q_j$, in matrix notation $H_0 : R\beta = q$, with J linear restrictions $j = 1 \ldots J$ is jointly tested with the F-**test**.

Tests in Small Samples II

A null hypotheses of the form $H_0 : r_{j1}\beta_1 + \ldots + r_{jK}\beta_K = q_j$, in matrix notation $H_0 : R\beta = q$, with J linear restrictions $j = 1 \ldots J$ is jointly tested with the F-**test**.

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If the null hypotheses is true, the F-statistic follows an F distribution with J numerator degrees of freedom and $N - K - 1$ denominator degrees of freedom:

$$F = \frac{\left(R\hat{\beta} - q\right)' \left[R\hat{V}(\hat{\beta}|X)R'\right]^{-1} \left(R\hat{\beta} - q\right)}{J} \sim F_{J,N-K-1}.$$

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For example, to perform a two-sided test of H_0 against the alternative hypotheses $H_A: r_{j1}\beta_1 + \ldots + r_{jK}\beta_K \neq q_j$ for all j at the 5% significance level, we calculate the F-statistic and compare it to the 0.95-quantile of the F-distribution.

With $N = 30, K = 2$ and $J = 2$, H_0 is rejected if $F > 3.35$. We cannot perform two-sided F-tests because the F distribution has one tail.

Tests in Small Samples III

Only under homoskedasticity (LRM4a), the F-statistic can also be computed as

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$$F = \frac{(R^2 - R^2_{\text{restricted}})/J}{(1 - R^2)/(N - K - 1)} \sim F_{J,N-K-1},$$

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Exclusionary restrictions of the form $H_0 : \beta_k = 0, \beta_m = 0, \ldots$ are a special case of $H_0 : r_{j1}\beta_1 + \ldots + r_{jK}\beta_K = q_j$ for all j. In this case, restricted least squares is simply estimated as a regression were the explanatory variables $k, m, \ldots$ are excluded, e.g. a regression with a constant only.

If the F distribution has degrees of freedom (df) 1 as the numerator df, and $N - K - 1$ as the denominator df, then it can be shown that $t^2 = F(1, N - K - 1)$.

Assuming LRM1,

Confidence Intervals in Small Samples

Assuming LRM1, LRM2,

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$$\left(\hat{\beta}_k - t_{(1-\alpha/2),(N-K-1)}\widehat{se}(\hat{\beta}_k), \hat{\beta}_k + t_{(1-\alpha/2),(N-K-1)}\widehat{se}(\hat{\beta}_k)\right),$$

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where $t_{(1-\alpha/2),(N-K-1)}$ is the $(1-\alpha/2)$ quantile of the t-distribution with $(N-K-1)$ degrees of freedom. For example, the 95% confidence interval with $N=30$ and $K=2$ is $\left(\hat{\beta}_k - 2.052\widehat{se}(\hat{\beta}_k), \hat{\beta}_k + 2.052\widehat{se}(\hat{\beta}_k)\right)$.

Recall: α is the maximum acceptable probability of a Type I error.

Null hypothesis (H_0)	is valid (Innocent)	is invalid (Guilty)
Reject H_0	**Type I ($\alpha = 0.05$) error**	Correct outcome
I think he is guilty!	False positive Convicted!	True positive Convicted!
Don't reject H_0	Correct outcome	**Type II (β) error**
I think he is innocent!	True negative Freed!	False negative Freed!

Assume LRM1,

Asymptotic Tests

Assume LRM1, LRM2,

Assume LRM1, LRM2, LRM3d,

Asymptotic Tests

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Assume LRM1, LRM2, LRM3d, LRM4a or LRM4b, and LRM5. A simple null hypotheses of the form $H_0 : \beta_k = q$ is tested with the z-**test**. If the null hypotheses is true, the z-statistic

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For example, to perform a two sided test of H_0 against the alternative hypotheses $H_A : \beta_k \neq q$ on the 5% significance level, we calculate the z-statistic and compare its absolute value to the 0.975-quantile of the standard normal distribution. H_0 is rejected if $|z| > 1.96$.

We talk about the Wald test later...

Confidence Intervals in Large Samples

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Confidence Intervals in Large Samples

Assuming LRM1, LRM2,

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Confidence Intervals in Large Samples

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For example, the 95% confidence interval is $\left(\hat{\beta}_k - 1.96\widehat{se}(\hat{\beta}_k), \hat{\beta}_k + 1.96\widehat{se}(\hat{\beta}_k)\right)$.

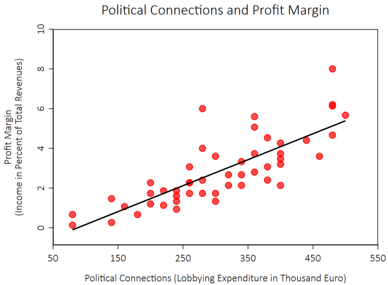
OLS Properties in the Small and in the Large

Set of assumptions	(1)	(2)	(3)	(4)	(5)	(6)
LRM1: linearity	fulfilled					
LRM2: simple random sampling	fulfilled					
LRM5: identifiability	fulfilled					
LRM4: error variance						
- LRM4a: homoskedastic	✓	✓	✓	×	×	×
- LRM4b: heteroskedastic	×	×	×	✓	✓	✓
LRM3: exogeneity						
- LRM3a: normality	✓	×	×	✓	×	×
- LRM3b: independent	✓	✓	×	×	×	×
- LRM3c: mean indep.	✓	✓	✓	✓	✓	×
- LRM3d: uncorrelated	✓	✓	✓	✓	✓	✓
Small sample properties of $\hat{\beta}$						
- unbiased	✓	✓	✓	✓	✓	×
- normally distributed	✓	×	×	✓	×	×
- efficient	✓	✓	✓	×	×	×
t-test, F-test	✓	×	×	×	×	×
Large sample properties of $\hat{\beta}$						
- consistent	✓	✓	✓	✓	✓	✓
- approx. normal	✓	✓	✓	✓	✓	✓
- asymptotically efficient	✓	✓	✓	×	×	×
z-test, Wald test	✓	✓	✓	✓*	✓*	✓*

- *Notes:* ✓ = fulfilled, × = violated, * = corrected standard errors.

Content

Arguments **For** Causality of Effect



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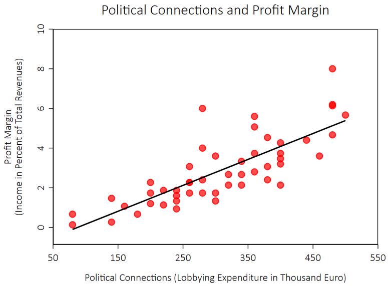

Econometric methods need to address concerns, including:

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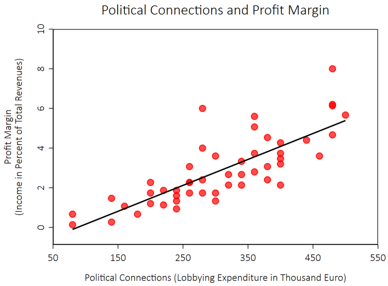

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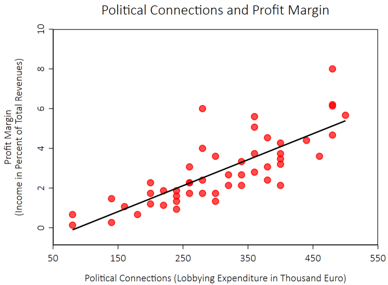

Econometric methods need to address concerns, including:

- **Misspecification:** Results robust to different functional forms
- **Errors-in-variables:** little concern with administrative data
- **External validity:** Similar effect found in independent studies.

- **Omitted variable bias:**
  e.g., business acumen
  $\rightarrow$ Panel data models

Arguments **Against** Causality of Effect

- **Omitted variable bias:**
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All of those concerns may be addressed with $\rightarrow$instrumental variable models. What would be a good instrument/experiment?

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