**MIE 1621 HW#1 Winter 2024**

**Due at the START of lecture on Jan 29 (hand in by 3:15PM in HA 401).**

In general, SHOW ALL WORK. You must use the material from the class slides only to do these problems. For example, the SONC mentioned in Problem 4 below is the one from the class slides and you must not use any other version of SONC found elsewhere.

**Problem 1 (20 points)**

Consider the following function

$$f(x) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$$

Find all stationary points and classify them (e.g. is a stationary point a (strict) local or (strict) global min or saddle point.)

**Problem 2 (20 points)**

Consider the problem

$$\min_{x \in R^n} \|Tx - t\|^2 + \theta \|Lx\|^2$$

where $T$ is an $m \times n$ matrix, $t$ is an $m$ dimensional vector, $\theta$ is a positive constant and $x$ is an $n$ dimensional vector. $L$ is a $p \times n$ matrix.

(a) (8 points) Solve for the vector $x$ that satisfies the first order necessary condition for the problem above. (hint: recall that $\|y\|^2 = y \cdot y$)

(b) (3 points) Under what conditions is the solution from (a) an optimal solution to the problem?

(c) (4 points) Solve the problem for the instance

$$T = \begin{bmatrix} 2 + 10^{-3} & 3 & 4 \\ 3 & 5 + 10^{-3} & 7 \\ 4 & 7 & 10 + 10^{-3} \end{bmatrix}, \; t = \begin{bmatrix} 20.0019 \\ 34.0004 \\ 48.0202 \end{bmatrix},$$

$L = I_3$ (3 by 3 identity matrix), and $\theta = 1$.

(d) (5 points) Now solve the optimization problem again using the data in (c) but without the second term i.e. remove $\theta \|Lx\|^2$. Now suppose that the vector $t$ is in reality a noisy estimate of $Ax_{true}$ where the true vector $x_{true}$ $= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Which of the two model formulations the one with $\theta \|Lx\|^2$ or without does a better job of estimating $x_{true}$ and why?

**Problem 3 (15 points)**

Consider the optimization problem in Problem 2

$$\min_{x \in R^n} \|Tx - t\|^2 + \theta \|Lx\|^2$$

where $T$ is an $m \times n$ matrix, $t$ is an $m$ dimensional vector, $\theta$ is a positive constant and $x$ is an $n$ dimensional vector. $L$ is a $p \times n$ matrix.

Prove that this problem has a unique solution if and only if $Null(T) \cap Null(L) = \{0\}$ where here for a matrix $B$, $Null(B)$ is the null space of $B$ which is the set $\{x | Bx = 0\}$ i.e. it is the set of all vectors $x$ such that $Bx = 0$.

**Problem 4 (15 points)**

Let $f(x) = x_1^3 + x_2^2$. Use $f(x)$ to illustrate that the (Second-Order Necessary Condition) SONC is not sufficient for determining local minimality of an unconstrained minimization problem.

**Problem 5 (15 points)**

Prove that the function $f(x) = e^{(x_1^2 + x_2^2 + x_3^2)}$ is a strictly convex function on $\mathbb{R}^3$.

**Problem 6 (15 points)**

Let $f(x)$ be a twice differentiable function over $\mathbb{R}^n$. Suppose that the Hessian $H(x)$ is positive definite for any $x$. Then, prove that a stationary point of $f(x)$ is necessarily a strict global minimum point.