Q3. $\cancel{2} \rightarrow \quad I_1 = \int_0^3 \left(\frac{x^3}{12} - 8x\right) dx$

$$I_2 = \int_1^2 \left(e^{3x+1}\right) dx$$

a. Computing the exact value of integral I_1

$$I_1 = \int_0^3 \left(\frac{x^3}{12} - 8x\right) dx$$

$$= \left[\frac{x^4}{4\times 12} - \frac{8x^2}{2}\right]_0^3$$

$$\left(\frac{3^4}{4\times 12} - 0\right) - \left(4(3)^2 - 0\right)$$

$$= 1.6875 - 36$$

$$= -34.3125$$

Using Gauss quadrature,

$$x = a\left(\frac{1-\varepsilon}{2}\right) + b.\left(\frac{\varepsilon+1}{2}\right)$$

where a, b are the limits of the integral

$$x = 0\left(\frac{1-\varepsilon}{2}\right) + 3\left(\frac{\varepsilon+1}{2}\right)$$

$$= 1.5\varepsilon + 1.5$$

determinant of Jacobian $|J| = \frac{\partial x}{\partial \varepsilon} = 1.5$

Use $N=2$ gauss points,

$$W_1 = W_2 = 1$$

$$\varepsilon_1 = -\frac{1}{\sqrt{3}} \qquad \varepsilon_2 = \frac{1}{\sqrt{3}}$$

Corresponding values of x

$$x_1 = 1.5(\varepsilon_1+1)$$
$$= 0.634$$

$$x_2 = 1.5(\varepsilon_2+1)$$
$$= 2.366$$

the function values at the respective x,

$$f(x_1) = \frac{(0.634)^3}{12} - 8(0.634)$$
$$= -5.0508$$

$$f(x_2) = \frac{(2.366)^3}{12} - 8(2.366)$$
$$= -17.8243$$

Expression to evaluate

$$I \approx |J|\hat{I} = |J| \sum_{i=1}^{2} w_i f(x_i)$$

$$= 1.5\left((1)(-5.0508) + (1)(-17.8243)\right)$$

$$= -34.3126$$

hence the % error in calculation $= \left[\frac{(-34.3125) - (-34.3126)}{(-34.3126)}\right] \times 100$

$= -0.00029\%$

which is well within the expected 0.5% error limit.

b. Computing the exact value of the integral I_2

$$I_2 = \int_1^2 (e^{3x+1})\,dx$$

$\Rightarrow u = 3x+1$ limits $u = 3(1)+1$ to $u = 3(2)+1$

$du = 3dx$ $= 4$ $= 7$

$$I_2 = \int_4^7 e^u \cdot \frac{du}{3}$$

$$= \left[\frac{e^u}{3}\right]_4^7$$

$$= \frac{e^7}{3} - \frac{e^4}{3}$$

$$= 347.345$$

Using gauss quadrature

$$x = a\left(\frac{1-\varepsilon}{2}\right) + b\left(\frac{1+\varepsilon}{2}\right)$$

$$= (1)\left(\frac{1-\varepsilon}{2}\right) + 2\left(\frac{1+\varepsilon}{2}\right)$$

$$= 0.5 - 0.5\varepsilon + 1 + \varepsilon$$

$$= 1.5 + 0.5\varepsilon$$

determinant of Jacobian

$$|J| = \frac{\partial x}{\partial \xi}$$

$$= 0.5$$

use $N=2$ gauss points.

$$w_1 = w_2 = 1$$

$$\xi_1 = \frac{-1}{\sqrt{3}} = -0.5774 \qquad \xi_2 = \frac{1}{\sqrt{3}} = 0.5774$$

Corresponding values of x

$$x_1 = 1.5 + 0.5(-0.5774)$$

$$= 1.2113$$

$$x_2 = 1.5 + 0.5(0.5774)$$

$$= 1.7887$$

the function values at respective x_i

$$f(x_1) = e^{3x_1+1}$$

$$= 102.9146$$

$$f(x_2) = e^{3x_2+1}$$

$$= 581.7844$$

Expression to evaluate

$$I \approx |J| \hat{I} = |J| \sum_{i=1}^{2} w_i f(x_i)$$

$$= 0.5 \Big((1)(102.7146) + (1)(581.7844) \Big)$$

$$= 342.3495$$

% change in the values $\Rightarrow \left(\dfrac{347.345 - 342.3495}{347.345} \right) \times 100$

$$= 1.438\,\%$$

which is greater than the accepted error limit,

hence we compute $N=3$

$w_1 = 0.5556 \qquad \varepsilon_1 = 0.7746$

$w_2 = 0.5556 \qquad \varepsilon_2 = -0.7746$

$w_3 = 0.8889 \qquad \varepsilon_3 = 0$

Corresponding values of x

$$x_1 = 1.5 + 0.5\,\varepsilon_1$$

$$= 1.5 + 0.5(0.7746)$$

$$= 1.8873$$

$x_2 = 1.5 + 0.5\,\varepsilon_2$

$= 1.5 + 0.5(-0.7746)$

$= 1.1127$

$x_3 = 1.5 + 0.5\,\varepsilon_3$

$= 1.5 + 0.5(0)$

$= 1.5$

finding corresponding $f(x)$

$f(x_1) = e^{3x_1+1}$

$= e^{3(1.8873)+1}$

$= 782.0354$

$f(x_2) = e^{3x_2+1}$

$= e^{3(1.1127)+1}$

$= 76.5619$

$f(x_3) = e^{3x_3+1}$

$= e^{3(1.5)+1}$

$= 244.6919$

expression to evaluate

$$\Rightarrow I \approx |J| \hat{I} = |J| \sum_{i=1}^{3} W_i P(x_i)$$

$$= 0.5\left(0.5556 \times 782.0354 + 0.5556 \times 76.5619 + 0.8889 \times 244.6919 \right)$$

$$= 694.5433 \times 0.5$$

$$= 347.2716$$

% change in calculated value $= \left(\frac{347.345 - 347.2716}{347.345} \right) \times 100$

$$= 0.0213\%$$

which is well within the $\nless 0.5\%$ requirement.