MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

American Mathematics Competitions

16th Annual

AMC 10 A

American Mathematics Contest 10A

Tuesday, February 3, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs conceptual, elementary vs advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Silvia Fernandez

1. **Answer (C):**

$$(1-1+25+0)^{-1}\times 5=\frac{1}{25}\times 5=\frac{1}{5}$$

2. **Answer (D):** Counting 3 edges per tile gives a total of $3\cdot 25=75$ edges, and exactly 1 edge per square tile is missing. So there are exactly $84-75=9$ square tiles.

OR

Let x be the number of square tiles in the box. Then there are $25-x$ triangular tiles and $4x+3(25-x)=84$ edges. Solving for x gives $x=9$ square tiles.

3. **Answer (D):** Five vertical and five horizontal toothpicks must be added to complete the fourth step. Six vertical and six horizontal toothpicks must be added to complete the fifth step. This is a total of 22 toothpicks added.

4. **Answer (B):** Let m be the number of eggs that Mia has. Then Sofia has $2m$ eggs and Pablo has $6m$ eggs. If the total of $9m$ eggs is to be divided equally, each person will have $3m$ eggs. Therefore Pablo should give $2m$ eggs to Mia and m eggs to Sofia. The fraction of his eggs he should give to Sofia is $\frac{m}{6m}=\frac{1}{6}$.

5. **Answer (E):** The sum of the 14 test scores was $14\cdot 80=1120$. The sum of all 15 test scores was $15\cdot 81=1215$. Therefore Payton's score was $1215-1120=95$.

OR

To bring the average up to 81, the total must include 1 more point for each of the 14 students, in addition to 81 points for Payton. Therefore Payton's score was $81+14=95$.

6. **Answer (B):** Let x and y be the two positive numbers, with $x>y$. Then $x+y=5(x-y)$. Thus $4x=6y$, so $\frac{x}{y}=\frac{3}{2}$.

7. **Answer (B):** The difference between consecutive terms is 3, and the difference between the first and last terms is $73-13=60=20\cdot 3$. Therefore the number of terms is $20+1=21$.

   Note: The kth term in the sequence is $3k+10$.

8. **Answer (B):** Let p be Pete's present age, and let c be Claire's age. Then $p-2=3(c-2)$ and $p-4=4(c-4)$. Solving these equations gives $p=20$ and $c=8$. Thus Pete is 12 years older than Claire, so the ratio of their ages will be $2:1$ when Claire is 12 years old. That will occur $12-8=4$ years from now.

9. **Answer (D):** Let r, h, R, H be the radii and heights of the first and second cylinders, respectively. The volumes are equal, so $\pi r^2 h = \pi R^2 H$. Also $R = r + 0.1r = 1.1r$. Thus $\pi r^2 h = \pi(1.1r)^2 H = \pi(1.21r^2)H$. Dividing by πr^2 yields $h = 1.21H = H + 0.21H$. Thus the first height is 21% more than the second height.

10. **Answer (C):** In the alphabet the letter b is adjacent to both a and c. So in any rearrangement, b can only be adjacent to d, and thus b must be the first or last letter in the rearrangement. Similarly, the letter c can only be adjacent to a, so c must be the first or last letter in the rearrangement. Thus the only two acceptable rearrangements are $bdac$ and $cadb$.

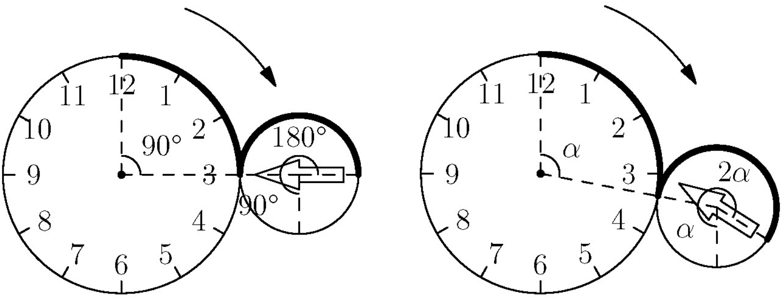
11. **Answer (C):** Let the sides of the rectangle have lengths $3a$ and $4a$. By the Pythagorean Theorem, the diagonal has length $5a$. Because $5a = d$, the side lengths are $\frac{3}{5}d$ and $\frac{4}{5}d$. Therefore the area is $\frac{3}{5}d \cdot \frac{4}{5}d = \frac{12}{25}d^2$, so $k = \frac{12}{25}$.

12. **Answer (C):** The equation is equivalent to $1 = y^2 - 2x^2y + x^4 = (y - x^2)^2$, or $y - x^2 = \pm 1$. The graph consists of two parabolas, $y = x^2 + 1$ and $y = x^2 - 1$. Thus a and b are $\pi + 1$ and $\pi - 1$, and their difference is 2. Indeed, the answer would still be 2 if $\sqrt{\pi}$ were replaced by any real number.

13. **Answer (C):** If Claudia only has 10-cent coins, then she can make 12 different values. Otherwise, suppose that the number of 10-cent coins is d and thus the number of 5-cent coins is $12-d$. Then she can make any value that is a multiple of 5 from 5 to $10d + 5(12-d) = 5(d+12)$. Therefore $d + 12 = 17$, and $d = 5$.

14. **Answer (C):** The circumference of the disk is half the circumference of the clock face. As the disk rolls $\frac{1}{4}$ of the way around the circumference of the clock face (from 12 o'clock to 3 o'clock), the disk rolls through $\frac{1}{2}$ of its own circumference. At that point, the arrow of the disk is pointing at the point of tangency, so the arrow on the disk will have turned $\frac{3}{4}$ of one revolution. In general, as the disk rolls through an angle α around the clock face, the arrow on the disk turns through an angle 3α on the disk. The arrow will again be

pointing in the upward vertical direction when the disk has turned through 1 complete revolution, and the angle traversed on the clock face is $\frac{1}{3}$ of the way around the face. The point of tangency will be at 4 o'clock.

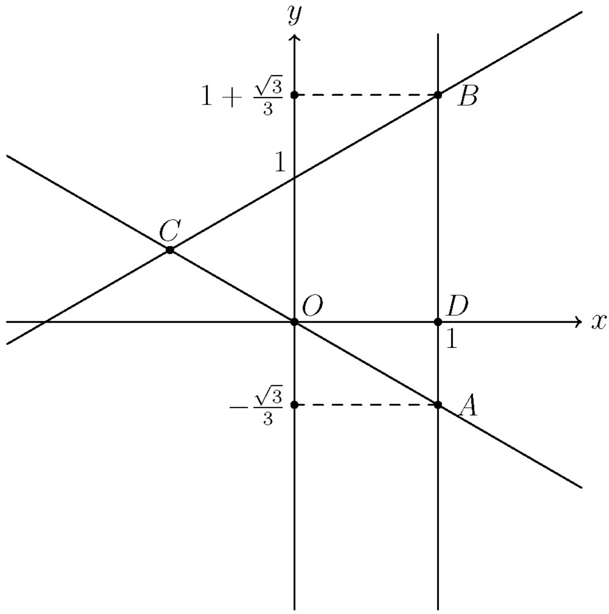


15. **Answer (B):** Because $\frac{x+1}{y+1} = \frac{11}{10} \cdot \frac{x}{y}$, it follows that $10y - 11x - xy = 0$ and so $(10 - x)(11 + y) = 110 = 2 \cdot 5 \cdot 11$. The only possible values of $10 - x$ are 5, 2, and 1 because x and y are positive integers. Thus the possible values of x are 5, 8, and 9. Of the resulting fractions $\frac{5}{11}$, $\frac{8}{44}$, and $\frac{9}{99}$, only the first is in simplest terms.

16. **Answer (B):** Expanding the binomials and subtracting the equations yields $x^2 - y^2 = 3(x - y)$. Because $x - y \neq 0$, it follows that $x + y = 3$. Adding the equations gives $x^2 + y^2 = 5(x + y) = 5 \cdot 3 = 15$.

    Note: The two solutions are $(x, y) = (\frac{3}{2} + \frac{\sqrt{21}}{2}, \frac{3}{2} - \frac{\sqrt{21}}{2})$ and $(\frac{3}{2} - \frac{\sqrt{21}}{2}, \frac{3}{2} + \frac{\sqrt{21}}{2})$.

17. **Answer (D):** Label the vertices of the equilateral triangle A, B, and C so that A is on the line $x = 1$ and B is on both lines $x = 1$ and $y = 1 + \frac{\sqrt{3}}{3}x$. Then $B = (1, 1 + \frac{\sqrt{3}}{3})$. Let O be the origin and $D = (1, 0)$. Because $\triangle ABC$ is equilateral, $\angle CAB = 60°$, and $\triangle OAD$ is a $30-60-90°$ triangle. Because $OD = 1$, $AD = \frac{\sqrt{3}}{3}$ and $AB = AD + DB = \frac{\sqrt{3}}{3} + (1 + \frac{\sqrt{3}}{3}) = 1 + \frac{2\sqrt{3}}{3}$. The perimeter of $\triangle ABC$ is $3 \cdot AB = 3 + 2\sqrt{3}$. Indeed, $\triangle ABC$ is equilateral with $C = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$.



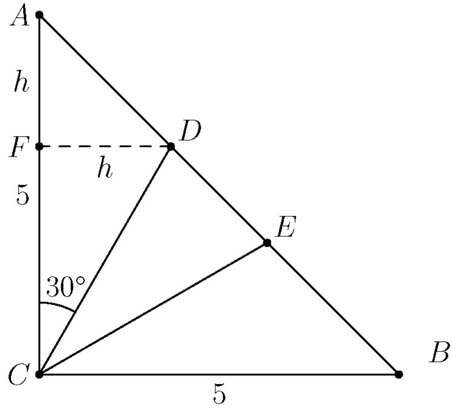
18. **Answer (E):** Because $1000 = 3 \cdot 16^2 + 14 \cdot 16 + 8$, the largest number less than 1000 whose hexadecimal representation contains only numeric digits is $3 \cdot 16^2 + 9 \cdot 16 + 9$. Thus the number of such positive integers is $n = 4 \cdot 10 \cdot 10 - 1 = 399$ ($0 \cdot 16^2 + 0 \cdot 16 + 0 = 0$ is excluded), and the sum of the digits of n is 21.

19. **Answer (D):** Because the area is 12.5, it follows that $AC = BC = 5$. Label D and E so that D is closer to A than to B. Let F be the foot of the perpendicular to $\overline{AC}$ passing through D. Let $h = FD$. Then $AF = h$ because $\triangle ADF$ is an isosceles right triangle, and $CF = h\sqrt{3}$ because $\triangle CDF$ is a $30-60-90^\circ$ triangle. So $h + h\sqrt{3} = AC = 5$ and

$$h = \frac{5}{1+\sqrt{3}} = \frac{5\sqrt{3}-5}{2}.$$

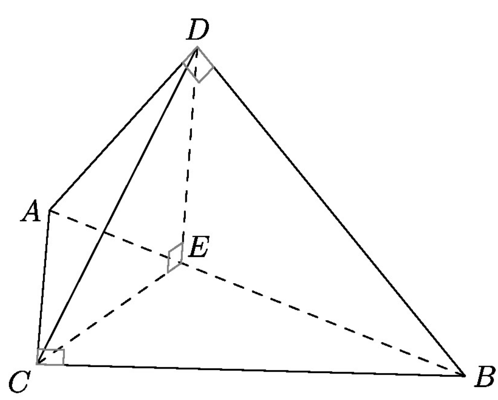
Thus the area of $\triangle CDE$ is

$$\frac{25}{2} - 2 \cdot \frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}-5}{2} = \frac{50-25\sqrt{3}}{2}.$$



20. **Answer (B):** Let x and y be the lengths of the sides of the rectangle. Then $A+P = xy + 2x + 2y = (x+2)(y+2) - 4$, so $A+P+4$ must be the product of two factors, each of which is greater than 2. Because the only factorization of $102+4 = 106$ into two factors greater than 1 is $2 \cdot 53$, $A+P$ cannot equal 102. Because $100+4 = 104 = 4 \cdot 26$, $104+4 = 108 = 3 \cdot 36$, $106+4 = 110 = 5 \cdot 22$, and $108+4 = 112 = 4 \cdot 28$, the other choices equal $A+P$ for rectangles with dimensions 2×24, 1×34, 3×20, and 2×26, respectively.

21. **Answer (C):** Triangles ABC and ABD are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE = \frac{12}{5}$. Likewise in $\triangle ABD$, $DE = \frac{12}{5}$. Triangle CDE has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle CED. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base ABC. The tetrahedron's volume is $\frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$.



22. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1 + 8 + (28 - 8) + (56 - 8 - 32) + 2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

23. **Answer (C):** The zeros of f are integers and their sum is a, so a is an integer. If r is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of r are 6, 4, 3, 1, 0, and -2. The possible values of a are 9, 8, 0, and -1, all of which yield integer zeros of f, and their sum is 16.

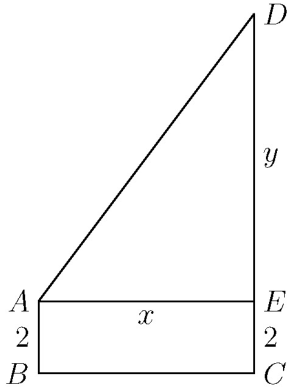
OR

As above, a must be an integer. The function f has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer w. Solving for a in terms of w gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for w are 0 and ± 3, from which it follows that the values of a are 0, 8, 9, and -1, all of which yield integer values of x. The requested sum is 16.

24. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2 + y$. By the Pythagorean Theorem, $x^2 + y^2 = (2+y)^2$, or $x^2 = 4 + 4y$. Therefore x is even, say $x = 2z$, and $z^2 = 1 + y$. The perimeter of the quadrilateral is $x + 2y + 6 = 2z^2 + 2z + 4$. Increasing positive integer values of z give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.

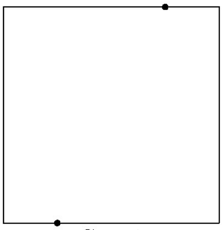


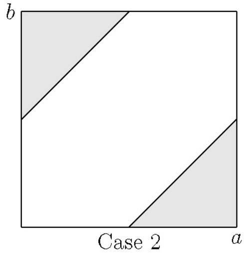
25. **Answer (A):** Let the square have vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

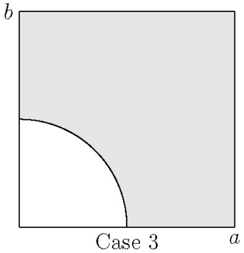
Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a, 0)$ and $(b, 0)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a, 0)$ and $(0, b)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2 + b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1 - \frac{1}{4}\pi(\frac{1}{2})^2 = 1 - \frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.


Case 1



Case 2



Case 3

Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\left(1-\frac{\pi}{16}\right)=\frac{26-\pi}{32},$$

and $a+b+c=59$.

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