Implementing non-ideal HWPs

Adri Duivenvoorden

September 9, 2021

Goal This document aims to provide extra information about the method for beam convolution with non-ideal HWPs described in [1]. We will focus on the code implementation.

General note on "spin" formalism versus E-, B-mode formalism In essence, the harmonic convolution method evaluates the following expression to produce time-ordered data:

$$d_t = \sum_{s=-s_{\max}}^{s_{\max}} \Big[\sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} {}_s f_{\ell m} \, {}_s Y_{\ell m}(\theta_t, \phi_t) \Big] \mathrm{e}^{-\mathrm{i} s \psi_t} \, . \tag{1}$$

The ${}_s f_{\ell m}$ coefficients obey the "reality condition":

$$({}_s f_{\ell m})^* = {}_{-s} f_{\ell -m} (-1)^{s+m} \, . \tag{2}$$

Equivalently, they are the spin-weighed SH coefficients of spin-weighted fields that obey $({}_s f)^* (\theta, \phi) = {}_{-s} f(\theta, \phi)$. This follows from the requirement that d_t is real.

In cases with a spin-0 beam and sky we have:

$${}_s f_{\ell m} = b_{\ell s} \, a_{\ell m} \, . \tag{3}$$

In this case the reality condition is clearly met:

$$\begin{aligned} ({}_s f_{\ell m})^* &= b_{\ell -s} \, a_{\ell -m} (-1)^{s+m} \, , &(4)\\ &= {}_{-s} f_{\ell m} (-1)^{s+m} \, . &(5) \end{aligned}$$

In cases with a spin-2 beam and sky we have:

$${}_s f_{\ell m} = \frac{1}{2} \left({}_{-2} b_{\ell s} \, {}_2 a_{\ell m} + {}_2 b_{\ell s} \, {}_{-2} a_{\ell m} \right) \, . \tag{6}$$

In this case the reality condition still holds:

$$({}_sf_{\ell m})^* = \frac{1}{2}\left({}_2b_{\ell-s}\,{}_{-2}a_{\ell-m} + {}_{-2}b_{\ell-s}\,{}_2a_{\ell-m}\right)(-1)^{s+m}\,, \tag{7}$$

$$= {}_{-s}f_{\ell m}(-1)^{s+m}\,. \tag{8}$$

But note that the reality condition does not hold for the individual terms.

By making use of the following relations:

$${}_{-2}x_{\ell m} = -(x^E_{\ell m} - \mathrm{i}x^B_{\ell m})\,, \tag{9}$$

$${}_2x_{\ell m} = -(x^E_{\ell m} + \mathrm{i}x^B_{\ell m})\,, \tag{10}$$

we can rewrite the spin-2 case in terms of the E- and B-mode coefficients of the beam and sky:

$${}_sf_{\ell m} = \frac{1}{2}\left({}_{-2}b_{\ell s}\,{}_2a_{\ell m} + {}_2b_{\ell s}\,{}_{-2}a_{\ell m}\right)\,, \tag{11}$$

$$= b^E_{\ell s}\,a^E_{\ell m} + b^B_{\ell s}\,a^B_{\ell m}\,. \tag{12}$$

Note that the reality condition now holds for both terms independently since we have $\left(x^{E/B}_{\ell m}\right)^* = x^{E/B}_{\ell -m}(-1)^m$.

Note on the implementation in `beamconv` In `beamconv` Eq. (1) is evaluated in two different ways depending on whether the ${}_sf_{\ell m}$ coefficients consist of spin-0 beams/skies or spin-2 beams/skies.

For the first case, i.e. when ${}_sf_{\ell m} = b_{\ell s}\,a_{\ell m}$, we make use of the fact that Eq. (1) can be written as:

$$d_t = {}_0f(\theta_t,\phi_t) + \sum_{s>0}^{s_{\max}} {}_sf(\theta_t,\phi_t)\mathrm{e}^{-\mathrm{i}s\psi_t} + {}_{-s}f(\theta_t,\phi_t)\mathrm{e}^{\mathrm{i}s\psi_t}\,, \tag{13}$$

$$= {}_0f(\theta_t,\phi_t) + \sum_{s>0}^{s_{\max}} \mathrm{Re}\left[{}_sf(\theta_t,\phi_t)\mathrm{e}^{-\mathrm{i}s\psi_t}\right]\,, \tag{14}$$

where:

$${}_sf(\theta_t,\phi_t) = \sum_{\ell=0}^{\ell_{\max}}\sum_{m=-\ell}^{\ell} {}_sf_{\ell m}\,{}_sY_{\ell m}(\theta_t,\phi_t)\,. \tag{15}$$

Eq. (15) is calculated for $s \geq 0$; the resulting (complex) maps are plugged into Eq. (14) where they are modulated by the complex exponential $\exp(-\mathrm{i}s\psi_t)$ before the real part is taken.

For the second case, i.e. when ${}_sf_{\ell m} = \left({}_{-2}b_{\ell s}\, {}_2a_{\ell m} + {}_2b_{\ell s}\, {}_{-2}a_{\ell m}\right)/2$, `beamconv` evaluates Eq. (1) by instead using ${}_sf'_{\ell m} = {}_{-2}b_{\ell s}\, {}_2a_{\ell m}$. These are the coefficients of a field ${}_sf'(\theta,\phi)$ that does not obey the reality condition: $\left({}_sf'\right)^*(\theta,\phi) \neq {}_{-s}f'(\theta,\phi)$. As a result, the maps in Eq. (15) have to be computed for $-s_{\max} \leq s \leq s_{\max}$. They are inserted into Eq. 13 without making use of the trick in Eq. 14. The result is still real, so d_t is still real valued.

In retrospect this seems an overly complicated method because it does not match well with `libsharp`, which does not support SH transforms for fields that do not obey the reality condition. As a result, `beamconv` has to do some relatively opaque operations to achieve the transforms for these fields.

Data model with non-ideal HWPs The data model is given by Eq. (11) in [1].

$$
\begin{aligned}
d_t = \int \mathrm{d}\nu F(\nu) \sum_{\ell,m,s} \Big\{ & b_{\ell s}^{\widetilde{I}_{\mathrm{i}}^{(0)}}(\nu,\alpha_t) a_{\ell m}^{I}(\nu) + b_{\ell s}^{\widetilde{V}_{\mathrm{i}}^{(0)}}(\nu,\alpha_t) a_{\ell m}^{V}(\nu) \\
& + \frac{1}{2}\Big[{}_{-2}b_{\ell s}^{\widetilde{P}_{\mathrm{i}}^{(0)}}(\nu,\alpha_t)\, {}_2a_{\ell m}^{P}(\nu) + {}_2b_{\ell s}^{\widetilde{P}_{\mathrm{i}}^{(0)}}(\nu,\alpha_t)\, {}_{-2}a_{\ell m}^{P}(\nu) \Big] \Big\} \\
& \times \sqrt{\frac{4\pi}{2\ell+1}} \mathrm{e}^{-\mathrm{i}s\psi_t}\, {}_sY_{\ell m}(\theta_t,\phi_t) + n_t \,,
\end{aligned}
\tag{16}
$$

Here, α_t is the time-dependent HWP angle.

Frequency passband We start with the integral over frequency and the frequency passband $F(\nu)$:

$$
d_t = \int \mathrm{d}\nu F(\nu) \Big(\dots \Big)_t(\nu) + n_t \,. \tag{17}
$$

In the code this integral is replaced by a sum over approximately 10 frequency subbands:

$$
d_t = \sum_{\nu=\nu_0}^{\nu_N} \Delta\nu F(\nu) \Big(\dots \Big)_t(\nu) + n_t \,. \tag{18}
$$

To produce the final time-ordered data, we thus have to compute the convolution for each frequency subband and add the result to a single array of time-ordered data. In the current version of `beamconv` the time-ordered data for each subband is also mapped onto the sky before all subbands are combined. This is valid because our map-making scheme is linear, but it is wasteful and should ideally be avoided.

Convolution of the Stokes I sky Let us focus on the part of the data model describing the coupling to the Stokes I component of the sky:

$$d_t = \int \mathrm{d}\nu F(\nu) \sum_{\ell,m,s} b_{\ell s}^{\widetilde{I}_\mathrm{i}^{(0)}}(\nu, \alpha_t) a_{\ell m}^I(\nu) \sqrt{\frac{4\pi}{2\ell+1}} \mathrm{e}^{-\mathrm{i}s\psi_t} {}_sY_{\ell m}(\theta_t, \phi_t)\,. \tag{19}$$

The SH coefficients of the beam $b_{\ell s}^{\widetilde{I}_\mathrm{i}^{(0)}}(\nu, \alpha_t)$ are given in Eq. (15) of [1]. For the present discussion the detailed expressions are not important (they are derived later in this document), the important part is that the coefficients can be written as:

$$\begin{aligned} b_{\ell s}^{\widetilde{I}_\mathrm{i}^{(0)}}(\nu, \alpha_t) &= b_{\ell s}^{\widetilde{I}_\mathrm{b}, II}(\nu) + b_{\ell s}^{\widetilde{V}_\mathrm{b}, VI}(\nu) \\ &\quad + b_{\ell s}^{\mathrm{Re}(\widetilde{P}_\mathrm{b}, P^*I)}(\nu) \cos(2\alpha_t) + b_{\ell s}^{\mathrm{Im}(\widetilde{P}_\mathrm{b}, P^*I)}(\nu) \sin(2\alpha_t)\,. \end{aligned} \tag{20}$$

The first two elements on the right hand side of the equation describe the coupling between the Stokes I and V beam of the instrument and the II and VI elements of the HWP Mueller matrix, respectively. These are coefficients of real-valued spin-0 fields, so the usual symmetry relations hold. For efficiency, the combination $b_{\ell s}^{\widetilde{I}_\mathrm{b}, II}(\nu) + b_{\ell s}^{\widetilde{V}_\mathrm{b}, VI}(\nu)$ can be treated as a single set of SH coefficients during the convolution.

The last two elements on the right hand side of the equation describe the coupling between the $\widetilde{P}$ beam and the P^*I element of the HWP. These are both complex fields, but the coupling can be separated into the real and imaginary parts of the coupling. The $b_{\ell s}^{\mathrm{Re}(\widetilde{P}_\mathrm{b}, P^*I)}$ and $b_{\ell s}^{\mathrm{Im}(\widetilde{P}_\mathrm{b}, P^*I)}$ coefficients are both SH coefficients of real-valued spin-0 fields, so they can be treated in the usual case. The modulation by α_t can be applied after the time-ordered data is generated. For example, the contribution to the total time-ordered data from the $\mathrm{Re}(\widetilde{P}_\mathrm{b}, P^*I)$ term would be given by (ignoring the integral over ν for simplicity):

$$d_t^{\mathrm{Re}(\widetilde{P}_\mathrm{b}, P^*I)} = \left[\sum_{\ell,m,s} b_{\ell s}^{\mathrm{Re}(\widetilde{P}_\mathrm{b}, P^*I)} a_{\ell m}^I \sqrt{\frac{4\pi}{2\ell+1}} \mathrm{e}^{-\mathrm{i}s\psi_t} {}_sY_{\ell m}(\theta_t, \phi_t) \right] \cos(4\alpha_t)\,. \tag{21}$$

Note that that the term inside the brackets does not depend on the HWP angle α. The $\cos(4\alpha_t)$ modulation can be done after the time-ordered data inside the square brackets have been generated.

Convolution of the Stokes V sky This is completely analogous to the case for the Stokes I sky (just swapping I for V). The data model is given by:

$$d_t = \int \mathrm{d}\nu F(\nu) \sum_{\ell,m,s} b_{\ell s}^{\widetilde{V}_\mathrm{i}^{(0)}}(\nu,\alpha_t) a_{\ell m}^{V}(\nu) \sqrt{\frac{4\pi}{2\ell+1}} \mathrm{e}^{-\mathrm{i}s\psi_t} {}_sY_{\ell m}(\theta_t,\phi_t)\,. \tag{22}$$

with coefficients that can be written as:

$$\begin{aligned} b_{\ell s}^{\widetilde{V}_\mathrm{i}^{(0)}}(\nu,\alpha_t) &= b_{\ell s}^{\widetilde{I}_\mathrm{b},IV}(\nu) + b_{\ell s}^{\widetilde{V}_\mathrm{b},VV}(\nu) \\ &\quad + b_{\ell s}^{\mathrm{Re}(\widetilde{P}_\mathrm{b},P^*V)}(\nu)\cos(2\alpha_t) + b_{\ell s}^{\mathrm{Im}(\widetilde{P}_\mathrm{b},P^*V)}(\nu)\sin(2\alpha_t)\,. \end{aligned} \tag{23}$$

Convolution of the Stokes Q, U sky The part of the data model that describes the coupling to the Stokes Q and U part of the sky is given by:

$$\begin{aligned} d_t = \int \mathrm{d}\nu F(\nu) \sum_{\ell,m,s} \frac{1}{2}\Big[{}_{-2}b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t) {}_2a_{\ell m}^{P}(\nu) + {}_2b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t) {}_{-2}a_{\ell m}^{P}(\nu) \Big]\Big\} \\ \times \sqrt{\frac{4\pi}{2\ell+1}} \mathrm{e}^{-\mathrm{i}s\psi_t} {}_sY_{\ell m}(\theta_t,\phi_t)\,. \end{aligned} \tag{24}$$

The SH coefficients of the beam ${}_2b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t)$ are given by Eq. (16) in [1]. We can divide the coefficients into 4 parts again:

$$\begin{aligned} {}_2b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t) &= {}_2b_{\ell s}^{\widetilde{I}_\mathrm{b},IP}(\nu)\mathrm{e}^{-2\mathrm{i}\alpha_t} + {}_2b_{\ell s}^{\widetilde{V}_\mathrm{b},VP}(\nu)\mathrm{e}^{-2\mathrm{i}\alpha_t} \\ &\quad + {}_2b_{\ell s}^{\widetilde{P^*}_\mathrm{b},P^*P}(\nu)\mathrm{e}^{-4\mathrm{i}\alpha_t} + {}_2b_{\ell s}^{\widetilde{P}_\mathrm{b},PP}(\nu)\,. \end{aligned} \tag{25}$$

The coefficients of the ${}_{-2}b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t)$ are similar:

$$\begin{aligned} {}_{-2}b_{\ell s}^{\widetilde{P}_\mathrm{i}^{(0)}}(\nu,\alpha_t) &= {}_{-2}b_{\ell s}^{\widetilde{I}_\mathrm{b},IP}(\nu)\mathrm{e}^{2\mathrm{i}\alpha_t} + {}_{-2}b_{\ell s}^{\widetilde{V}_\mathrm{b},VP}(\nu)\mathrm{e}^{2\mathrm{i}\alpha_t} \\ &\quad + {}_{-2}b_{\ell s}^{\widetilde{P}_\mathrm{b},PP^*}(\nu)\mathrm{e}^{4\mathrm{i}\alpha_t} + {}_{-2}b_{\ell s}^{\widetilde{P^*}_\mathrm{b},P^*P^*}(\nu)\,. \end{aligned} \tag{26}$$

If we combine with the ${}_{\pm 2}a_{\ell m}^{P}$ coefficients, we thus get 4 different terms:

$${}_sf_{\ell m}^{\widetilde{I}_\mathrm{b},IP} = \frac{1}{2}\left({}_{-2}b_{\ell s}^{\widetilde{I}_\mathrm{b},IP}\,{}_2a_{\ell m}^{P}\mathrm{e}^{2\mathrm{i}\alpha_t} + {}_2b_{\ell s}^{\widetilde{I}_\mathrm{b},IP}\,{}_{-2}a_{\ell m}^{P}\mathrm{e}^{-2\mathrm{i}\alpha_t} \right)\,, \tag{27}$$

$${}_sf_{\ell m}^{\widetilde{V}_\mathrm{b},VP} = \frac{1}{2}\left({}_{-2}b_{\ell s}^{\widetilde{V}_\mathrm{b},VP}\,{}_2a_{\ell m}^{P}\mathrm{e}^{2\mathrm{i}\alpha_t} + {}_2b_{\ell s}^{\widetilde{V}_\mathrm{b},VP}\,{}_{-2}a_{\ell m}^{P}\mathrm{e}^{-2\mathrm{i}\alpha_t} \right)\,, \tag{28}$$

$${}_sf_{\ell m}^{\widetilde{P}_\mathrm{b},PP^*} = \frac{1}{2}\left({}_{-2}b_{\ell s}^{\widetilde{P}_\mathrm{b},PP^*}\,{}_2a_{\ell m}^{P}\mathrm{e}^{4\mathrm{i}\alpha_t} + {}_2b_{\ell s}^{\widetilde{P^*}_\mathrm{b},P^*P}\,{}_{-2}a_{\ell m}^{P}\mathrm{e}^{-4\mathrm{i}\alpha_t} \right)\,, \tag{29}$$

$${}_sf_{\ell m}^{\widetilde{P}_\mathrm{b},PP} = \frac{1}{2}\left({}_{-2}b_{\ell s}^{\widetilde{P^*}_\mathrm{b},P^*P^*}\,{}_2a_{\ell m}^{P} + {}_2b_{\ell s}^{\widetilde{P}_\mathrm{b},PP}\,{}_{-2}a_{\ell m}^{P} \right)\,. \tag{30}$$

Note that the coefficients that describe the coupling between the I beam and the HWP and the V beam with the HWP can treated as a single set of coefficients: ${}_{-2}b_{\ell s}^{\widetilde{I}_{\mathrm{b}},IP} + {}_{-2}b_{\ell s}^{\widetilde{V}_{\mathrm{b}},VP}$ and similar for the +2 versions.

The ${}_s f_{\ell m}$ coefficients in Eq. (27 - 30) all obey the reality condition and thus they can be treated in a similar way as the usual case for Q, U beams and skies. The last line, Eq. (30), is exactly the same as the usual case. For the first three lines we have to take care of the α_t terms. As explained earlier, in `beamconv` these terms are handled by only considering the first term, e.g. ${}_{-2}b_{\ell s}^{\widetilde{I}_{\mathrm{b}},IP}\,{}_2a_{\ell m}^{P}$ and treating the resulting complex field that does not obey the reality condition. In this case it is straightforward to apply the complex exponential after the complex time-ordered data have been generated.

For the case where the E- and B-mode coefficients of the beam and sky are used Eq. (27 - 30) can be rewritten as follows:

$$
\begin{aligned}
{}_s f_{\ell m}^{\widetilde{I}_{\mathrm{b}},IP} &= b_{\ell s}^{E,\widetilde{I}_{\mathrm{b}},IP} a_{\ell m}^{E} \cos(2\alpha_t) + b_{\ell s}^{B,\widetilde{I}_{\mathrm{b}},IP} a_{\ell m}^{B} \cos(2\alpha_t) \\
&\quad + b_{\ell s}^{E,\widetilde{I}_{\mathrm{b}},IP} a_{\ell m}^{B} \sin(2\alpha_t) + b_{\ell s}^{B,\widetilde{I}_{\mathrm{b}},IP} a_{\ell m}^{E} \sin(2\alpha_t)\,,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
{}_s f_{\ell m}^{\widetilde{V}_{\mathrm{b}},VP} &= b_{\ell s}^{E,\widetilde{V}_{\mathrm{b}},VP} a_{\ell m}^{E} \cos(2\alpha_t) + b_{\ell s}^{B,\widetilde{V}_{\mathrm{b}},VP} a_{\ell m}^{B} \cos(2\alpha_t) \\
&\quad + b_{\ell s}^{E,\widetilde{V}_{\mathrm{b}},VP} a_{\ell m}^{B} \sin(2\alpha_t) + b_{\ell s}^{B,\widetilde{V}_{\mathrm{b}},VP} a_{\ell m}^{E} \sin(2\alpha_t)\,,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
{}_s f_{\ell m}^{\widetilde{P}_{\mathrm{b}},PP^*} &= b_{\ell s}^{E,\widetilde{P}_{\mathrm{b}},PP^*} a_{\ell m}^{E} \cos(4\alpha_t) + b_{\ell s}^{B,\widetilde{P}_{\mathrm{b}},PP^*} a_{\ell m}^{B} \cos(4\alpha_t) \\
&\quad + b_{\ell s}^{E,\widetilde{P}_{\mathrm{b}},PP^*} a_{\ell m}^{B} \sin(4\alpha_t) + b_{\ell s}^{B,\widetilde{P}_{\mathrm{b}},PP^*} a_{\ell m}^{E} \sin(4\alpha_t)\,,
\end{aligned} \tag{33}
$$

$$
{}_s f_{\ell m}^{\widetilde{P}_{\mathrm{b}},PP} = b_{\ell s}^{E,\widetilde{P}_{\mathrm{b}},PP} a_{\ell m}^{E} + b_{\ell s}^{B,\widetilde{P}_{\mathrm{b}},PP} a_{\ell m}^{B}\,. \tag{34}
$$

Again, the α_t modulation can be applied after the time-ordered data for each term is generated.

Computing the HPW + beam SH coefficients We now focus on computing the spherical harmonic coefficients of the product of the I, P and V beams with the different elements of the HPW Mueller matrix. These are the coefficients that are needed to compute Eq. (20) and Eq. (25).

We start by transforming the HWP Mueller matrix $\mathbf{M}_{\mathrm{HWP}}$ using the complex transformation:

$$
\mathbf{C} = \mathbf{T}\mathbf{M}_{\mathrm{HWP}}\mathbf{T}^{\dagger}\,, \tag{35}
$$

where:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{36}$$

The advantage of this representation is that the elements of $\mathbf{C}$ transform as spin-weighted fields under rotations.

The second ingredient we need is a way to represent a tensor product of two spin-weighted spherical harmonics as a direct sum of a single spin-weighted spherical harmonic:

$$\begin{aligned} {}_{s_1}Y_{\ell_1 m_1}(\hat{\mathbf{n}})\, {}_{s_2}Y_{\ell_2 m_2}(\hat{\mathbf{n}}) = \sum_{\ell_3=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{m_3=-\ell_3}^{\ell_3} \sum_{s_3=-\ell_3}^{\ell_3} & J^{-s_1-s_2-s_3}_{\ell_1\ell_2\ell_3} \\ & \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} {}_{s_3}Y^*_{\ell_3 m_3}(\hat{\mathbf{n}}) , \end{aligned} \tag{37}$$

with:

$$J^{s_1 s_2 s_3}_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1 & s_2 & s_3 \end{pmatrix} . \tag{38}$$

We start with the terms that make up Eq. (20). They are given in Eq. (15) of [1]:

$$\begin{aligned} b^{\widetilde{I}^{(0)}_{\mathrm{i}}}_{\ell s}(\nu,\alpha) = \int_{S^2} \mathrm{d}\Omega(\hat{\mathbf{n}}) \Big[& \widetilde{I}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{II}(\nu) \\ & + \widetilde{V}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{VI}(\nu) \\ & + \sqrt{2}\mathrm{Re}\Big(\widetilde{P}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{P^*I}(\nu) \mathrm{e}^{-2\mathrm{i}\alpha} \Big) \Big] Y^*_{\ell s}(\hat{\mathbf{n}}) , \end{aligned} \tag{39}$$

$$\begin{aligned} = & \, b^{\widetilde{I}_{\mathrm{b}},II}_{\ell s}(\nu) + b^{\widetilde{V}_{\mathrm{b}},VI}_{\ell s}(\nu) \\ & + b^{\mathrm{Re}(\widetilde{P}_{\mathrm{b}},P^*I)}_{\ell s}(\nu) \cos(2\alpha_t) + b^{\mathrm{Im}(\widetilde{P}_{\mathrm{b}},P^*I)}_{\ell s}(\nu) \sin(2\alpha_t) . \end{aligned} \tag{40}$$

Lets go through the individual terms:

1. $\widetilde{I}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{II}(\nu)$ is a product of the I beam: $\widetilde{I}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu)$ and the II component of the complex HWP matrix (which for II is the same as the normal real-valued Mueller matrix). Decomposing both quantities into spherical harmonics yields:

$$\widetilde{I}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}}) C_{II} = \sum_{\ell_1,m_1} \sum_{\ell_2,m_2} b^{\widetilde{I}}_{\ell_1 m_1} c^{II}_{\ell_2 m_2} Y_{\ell_1 m_1}(\hat{\mathbf{n}}) Y_{\ell_2 m_2}(\hat{\mathbf{n}}) . \tag{41}$$

This is valid because both are spin 0 quantities. Now we use Eq. (37) and make use of the fact that we model the II element of the HWP as a monopole, i.e. $c^{II}_{\ell m} = \sqrt{4\pi} C_{II} \delta_{\ell,0}\delta_{m,0}$:

$$\widetilde{I}_{\rm b}^{(0)}(\hat{\mathbf{n}}) C_{II} = \sqrt{4\pi} C_{II} \sum_{\ell_1,m_1}\sum_{\ell_3,m_3} b^{\widetilde{I}}_{\ell_1 m_1} J^{000}_{\ell_1 0 \ell_3} \times \begin{pmatrix} \ell_1 & 0 & \ell_3 \\ m_1 & 0 & m_3 \end{pmatrix} Y^*_{\ell_3 m_3}(\hat{\mathbf{n}}) \,, \tag{42}$$

$$= \sqrt{4\pi} C_{II} \sum_{\ell_1,m_1} b^{\widetilde{I}}_{\ell_1 m_1} \frac{2\ell_1+1}{\sqrt{4\pi}} \times \begin{pmatrix} \ell_1 & 0 & \ell_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & \ell_1 \\ m_1 & 0 & -m_1 \end{pmatrix} Y^*_{\ell_1 -m_1}(\hat{\mathbf{n}}) \,, \tag{43}$$

$$= C_{II} \sum_{\ell_1,m_1} b^{\widetilde{I}}_{\ell_1 m_1} (-1)^{m_1} Y^*_{\ell_1 -m_1}(\hat{\mathbf{n}}) \,, \tag{44}$$

$$= C_{II} \sum_{\ell_1,m_1} b^{\widetilde{I}}_{\ell_1 m_1} Y_{\ell_1 m_1}(\hat{\mathbf{n}}) \,. \tag{45}$$

In the first step we have used the triangle constraint of the 3-j symbols. In the second step, we have used:

$$\begin{pmatrix} \ell & 0 & \ell \\ m & 0 & -m \end{pmatrix} = \frac{(-1)^{\ell+m}}{\sqrt{2\ell+1}} \,. \tag{46}$$

Plugging in this result then yields:

$$b^{\widetilde{I}_{\rm b},II}_{\ell s} = \int_{S^2} {\rm d}\Omega(\hat{\mathbf{n}}) \widetilde{I}_{\rm b}^{(0)}(\hat{\mathbf{n}}) C_{II} Y^*_{\ell s}(\hat{\mathbf{n}}) \,, \tag{47}$$

$$= \int_{S^2} {\rm d}\Omega(\hat{\mathbf{n}}) C_{II} \sum_{\ell_1,m_1} b^{\widetilde{I}}_{\ell_1 m_1}(\hat{\mathbf{n}}) Y_{\ell_1 m_1} Y^*_{\ell s}(\hat{\mathbf{n}}) \,, \tag{48}$$

$$= C_{II} b^{\widetilde{I}}_{\ell s} \,, \tag{49}$$

which is of course what we expected for this simple case.

2. The $\widetilde{V}_{\rm b}^{(0)}(\hat{\mathbf{n}},\nu) C_{VI}(\nu)$ case is completely analogous to the previous case. One finds that $b^{\widetilde{V}_{\rm b},VI}_{\ell s} = C_{VI} b^{\widetilde{V}}_{\ell s}$.

3. The $\sqrt{2}{\rm Re}\Big(\widetilde{P}_{\rm b}^{(0)}(\hat{\mathbf{n}},\nu) C_{P^*I}(\nu) {\rm e}^{-2{\rm i}\alpha}\Big)$ case is a bit more complicated.

We start by writing:

$$\sqrt{2}\mathrm{Re}\Big(\widetilde{P}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}})C_{P^*I}\mathrm{e}^{-2\mathrm{i}\alpha}\Big) = \frac{1}{\sqrt{2}}\Big(\widetilde{P}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}})C_{P^*I}\mathrm{e}^{-2\mathrm{i}\alpha} + \widetilde{P^*}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}})C_{PI}\mathrm{e}^{2\mathrm{i}\alpha}\Big)\,. \tag{50}$$

Lets calculate the SH coefficients for the first term:

$$b^1_{\ell m} = \frac{1}{\sqrt{2}}\int_{S^2}\mathrm{d}\Omega(\hat{\mathbf{n}})\widetilde{P}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}})C_{P^*I}\mathrm{e}^{-2\mathrm{i}\alpha}Y^*_{\ell m}(\hat{\mathbf{n}})\,. \tag{51}$$

We note that $\widetilde{P}_{\mathrm{b}}^{(0)}$ is spin 2 and that C_{P^*I} is spin -2. Furthermore, we assume that C_{P^*I} has no spatial variation (just like we did above where we assumed that C_{II} was described by a monopole). A constant spin -2 field cannot be a monopole but must proportional to $\exp 2\mathrm{i}\phi$ near the north pole of the spherical coordinate system (see Eq. (31) in [2]), so we assume that we can decompose C_{P^*I} into SH coefficients that are nonzero only for $m = 2$: $C_{P^*I} = \sum_{\ell m} c_{\ell 2}\,{}_2Y_{\ell 2}$. We can therefore write:

$$\begin{aligned}
b^1_{\ell m} &= \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\sum_{\ell' m'}\sum_{\ell''}{}_2b^{\widetilde{P}}_{\ell' m'}c_{\ell'' 2}(-1)^m \\
&\qquad\times\int_{S^2}\mathrm{d}\Omega(\hat{\mathbf{n}})\,{}_2Y_{\ell' m'}(\hat{\mathbf{n}})\,{}_{-2}Y_{\ell'' 2}(\hat{\mathbf{n}})Y_{\ell -m}(\hat{\mathbf{n}})\,, && (52)\\
&= \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\sum_{\ell' m'}\sum_{\ell''}{}_2b^{\widetilde{P}}_{\ell' m'}c_{\ell'' 2}(-1)^m J^{02-2}_{\ell\ell'\ell''}\begin{pmatrix}\ell & \ell' & \ell''\\ -m & m' & 2\end{pmatrix}\,, && (53)\\
&= \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\sum_{\ell'}\sum_{\ell''}{}_2b^{\widetilde{P}}_{\ell' m-2}c_{\ell'' 2}(-1)^m \\
&\qquad\times J^{02-2}_{\ell\ell'\ell''}\begin{pmatrix}\ell & \ell' & \ell''\\ -m & m-2 & 2\end{pmatrix}\,. && (54)
\end{aligned}$$

Now we use the approximation from Appendix G from [3]: we note that $c_{\ell'' 2}$ peaks at $\ell'' = 2$. The triangle constraint of the 3-j symbols thus constrains ℓ' to be very close to ℓ. This means that we can safely move ${}_2b^{\widetilde{P}}_{\ell' m-2}$ outside the sum over ℓ' (assuming that ${}_2b^{\widetilde{P}}_{\ell' m-2}$ is slowly varying with ℓ', i.e. a narrow beam) and that we can approximate the prefactor of $J^{02-2}_{\ell\ell'\ell''}$ as $(2\ell'+1)\sqrt{2\ell''+1}/\sqrt{4\pi}$. This yields:

$$\begin{aligned} b^1_{\ell m} \approx \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\, {}_2b^{\widetilde{P}}_{\ell m-2}(-1)^m \sum_{\ell''} c_{\ell''2}\frac{\sqrt{2\ell''+1}}{\sqrt{4\pi}} \\ \times \sum_{\ell'}(2\ell'+1)\begin{pmatrix}\ell & \ell' & \ell'' \\ 0 & -2 & 2\end{pmatrix}\begin{pmatrix}\ell & \ell' & \ell'' \\ -m & m-2 & 2\end{pmatrix}. \end{aligned} \tag{55}$$

Then, using Eq. (C9) from [3], this can be written as:

$$b^1_{\ell m} \approx \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\, {}_2b^{\widetilde{P}}_{\ell m-2} \sum_{\ell''} c_{\ell''2}\frac{\sqrt{2\ell''+1}}{\sqrt{4\pi}}\,. \tag{56}$$

Finally, in Appendix G from [3] it is shown that $\sum_{\ell''} c_{\ell''2}\sqrt{2\ell''+1}/\sqrt{4\pi}$ is very close to C_{P^*I}. So we end up with:

$$b^1_{\ell m} \approx \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{\sqrt{2}}\, {}_2b^{\widetilde{P}}_{\ell m-2}\, C_{P^*I}\,. \tag{57}$$

The same derivation can be applied to the second term, giving:

$$b^2_{\ell m} = \frac{1}{\sqrt{2}}\int_{S^2} \mathrm{d}\Omega(\hat{\mathbf{n}}) \widetilde{P^*}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}}) C_{PI}\mathrm{e}^{2\mathrm{i}\alpha} Y^*_{\ell m}(\hat{\mathbf{n}})\,, \tag{58}$$

$$\approx \frac{\mathrm{e}^{2\mathrm{i}\alpha}}{\sqrt{2}}\, {}_{-2}b^{\widetilde{P^*}}_{\ell m+2}\, C_{PI}\,. \tag{59}$$

The SH coefficients corresponding to $\sqrt{2}\mathrm{Re}\Big(\widetilde{P}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{P^*I}(\nu)\mathrm{e}^{-2\mathrm{i}\alpha}\Big)$ are thus approximately given by:

$$\begin{aligned} \int_{S^2} \mathrm{d}\Omega(\hat{\mathbf{n}}) \Big[\sqrt{2}\mathrm{Re}\Big(\widetilde{P}^{(0)}_{\mathrm{b}}(\hat{\mathbf{n}},\nu) C_{P^*I}(\nu)\mathrm{e}^{-2\mathrm{i}\alpha}\Big)\Big] Y^*_{\ell s}(\hat{\mathbf{n}}) \\ \approx b^1_{\ell m} + b^2_{\ell m}\,, \end{aligned} \tag{60}$$

$$= \frac{1}{\sqrt{2}}\Big(\mathrm{e}^{-2\mathrm{i}\alpha}\, {}_2b^{\widetilde{P}}_{\ell m-2}\, C_{P^*I} + \mathrm{e}^{2\mathrm{i}\alpha}\, {}_{-2}b^{\widetilde{P^*}}_{\ell m+2}\, C_{PI}\Big)\,. \tag{61}$$

Which satisfy the symmetry relation for the SH coefficients of a real-valued field. We can also write this as:

$$\begin{aligned} b^1_{\ell m} + b^2_{\ell m} = \frac{1}{\sqrt{2}}\Big[\cos(2\alpha)\Big({}_{-2}b^{\widetilde{P^*}}_{\ell m+2} C_{PI} + {}_2b^{\widetilde{P}}_{\ell m-2} C_{P^*I} \\ \mathrm{i}\sin(2\alpha)\Big({}_{-2}b^{\widetilde{P^*}}_{\ell m+2} C_{PI} - {}_2b^{\widetilde{P}}_{\ell m-2} C_{P^*I}\Big)\Big]\,, \end{aligned} \tag{62}$$

which means that we have found the expressions for $b_{\ell s}^{\mathrm{Re}(\widetilde{P}_{\mathrm{b}},P^*I)}$ and $b_{\ell s}^{\mathrm{Im}(\widetilde{P}_{\mathrm{b}},P^*I)}$ in Eq. (20) and Eq. (40):

$$b_{\ell m}^{\mathrm{Re}(\widetilde{P}_{\mathrm{b}},P^*I)} = \frac{1}{\sqrt{2}}\Big({}_{-2}b_{\ell m+2}^{\widetilde{P^*}}\,C_{PI} + {}_{2}b_{\ell m-2}^{\widetilde{P}}\,C_{P^*I}\Big)\,, \tag{63}$$

$$b_{\ell m}^{\mathrm{Im}(\widetilde{P}_{\mathrm{b}},P^*I)} = \frac{\mathrm{i}}{\sqrt{2}}\Big({}_{-2}b_{\ell m+2}^{\widetilde{P^*}}\,C_{PI} - {}_{2}b_{\ell m-2}^{\widetilde{P}}\,C_{P^*I}\Big)\,. \tag{64}$$

Here, ${}_{-2}b_{\ell m+2}^{\widetilde{P^*}}$ and ${}_{2}b_{\ell m-2}^{\widetilde{P}}$ are the spin-weighed spherical harmonic coefficients of the normal (i.e. no HWP) $\widetilde{Q} - \mathrm{i}\widetilde{U}$ and $\widetilde{Q} + \mathrm{i}\widetilde{U}$ beams respectively, while C_{P^*I} and C_{PI} are elements of the complex HWP Mueller matrix.

We now turn our attention to the SH coefficients that make up Eq. (23). We will not repeat the above derivation, we can obtain the result by swapping the I and V indices in the appropriate places. We obtain:

$$b_{\ell s}^{\widetilde{I}_{\mathrm{b}},IV} = b_{\ell s}^{\widetilde{I}}C_{IV}\,, \tag{65}$$

$$b_{\ell s}^{\widetilde{V}_{\mathrm{b}},VV} = b_{\ell s}^{\widetilde{V}}C_{VV}\,, \tag{66}$$

$$b_{\ell s}^{\mathrm{Re}(\widetilde{P}_{\mathrm{b}},P^*V)} = \frac{1}{\sqrt{2}}\Big({}_{-2}b_{\ell m+2}^{\widetilde{P^*}}\,C_{PV} + {}_{2}b_{\ell m-2}^{\widetilde{P}}\,C_{P^*V}\Big)\,, \tag{67}$$

$$b_{\ell s}^{\mathrm{Im}(\widetilde{P}_{\mathrm{b}},P^*V)} = \frac{\mathrm{i}}{\sqrt{2}}\Big({}_{-2}b_{\ell m+2}^{\widetilde{P^*}}\,C_{PV} - {}_{2}b_{\ell m-2}^{\widetilde{P}}\,C_{P^*V}\Big)\,. \tag{68}$$

Now we move on to the SH coefficients in Eq. (25). Again, the derivation is so similar that I do not want to repeat it, the only new things are realizing that C_{PP^*} and C_{P^*P} are spin -4 and 4 respectively and that C_{PP} and $C_{P^*P^*}$ are both spin 0. I also found a mistake in our paper unfortunately: the SH coefficients in Eq. (16) should be:

$$\begin{aligned}{}_{2}b_{\ell s}^{\widetilde{P}_{\mathrm{i}}^{(0)}}(\nu,\alpha) = \int_{S^2}\mathrm{d}\Omega(\hat{\mathbf{n}})\Big[&\widetilde{I}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}},\nu)C_{IP}(\nu)\sqrt{2}\,\mathrm{e}^{-2\mathrm{i}\alpha}\\ &+\widetilde{V}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}},\nu)C_{VP}(\nu)\sqrt{2}\mathrm{e}^{-2\mathrm{i}\alpha}\\ &+\widetilde{P}_{\mathrm{b}}^{(0)*}(\hat{\mathbf{n}},\nu)C_{P^*P}(\nu)\mathrm{e}^{-4\mathrm{i}\alpha}\\ &+\widetilde{P}_{\mathrm{b}}^{(0)}(\hat{\mathbf{n}},\nu)C_{PP}(\nu)\Big]{}_{2}Y_{\ell s}^{*}(\hat{\mathbf{n}})\,,\end{aligned} \tag{69}$$

(I accidentally swapped around $\widetilde{P}_b^{(0)}$ and $\widetilde{P}_b^{(0)*}$). Using the correct expression we obtain:

$$
\begin{aligned}
{}_2b_{\ell m}^{\widetilde{I}_{\rm b},IP} &= \sqrt{2}C_{IP}\, b_{\ell m+2}^{\widetilde{I}}\,, && (70)\\
{}_2b_{\ell m}^{\widetilde{V}_{\rm b},VP} &= \sqrt{2}C_{VP}\, b_{\ell m+2}^{\widetilde{V}}\,, && (71)\\
{}_2b_{\ell m}^{\widetilde{P^*}_{\rm b},P^*P} &= C_{P^*P}\; {}_{-2}b_{\ell m+4}^{\widetilde{P}}\,, && (72)\\
{}_2b_{\ell m}^{\widetilde{P}_{\rm b},PP} &= C_{PP}\; {}_2b_{\ell m}^{\widetilde{P}}\,. && (73)
\end{aligned}
$$

and similarly, the coefficients in Eq. (26) are given by:

$$
\begin{aligned}
{}_{-2}b_{\ell m}^{\widetilde{I}_{\rm b},IP} &= \sqrt{2}C_{IP^*}\, b_{\ell m-2}^{\widetilde{I}}\,, && (74)\\
{}_{-2}b_{\ell m}^{\widetilde{V}_{\rm b},VP} &= \sqrt{2}C_{VP^*}\, b_{\ell m-2}^{\widetilde{V}}\,, && (75)\\
{}_{-2}b_{\ell m}^{\widetilde{P}_{\rm b},PP^*} &= C_{PP^*}\; {}_2b_{\ell m-4}^{\widetilde{P}}\,, && (76)\\
{}_{-2}b_{\ell m}^{\widetilde{P^*}_{\rm b},P^*P^*} &= C_{P^*P^*}\; {}_{-2}b_{\ell m}^{\widetilde{P}}\,. && (77)
\end{aligned}
$$

These can be converted to the E and B harmonic modes used in Eq. (31)-(34) by making use of Eq. (9)-(10).

References

[1] Adriaan J. Duivenvoorden, Alexandre E. Adler, Matteo Billi, Nadia Dachlythra, and Jon E. Gudmundsson. Probing frequency-dependent half-wave plate systematics for CMB experiments with full-sky beam convolution simulations. MNRAS, 502(3):4526–4539, April 2021.

[2] Anthony Challinor, Pablo Fosalba, Daniel Mortlock, Mark Ashdown, Benjamin Wandelt, and Krzysztof Górski. All-sky convolution for polarimetry experiments. Phys. Rev. D, 62(12):123002, December 2000.

[3] E. Hivon, S. Mottet, and N. Ponthieu. QuickPol: Fast calculation of effective beam matrices for CMB polarization. *A&A*, 598:A25, February 2017.