1 P, NPC, and NP

Decision Problem A problem whose answer is **yes** or **no**, if decision problem can be answered in poly time, then a optimal solution can be found with a few decision problem by binary search.

P Class of decision problems that can be solved in polynomial time.

NP Class of problem for which there is a “Yes certification”, that is, for a yes instance, there exists a short certification that the answer is yes. (Given a TSP path that less than K, we know that there exist a path less than K, however, given a path greater than K, we don't no the answer is no in poly time).

Co-NP Class of problem for which there is a no-certificate. That is for no-instances a short proof/certificate that the answer is no. (The complement of the problem is NP).

NP Complete Class of decision problem for which the following properties hold:

- The problem is in the class NP
- If you could solve the problem in polynomial time, then all the problem in NP can be solved in polynomial time.

Well Characterized Problem that has both YES and NO certificate.

Problems that are well-characterized typically have an associated min-max relation.

1.1 Linear Problem

Primal (P)

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Dual (D)

$$\begin{aligned} \text{Minimize} \quad & \mathbf{b}^y\mathbf{y} \\ \text{Subject to} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Theorem 1. *Weak Duality Theorem*

- $\mathbf{x}$ – *any feasible solution of (P)*
- $\mathbf{y}$ – *any feasible solution of (D)*

Then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$. *That is, primal bounded above by the dual*

Theorem 2. *Strong Duality Given Primal and Dual, exactly one of the following holds:*

1. *Neither (P) nor (D) has feasible solution*
2. *(P) is unbounded and (D) has no feasible solution*
3. *(D) is unbounded and (P) has has no feasible solution.*
4. *Both (P) and (D) has feasible solution, and* $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$

Proof. We will show that if (P) has an optimal $\mathbf{x}^*$, then (D) has optimal solution $\mathbf{y}^T$, and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$

Let $\gamma = \mathbf{c}^T\mathbf{x}^*$, then the following is feasible:

$$\begin{aligned} \mathbf{A}\mathbf{x} &\le \mathbf{b} \\ \mathbf{c}^T\mathbf{x} &\ge \gamma \\ \mathbf{x} &\ge \mathbf{0} \end{aligned}$$

The following is not feasible:

$$\begin{aligned} &\mathbf{A}\mathbf{x} \le \mathbf{b} \\ &\mathbf{c}^T\mathbf{x} \ge \gamma + \epsilon \equiv -\mathbf{c}^T\mathbf{x} \le -\gamma - \epsilon \\ &\mathbf{x} \ge \mathbf{0} \end{aligned}$$

Let

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ -\mathbf{c}^T \end{bmatrix}, \quad \hat{b_\epsilon} = \begin{bmatrix} \mathbf{b} \\ -\gamma - \epsilon \end{bmatrix}$$

Then $\hat{\mathbf{A}}\mathbf{x} = \mathbf{A} - \mathbf{c}^T\mathbf{x}$

We have the follow:

1. $\hat{\mathbf{A}}\mathbf{x} \le \hat{\mathbf{b}_\epsilon}$ is infeasible, so $\exists \hat{\mathbf{y}} \ge 0, \hat{\mathbf{A}^T}\hat{\mathbf{y}} \ge 0, \hat{\mathbf{b}_\epsilon}^T \hat{y} < 0$
2. $\hat{\mathbf{A}}\mathbf{x} \le \hat{\mathbf{b}_0}$ is feasible.

□

Lemma 1. *Farkve's Lemma*

- $\mathbf{A}$*, real matrix with* m *rows and* n *cols.*
- $\mathbf{b} \in \mathbb{R}^m$ *a vector*

Exactly one of the following occurs:

1.

$$\exists \mathbf{x} \in \mathbb{R}^n \quad s.\ t.\ \mathbf{A}\mathbf{x} = \mathbf{b} \quad and \quad \mathbf{x} \ge 0$$

2.

$$\exists \mathbf{y} \in \mathbb{R}^m \quad s.\ t.\ \mathbf{y}^T\mathbf{A} \ge 0 \quad and \quad \mathbf{y}^T\mathbf{b} < 0$$

Theorem 3. *Let* $\mathbf{A}, \mathbf{b}$ *be as in Farkvs Lemma, then exactly one of the following possibilities occurs:*

1.

$$\exists \mathbf{x} \ge \mathbf{0} \quad and \quad \mathbf{A}\mathbf{x} \le \mathbf{b}$$

2.

$$\exists \mathbf{y} \ge \mathbf{0}, \mathbf{y}^T\mathbf{A} \ge 0 \quad and \quad \mathbf{y}^T\mathbf{b} < 0$$

Proof. Using Farkvs Lemma for equations. Introduce slack variables in each inequality of $\mathbf{A}\mathbf{x} \le \mathbf{b}$, **turning them into equation**.

Let

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{I}_m \end{bmatrix}$$

Introduce slack variable $\mathbf{S} \ge \mathbf{0}$ such that

$$\bar{\mathbf{A}} \begin{bmatrix} \mathbf{x} \\ \mathbf{b} \end{bmatrix} = \mathbf{b}$$

by Farkv Lemma, we conclude that exactly one of the following holds:

1. holds
2. $$\mathbf{y}^T \begin{bmatrix} \mathbf{A} & \mathbf{I}_m \end{bmatrix} \geq 0 \implies \mathbf{y}^T \mathbf{A} \geq 0 \quad \text{and} \quad \mathbf{y} \geq \mathbf{00}$$

□

1.2 Matching and Vertex Cover

1.2.1 Definitions

Definition 1. *Incident $e = (u, v)$, e is incident to u and v, u and v are endpoints of e.*
Definition 2. *Bipartite Graph $G = (V, E)$, bipartition $V = A \cup B$*
Definition 3. *Matching $M \subseteq E$, every $v \in V$ is incident to at most one $e \in M$, size of matching: $|M|$*
Definition 4. *Vertex Cover $C \subseteq V$, $\forall e \in E$ incident to some vertices of C.*

Vertex Cover $\leq$ Size of Matching

1.2.2 Koning's Theorem

Theorem 4. *Konig's Theorem For any **bipartite** graph, maximum size of a matching is equal to the minimum size of vertex cover.*

Integer LP Problem

Label edges:

$$e_1, e_2, \ldots, e_m$$

Label vertices:

$$v_1, v_2, \ldots, v_n$$

Variable x_j corresponding to edge e_j

Integer LP for max matching

$$\max \quad \sum_{j=1}^{m} x_j e_m$$

$$\text{Subject to} \quad \sum_{e_j \text{incident to} v_i} x_j \leq 1 \quad \forall 1 \leq i \leq n (\text{each vertex is endpoint of at most one edge})$$

$$x_j \in \{0, 1\} \quad \text{relaxation} x_j \geq 0, x_j \text{is integer}$$

Incident matrix $\mathbf{A}$, $\mathbf{A}[i, j] = 1$ if v_i is endpoint of e_j, e_j incident to v_i

Integer LP for Min Vertex Cover Introduce y_i for vertex v_i:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} y_i \\ \text{subject to} \quad & y_a + y_b \geq 1 \text{for each edge } e_j = (v_a, v_b) \\ & \mathbf{y}_i \in \{0, 1\} \equiv \mathbf{y}_i \geq 0 \end{aligned}$$

Definition 5. *Totally Unimodular Matrix* $\mathbf{A}$ *is totally unimodular matrix if every square matrix of* $\mathbf{A}$ *obtained by deleting rows and cols from* $\mathbf{A}$ *has determinant* $0, 1$ *or* -1.
Theorem 5. $\mathbf{A}$ *be* $m \times n$ *totally unimodular,* $\mathbf{b} \in \mathbb{Z}^n$ *be a vector, then each vertex of the polyhedron*

$$P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$$

is an integer. (We can plot the constraints with lines, and those lines form a polyhedron, we are taking about the vertices of this polyhedron.)
Theorem 6. *Incident matrix of bipartite graph is totally unimodular.*
Lemma 2. *Let* A *be a totally unimodular matrix, and consider the matrix* $\bar{A}$ *obtained from* A *by appending a unit vector as the last row or column, then* $\bar{A}$ *is totally unimodular.*
Theorem 7. *Hull's Theorem Let* $G = \{V, E\}$ *be a bipartite with bipartition* X *and* Y. G *has a matching that covers all vertices in* X *if and only if for every* $T \in X$, $N(T)| \geq |T|$

$N(T) = \{y \in Y : \exists t \in T with (y, t) \in E\}$