Pure Mathematics 1

Lecture Notes

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Content

Chapter 1 Quadratics

1.1 Completing the Square

Rearranging $(x+d)^2 = x^2 + 2dx + d^2$ and $(x-d)^2 = x^2 - 2dx + d^2$:

Key Point 1.1

$x^2 + 2dx = (x+d)^2 - d^2$ and $x^2 - 2dx = (x-d)^2 - d^2$.

Example 1.1 Express $4x^2 - 20x + 5$ in the form $a(x+b)^2 + c$.

Solution. $4x^2 - 20x + 5 = 4(x^2 - 5x) + 5 = 4\left[\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right] + 5 = 4\left(x - \frac{5}{2}\right)^2 - 20;$

Example 1.2 Express $4x^2 - 20x + 5$ in the form $(px+q)^2 + r$, where $p > 0$.

Solution. Let $4x^2 - 20x + 5 = (px+q)^2 + r = p^2x^2 + 2pqx + q^2 + r$. Comparing each coefficient:

$$\begin{aligned} 4 &= p^2 \\ -20 &= 2pq \\ 5 &= q^2 + r \end{aligned}$$

we solve for $p = 2$, $q = -5$ and $r = -20$. Then $4x^2 - 20x + 5 = (2x-5)^2 - 20$.

1.2 Solving Quadratic Equations

1.2.1 By Factorisation

Example 1.3 Solve $\dfrac{21}{2x} - \dfrac{2}{x+3} = 1$.

Solution. Multiplying both sides by $2x(x+3)$ gives $21(x+3) - 4x = 2x(x+3)$. Rearranging and factorising it as:

$$2x^2 - 11x - 63 = (2x+7)(x-9) = 0.$$

So $2x + 7 = 0$ or $x - 9 = 0$. We solve for $x = -\dfrac{7}{2}$ or $x = 9$.

1.2.2 By Completing the Square

Example 1.4 Solve $4x^2 - 20x + 5 = 0$.

Solution. Completing the square $4x^2 - 20x + 5 = (2x-5)^2 - 20 = 0$, we have

$$\begin{aligned}
&(2x-5)^2 = 20 \\
\Leftrightarrow\ & 2x - 5 = \pm 2\sqrt{5} \\
\Leftrightarrow\ & x = \frac{5}{2} \pm \sqrt{5}.
\end{aligned}$$

1.2.3 By Quadratic Formula

If $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$, then

Key Point 1.2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.5 Solve $2x^2 - 11x - 63 = 0$.

Solution. Using $a = 2, b = -11$ and $c = -63$ in the quadratic formula gives:

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \times 2 \times (-63)}}{2 \times 2} = \frac{11 \pm \sqrt{625}}{4} = \frac{11 \pm 25}{4},$$

and thus $x = -\dfrac{7}{2}$ or $x = 9$.

1.2.4 Solving More Complex Quadratic Equations

Some equations that are **quadratic in some function of** x. For example, $x^4 - 5x^2 + 4 = 0$ or $6x + \sqrt{x} - 1 = 0$.

Example 1.6 Solve $2x - 11\sqrt{x} - 63 = 0$.

Solution. Let $y = \sqrt{x}$, then $2x - 11\sqrt{x} - 63 = 0$ can be writen as $2y^2 - 11y - 63 = 0$. We solve for $y = -\dfrac{7}{2}$ or $y = 9$. So $\sqrt{x} = -\dfrac{7}{2}$ or $\sqrt{x} = 9$. We can **ONLY** solve for $x = 81$. (why?)

Remark. 在三角函数（Trigonometric Functions）中，我们会遇到形如 $\tan^2 x = 1 + \tan x$ 的方程求解问题，我们仍然是利用替代法（Substitution）解决.

1.3 Solving Simultaneous Equations

In this section, we use **substitution** to solve simultaneous equations where one equation is linear and the second equation is quadratic.

Example 1.7 Solve the simultaneous equations $y = x^2 - 4$ and $y = 2x - 1$.

Solution. Consider

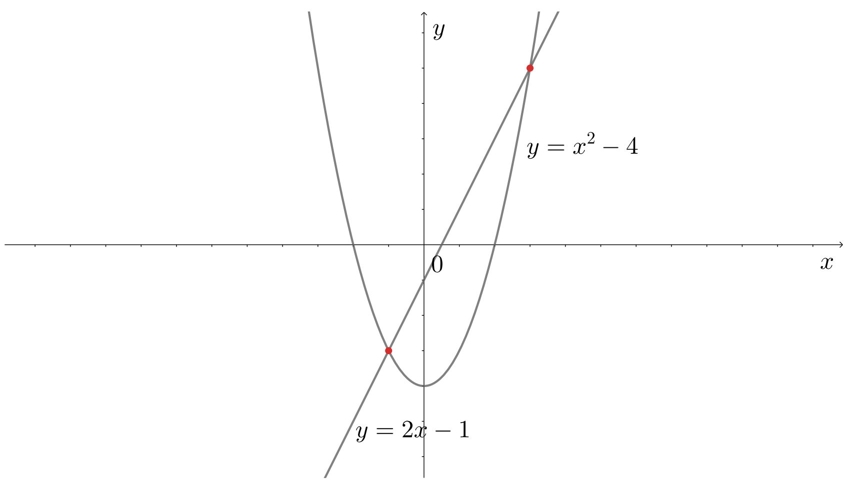
$$y = x^2 - 4 \tag{1}$$
$$y = 2x - 1 \tag{2}$$

Substituting for y from (2) into (1):

$$\begin{aligned} 2x - 1 &= x^2 - 4 \\ \Leftrightarrow x^2 - 2x - 3 &= 0 \\ \Leftrightarrow (x+1)(x-3) &= 0 \end{aligned}$$

and we solve for $x = -1$ (with $y = -3$) or $x = 3$ (with $y = 5$).

Remark. We can observe that the line intersects the curve at **two distinct points**.



Example 1.8 Solve the simultaneous equations $y = x^2 - 4$ and $2x + y + 5 = 0$.

Solution. Consider

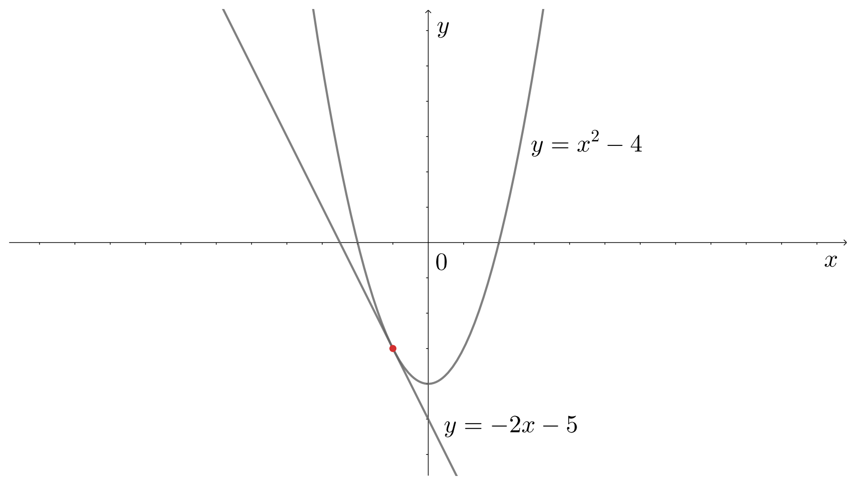
$$y = x^2 - 4 \tag{1}$$
$$2x + y + 5 = 0 \tag{2}$$

We can rewrite (2) as $y = -2x - 5$ and substitute for y into (1):

$$\begin{aligned} -2x - 5 &= x^2 - 4 \\ \Leftrightarrow x^2 + 2x + 1 &= 0 \\ \Leftrightarrow (x-1)^2 &= 0 \end{aligned}$$

and we solve for $x = -1$ (with $y = -3$).

Remark. We can observe that the line intersects the curve at **one point**, i.e., the line is a **tangent** to the curve.



Example 1.9 Solve the simultaneous equations $y = x^2 - 4$ and $y = -x - 5$.

Solution. Consider

$$y = x^2 - 4 \tag{1}$$

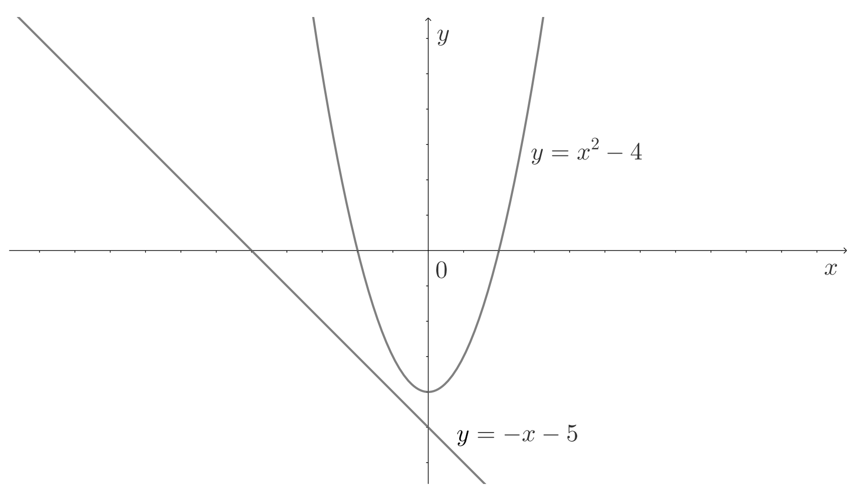
$$y = -x - 5 \tag{2}$$

Substituting for y from (2) into (1):

$$\begin{aligned} -x - 5 &= x^2 - 4 \\ \Leftrightarrow x^2 + x + 1 &= 0 \end{aligned}$$

Using quadratic formula, we have $x = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} = \dfrac{-1 \pm \sqrt{-3}}{2}$, and thus there is no (real) solution.

Remark. We can observe that the line does not intersect the curve.



In general, an equation in x and y is called **quadratic** if it has the form

Key Point 1.3

$ax^2 + bxy + cy^2 + dx + ey + f = 0$, where **at least one of** a, b **and** c **is non-zero**.

Remark. The technique for solving one linear and one quadratic equation will work for more general quadratics.

✍ **Exercise 1.1** Solve the simultaneous equations $2x + y = 8$ and $xy = 8$. (Is $xy = 8$ quadratic?)

1.4 Quadratic Functions

The general form of a **quadratic function** is

Key Point 1.4

$y = f(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

The shape of $y = f(x)$ is called **parabola**. When we sketch the graph of a quadratic function, three key features are:

- the general shape of the graph
  - if $a > 0$, the parabola opens upwards
  - if $a < 0$, the parabola opens downwards
- the axis intercepts (when $x = 0$ and $y = f(x) = 0$)
- the coordinates of the vertex
  - if $a > 0$, the vertex is the minimum point
  - if $a < 0$, the vertex is the maximum point

Note. We can sketch a quadratic function $f(x) = ax^2 + bx + c$ by completing the square:

1. $f(x) = ax^2 + bx + c$ can be written as $f(x) = a(x - h)^2 + k$.
2. Determine the shape by a.
3. Find the axis intercepts, i.e., when $x = 0$ and when $y = f(x) = 0$.
4. The line of symmetry is $x = h = -\dfrac{b}{2a}$.
5. The coordinates of the vertex is (h, k).

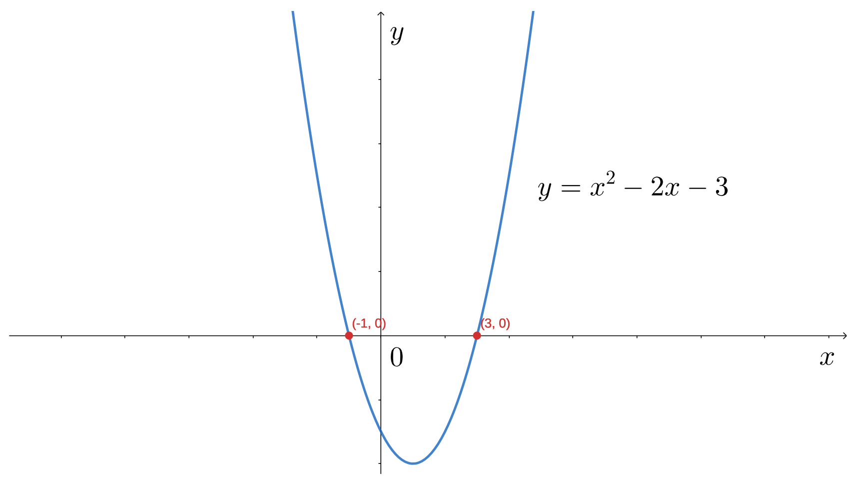
Exercise 1.2 Sketch the graph of $y = f(x) = -x^2 + 2x + 3$.

1.5 Solving Quadratic Inequalities

Quadratic inequalities can be solved by sketching a graph and considering when the graph is above or below the x-axis.

Example 1.10 Solve $-x^2 + 2x + 3 < 0$.

Solution. $-x^2 + 2x + 3 < 0$ is equivalent to $x^2 - 2x - 3 > 0$. When $x^2 - 2x - 3 = 0$, $x = -1$ or $x = 3$. So the x-axis crossing points are -1 and 3. The range of values of x where the curve is positive (above the x-axis) is $x < -1$ or $x > 3$.



Remark. If we multiply or divider both sides of an inequality by a negative number, then the inequality sign must be reversed.

1.6 Discriminant

Consider solving the following three quadratic equations of the form $ax^2 + bx + c = 0$ using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 1.11 For $x^2 + 2x - 8 = 0$, $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-8)}}{2 \times 1} = \dfrac{-2 \pm \sqrt{36}}{2}$, and there are two distinct solutions.

Example 1.12 For $x^2+6x+9 = 0$, $x = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1} = \dfrac{-6 \pm \sqrt{0}}{2}$, and there are two equal solutions.

Example 1.13 For $x^2+2x+6 = 0$, $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 6}}{2 \times 1} = \dfrac{-2 \pm \sqrt{-20}}{2}$, and there is no (real) solution.

As we can see in those three quadratic equations, the sign of $b^2 - 4ac$ determines the case of the solution of the equation. $b^2 - 4ac$ is called the **discriminant** of $ax^2 + bx + c = 0$.

1.6.1 Number of Roots

The sign of the discriminant tells us the number of roots (the solutions to $y = f(x) = 0$) for quadratic equations.

$b^2 - 4ac$	Number of roots of $ax^2 + bx + c = 0$	Shape of curve $y = f(x) = ax^2 + bx + c$
> 0	Two **distinct** real roots	
$= 0$	One **repeated** real root	
< 0	**No** real roots	

Exercise 1.3 Find the values of k for which the equation $kx^2 - 2kx + 8 = 0$ (a) has two equal roots, (b) has two distinct roots, and (c) has no roots.

1.6.2 Position of Line and Quadratic Curve

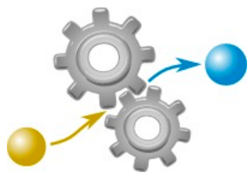
$b^2 - 4ac$	Number of roots of $ax^2 + bx + c = 0$	Position of line and curve
> 0	Two **distinct** real roots	**Two distinct** points of intersection
$= 0$	One **repeated** real root	**One point** of intersection (line is a **tangent**)
< 0	**No** real roots	**No** points of intersection

Remark. The technique for finding the conditions for intersection of a line and a quadratic equation will work for more general quadratics.

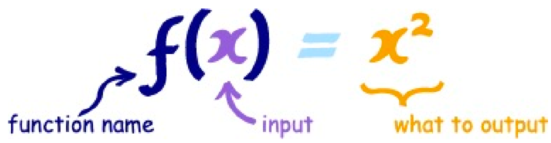
Exercise 1.4 Find the values of k for which the line $2x + y = k$ (a) does not intersect, and (b) is a tangent to the curve $xy = 8$.

Chapter 2 Functions

2.1 Basic Concepts of Functions



A **function** is like a machine that has an input and an output, and the output is related somehow to the input. We can write a function as: $y = x^2$, $f : x \mapsto x^2$, or equivalently, $f(x) = x^2$.



We should notice that

1. f is the most common **function name**, but we can use g, h, or anything we want;
2. x is the **input**, but we can also use anything we want, e.g., $f(x) = x^2$ is the same function as $f(a) = a^2$, $w(\theta) = \theta^2$ or $h : y \mapsto y^2$;
3. sometimes a function has no name, such as $y = x^2$;
4. $f(x)$ is the **output** of the function f when the input is x.

For any function, we care about three things at first: input, relationship, and output. We take $f(x) = 5x$ as an example:

Input	Relationship	Output
0	$\times 5$	0
1	$\times 5$	5
2	$\times 5$	10
3	$\times 5$	15
x	$\times 5$	$5x$
$\cdots$	$\times 5$	$\cdots$

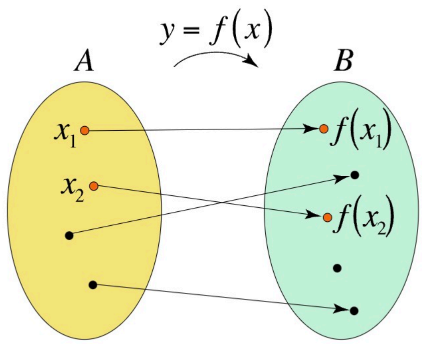
Beyond The Syllabus. 映射（Mapping）是集合与集合之间的一种对应关系，其中集合里的元素可以是任意的数学对象. 函数（Function）是一种特殊的映射，集合里的元素只能为数. 我们有时会称函数为映射.

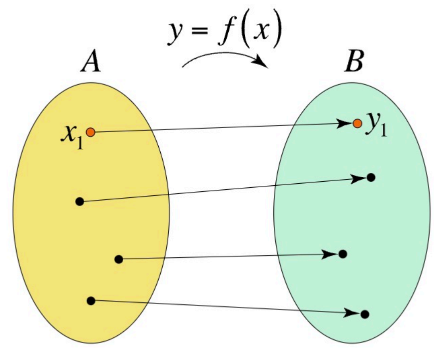
Definition 2.1

A **function** relates **each element** of a set with **exactly one** element of another set.

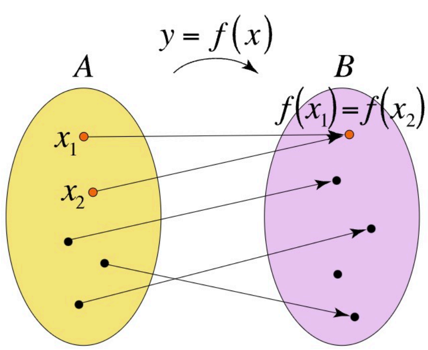
Remark. Exactly one means a function will not give back two or more results for the same input. For example, $f(5) = 2$ or 4 is incorrect! So a function can **ONLY** be either a **one-one** function or a **many-one** function.

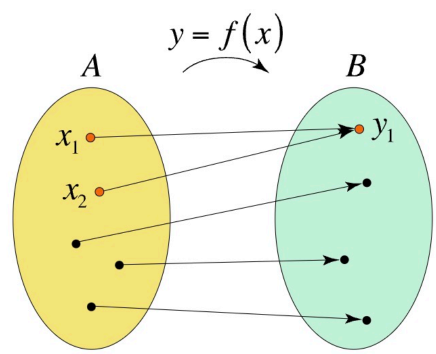
We give some illustrations of one-one:



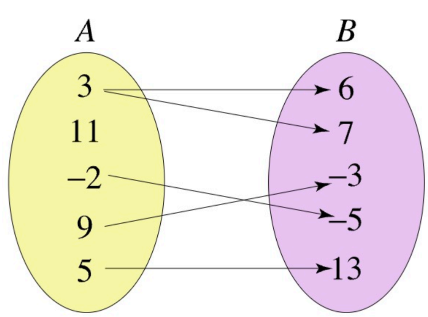


and of many-one:





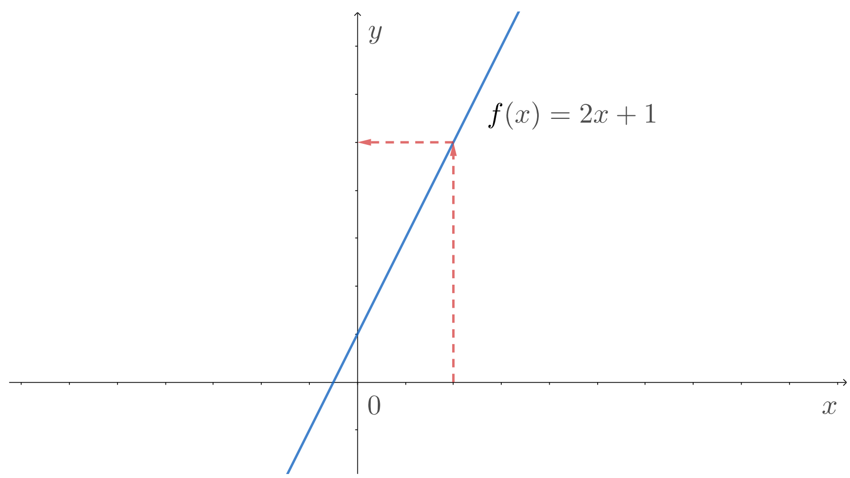
We also give an illustration of neither one-one nor many-one:



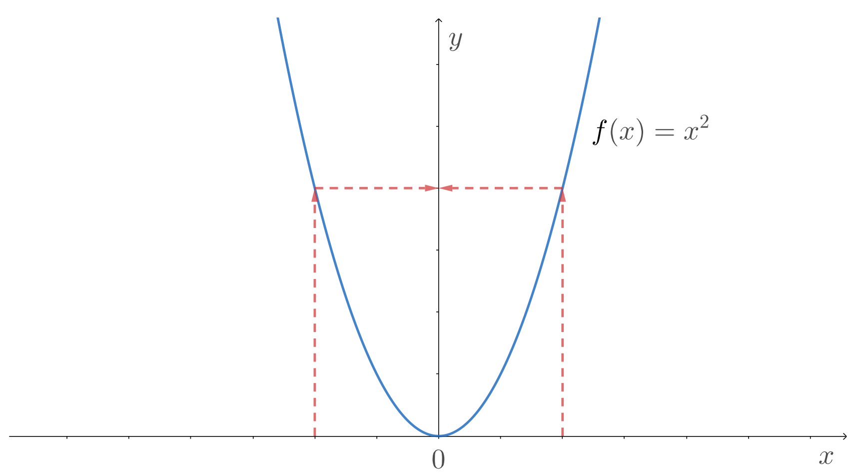
Note. We can use **Vertical Line Test** and **Horizontal Line Test** to determine if an expression is a (one-one or many-one) function.

1. Vertical Line Test: On a graph of a function, no vertical line crosses more than one value. If it crosses more than once, it is not a function.
2. Horizontal Line Test: On a graph of a function, if the horizontal line can cross more than one value, then the function is a many-one function; otherwise, it is a one-one function.

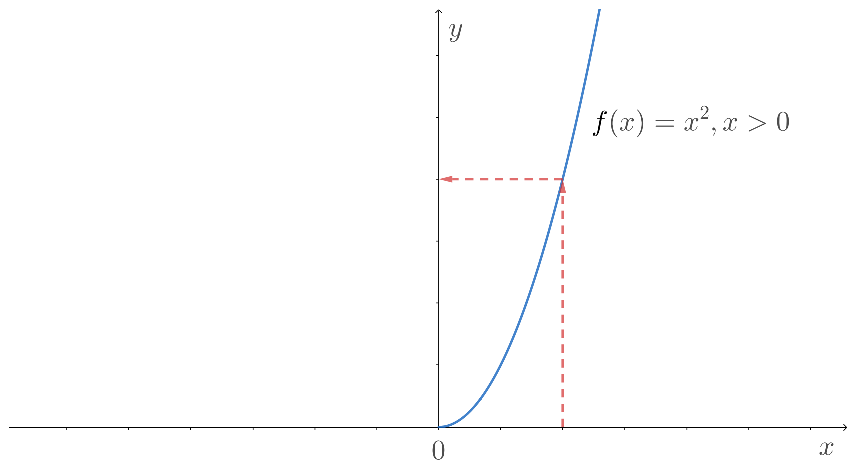
Example 2.1 The function $f(x) = 2x + 1$ where $x \in \mathbb{R}$ is a one-one function.



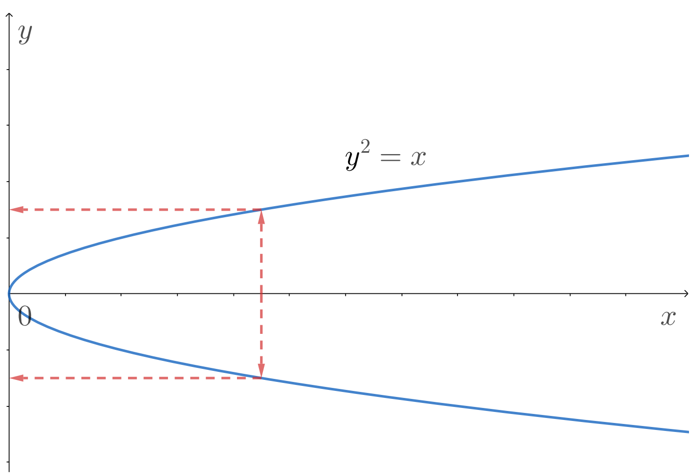
Example 2.2 The function $f(x) = x^2$ where $x \in \mathbb{R}$ is a many-one function.



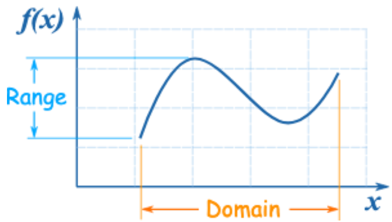
Example 2.3 The function $f(x) = x^2$ where $x > 0$ is a one-one function.



Example 2.4 Consider $y^2 = x$. The input value has two output values, which means this **relation** is **not a function**.



Beyond The Syllabus. 数学中的关系（Relation）需要更严格的概念来定义. 以 $y^2 = x$ 为例，我们可以理解成 $y^2 = x$ 描述了两个变量 x 和 y 之间的关系. 函数是一种特殊的关系.

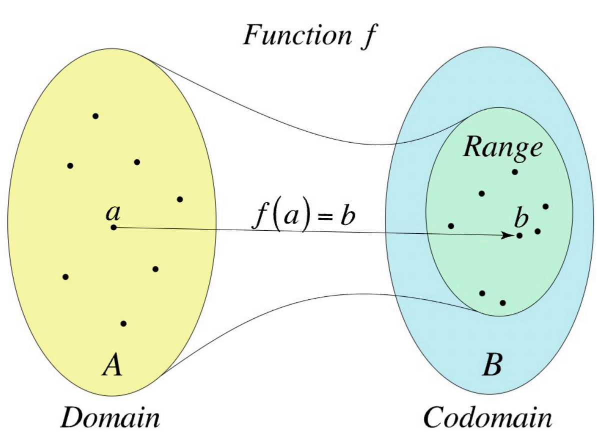


The domain of a function is the set of input values, and the **range** of a function is the set of output values (given input values).

Example 2.5 Consider $f(x) = 2x + 3$, where $-1 < x < 1$. The domain of function f is $-1 < x < 1$, and the range of function f is $1 < f(x) < 5$.

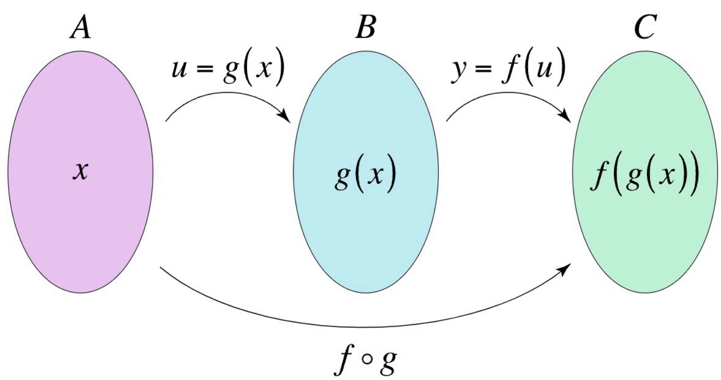
✍ **Exercise 2.1** The function f is defined by $f(x) = (x-1)^2 + 4$ for $-2 \leq x \leq 3$. Find the range of f.

Beyond The Syllabus. 在部分教材中，值域（Range）和培域（Codomain）没有被区分开来. 严格来说，我们称函数的可能输出为培域，而函数的由定义域（Domain）确定的实际输出叫值域.



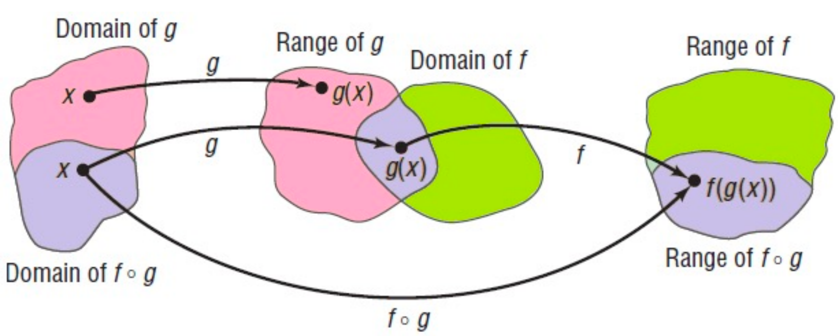
2.2 Composite Functions

A function is like a machine that has an input and an output, and a **composite function** is like a assembly line that puts an input and gets the output after several processes. For example, $x \mapsto 2x+3 \mapsto (2x+3)^2$. We can write $f(x) = x^2$, $g(x) = 2x+3$, and $fg(x) = f(g(x)) = (2x+3)^2$, i.e., first do g then do f.



Key Point 2.1

fg only exists if the range of g is contained within the domain of f.



Example 2.6 Let $f(x) = \sqrt{x}$ for $x > 0$, and $g(x) = x + 5$ for $-10 \le x \le -6$. fg does not exist since the range of f ($-5 \le f(x) \le -1$) is not contained within the domain of g ($x > 0$).

Remark. 在实际解题过程中，（以计算 fg 为例），我们应该时刻关注“先运算的” g 的值域（Range）是否被包含于“后运算的” f 的定义域（Domain）中，尤其要注意 $fg(x) = \sqrt{g(x)}$, $fg(x) = 1/g(x)$ 或一些三角函数.

Example 2.7 Let $f(x) = 2x + 3$ for $x \in \mathbb{R}$, and $g(x) = x^2 - 1$ for $x \in \mathbb{R}$. Find (a) $fg(x)$, (b) $gf(x)$, (c) $ff(x)$, and (d) $f(x)g(x)$.

Solution.

(a) $fg(x) = f(x^2 - 1) = 2(x^2 - 1) + 3 = 2x^2 + 1$.

(b) $gf(x) = g(2x + 3) = (2x + 3)^2 - 1 = 4x^2 + 12x + 8$.

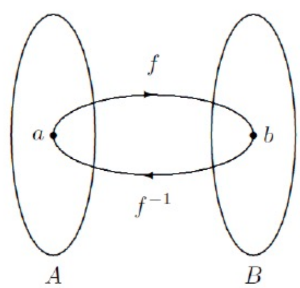
(c) $ff(x) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$.

(d) $f(x)g(x) = (2x + 3)(x^2 - 1) = 2x^3 + 3x^2 - 2x - 3$.

Remark. In general, $fg(x) \neq gf(x)$ and $fg(x) \neq f(x)g(x)$. $ff(x)$ means we apply the function f twice.

2.3 Inverse Functions

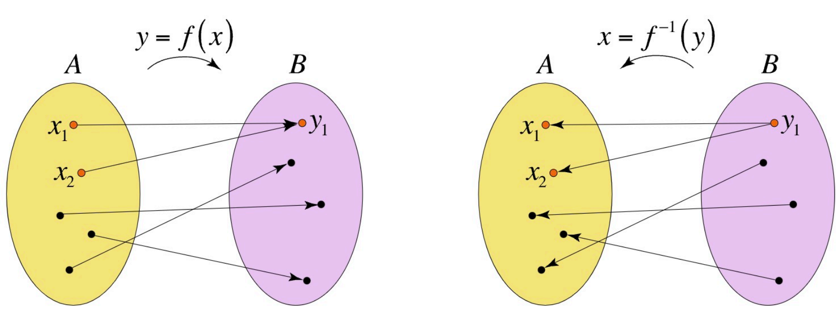
The **inverse of a function** is the function that undoes what $f(x)$ has done. We denote inverse function as f^{-1}. For example, when the function f turns the apple into a banana, then the inverse function f^{-1} turns the banana back to the apple. We give another example: if $f(a) = b$, then $f^{-1}(b) = a$.



Key Point 2.2

$ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

We should notice that not every function has an inverse. Suppose we have a many-one function, and its inverse existed:



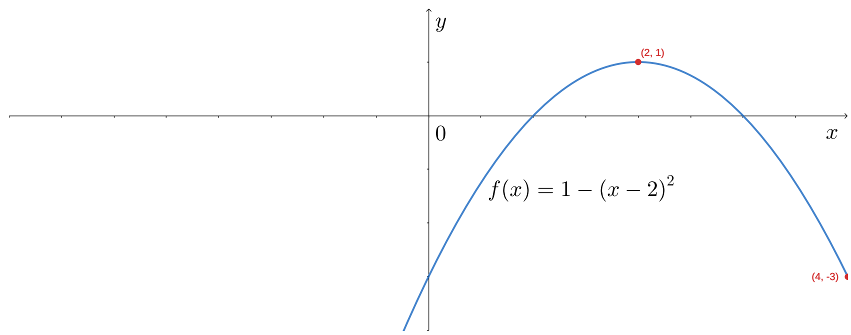
The inverse of the many-one function even is not a function.

Key Point 2.3

$f^{-1}(x)$ exists if, and only if, $f(x)$ is a **one-one** function. $f^{-1}(x)$ is also a one-one function.

Example 2.8 Let $f(x) = 1 - (x-2)^2$ for $k \le x \le 4$. Find the smallest value of k for which f has an inverse.

Solution. When $2 \le 4$, f is a one-one function, and so the smallest value of k for which f has an inverse is $k = 2$.



Note. We can find the inverse of function by following steps:

1. Write the function as $y = f(x)$.
2. Interchange the x and y variables.
3. Rearrange to make y the subject.

Example 2.9 Find the inverse function of $f(x) = \sqrt{x+2} - 7$ for $x \ge -2$.

Solution. We follow three steps:

1. Write the function as $y = f(x)$: $y = \sqrt{x+2} - 7$.
2. Interchange the x and y variables: $x = \sqrt{y+2} - 7$.
3. Rearrange to make y the subject:

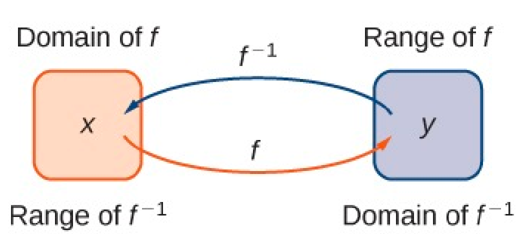
$$\begin{aligned} x + 7 &= \sqrt{y+2} \\ \Rightarrow (x+7)^2 &= y + 2 \\ \Rightarrow y &= (x+7)^2 - 2 \end{aligned}$$

So $f^{-1}(x) = (x+7)^2 - 2$ for $x \ge -7$.

Remark. 在上例中，我们发现 $f(x)$ 的定义域是 $x \ge -2$，而 $f^{-1}(x)$ 的值域是 $f^{-1}(x) \ge -2$. 此外，当 $x \ge -7$ 时，$f^{-1}(x)$ 是一个一对一函数，正好对应 $f(x)$ 的值域 $f(x) \ge -7$.

Key Point 2.4

The domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$.



Example 2.10 Let $f(x) = \sqrt{x+2} - 7$ for $x \geq -2$, solve $f^{-1}(x) = f(62)$.

Solution. We know $f^{-1}(x) = (x+7)^2 - 2$ for $x \geq -7$, and $f(62) = \sqrt{62+2} - 7 = 1$. Then

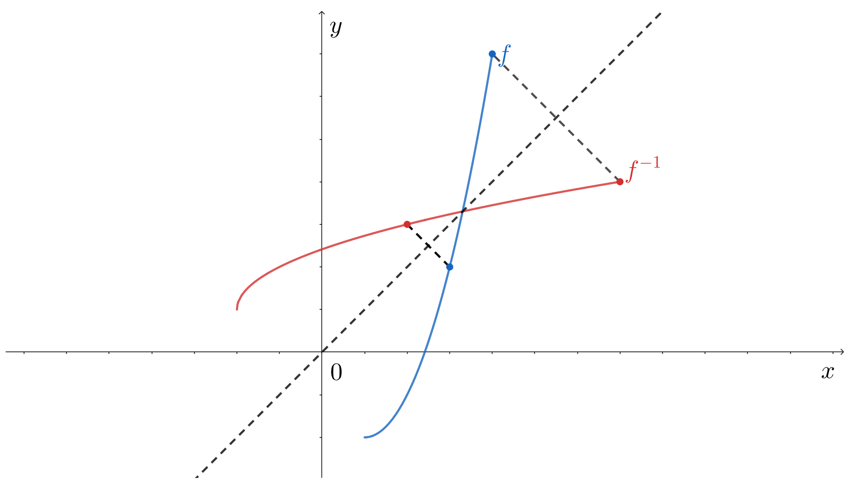
$$\begin{aligned} & f^{-1}(x) = f(62) \\ \Leftrightarrow\ & (x+7)^2 - 2 = 1 \\ \Leftrightarrow\ & x = -7 \pm \sqrt{3} \end{aligned}$$

Since the domain of f^{-1} is $x \geq -7$, then we solve for $x = -7 + \sqrt{3}$.

In some cases, f and f^{-1} are the same function, then f is called a **self-inverse function**. For example, if $f(x) = x$, then $f^{-1}(x) = x$, and thus $f(x) = x$ is a self-inverse function.

2.3.1 Graph of a Function and its Inverse

Consider the graph of f and any point (a, b) on f. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, point (b, a) is on f^{-1}. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.



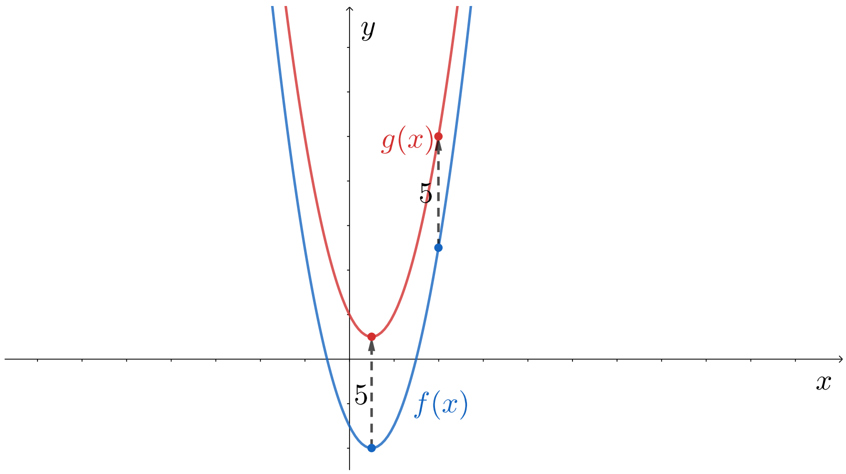
Key Point 2.5

For each one-one function, then the graphs of f and f^{-1} are reflections of each other in the line $y = x$.

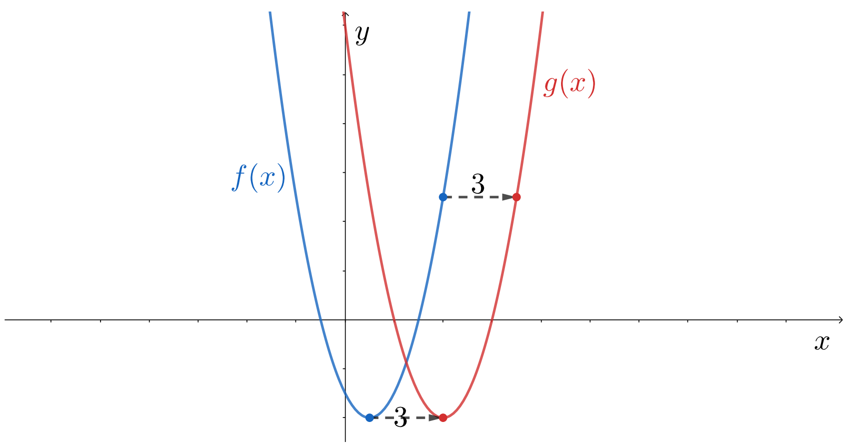
2.4 Transformations of Functions

2.4.1 Translations

Example 2.11 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = f(x) + 5 = x^2 - 2x + 2$. Two curves have exactly the same shape but they are separated by 5 units in the positive y direction.



Example 2.12 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = f(x-3) = (x-3)^2 - 2(x-3) - 3$. Two curves have exactly the same shape but they are separated by 3 units in the positive x direction.



Key Point 2.6

The graph of $y = f(x) + a$ is a **translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

The graph of $y = f(x - a)$ is a **translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

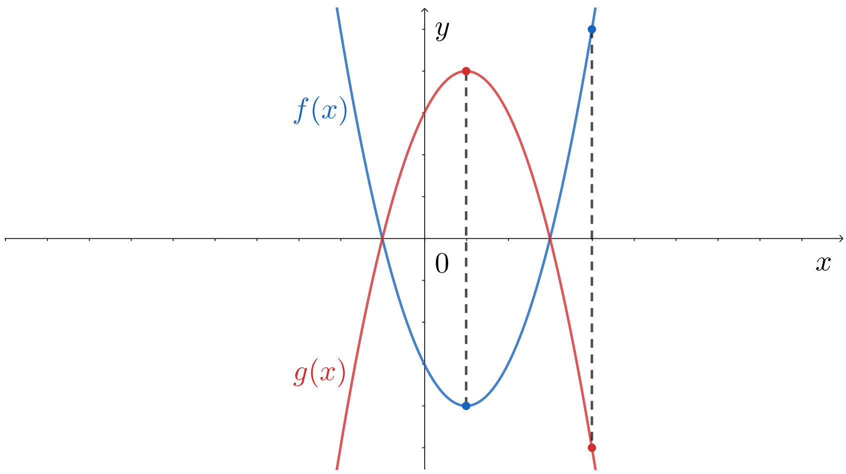
The graph of $y = f(x - a) + b$ is a **translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 2.13 The graph of $y = \sqrt{2x}$ is translated by the vector $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$. Find the equation of the resulting graph.

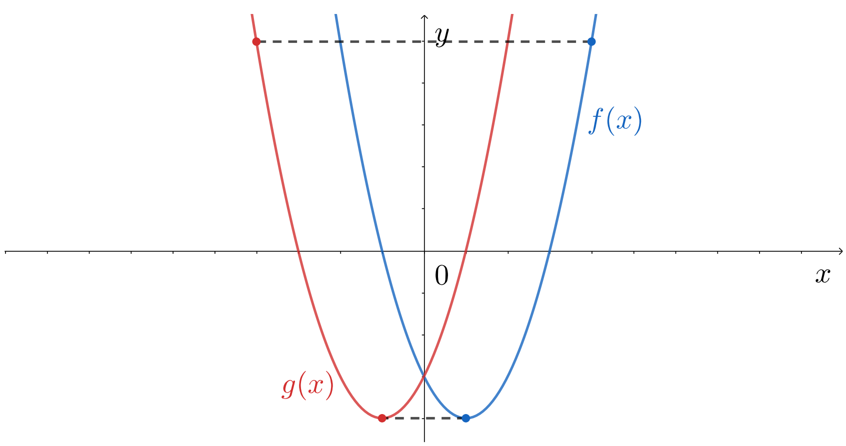
Solution. Let $f(x) = \sqrt{2x}$, the resulting equation is $f(x+5) + 3 = \sqrt{2(x+5)} + 3 = \sqrt{2x+10} + 3$.

2.4.2 Reflections

Example 2.14 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = -f(x) = -(x^2 - 2x - 3)$. The graph of $g(x) = -f(x)$ is a reflection of the graph of $f(x)$ in the x-axis.



Example 2.15 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = f(-x) = (-x)^2 - 2(-x) - 3$. The graph of $g(x) = f(-x)$ is a reflection of the graph of $f(x)$ in the y-axis.



Key Point 2.7

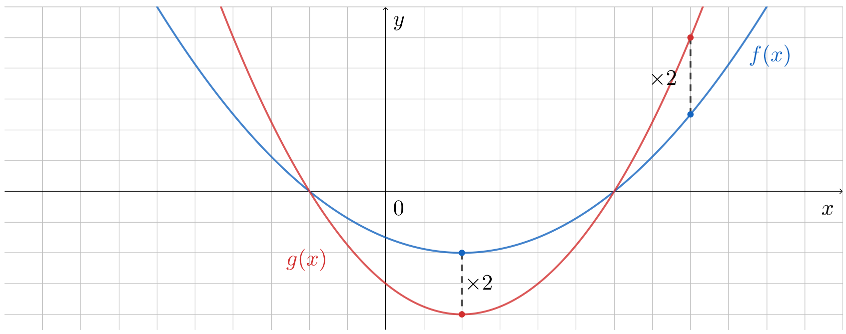
The graph of $y = -f(x)$ is a **reflection** of the graph $y = f(x)$ in the x-axis.

The graph of $y = f(-x)$ is a **reflection** of the graph $y = f(x)$ in the y-axis.

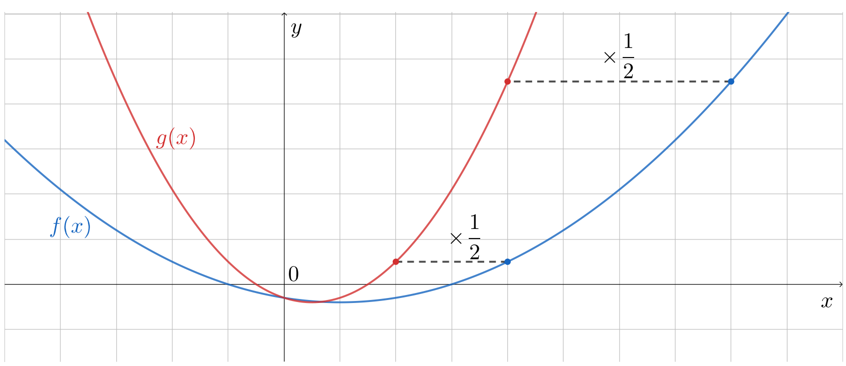
✍ **Exercise 2.2** The quadratic graph $y = f(x)$ has a minimum at the point $(1, -4)$. Find the coordinates of the vertex of (a) $y = -f(x)$, and (b) $y = f(-x)$. State whether it is a maximum or minimum of the graph.

2.4.3 Stretches

Example 2.16 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = 2f(x) = 2(x^2 - 2x - 3)$. When the x-coordinate is the same, the y-coordinate on $g(x)$ is double the y-coordinate on $f(x)$.



Example 2.17 Consider the graph of $f(x) = x^2 - 2x - 3$ and $g(x) = f(2x) = (2x)^2 - 2(2x) - 3$. When the y-coordinate is the same, the x-coordinate on $g(x)$ is half the x-coordinate on $f(x)$.



Key Point 2.8

The graph of $y = af(x)$ is a **(vertical) stretch** of the graph $y = f(x)$ with stretch factor a in y-direction.
The graph of $y = f(ax)$ is a **(horizontal) stretch** of the graph $y = f(x)$ with stretch factor $\frac{1}{a}$ in x-direction.

Example 2.18 Describe the single transformation that maps the graph of $y = x^2 - 3x - 5$ to the graph of $y = 4x^2 - 6x - 5$.

Solution. Let $f(x) = x^2 - 3x - 5$. Since $4x^2 - 6x - 5 = (2x)^2 - 3(2x) - 5 = f(2x)$, then the transformation is a stretch with factor $\frac{1}{2}$ in x-direction.

2.4.4 Combined Transformations

We can categorise the transformations in this section as either vertical or horizontal transformations.

Category	Formation	Implication
Vertical transformations	$y = f(x) + a$	Translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$
	$y = -f(x)$	Reflection in the x-axis
	$y = af(x)$	Stretch with factor a in y-direction
Horizontal transformations	$y = f(x + a)$	Translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
	$y = f(-x)$	Reflection in the y-axis
	$y = f(ax)$	Stretch with factor $\frac{1}{a}$ in x-direction

2.4.4.1 Combining Two Vertical Transformations

Example 2.19 We now consider how the graph of $f(x) = x^2 - 2x + 1$ is transformed to the graph $g(x) = 3f(x) + 4 = 3x^2 - 6x + 7$. Suppose there are two options:

1. First stretch with factor 3 in y-direction, then translate by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$. So
$$f(x) = x^2 - 2x + 1 \to 3f(x) = 3x^2 - 6x + 3 \to (3f(x)) + 4 = 3x^2 - 6x + 7 = g(x).$$
2. First translate by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, then stretch with factor 3 in y-direction. So
$$f(x) = x^2 - 2x + 1 \to f(x) + 4 = x^2 - 2x + 5 \to 3(f(x) + 4) = 3x^2 - 6x + 15 \neq g(x).$$

Hence, option 2 is incorrect.

Key Point 2.9

Vertical transformations follow the normal order of operations, as used in arithmetic.

✍ **Exercise 2.3** The graph of $y = f(x)$ is transformed to the graph of $y = 2f(x) - 3$. Describe fully the two single transformations which have been combined to give the resulting transformation.

2.4.4.2 Combining Two Horizontal Transformations

Example 2.20 We now consider how the graph of $f(x) = x^2 - 2x + 1$ is transformed to the graph $g(x) = f(2x+1) = 4x^2$. Suppose there are two options:

1. First stretch with factor $\frac{1}{2}$ in x-direction, then translate by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. So

$$\begin{aligned} & f(x) = x^2 - 2x + 1 \\ \rightarrow\ & f(2x) = (2x)^2 - 2(2x) + 1 = 4x^2 - 4x + 1 \\ \rightarrow\ & f(2(x+1)) = 4(x+1)^2 - 4(x+1) + 1 = 4x^2 + 4x + 1 \neq g(x). \end{aligned}$$

2. First translate by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then stretch with factor $\frac{1}{2}$ in x-direction. So

$$\begin{aligned} & f(x) = x^2 - 2x + 1 \\ \rightarrow\ & f(x+1) = (x+1)^2 - 2(x+1) + 1 = x^2 \\ \rightarrow\ & f(2x+1) = (2x)^2 = 4x^2 = g(x). \end{aligned}$$

Hence, option 1 is incorrect.

Key Point 2.10

Horizontal transformations follow the **opposite** order to the normal order of operations, as used in arithmetic.

Exercise 2.4 The graph of $y = f(x)$ is transformed to the graph of $y = f(3x - 4)$. Describe fully the two single transformations which have been combined to give the resulting transformation.

2.4.4.3 Combining One Horizontal and One Vertical Transformation

Example 2.21 We now consider how the graph of $f(x) = x^2 - 2x + 1$ is transformed to the graph $g(x) = f(3x) + 2 = 9x^2 - 6x + 3$. Suppose there are two options:

1. First stretch with factor $\dfrac{1}{3}$ in x-direction, then translate by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. So

$$\begin{aligned} &f(x) = x^2 - 2x + 1 \\ \to\ &f(3x) = (3x)^2 - 2(3x) + 1 = 9x^2 - 6x + 1 \\ \to\ &f(3x) + 2 = 9x^2 - 6x + 3 = g(x). \end{aligned}$$

2. First translate by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, then stretch with factor $\dfrac{1}{3}$ in x-direction. So

$$\begin{aligned} &f(x) = x^2 - 2x + 1 \\ \to\ &f(x) + 2 = x^2 - 2x + 3 \\ \to\ &f(3x) + 2 = (3x)^2 - 2(3x) + 3 = 9x^2 - 6x + 3 = g(x). \end{aligned}$$

Key Point 2.11

When one horizontal and one vertical transformation are combined, the order in which they are applied does **NOT** affect the outcome.

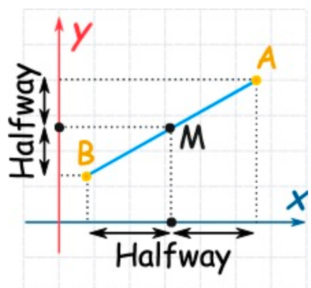
✍ **Exercise 2.5** The graph of $y = f(x)$ is transformed to the graph of $y = 3f(x-4)$. Describe fully the two single transformations which have been combined to give the resulting transformation.

Chapter 3 Coordinate Geometry

3.1 Straight Lines

3.1.1 Length and Midpoint of a Line Segment

Let the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

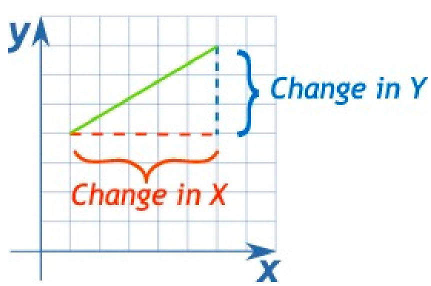


Key Point 3.1

The **midpoint** M of the line segment AB is $M = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

The **length** of AB is $AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

The **gradient** or **slope** of a line is calculated by finding the ratio of the vertical change to the horizontal change between (any) two distinct points on a line.



Key Point 3.2

Gradient of $AB = \dfrac{y_1-y_2}{x_1-x_2}$.

3.1.2 Parallel and Perpendicular Lines

Two lines are **parallel** if they will never meet. **If two lines are parallel, then their gradients are equal.**

Remark. 两条直线的梯度（Gradient）相等并不能得出两条直线平行（Parallel），两条直线也有可能完全重合.

Two lines are **perpendicular** when they meet at a right angle ($90°$). **If a line has gradient m, then the perpendicular line has gradient $-\frac{1}{m}$.**

Key Point 3.3

If the gradients of two perpendicular lines are m_1 and m_2, then $m_1 m_2 = -1$.

Remark. For a vertical line, the gradient is undefined as we cannot divide by 0, and so we cannot use the gradient to find the parallel and perpendicular line. It is noted that a vertical line is parallel to another vertical line, and a vertical line is perpendicular to a horizontal line (and vice versa).

3.1.3 Equations of Straight Lines

The **equation of a straight line** is usually written as

$$y = mx + c$$

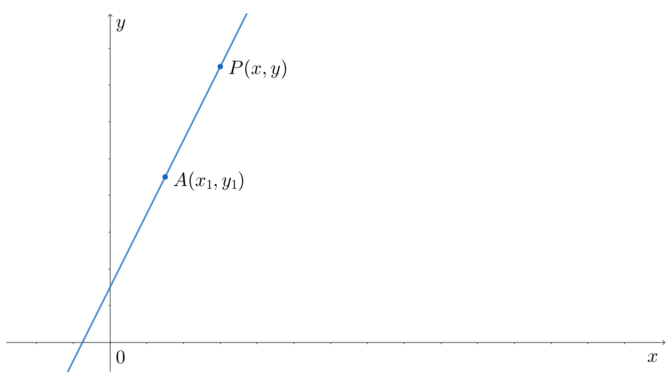
where m is the gradient and c is the y-intercept, when the line is non-vertical.

When the line is vertical, we write

$$x = b$$

where b is the x-intercept.

There is an alternative formula that we can use when we know the gradient of a straight line and a point on the line.



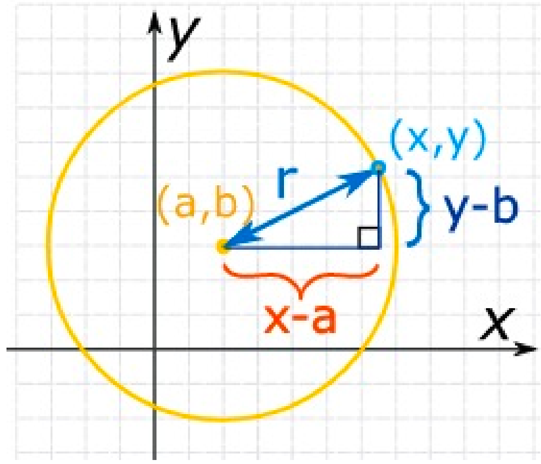
Consider a line, with gradient m that passes through the known point $A(x_1, y_1)$. Assume an arbitrary point $P(x, y)$ on the line. The gradient of AP is m and so $\frac{y - y_1}{x - x_1} = m$, leading to $y - y_1 = m(x - x_1)$.

Key Point 3.4

The equation of a straight line, with gradient m, that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

3.2 Equations of Circles

A **circle** is defined as the collection of all points on a plane that are a fixed distance (the **radius**) from a **centre**. Assume the centre is (a, b) and an arbitrary point (x, y) on the circle.



Using Pythagoras' Theorem on triangle, we have $(x-a)^2+(y-b)^2=r^2$, which is the equation of the circle.

Key Point 3.5

The equation of a circle with centre (a, b) and radius r can be written in completed square form as

$$(x-a)^2+(y-b)^2=r^2.$$

Example 3.1 A is the point $(3,0)$ and B is the point $(7,-4)$. Find the equation of the circle that has AB as a **diameter** (any straight line segment that passes through the centre of the circle and whose endpoints lie on the circle).

Solution. The centre of the circle C is the midpoint of AB:

$$C=\left(\frac{3+7}{2}, \frac{0+(-4)}{2}\right)=(5,-2).$$

The radius of circle r is equal to CA, where

$$r=\sqrt{(5-3)^2+(-2-0)^2}=\sqrt{8}.$$

So the equation of circle is $(x-a)^2+(y-b)^2=r^2$ where $a=5, b=-2$ and $r=\sqrt{8}$, leading to

$$(x-5)^2+(y+2)^2=8.$$

Remark. 直径（Diameter）的长度是半径（Radius）长度的两倍.

Expanding the equation $(x-a)^2+(y-b)^2=r^2$ gives $x^2-2ax+a^2+y^2-2by+b^2=r^2$, and rearranging gives $x^2+y^2-2ax-2by+(a^2+b^2-r^2)=0$. In this form, we should notice that for any circle, the coefficients of x^2 and y^2 are equal, and there is no xy term. We can rewrite it as a general form: $x^2+y^2+2gx+2fy+c=0$, where $(-g,-f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius.

Remark. 我们不需要去记忆圆的一般形式，但我们需要根据一般形式计算圆心、半径等. 计算的常用方法是配方法.

Example 3.2 Find the centre and the radius of the circle $x^2+y^2+10x-8y-40=0$.

Solution. Using completing the square:

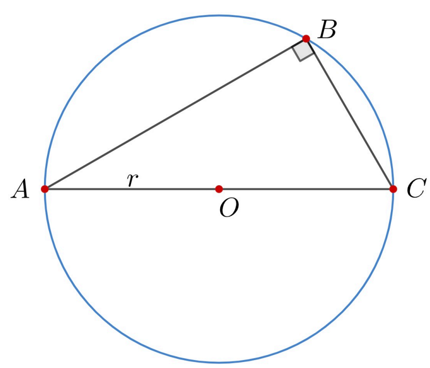
$$\begin{aligned} & x^2+10x+y^2-8y-40=0 \\ \Leftrightarrow\ & (x+5)^2-5^2+(y-4)^2-4^2-40=0 \\ \Leftrightarrow\ & (x+5)^2+(y-4)^2=81. \end{aligned}$$

So the centre is $(-5,4)$ and the radius is 9.

It is useful to remember the following right angle properties for circles:

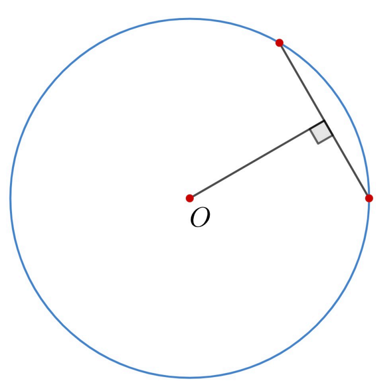
1. (a) The angle in a semicircle is a right angle.

   (b) If triangle ABC is right angled at B, then the points A, B and C lie on the circumference of a circle with AC as diameter.

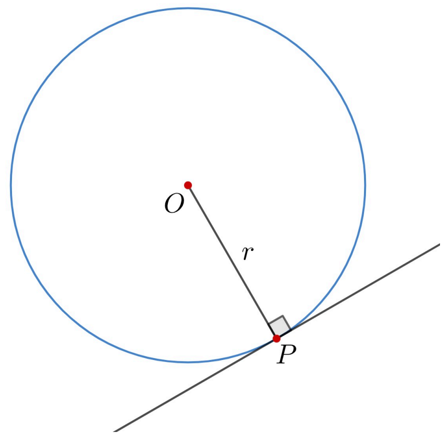


2. (a) The perpendicular from the centre of a circle to a chord bisects the chord.

   (b) The perpendicular bisector of a chord passes through the centre of the circle.



3. (a) The tangent to a circle at a point P is perpendicular to the radius at P.
(b) If a radius and a line at a point P, on the circumference are at right angles, then the line must be a tangent to the curve.



Beyond The Syllabus. 我们给出关于上面三组性质的证明思路（不唯一）:（1a）连接圆心和直角顶点，利用等腰三角形底角相等和三角形的内角和；（1b）利用两个向量的夹角成直角时，两个向量的内积为零.（2a）连接圆心和弦（Chord）的两个顶点，利用勾股定理；（2b）利用中垂线的性质.（3a）假设半径与切线垂直于另外一点；（3b）假设直线不是切线.

Example 3.3 A circle passes through the points $O(0,0)$, $A(8,4)$ and $B(6,6)$. Find the equation of this circle.

Solution. Since $OA = \sqrt{8^2+4^2} = \sqrt{80}, OB = \sqrt{6^2+6^2} = \sqrt{72}$ and $AB = \sqrt{(8-6)^2+(4-6)^2} = \sqrt{8}$, then $AB^2 + OB^2 = OA^2$ and thus triangle OAB is right angled at B. Hence, OA is the diameter. The centre, say D is the midpoint of OA:

$$D = \left(\frac{0+8}{2}, \frac{0+4}{2}\right) = (4,2)$$

and the radius is the half length of OA:

$$r = \frac{1}{2}\sqrt{80} = \sqrt{20}.$$

The equation of the circle is $(x-4)^2 + (y-2)^2 = 20$.

3.3 Position of Line and Circle

Recall that we can use discriminant to determine the position of line and (general) quadratic curve:

$b^2 - 4ac$	Number of roots	Position of line and curve
> 0	Two **distinct** real roots	**Two distinct** points of intersection
$= 0$	One **repeated** real root	**One point** of intersection (line is a **tangent**)
< 0	**No** real roots	**No** points of intersection

Remark. A general quadratic has the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$, where at least a, b and c is non-zero. The equation of a circle is a general quadratic!

Example 3.4 Show that the line $y = x - 13$ is a tangent to the circle $x^2 + y^2 - 8x + 6y + 7 = 0$.

Proof. Substituting $x - 13$ for y in the equation of circle:

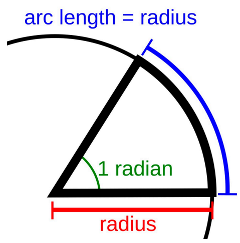
$$\begin{aligned} & x^2 + (x-13)^2 - 8x + 6(x-13) + 7 = 0 \\ \Leftrightarrow\ & x^2 - 14x + 49 = 0. \end{aligned}$$

The discriminant of $x^2 - 14x + 49 = 0$ is $14^2 - 4 \times 1 \times 49 = 196 - 196 = 0$, and thus the line is a tangent.

Remark. We can also solve for the coordinates of the intersection point(s).

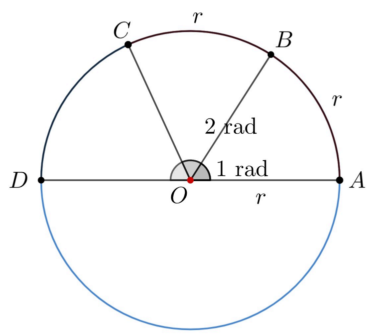
Chapter 4 Circular Measure

4.1 Radians



The **radian** is a pure measure based on the radius of the circle. An arc of a circle with the same length as the radius of that circle subtends an angle of 1 radian.

According to the definition, in the figure, the angle AOB is 1 radian, and the angle AOC is 2 radians.



More importantly, angle AOD is π radian. Therefore,

Key Point 4.1

π radians$=180^\circ$ and 2π radians$=360^\circ$.

Remark. When an angle is written in terms of π, we usually omit the word radian, and so $\pi = 180^\circ$ and $2\pi = 360^\circ$.
Since $180^\circ = \pi$, then $90^\circ = \dfrac{\pi}{2}$, $45^\circ = \dfrac{\pi}{4}$, etc.

Key Point 4.2

To change from degrees to radians, multiply by $\dfrac{\pi}{180}$.

Since $\pi = 180°$, then $\dfrac{\pi}{6} = 30°$, $\dfrac{\pi}{10} = 18°$, etc.

Key Point 4.3

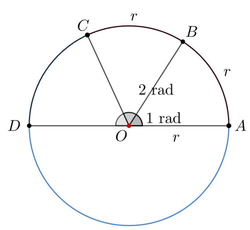
To change from radians to degrees, multiply by $\dfrac{180}{\pi}$.

Remark. It is useful to remember that 1 radian$= 1 \times \dfrac{180}{\pi} \approx 57°$.

We should remember some important angles in degrees and radians.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π

4.2 Length of an Arc



In the figure, an arc that subtends an angle of 1 radian at the centre of the circle is of length r; an arc that subtends an angle of 2 radians at the centre of the circle is of length $2r$.

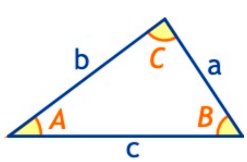
Key Point 4.4

Suppose an arc subtends an angle of θ radians at the centre, the **length of the arc** is $r\theta$.

Example 4.1 A sector has an angle of 1.5 radians and an arc length of 12 cm. Find the radius of the sector.

Solution. Since arc length is $r\theta$, then $12 = 1.5r$ leading to $r = 8$ cm.

4.2.1 Recall: Sine Law and Cosine Law



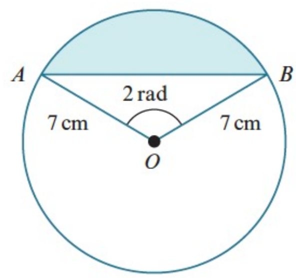
For any triangle, if we denote a, b and c as sides, denote A, B and C as angles, then the **Sine Law** is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The **Cosine Law** is

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

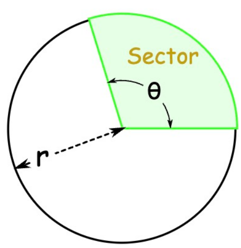
Example 4.2 Suppose the circle has radius 7 cm and centre O. AB is chord and angle $AOB = 2$ radians. Find: (a) the length of arc AB, (b) the length of chord AB, and (c) the perimeter of the shaded segment.



Solution.
(a) The length of arc AB is $r\theta = 7 \times 2 = 14$ cm.
(b) Using the cosine rule $AB^2 = OA^2 + OB^2 - 2OA \times OB\cos(\angle AOB) = 7^2 + 7^2 - 2 \times 7 \times 7\cos 2 = 98 - 98\cos 2$. So $AB \approx 11.8$ cm.
(c) The perimeter of the shaded segment is

$$\text{Arc } AB + \text{Chord } AB = 14 + 11.8 = 25.8 \text{ cm}.$$

4.3 Area of a Sector



To find the formula for the **area of a sector**, we use the ratio:

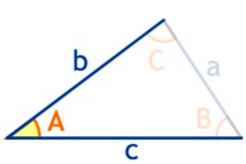
$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Angle in the sector}}{\text{Complete angle at the centre}}$$
$$\Leftrightarrow \frac{\text{Area of sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

leading to

Key Point 4.5

Area of sector $= \frac{1}{2}r^2\theta$.

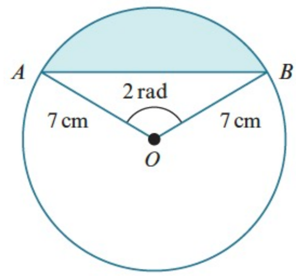
4.3.1 Recall: Area of a Triangle



When we know two sides and the included angle in a triangle, we can find the area using

$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}ac\sin B.\end{aligned}$$

Example 4.3 Suppose the circle has radius 7 cm and centre O. AB is chord and angle $AOB = 2$ radians. Find: (a) the area of sector AOB, (b) the area of triangle AOB, and (c) the area of the shaded segment.



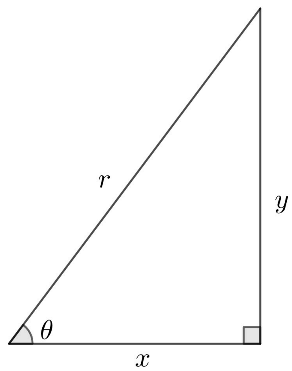
Solution.
(a) The area of sector AOB is $\frac{1}{2}r^2\theta = \frac{1}{2} \times 7^2 \times 2 = 49$ cm^2.
(b) The area of triangle AOB is $\frac{1}{2}OA \times OB \sin(\angle AOB) = \frac{1}{2} \times 7 \times 7 \sin 2 \approx 22.3$ cm^2.
(c) The area of shaded segment is

$$\text{Sector } AOB - \text{Triangle } AOB = 49 - 22.3 = 26.7 \text{ cm}^2.$$

Chapter 5 Trigonometry

5.1 Trigonometric Functions

5.1.1 Angles between 0° and 90°

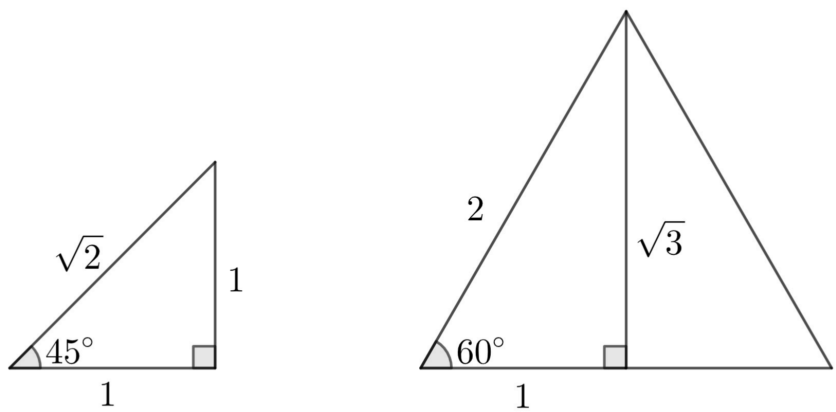


We define three important trigonometric functions in a right triangle:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{r}$$
$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{r}$$
$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y}{x}.$$

✍ **Exercise 5.1** Suppose $\cos\theta = \dfrac{\sqrt{5}}{3}$, where $0^\circ \le \theta \le 90^\circ$. Find (a) $\sin\theta$, and (b) $\tan\theta$.

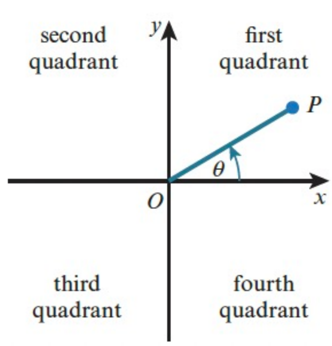
We should remember the exact values of the sine, cosine and tangent of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ (or $30°$, $45°$ and $60°$). Considering two special triangles: right-angled isosceles (whose two equal sides are of length 1 unit) and equilateral triangle (whose sides are of length 2 units).



Then we have

	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\theta = 30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\theta = 45° = \frac{\pi}{6}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\theta = 60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

5.1.2 General Definition of an Angle



To be able to use the three basic trigonometric functions for any angle, we need a general definition of an angle.

An **angle** is a measure of the rotation of a line segment OP about a fixed point O. The angle is measured from the positive x-direction. An anticlockwise rotation is taken as positive and a clockwise rotation is taken as negative.

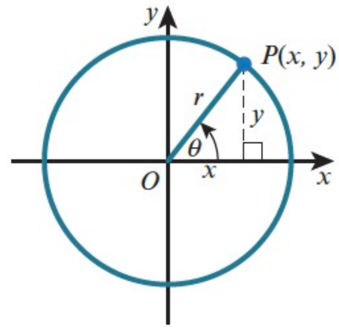
The Cartesian plane is divided into four **quadrants**, and the angle θ is said to be in the quadrant where OP lies.

✍ **Exercise 5.2** Draw a diagram showing the quadrant in which the rotation line OP lies for each of the following angles: (a) $120°$, (b) $430°$, (c) $\dfrac{3\pi}{4}$, and (d) $-\dfrac{2\pi}{3}$. In each case, find the acute angle that the line OP makes with the x-axis.

Remark. The acute angle made with the x-axis is called the **basic angle** or the **reference angle**.

5.1.3 Trigonometric Ratios of General Angles

In general, trigonometric ratios of any angle θ in any quadrant are defined as:



Key Point 5.1

$\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$ and $\tan\theta = \dfrac{y}{x}$ when $x \neq 0$.

We should care about the signs of the three trigonometric ratios in each of the four quadrants.

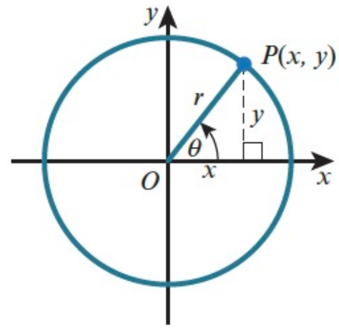
Exercise 5.3 Determine the signs of the three trigonometric ratios in each of the four quadrants.

Remark. 我们可以用一个简单的口诀来记忆三个三角函数的符号情况：**一全二正三切四余**，意思是：当 OP 落在**第一象限**时，三个三角函数的取值**全**为正数；落在**第二象限**时，**正弦函数**（Sine）的取值为正数；落在**第三象限**时，**正切函数**（Tangent）的取值为正数；落在**第四象限**时，**余弦函数**（Cosine）的取值为正数.

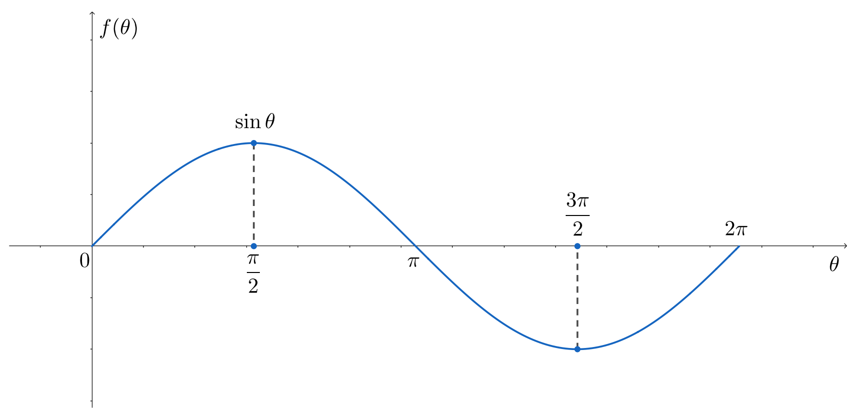
Exercise 5.4 Express in terms of trigonometric ratios of acute angles: (a) $\sin 120°$, (b) $\cos(-130°)$, and (c) $\tan 220°$.

5.1.4 Graphs of Trigonometric Functions

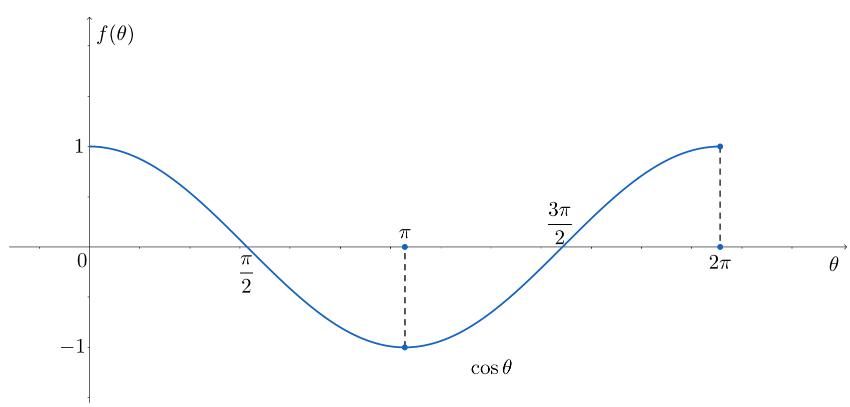
Suppose that OP makes an angle of θ with the positive horizontal axis and that P moves around the unit circle (i.e., $x^2 + y^2 = 1$), then the coordinates of $P(x, y)$ can be written as $P(\cos x, \sin x)$.



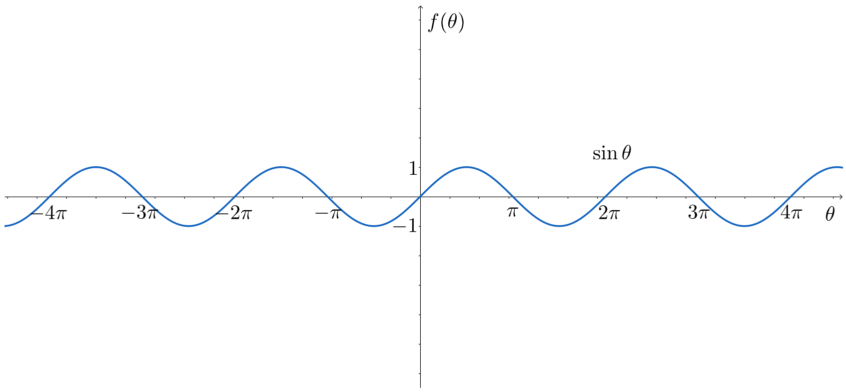
We observe that the height of P above the horizontal axis changes from $0 \to 1 \to 0 \to -1 \to 0$. After we record every value of the height of P above the horizontal axis with the change in θ, we can draw the graph of $f(\theta) = \sin\theta$ ($0° \le \theta \le 360°$) as:

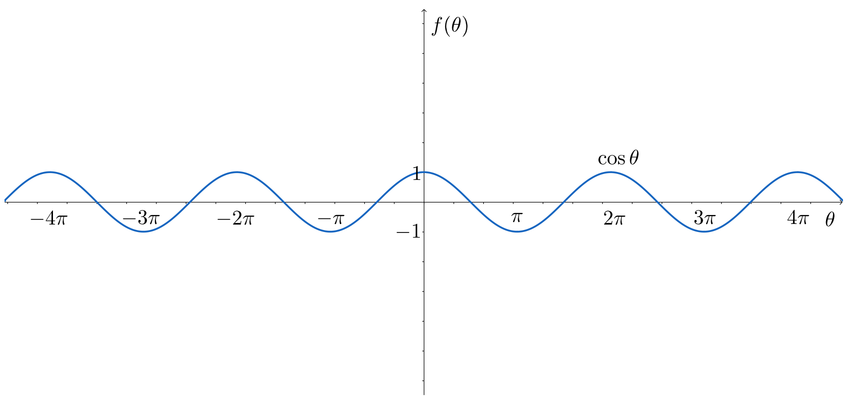


We also observe that the displacement of P from the vertical axis changes from $1 \to 0 \to -1 \to 0 \to 1$. After we record every value of the displacement of P from vertical axis with the change in θ, we can draw the graph of $f(\theta) = \cos\theta$ ($0° \le \theta \le 360°$) as:



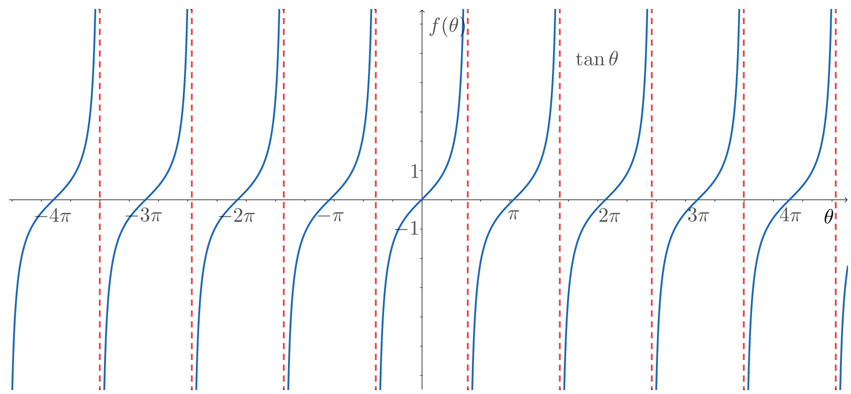
Remark. The graphs of $y = \sin x$ and $y = \cos x$ can be continued beyond $0° \leq \theta \leq 360°$.





We can see that the sine and cosine functions repeat themselves over and over again, and we call them **periodic functions**. For periodic functions, we pay attention to:

1. **Period**: the length of one repetition or cycle. The sine and cosine functions repeat every $360°$ and they have a period of $360°$ (or 2π).
2. **Amplitude**: the height from the center line to the peak (or to the trough) (or we can measure the height from highest to lowest points and divide that by 2). The sine and cosine functions have amplitude 1.



The tangent function behaves differently to the sine and cosine functions:

1. The tangent function repeats its cycle every $180°$ so its period is $180°$ (or π).
2. The red dashed lines at $\theta = \dfrac{\pi}{2} + n\pi$ for n is integer are called **asymptotes**. The branches of the graph get closer and closer to the asymptotes without ever reaching them, since at such θ, $\cos\theta = 0$ and $\tan\theta$ is undefined.
3. The tangent function does not have an amplitude.

5.1.5 Transformations of Trigonometric Functions

Recall the transformations of any functions:

Category	Formation	Implication
Vertical transformations	$y = f(x) + a$	Translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$
	$y = -f(x)$	Reflection in the x-axis
	$y = af(x)$	Stretch with factor a in y-direction
Horizontal transformations	$y = f(x+a)$	Translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
	$y = f(-x)$	Reflection in the y-axis
	$y = f(ax)$	Stretch with factor $\dfrac{1}{a}$ in x-direction

At first, we consider some important transformations. Because of the symmetry of the curve, we have

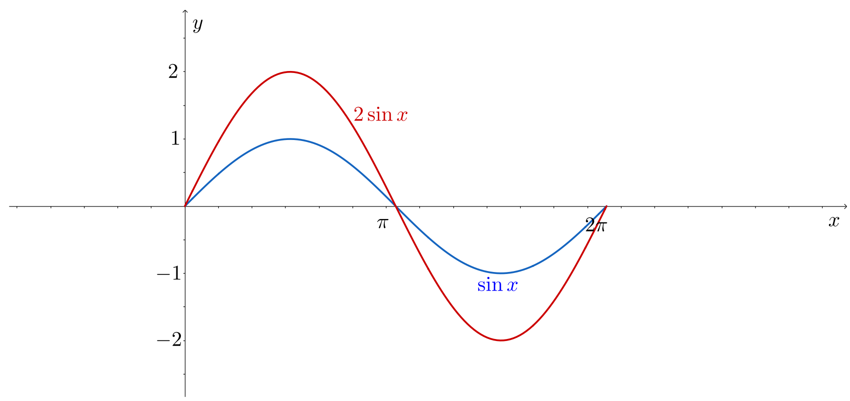
$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\sin(180° - x) = \sin x$	$\cos(180° - x) = -\cos x$	$\tan(180° - x) = -\tan x$
$\sin(180° + x) = -\sin x$	$\cos(180° + x) = -\cos x$	$\tan(180° + x) = \tan x$
$\sin(360° - x) = -\sin x$	$\cos(360° - x) = \cos x$	$\tan(360° - x) = -\tan x$
$\sin(360° + x) = \sin x$	$\cos(360° + x) = \cos x$	$\tan(360° + x) = \tan x$

Beyond The Syllabus. 令 n 为任意整数（$n \in \mathbb{Z}$），$\sin(n\pi \pm x)$、$\cos(n\pi \pm x)$ 及 $\tan(n\pi \pm x)$ 都会得到同名函数，即仍然是 sin、cos 及 tan，但它们的符号可以通过象限来判断，也即我们可以记住口诀：符号看象限. 以 $\tan(180° + x)$ 为例，因为 $180° = \pi = 1\pi$，所以 $\tan(180° + x)$ 得到同名函数. 在这里我们假设 x 是任意锐角（Acute Angle），因为 $180° + x$ 落在了第三象限，根据“一全二正三切四余”的口诀，我们知道符号为正，因此 $\tan(180° + x) = \tan x$. 再以 $\cos(360° - x)$ 为例，因为 $360° = 2\pi$，所以 $\cos(360° - x)$ 得到同名函数. 因为 $360° - x$ 落在了第四象限，根据“一全二正三切四余”的口诀，我们知道符号为正，因此 $\cos(360° - x) = \cos x$.

5.1.5.1 Stretch

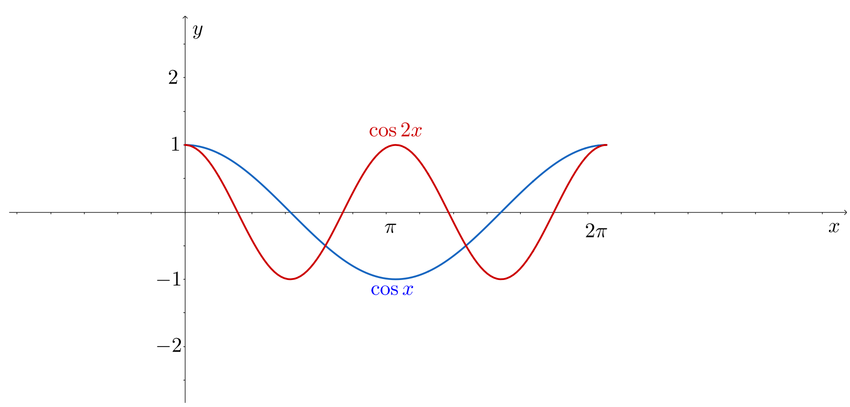
Consider the graph of $y = a\sin x$ or $y = a\cos x$. It is a stretch with factor a in y-direction, so the amplitude of $y = a\sin x$ or $y = a\cos x$ is a, and the period is 2π.

Example 5.1 The graph of $y = 2\sin x$ is



Consider the graph of $y = \sin ax$ or $y = \cos ax$. It is a stretch with factor $\frac{1}{a}$ in x-direction, so the amplitude of $y = \sin ax$ or $y = \cos ax$ is 1, and the period is π.

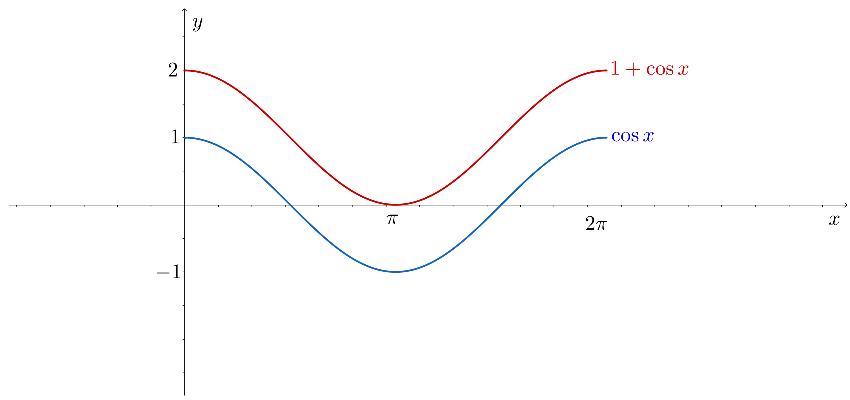
Example 5.2 The graph of $y = \cos 2x$ is



5.1.5.2 Translation

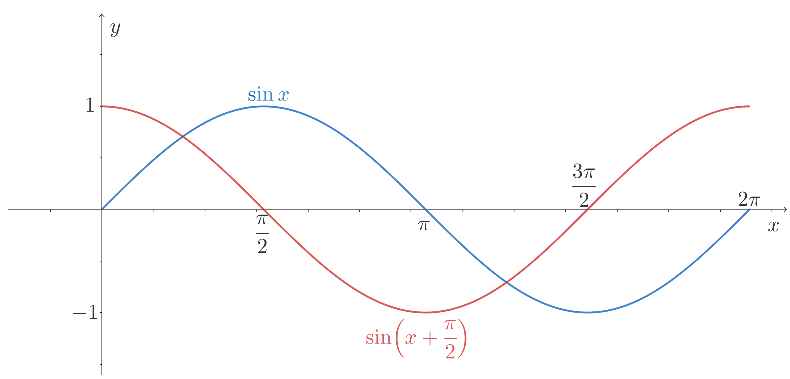
Consider the graph of $y = a + \sin x$ or $y = a + \cos x$. It is a translation of $\begin{pmatrix} 0 \\ a \end{pmatrix}$, so the amplitude of $y = a + \sin x$ or $y = a + \cos x$ is 1, and the period is 2π.

Example 5.3 The graph of $y = 1 + \cos x$ is



Consider the graph of $y = \sin(x + a)$ or $y = \cos(x + a)$. It is a translation of $\begin{pmatrix} -a \\ 0 \end{pmatrix}$, so the amplitude of $y = \sin(x + a)$ or $y = \cos(x + a)$ is 1 and the period is 2π.

Example 5.4 The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is



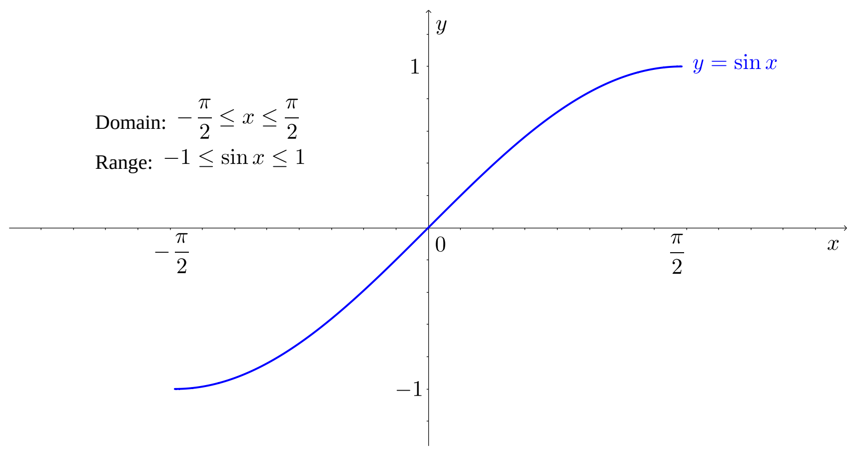
5.2 Inverse Trigonometric Functions

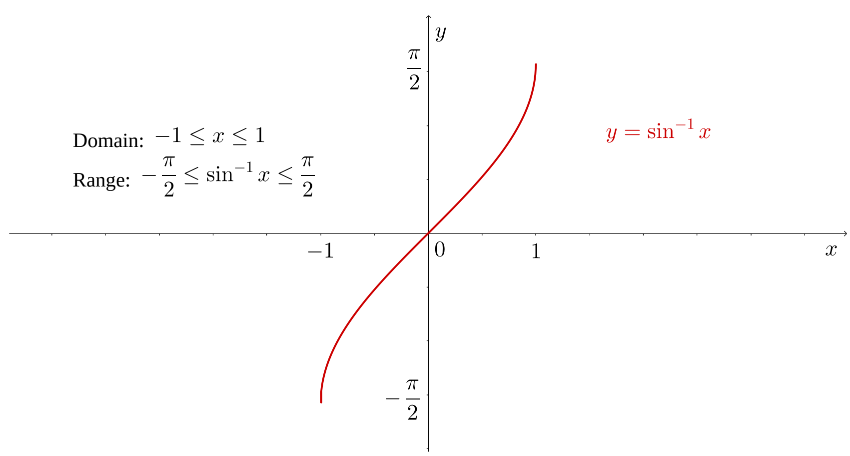
The functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $x \in \mathbb{R}$ are many-one functions. Recall that $f^{-1}(x)$ exists if, and only if, $f(x)$ is a one-one function. If we suitably restrict the domain of each of these functions, it is possible to make them one-one and hence we can define each inverse function.

After we define an inverse function, we should recall that the domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$.

5.2.1 Inverse Sine Function

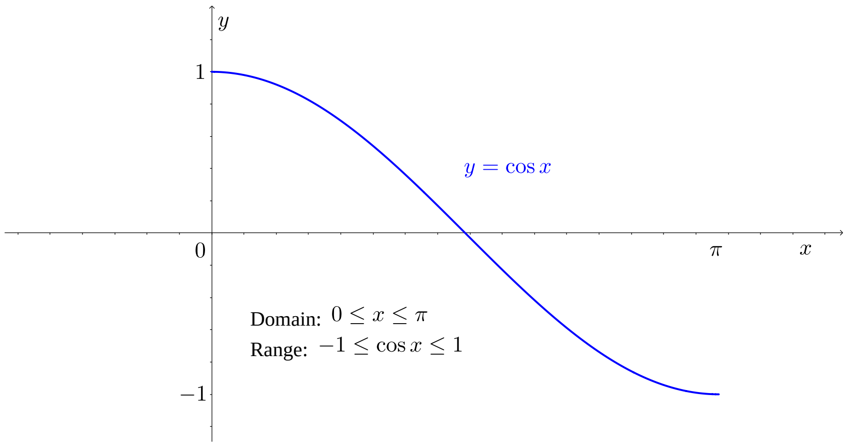
We define $y = \sin x$ for $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$, which is a one-one function, and hence we can define its inverse function as $y = \sin^{-1} x$.

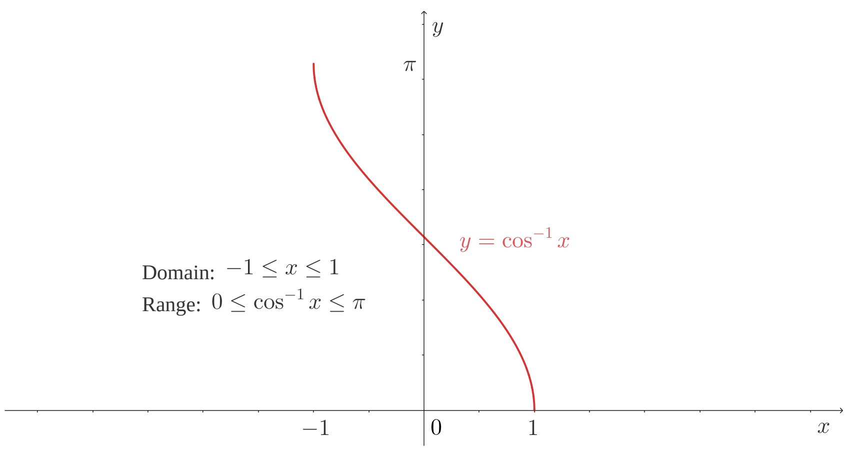




5.2.2 Inverse Cosine Function

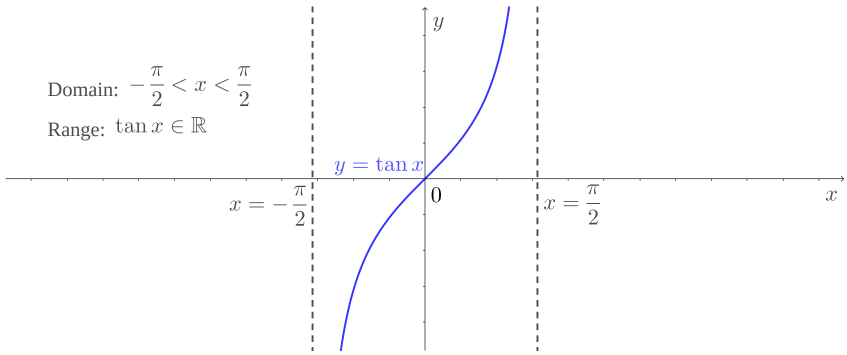
We define $y = \cos x$ for $0 \leq x \leq \pi$, which is a one-one function, and hence we can define its inverse function as $y = \cos^{-1} x$.

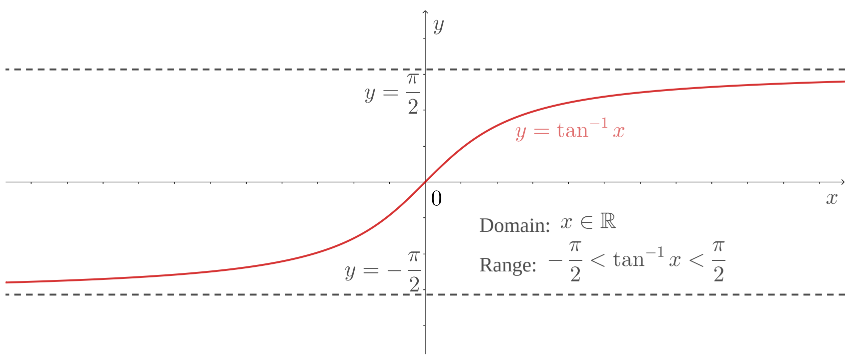




5.2.3 Inverse Tangent Function

We define $y = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, which is a one-one function, and hence we can define its inverse function as $y = \tan^{-1} x$.





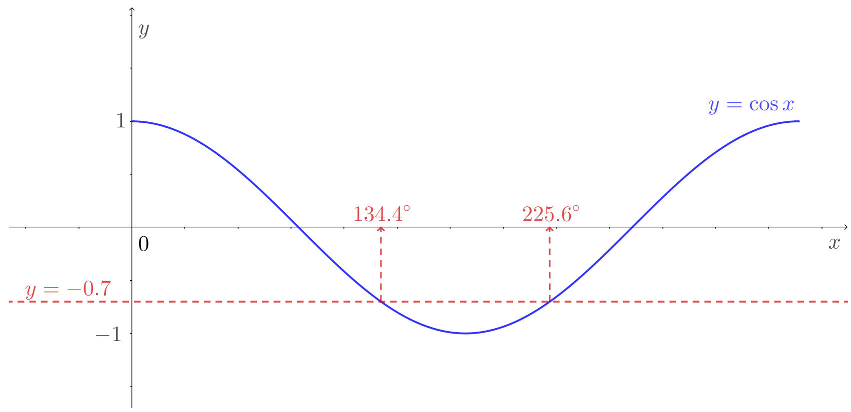
5.3 Trigonometric Equations

When solving the equation $sinx = 0.5$ for $0 \le x \le \pi$, we can find one solution using the inverse function: $x = \sin^{-1} 0.5$, leading to $x = \dfrac{\pi}{6}$. The second angle is $x = \dfrac{5\pi}{6}$ that satisfies $\sin x = 0.5$ with $0 \le x \le \pi$. We can find this second angle either by using the symmetry of the curve $y = \sin x$ or by using $\sin(\pi - x) = \sin x$.

The angle (that the calculator gives) is the one that lies in the range of the inverse trigonometric function, and we call it **principal angle**.

In more general trigonometric equations, we can use calculator to find the principal angle, and then use the symmetry of the curve to find all the other angles, given the domain of the function.

Example 5.5 Solve $\cos x = -0.7$ for $0^\circ \leq x \leq 360^\circ$.
Solution. Using a calculator, we know $\cos^{-1}(-0.7) \approx 134.4^\circ$.



The graph shows that there are two values of x between 0° and 360°, for which $\cos x = -0.7$. Using the symmetry of the curve, the second value is $360^\circ - 134.4^\circ = 225.6^\circ$. Hence the solution is $x = 134.4^\circ$ or $x = 225.6^\circ$.

5.4 Trigonometric Identities

Two commonly used trigonometric identities are

Key Point 5.2

$\sin^2\theta + \cos^2\theta \equiv 1$ and $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$. For $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, we should notice that $\cos\theta \neq 0$.

We can prove these two identities using a right-angled triangle:

Note. We can use trigonometric identities to solve more complicated trigonometric equations. When proving an identity, it is usual to start with the more complicated side of the identity, and prove that it simplifies to the less complicated side. We can use trigonometric identities to simplify expressions that involve $\sin x, \cos x$ and $\tan x$.

Example 5.6 Prove the identity $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = 2\cos^2\theta - 1$.

Proof. We have

$$\begin{aligned}
\text{LHS} &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \\
&= \frac{1-\left(\dfrac{\sin\theta}{\cos\theta}\right)^2}{1+\left(\dfrac{\sin\theta}{\cos\theta}\right)^2} \\
&= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} \\
&= \cos^2\theta-\sin^2\theta \\
&= \cos^2\theta-(1-\cos^2\theta) \\
&= 2\cos^2\theta-1 \\
&= \text{RHS}.
\end{aligned}$$

Example 5.7 Solve the equation $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = 5\cos\theta - 3$ for $0^\circ \le \theta \le 360^\circ$.

Solution. Since $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = 2\cos^2\theta - 1$, then

$$\begin{aligned}
&\frac{1-\tan^2\theta}{1+\tan^2\theta} = 5\cos\theta-3 \\
\Leftrightarrow\ & 2\cos^2\theta-1 = 5\cos\theta-3 \\
\Leftrightarrow\ & 2\cos^2\theta-5\cos\theta+2 = 0 \\
\Leftrightarrow\ & (2\cos\theta-1)(\cos\theta-2) = 0
\end{aligned}$$

Hence, $\cos\theta = \dfrac{1}{2}$ or $\cos\theta = 2$ ($\cos\theta = 2$ has no solution), and thus $\theta = \cos^{-1}\left(\dfrac{1}{2}\right)$. We then solve for $\theta = 60^\circ$ or $\theta = 360^\circ - 60^\circ = 300^\circ$.

Chapter 6 Series

6.1 Binomial Expansion

6.1.1 Pascal's Triangle

Binomial means two terms. The word is used in algebra for expressions such as $x+3$ and $5x-2y$. In algebra, we can show

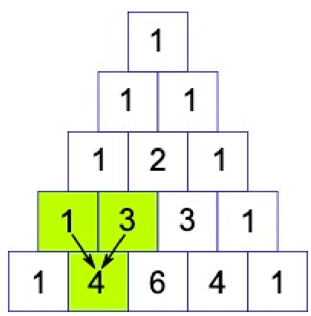
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = (a+b)(a^2+2ab+b^2) = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If $(a+b)^4$, we notice that the powers of a and b from a pattern:

- the first term is a^4 and then the power of a decreases by 1 while the power of b increases by 1 in each successive term;
- all of the terms have a total index of 4 (a^4, a^3b, a^2b^2, ab^3 and b^4).

There is a similar pattern in the other expansions.

The coefficients also form a pattern that is known as **Pascal's triangle**.



The next row is then $(1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1)$, and so we can write down the expansion of $(a+b)^5$:

$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5.$$

Example 6.1 Use Pascal's triangle to find the expansion of $(1-2x)^5$.

Solution. The row for $n=5$ in Pascal's triangle is $(1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1)$, and so

$$\begin{aligned}(1-2x)^5 &= 1(1)^5 + 5(1)^4(-2x) + 10(1)^3(-2x)^2 + 10(1)^2(-2x)^3 + 5(1)(-2x)^4 + 1(-2x)^5 \\ &= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5.\end{aligned}$$

Example 6.2 Find the coefficient of x^3 in the expansion of $(3+5x)(1-2x)^5$.

Solution. We know that $(3+5x)(1-2x)^5 = (3+5x)(1-10x+40x^2-80x^3+80x^4-32x^5)$. The term in x^3 comes from the products $3\times(-80x^3) = -240x^3$ and $5x\times(40x^2) = 200x^3$, and thus the coefficient of x^3 is $-240+200=-40$.

6.1.2 Binomial Coefficients

Pascal's triangle can be used to expand $(a+b)^n$ for any positive integer n, but if n is large it takes a long time to write out all the rows in the triangle. Hence, we need a more efficient method to find the coefficients in the expansions. The coefficients in the binomial expansion of $(1+x)^n$ are known as **binomial coefficients**.

Beyond The Syllabus. 在介绍二项式系数（Binomial Coefficient）之前，我们需要定义并理解一些符号. 我们将“在 n 件物件中，不分先后地选取 r 件的方法总数”记为 $\binom{n}{r}$.

比如说，在 3 个人（x_1, x_2 和 x_3）中，我们不分先后地选取 1 个人的方法总数为 3（分别是 x_1、x_2 和 x_3），也即 $\binom{3}{1} = 3$（可以通过计算器验算，`3 nCr 1`）.

在 3 个人中，我们不分先后地选取 2 个人的方法总数为 3（分别是 x_1x_2、x_1x_3 和 x_2x_3），也即 $\binom{3}{2} = 3$（可以通过计算器验算，`3 nCr 2`）.

可以通过计算验证发现 $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$ 和 $\binom{3}{3}$ 正好对应了杨辉三角或帕斯卡三角（Pascal's Triangle）的 $n=3$ 行：$(1 \quad 3 \quad 3 \quad 1)$，也正好对应了 $(1+x)^3$ 展开后的系数：$1+3x+3x^2+1x^3$. 这不是一种巧合，假设 $n=3, r=2$，我们有

$$(1+x_1)(1+x_2)(1+x_3) = \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$$

所以从 3 个物件中，不分先后地选取 2 件的方法总数对应了 x^2 的系数 3.

关于 $\binom{n}{r}$ 的计算公式，我们需要学习关于排列组合（Combination and Permutation）的知识才能更好地理解，在此我们仅给出

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

其中 n 和 r 都是非负整数（$n, r \in \mathbb{N}$）. 若 n 是一个正整数，$n!$ 表示 n 的阶乘（**Factorial**），也即

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

我们定义 $0! = 1$.

We write the binomial expansion of $(1+x)^n$, where n is a positive integer as

Key Point 6.1

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$, where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

The result is known as the **Binomial theorem**.

Remark. For a positive integer n, $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. We define $0! = 1$.

We can use the Binomial theorem to expand $(a+b)^n$ since $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$ assuming $a \neq 0$.

Key Point 6.2

$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$

Example 6.3 Find the term independent of x in the expansion of $\left(x+\frac{5}{x^2}\right)^9$.

Solution. We first expand the $\left(x+\frac{5}{x^2}\right)^9$ as:

$$\left(x+\frac{5}{x^2}\right)^9 = \binom{9}{0}x^9 + \binom{9}{1}x^8\left(\frac{5}{x^2}\right)^1 + \binom{9}{2}x^7\left(\frac{5}{x^2}\right)^2 + \binom{9}{3}x^6\left(\frac{5}{x^2}\right)^3 + \cdots$$

The term is independent of x when the power of x is double the power of $\frac{5}{x^2}$, and the sum of these powers must be 9. Hence, we are looking for powers of 6 and 3, respectively, and the corresponding binomial coefficient is $\binom{9}{3}$. Therefore, the term independent of x is

$$\binom{9}{3}x^6\left(\frac{5}{x^2}\right)^3 = 84x^6 \times \frac{125}{x^6} = 10500.$$

6.2 Arithmetic Progressions

A **sequence** is a set of things (usually numbers) that are in order. Each number in the sequence is called a **term**.

In an **arithmetic progression**, the difference between one term and the next is a constant. In other words, we just add the same value (could be zero or negative value) each time. Such constant is called the **common difference**.

In arithmetic progression, we denote the **first term** as a, the common difference as d, and the **last term** as l.

The first four terms of an arithmetic progression whose first term is a and whose common difference is d are: $a, a+d, a+2d, a+3d$. From this pattern, the formula for the nth term is given by

Key Point 6.3

$a+(n-1)d.$

Remark. Suppose an arithmetic progression is given by $x, y, z, \ldots$, we can find an equation connecting x, y and z:

$$y - x = z - y.$$

6.2.1 The Sum of an Arithmetic Progression

When the terms in a sequence are added together, we call the resulting sum a **series**. We denote the sum of an arithmetic progression as S_n, and

Key Point 6.4

$$S_n = \frac{n}{2}(a + l) = \frac{n}{2}(2a + (n-1)d).$$

Beyond The Syllabus. 为了证明关于 S_n 的公式，我们先考虑下列两个等式：

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l \\ S_n &= l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a \end{aligned}$$

两式相加，我们得到

$$\begin{aligned} 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l) \\ &= n(a+l). \end{aligned}$$

因此 $S_n = \frac{n}{2}(a+l)$. 又因为 $l = a + (n-1)d$，所以 $S_n = \frac{n}{2}(2a + (n-1)d)$.

Using the sum, we can get another formula for the nth term:

Key Point 6.5

$$S_n - S_{n-1}.$$

Example 6.4 The sum of the first n terms S_n of a particular arithmetic progression is given by $S_n = 4n^2 + n$. Find the first term, the common difference, and an expression for the nth term.

Solution. We have $S_1 = 4(1)^2 + 1 = 5$ and $S_2 = 4(2)^2 + 2 = 18$, and so the first term is 5 and the common difference is $S_2 - S_1 = 8$. Therefore, the nth term is $a + (n-1)d = 5 + 8(n-1) = 8n - 3$.

6.3 Geometric Progressions

In a **geometric progression**, each term is found by multiplying the previous term by a constant. Such constant is called the **common ratio**.

In geometric progression, we denote the first term as a, and the common ratio as r.

The first four terms of a geometric progression whose first term is a and whose common ratio is r are: a, ar, ar^2, ar^3, ar^4. From this pattern, the formula for the nth term is given by

Key Point 6.6

$$ar^{n-1}.$$

Remark. Suppose an geometric progression is given by $x, y, z, \ldots$, we can find an equation connecting x, y and z:

$$\frac{y}{x} = \frac{z}{y}.$$

Example 6.5 The first, second and third terms of an arithmetic progression are x, y and x^2. The first, second and third terms of a geometric progression are x, x^2 and y. Given that $x < 0$, find the value of x and y.

Solution. For the arithmetic progression, we have $y - x = x^2 - y$. For the geometric progression, we have $\frac{x^2}{x} = \frac{y}{x^2}$, leading to $y = x^3$. Solving the simultaneous equations, we have

$$\begin{aligned}
& x^3 - x = x^2 - x^3 \\
\Leftrightarrow\ & 2x^2 - x - 1 = 0 \\
\Leftrightarrow\ & (2x+1)(x-1) = 0 \\
\Leftrightarrow\ & x = -\frac{1}{2} \text{ or } x = 1
\end{aligned}$$

Since $x < 0$, then we solve for $x = -\frac{1}{2}$, and thus $y = -\frac{1}{8}$.

6.3.1 The Sum of an Geometric Progression

The sum of a geometric progression S_n can be written as

Key Point 6.7

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \text{ for } r \neq 1.$$

Beyond The Syllabus. 为了证明关于 S_n 的公式，我们先考虑下列两个等式：

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1} \\
rS_n &= \phantom{a+{}} ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n
\end{aligned}$$

两式相减，我们得到

$$rS_n - S_n = ar^n - a.$$

因此 $S_n = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}$.

Remark. Either formula can be used, but it is easier to use $S_n = \frac{a(1-r^n)}{1-r}$ when $-1 < r < 1$, and to use $S_n = \frac{a(r^n-1)}{r-1}$ when $r > 1$ or $r \leq -1$. When $r = 1$, we cannot use formula above, but we use $S_n = na$.

Example 6.6 The third term of a geometric progression is nine times the first term. The sum of the first six terms is k times the sum of the first two terms. Find the value of k.

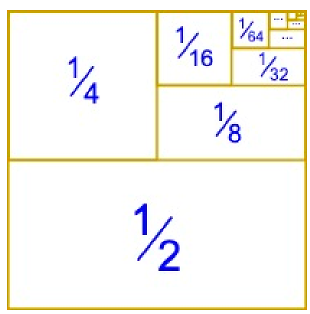
Solution. Since the third term of a geometric progression is nine times the first term, then $ar^2 = 9a$ leading to $r = \pm 3$. Since the sum of the first six terms is k times the sum of the first two terms, then

$$S_6 = kS_2$$
$$\Leftrightarrow \frac{a(r^6-1)}{r-1} = \frac{ka(r^2-1)}{r-1}$$
$$\Leftrightarrow k = \frac{r^6-1}{r^2-1}$$

When $r = 3, k = 91$, and when $r = -3, k = 91$. Hence, $k = 91$.

6.4 Infinite Geometric Series

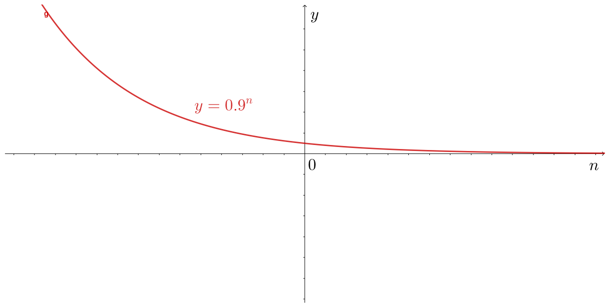
An infinite sequence is a sequence whose terms continue forever. Consider the infinite geometric progression where $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so it begins $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. We can work out the sum of the first n terms: $S_1 = 0.5, S_2 = 0.75, S_3 = 0.875, S_4 = 0.9375, S_5 = 0.96875$, and so on. These sums are getting closer and closer to 1. We can show this in a 1 by 1 square:



If the pattern of rectangles inside the square is continued, the total area of the rectangles approximates the area of the whole square (which is 1) increasingly well as more rectangles are included. We therefore say that the sum of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is 1, because the sum of the first n terms gets as close to 1 as n gets larger.

We write $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. We also say that the sum to infinity of this series is 1, and that the series converges to 1. A series that converges is also known as a **convergent series**.

Consider the geometric series $a+ar+ar^2+ar^3+\cdots+ar^{n-1}$, the sum is $S_n = \dfrac{a(1-r^n)}{1-r}$. If $-1 < r < 1$, then as n gets larger and larger, r^n gets closer and closer to 0. For example, let $r = 0.9$, the graph of r^n is



We say that as n tends to infinity, r^n tends to zero, and we write as $n \to \infty, r^n \to 0$. Hence, as $n \to \infty$,

$$\frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

Key Point 6.8

$S_\infty = \dfrac{a}{1-r}$ provided that $-1 < r < 1$.

If $r \geq 1$ or $r \leq -1$, then r^n does not converge, and so the series itself does not converge. For example, $1+1+1+\cdots = \infty$. Hence, an infinite geometric series converges when and only when $-1 < r < 1$.

Example 6.7 A geometric progression has a common ratio of $-\dfrac{2}{3}$ and the sum of the first three terms is 63. Find the first term of the progression and the sum to infinity.

Solution. Since the sum of the first three terms is 63, then

$$\begin{aligned} S_3 &= \frac{a(1-r^3)}{1-r} \\ \Leftrightarrow 63 &= \frac{a\left(1-\left(-\frac{2}{3}\right)^3\right)}{1-\left(-\frac{2}{3}\right)} \\ \Leftrightarrow 63 &= \frac{\frac{35}{27}a}{\frac{5}{3}} \\ \Leftrightarrow a &= 81. \end{aligned}$$

Hence,

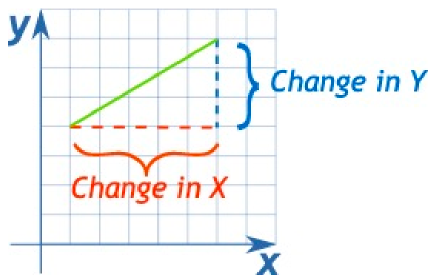
$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{81}{1-\left(-\frac{2}{3}\right)} \\ &= \frac{243}{5}. \end{aligned}$$

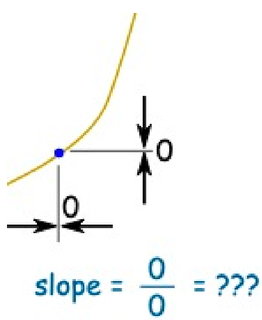
Chapter 7 Differentiation

7.1 Derivatives and Gradient Functions

We want to find the exact gradient of a curve at a point, and this exact method is called **differentiation**.

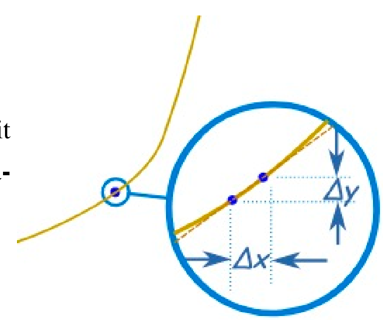
Recall that the gradient or the slope of a line is calculated by finding the ratio of the vertical change to the horizontal change between (any) two distinct points on a line.





The gradient of a curve at a point should be the ratio of the vertical change to the horizontal change, but there is nothing to measure at a point.

But to find the gradient, we use a small difference, then have it **shrink towards zero**, which is sometimes called **differentiation from first principles**.

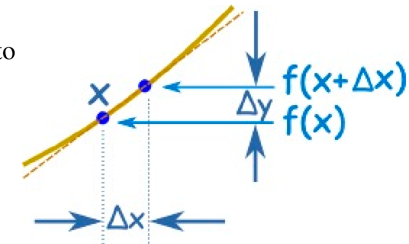


Remark. Formal use of the general method of differentiation from first principles is not required in P1, but it is useful to have an informal understanding.

7.1.1 Differentiation from First Principles

Recall the slope formula

$$\text{Slope} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}.$$



If x changes from x to $x + \Delta x$, then y changes from $f(x)$ to $f(x + \Delta x)$, and so

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Then make Δx shrink towards zero.

Example 7.1 Let $f(x) = x^2$, then

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2.$$

The slope formula is

$$\begin{aligned}\frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= 2x + \Delta x.\end{aligned}$$

Then as Δx tends to 0, we get the slope at x is $2x$, i.e., the gradient of the curve at x is $2x$.

7.1.2 Notation

There are three different notations that are used to describe the previous example.

1. If $y = x^2$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$.
2. If $f(x) = x^2$, then $f'(x) = 2x$.
3. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$, i.e., if we differentiate x^2 with respect to x, the result is $2x$.

If y is a function of x, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is called the **derivative** of y with respect to x. Likewise $f'(x)$ is called the derivative of $f(x)$. $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ of $f'(x)$ is sometimes called the **gradient function** of the the curve $y = f(x)$.

7.1.3 Differentiation of Power Functions

Using the differentiation from first principles, we can also know that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^4) = 4x^3 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^5) = 5x^4 \qquad \cdots$$

Hence the general rule for differentiating power functions is

Key Point 7.1

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ for any real power n.

Example 7.2 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x^2}\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}(x^{-2}) = (-2)x^{-2-1} = -2x^{-3} = -\dfrac{2}{x^3}$.

Example 7.3 $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) = \dfrac{\mathrm{d}}{\mathrm{d}x}(x^{1/2}) = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

Example 7.4 $\dfrac{\mathrm{d}}{\mathrm{d}x}(2) = \dfrac{\mathrm{d}}{\mathrm{d}x}(2x^0) = 0x^{0-1} = 0$.

7.1.4 Basic Rules

Key Point 7.2

Scalar Multiple Rule: If k is a constant and $f(x)$ is a function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf(x)) = k\frac{\mathrm{d}}{\mathrm{d}x}(f(x)).$$

Key Point 7.3

Addition/Subtraction Rule: If $f(x)$ and $g(x)$ are functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{d}x}(g(x)).$$

Example 7.5 Differentiate $3x^4 - \frac{1}{2x^2} + \frac{4}{\sqrt{x}} + 5$ with respect to x.

Solution. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4 - \frac{1}{2x^2} + \frac{4}{\sqrt{x}} + 5\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4 - \frac{1}{2}x^{-2} + 4x^{-1/2} + 5x^0\right) \\
&= 3\frac{\mathrm{d}}{\mathrm{d}x}(x^4) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}) + 4\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1/2}) + 5\frac{\mathrm{d}}{\mathrm{d}x}(x^0) \\
&= 3(4x^3) - \frac{1}{2}(-2x^{-3}) + 4\left(-\frac{1}{2}x^{-3/2}\right) + 5(0x^{-1}) \\
&= 12x^3 + x^{-3} - 2x^{-3/2} \\
&= 12x^3 + \frac{1}{x^3} - \frac{2}{\sqrt{x^3}}.
\end{aligned}$$

7.2 Chain Rule

To differentiate $y = (5x-2)^7$, we could expand the brackets and then differentiate each term separately, but it would take a long time to do. There is a more efficient method that allows us to find the derivative without expanding. Let $u = 5x - 2$, then $y = (5x-2)^7$ becomes $y = u^7$.

We can find the derivative of the composite function using the **chain rule**:

Key Point 7.4

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Beyond The Syllabus. 我们需要更多严格的数学定义去证明链式法则（Chain Rule），比如关于导数（Derivative）的定义等等.

Example 7.6 Find the derivative of $y = (5x-2)^7$.

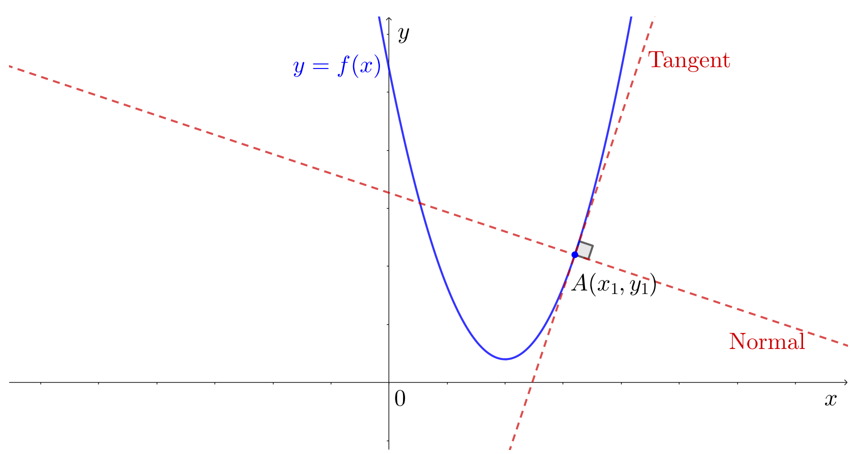
Solution. Let $u = 5x - 2$, so $y = u^7$. Then $\frac{\mathrm{d}y}{\mathrm{d}u} = 7u^6$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$. By chain rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}\\ &= 7u^6 \times 5\\ &= 7(5x-2)^6 \times 5\\ &= 35(5x-2)^6.\end{aligned}$$

Note. With practice we can do this mentally. We consider the inside of $(5x-2)^7$ to be $5x-2$. To differentiate $(5x-2)^7$:

1. Differentiate the outside: $7(5x-2)^6$;
2. Differentiate the inside: 5;
3. Multiply these two expressions: $35(5x-2)^6$.

7.3 Tangents and Normals



The line perpendicular to the tangent at the point A is called the **normal** at A. If the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $A(x_1, y_1)$ is m, then the equation of the tangent at A is given by

Key Point 7.5

$y - y_1 = m(x - x_1)$.

The normal at the point (x_1, y_1) is perpendicular to the tangent, so the gradient of the normal is $-\frac{1}{m}$ and the equation of the normal is given by

Key Point 7.6

$y - y_1 = -\frac{1}{m}(x - x_1)$.

Remark. The formula of normal only makes sense when $m \neq 0$. If $m = 0$, it means that the tangent is horizontal and the normal is vertical, so it has equation $x = x_1$.

7.4 Second Derivatives

$\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the **first derivative** of y with respect to x. If we then differentiate $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x, we obtain $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$, which is usually written as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is called the **second derivative** of y with respect to x.

Example 7.7 A curve has equation $y = x^3 + 3x^2 - 9x + 2$. Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solution. For $y = x^3 + 3x^2 - 9x + 2$,

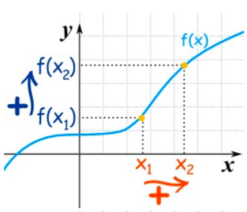
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 6x - 9$$

and thus

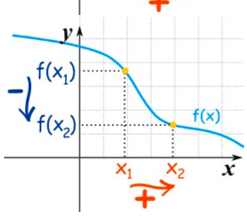
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 6.$$

Remark. In general, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

7.5 Increasing and Decreasing Functions

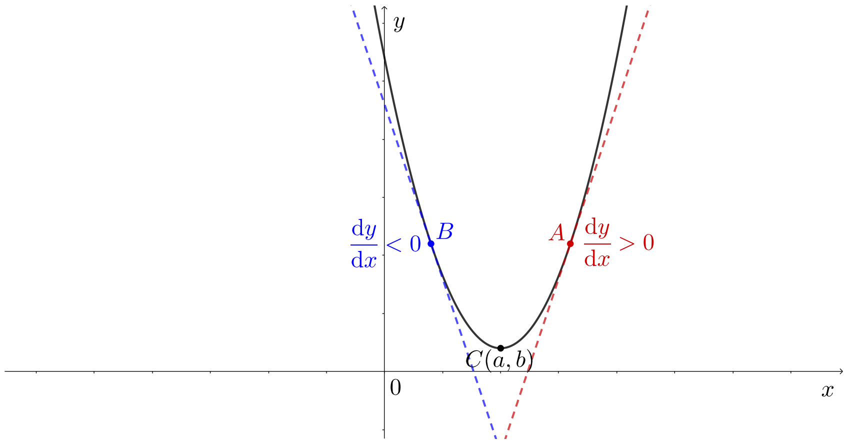


A function is **increasing** when the $f(x)$ value increases as the x value increases. More precisely, if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is an **increasing function**.



A function is **decreasing** when the $f(x)$ value decreases as the x value increases. More precisely, if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is an **decreasing function**.

Sometimes we talk about a function **increasing** (or **decreasing**) **at a point**, meaning that the function values are increasing (or decreasing) around that point. For example, a function is increasing at A, and is decreasing at B; the gradient of the function is positive at A, and the gradient of the function is negative at B.



Moreover, we can divide the graph into two distinct sections.

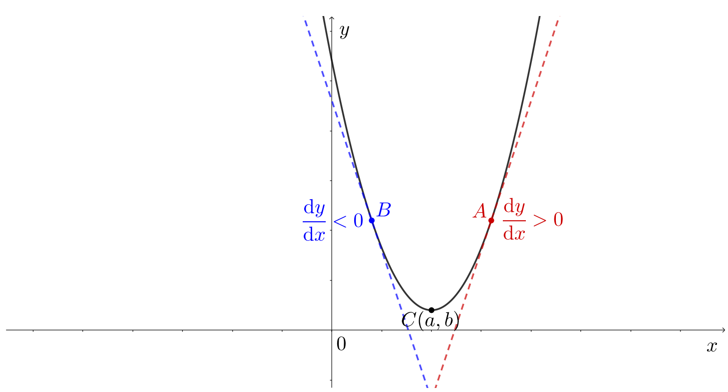
- The function is increasing when $x > a$, i.e., $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ for $x > a$.
- The function is decreasing. when $x < a$, i.e., $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ for $x < a$.

Example 7.8 Find the set of values of x for which $y = 8 - 3x - x^2$ is decreasing.

Solution. For $y = 8 - 3x - x^2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3 - 2x$. When $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$, y is decreasing, i.e.,

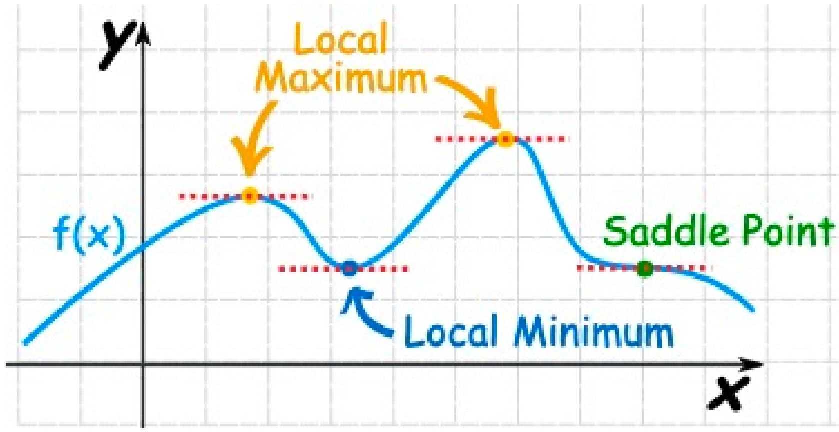
$$
\begin{aligned}
& -3 - 2x < 0 \\
\Leftrightarrow\ & 2x > -3 \\
\Leftrightarrow\ & x > -\frac{3}{2}.
\end{aligned}
$$

7.6 Stationary Points



The gradient of the curve above is zero at the point C. A point where the gradient is zero is called a **stationary point** or a **turning point**.

7.6.1 Types of Stationary Points



The stationary point is called a **(local) maximum point** when the value of y at this point is greater than the value of y at **nearby points**. At a maximum point:

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$
- the gradient is positive to the left of the maximum and negative to the right.

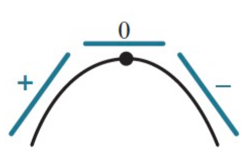
The stationary point is called a **(local) minimum point** when the value of y at this point is smaller than the value of y at **nearby points**. At a minimum point:

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$
- the gradient is negative to the left of the minimum and positive to the right.

There is a third type of stationary point called a **point of inflexion** (or **saddle point**). At a stationary point of inflexion:

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$
- the gradient changes
    - from positive to zero and then to positive again or
    - from negative to zero and then to negative again.

7.6.2 Second Derivatives and Stationary Points



Consider moving from left to right along a curve, passing through a maximum point. The gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ starts as a positive value, decreases to zero at the maximum point and then decreases to a negative value. We can consider the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function. Since $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ decreases as x increases, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0.$$

Key Point 7.7

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, then the point is a maximum point.

Similarly, consider moving from left to right along a curve, passing through a minimum point. The gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ starts as a negative value, increases to zero at the minimum point and then increases to a positive value. Since $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ increases as x increases, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is increasing. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0.$$

Key Point 7.8

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, then the point is a minimum point.

Remark. If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, then the nature of the stationary point can be found using the first derivative test in the previous section.

Example 7.9 Find the coordinates of the stationary points on the curve $y = \dfrac{x^2+9}{x}$ and use the second derivative to determine the nature of these points.

Solution. Since $y = \dfrac{x^2+9}{x} = x + 9x^{-1}$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 9x^{-2} = 1 - \dfrac{9}{x^2}$. For stationary points:

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ \Leftrightarrow\ &1 - \frac{9}{x^2} = 0 \\ \Leftrightarrow\ &x^2 - 9 = 0 \\ \Leftrightarrow\ &(x+3)(x-3) = 0 \end{aligned}$$

Then we solve for $x = -3$ (with $y = -6$) or $x = 3$ (with $y = 6$). The stationary points are $(-3, -6)$ and $(3, 6)$. The second derivative is

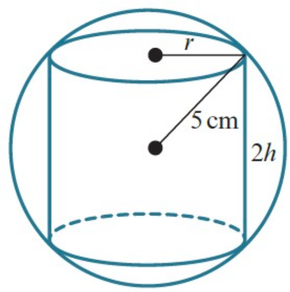
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 18x^{-3} = \frac{18}{x^3}.$$

When $x = -3$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, and so $(-3, -6)$ is a maximum point. When $x = 3$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, and so $(3, 6)$ is a minimum point.

Remark. The y value of a minimum point might be greater than that of a maximum point. We should always remember that a maximum point or a minimum point is local, meaning that the value of y at this point is greater or smaller than the value of y at nearby points.

There are many practical problems for which we need to find the maximum or minimum value of an expression. For example, we want to find the minimum amount of metal required to make a container for a given volume to minimise the cost.

Example 7.10 The diagram shows a solid cylinder of radius r cm and height $2h$ cm cut from a solid sphere of radius 5 cm. The volume of the cylinder is C cm^3. Express V in terms of h, find the value for h for which there is a stationary value of V, and determine the nature of this stationary value.



Solution. Since $r^2 + h^2 = 5^2$ and $r > 0$, then $r = \sqrt{25 - h^2}$. Then

$$\begin{aligned} V &= \pi r^2(2h) \\ &= \pi(25 - h^2)(2h) \\ &= 50\pi h - 2\pi h^3. \end{aligned}$$

Therefore, $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 50\pi - 6\pi h^2$. The stationary values occur when

$$\begin{aligned} &\frac{\mathrm{d}V}{\mathrm{d}h} = 0 \\ \Leftrightarrow\ & 50\pi - 6\pi h^2 = 0 \\ \Leftrightarrow\ & h^2 = \frac{50\pi}{6\pi} \end{aligned}$$

Since $h > 0$, then $h = \dfrac{5\sqrt{3}}{3}$. In addition, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = -12\pi h$. When $h = \dfrac{5\sqrt{3}}{3}$, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = -\dfrac{12(5\sqrt{3})\pi}{3} < 0$ and thus the stationary value is a maximum value.

7.7 Rates of Change

Recall that the **rate of change** is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. If we make Δx shrink towards zero, then we can find the rate of change at one point, i.e., $\dfrac{\mathrm{d}x}{\mathrm{d}y}$.

Example 7.11 Given that $h = \dfrac{1}{5}t^2$, find the rate of change of h with respect to t when $t = 2$.

Solution. Since $h = \dfrac{1}{5}t^2$, then $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{2}{5}t$. When $t = 2$, $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{2}{5}(2) = \dfrac{4}{5}$, i.e., the rate of change is $\dfrac{4}{5}$.

When two variables x and y both vary with a third variable t, we can connect the three variables using the chain rule:

Key Point 7.9

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

If we set $t = y$, then $\dfrac{\mathrm{d}y}{\mathrm{d}y} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}y}$. Since $\dfrac{\mathrm{d}y}{\mathrm{d}y} = 1$, then

Key Point 7.10

$\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{1}{\dfrac{\mathrm{d}y}{\mathrm{d}x}}$.

Example 7.12 A point with coordinates (x, y) moves along the curve $y = x + \sqrt{2x+3}$ in such a way that the rate of increase of x has the constant value 0.06 units per second. Find the rate of increase of y at the instant when $x = 3$.

Solution. We know $y = x + \sqrt{2x+3} = x + (2x+3)^{1/2}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0.06$. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{2}(2x+3)^{-1/2}(2) = 1 + \frac{1}{\sqrt{2x+3}}.$$

When $x = 3$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + \dfrac{1}{\sqrt{2(3)+3}} = \dfrac{4}{3}$. Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{4}{3} \times 0.06 = 0.08.$$

Hence, the rate of change of y is 0.08 units per second.

Example 7.13 A solid sphere has radius r cm and surface area A cm^2. The radius is increasing at a rate of $\dfrac{1}{5\pi}$ cm/s. Find the rate of increase of the surface area when $r = 3$.

Solution. Since $A = 4\pi r^2$, then $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$. When $r = 3$, $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 24\pi$. Since the radius is increasing at a rate of $\dfrac{1}{5\pi}$ cm/s, then $\dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{1}{5\pi}$. Using the chain rule:

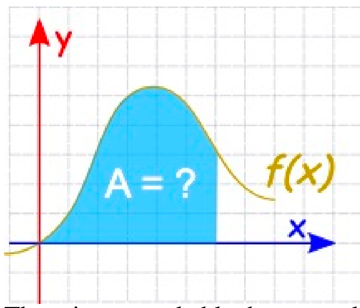
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{24\pi}{5\pi} = 4.8.$$

The surface area is increasing at a rate of 4.8 cm^2/s.

Chapter 8 Integration

In the previous chapter, we know if $y = x^3$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$. The process of obtaining $\frac{\mathrm{d}y}{\mathrm{d}x}$ where y is known is called differentiation.

In this chapter, we want to learn the reverse process, i.e., to obtain y when $\frac{\mathrm{d}y}{\mathrm{d}x}$ is known. We call such reverse process **antidifferentiation**.



In this chapter, we will also answer a question: what is the area between the graph $f(x)$ and x-axis from $x = a$ to $x = b$? The process used to answer that question is known as **integration**.

There is a remarkable theorem, also known as the Fundamental Theorem of Calculus, that says that **integration is essentially the same as antidifferentiation**. Because of this theorem, we will only talk about **integration**, whether we are reversing the process of differentiation or finding the area under a graph.

8.1 Indefinite Integration

Suppose we know $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$, we cannot conclude that $y = x^3$. We should notice that if $y = x^3 + 5$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$. Moreover, we find that if $y = x^3 + c$, where c is some constant, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$. Similarly, if $y = \frac{x^5}{5} + c$, where c is some constant, then $\frac{\mathrm{d}y}{\mathrm{d}x} = x^4$. We can give more similar examples.

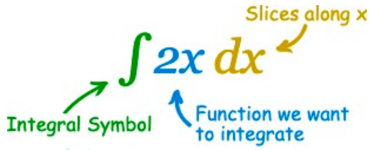
Key Point 8.1

If $\frac{\mathrm{d}y}{\mathrm{d}x} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c$, where c is an arbitrary constant and $n \neq -1$.

In function notation: if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c$, where c is an arbitrary constant and $n \neq 1$.

Remark. It is easier to remember: given $\frac{\mathrm{d}y}{\mathrm{d}x} = x^n$ for $n \neq 1$, we can get y by increasing the power n by 1 to obtain the new power, then dividing by the new power. We also need to remember to add a constant c at the end.

Beyond The Syllabus. 如果 $y = \ln|x| + c$，那么 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} = x^{-1}$. 因此如果 $\frac{\mathrm{d}y}{\mathrm{d}x} = x^{-1}$，那么 $y = \ln|x| + c$.



We can use a special symbol $\int$ to denote integration.

Example 8.1 When we need to integrate $2x$, we rite

$$\int 2x\mathrm{d}x = x^2 + c.$$

$\int 2x\mathrm{d}x$ is called the **indefinite integral** of $2x$ with respect to x. We call it indefinite because it has infinitely many solutions. Using this notation, we can write the rule for integrating powers as:

Key Point 8.2

$\int x^n\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$, where c is a constant and $n \neq -1$.

Consider a more complex example. We know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3\times 7}(3x-1)^7\right) = (3x-1)^6.$$

Hence,

$$\int (3x-1)^6\mathrm{d}x = \frac{1}{3\times 7}(3x-1)^7 + c.$$

We can give more similar examples ($n \neq -1$).

Key Point 8.3

If $n \neq -1$ and $a \neq 0$, then $\int (ax+b)^n\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$.

Remark. It is important to note that the rule only works for powers of linear functions. For example, we cannot apply the rule to $\int (ax^2+b)^6\mathrm{d}x$.

Example 8.2 Find $\int (2x-3)^4\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int (2x-3)^4\mathrm{d}x &= \frac{1}{2(4+1)}(2x-3)^{4+1} + c\\ &= \frac{1}{10}(2x-3)^5 + c.\end{aligned}$$

In general, if we know the original function $F(x)$ and its derivative $F'(x) = f(x)$, then we can get $\int f(x)\mathrm{d}x$ back.

Key Point 8.4

If $\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = f(x)$, then $\int f(x)\mathrm{d}x = F(x) + c$.

8.1.1 Basic Rules

The rule for integrating constant multiples of a function is:

Key Point 8.5

$\displaystyle\int kf(x)\mathrm{d}x = k\int f(x)\mathrm{d}x$, where k is a constant.

The rule for integrating sums and differences of two functions is:

Key Point 8.6

$\displaystyle\int (f(x) \pm g(x))\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x.$

Example 8.3 Find $\displaystyle\int x(2x-1)(2x+3)\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int x(2x-1)(2x+3)\mathrm{d}x &= \int (4x^3+4x^2-3x)\mathrm{d}x\\ &= \frac{4x^4}{4}+\frac{4x^3}{3}-\frac{3x^2}{2}+c\\ &= x^4+\frac{4x^3}{3}-\frac{3x^2}{2}+c.\end{aligned}$$

Example 8.4 Find $\displaystyle\frac{20}{(1-4x)^6}\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int \frac{20}{(1-4x)^6}\mathrm{d}x &= 20\int (1-4x)^{-6}\mathrm{d}x\\ &= \frac{20}{(-4)(-6+1)}(1-4x)^{-6+1}+c\\ &= (1-4x)^{-5}+c\\ &= \frac{1}{(1-4x)^5}+c.\end{aligned}$$

Example 8.5 Suppose we know $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}((3x^2-4)^8) = 48x(3x^2-4)^7$, find $\displaystyle\int 6x(3x^2-4)^7\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int 6x(3x^2-4)^7\mathrm{d}x &= \frac{1}{8}\int 48x(3x^2-4)^7\mathrm{d}x\\ &= \frac{1}{8}(3x^2-4)^8+c.\end{aligned}$$

8.1.2 Finding the Constant of Integration

The indefinite integral has infinitely many solutions, but if we have more information about the curve, we can find the exact constant of integration.

Example 8.6 The function f is such that $f'(x) = 15x^4 - 6x$ and $f(-1) = 1$. Find $f(x)$.

Solution. We have

$$\begin{aligned} f(x) &= \int f'(x)\mathrm{d}x \\ &= \int (15x^4 - 6x)\mathrm{d}x \\ &= 3x^5 - 3x^2 + c. \end{aligned}$$

Using $f(-1) = 1$ gives

$$\begin{aligned} & 1 = 3(-1)^5 - 3(-1)^2 + c \\ \Leftrightarrow\ & 1 = -3 - 3 + c \\ \Leftrightarrow\ & c = 7. \end{aligned}$$

Hence, $f(x) = 3x^5 - 3x^2 + 7$.

8.2 Definite Integration

We can integrate a function between two specified limits. For example, $\displaystyle\int_2^5 2x\mathrm{d}x$ is called the **definite integral** of $2x$ with respect to x between the limits 2 and 5. The evaluation of a definite integral is

Key Point 8.7

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a), \text{ where } \frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = f(x).$$

Beyond The Syllabus. 上述公式又称为微积分基本定理（Fundamental Theorem of Calculus）的第二部分. 关于这一公式的证明需要更多严格的数学定义.

Example 8.7 Find $\displaystyle\int_2^4 x^3\mathrm{d}x$.

Solution. We have

$$\int_2^4 x^3\mathrm{d}x = \left[\frac{1}{4}x^4 + c\right]_2^4 = \left(\frac{4^4}{4} + c\right) - \left(\frac{2^4}{4} + c\right) = 60.$$

Remark. We should note that the constant c can be cancel out when calculating definite integral, and so

$$\int_2^4 x^3\mathrm{d}x = \left[\frac{1}{4}x^4\right]_2^4.$$

8.2.1 Basic Rules

Some useful rules are:

Key Point 8.8

$\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$, where k is a constant.

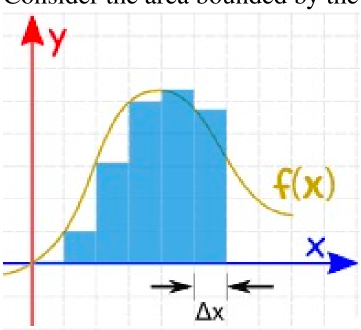
$\int_a^b (f(x) \pm g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x.$

$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x.$

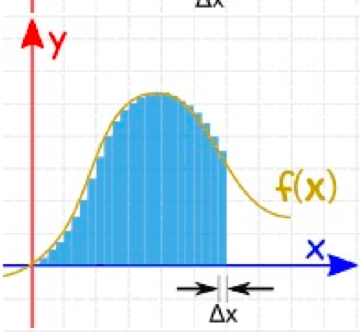
$\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x.$

8.2.2 Area under a Curve

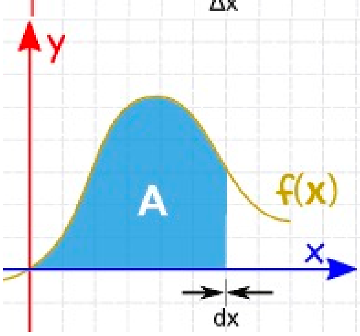
Consider the area bounded by the curve $f(x)$, the x-axis and the lines say $x = a$ and $x = b$ ($a < b$).



The area of the region can be approximated by a series of rectangular strips of thickness Δx and height $f(x)$. The approximated area is $\sum f(x)\Delta x$.



We can make Δx smaller and the region can be approximated better by more rectangular strips of thickness Δx and height $f(x)$. The approximated area is $\sum f(x)\Delta x$.



As the slices approach zero in width ($\Delta x \to 0$), the approximation approaches the true value. We now write $\mathrm{d}x$ to mean that Δx are approaching zero in width. As $\Delta x \to 0$, $\sum f(x)\Delta x \to \int_a^b f(x)\mathrm{d}x$.

Key Point 8.9

If $y = f(x)$ is a function with $y \geq 0$, then the area A bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is

$$A = \int_a^b y \mathrm{d}x.$$

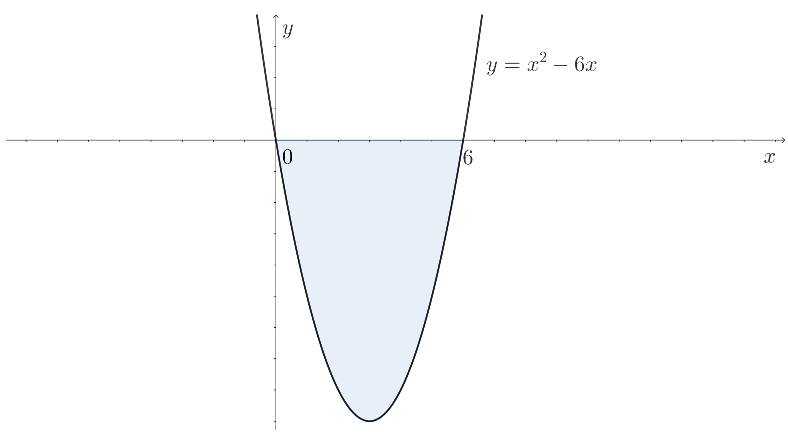
Example 8.8 Consider the area bounded by the curve $y = 3x^2$, the x-axis and the lines $x = 4$ and $x = 6$. Find the area.

Solution. The area is

$$\int_4^6 3x^2 \mathrm{d}x = \left[\frac{3x^3}{3}\right]_4^6 = 216 - 64 = 152 \text{ units}^2.$$

If the required area lies below the x-axis, then $\int_a^b f(x)\mathrm{d}x$ will have a negative value. This is beacuse the integral is summing the y values, and these are all negative.

Example 8.9 Consider the area bounded by the curve $y = x^2 - 6x$, the x-axis and the lines $x = 0$ and $x = 6$. Find the area.



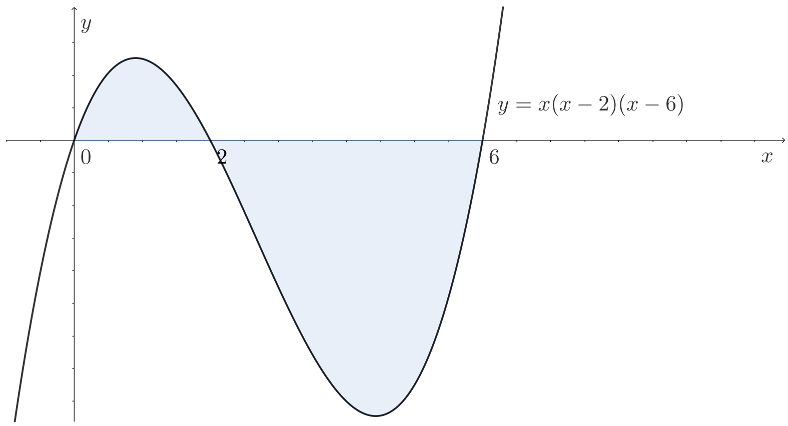
Solution. We have

$$\int_0^6 (x^2 - 6x)\mathrm{d}x = \left[\frac{1}{3}x^3 - \frac{6}{2}x^2\right]_0^6 = (72 - 108) - (0 - 0) = -36.$$

Hence the area is 36 units^2.

The required region could consist of a section above the x-axis and a section below the x-axis. If this happens, we must evaluate each area separately.

Example 8.10 Find the total area of the shaded regions.



Solution. We have

$$\begin{aligned}\int_0^2 x(x-2)(x-6)\mathrm{d}x &= \int_0^2 (x^3-8x^2+12x)\mathrm{d}x\\ &= \left[\frac{1}{4}x^4-\frac{8}{3}x^3+6x^2\right]_0^2\\ &= \left(\frac{1}{4}(2)^4-\frac{8}{3}(2)^3+6(2)^2\right)-\left(\frac{1}{4}(0)^4-\frac{8}{3}(0)^3+6(0)^2\right)\\ &= \frac{20}{3}\end{aligned}$$

and

$$\begin{aligned}\int_2^6 x(x-2)(x-6)\mathrm{d}x &= \left[\frac{1}{4}x^4-\frac{8}{3}x^3+6x^2\right]_2^6\\ &= \left(\frac{1}{4}(6)^4-\frac{8}{3}(6)^3+6(6)^2\right)-\left(\frac{1}{4}(2)^4-\frac{8}{3}(2)^3+6(2)^2\right)\\ &= -36-\frac{20}{3}\\ &= -\frac{128}{3}.\end{aligned}$$

Hence, the total area of the shaded regions is $\dfrac{20}{3}+\dfrac{128}{3}=\dfrac{148}{3}$ units^2.

Similarly, we can also use definite integral to find the area enclosed by a curve and the y-axis.

Key Point 8.10

If $x = f(y)$ is a function with $x \geq 0$, then the area A bounded by the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$ is

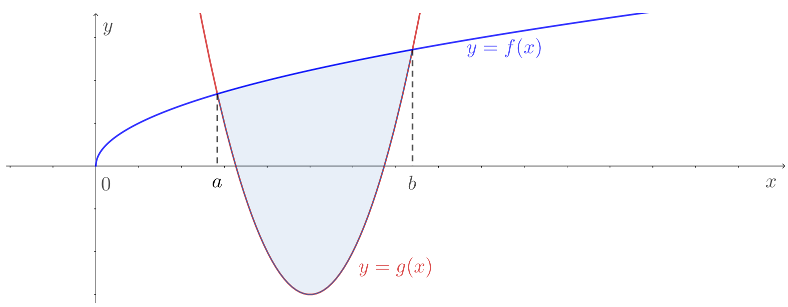
$$A = \int_a^b x\mathrm{d}y.$$

Example 8.11 Consider the area bounded by the curve $x = y(4-y)$, the y-axis and the lines $y = 0$ and $y = 4$. Find the area.

Solution. The area is

$$\int_0^4 x\mathrm{d}y = \int_0^4 (4y - y^2)\mathrm{d}y = \left[\frac{4}{2}y^2 - \frac{1}{3}y^3\right]_0^4 = \left(2(4)^2 - \frac{1}{3}(4)^3\right) - \left(2(0)^2 - \frac{1}{3}(0)^3\right) = \frac{32}{3} \text{ units}^2.$$

8.2.3 Area Bounded by a Curve and a Line or by Two Curves



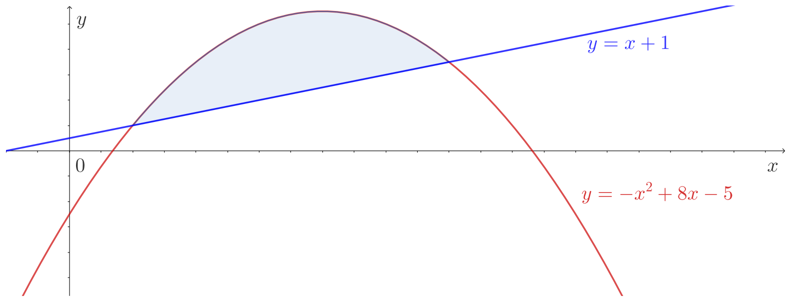
We introduce a general method to find the area bounded by a curve and a line or by two curves.

Key Point 8.11

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, then the area A enclosed between the two functions is given by

$$A = \int_a^b (f(x) - g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x.$$

Example 8.12 Find the area enclosed by $y = -x^2 + 8x - 5$ and $y = x + 1$.

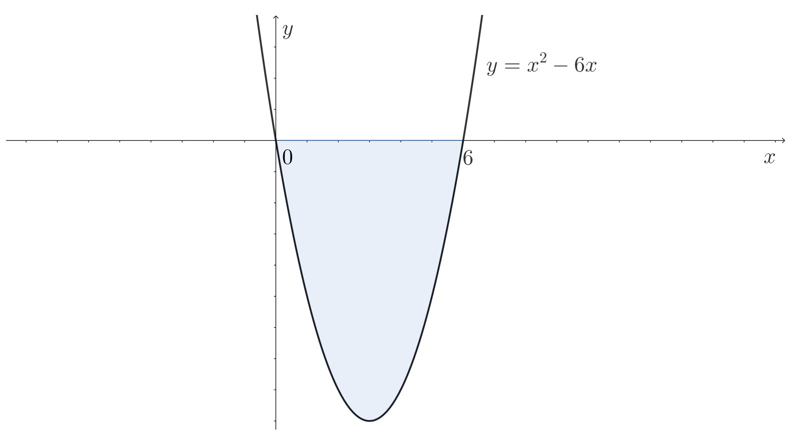


Solution. Using $f(x) = -x^2 + 8x - 5$ and $g(x) = x + 1$ gives the area is

$$\begin{aligned}\int_1^6 f(x)\mathrm{d}x - \int_1^6 g(x)\mathrm{d}x &= \int_1^6 (-x^2 + 8x - 5)\mathrm{d}x - \int_1^6 (x+1)\mathrm{d}x \\ &= \int_1^6 (-x^2 + 7x - 6)\mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 6x\right]_1^6 \\ &= \left(-\frac{1}{3}(6)^3 + \frac{7}{2}(6)^2 - 6(6)\right) - \left(-\frac{1}{3}(1)^3 + \frac{7}{2}(1)^2 - 6(1)\right) \\ &= \frac{125}{6} \text{ units}^2.\end{aligned}$$

Remark. We can also use the area bounded by the curve $y = -x^2 + 8x - 5$, the x-axis and the lines $x = 1$ and $x = 6$ minus the area of trapezium. But in principle, it is easiest to use the general method.

Example 8.13 Consider the area bounded by the curve $y = x^2 - 6x$, the x-axis and the lines $x = 0$ and $x = 6$. Find the area.



Solution. Using $f(x) = 0$ and $g(x) = x^2 - 6x$ gives the area is

$$\int_0^6 0\mathrm{d}x - \int_0^6 (x^2 - 6x)\mathrm{d}x = \int_0^6 -x^2 + 6x\mathrm{d}x = \left[-\frac{1}{3}x^3 + \frac{6}{2}x^2\right]_0^6 = -72 + 108 = 36 \text{ units}^2.$$

8.3 Improper Integrals

In this section, we consider what happens if some part of a definite integral becomes **infinite**. These are known as **improper integrals**.

At first, we consider definite integrals that have either one limit infinite or both limits infinite. Examples of these are $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ and $\int_{-\infty}^2 \frac{1}{x^3}\mathrm{d}x$.

Key Point 8.12

We can evaluate integrals of the form $\int_a^\infty f(x)\mathrm{d}x$ by replacing the infinite limit with a finite value X, and then taking the limit as $X \to \infty$, provided the limit exists.

Similarly, we can evaluate integrals of the form $\int_{-\infty}^b f(x)\mathrm{d}x$ by replacing the infinite limit with a finite value X, and then taking the limit as $X \to -\infty$, provided the limit exists.

Example 8.14 Find the value of improper integral $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ if it exists.

Solution. We first consider

$$\int_1^X \frac{1}{x^2}\mathrm{d}x = \int_1^X x^{-2}\mathrm{d}x = [-x^{-1}]_1^X = \left(-\frac{1}{X}\right) - \left(-\frac{1}{1}\right) = 1 - \frac{1}{X}.$$

As $X \to \infty$, $\frac{1}{X} \to 0$, and thus $\int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1 - 0 = 1$.

Example 8.15 Find the value of improper integral $\int_{-\infty}^0 \frac{1}{(1-x)^3}\mathrm{d}x$ if it exists.

Solution. We first consider

$$\int_X^0 \frac{1}{(1-x)^3}\mathrm{d}x = \int_X^0 (1-x)^{-3}\mathrm{d}x = \left[\frac{1}{(-2)(-1)}(1-x)^{-2}\right]_X^0 = \frac{1}{2} - \frac{1}{2(1-X)^2}.$$

As $X \to \infty$, $\frac{1}{2(1-X)^2} \to 0$, and thus $\int_{-\infty}^0 \frac{1}{(1-x)^3} = \frac{1}{2}$.

Then we consider definite integrals where the function to be integrated approaches an infinite value at either or both end points in the interval.

Remark. $\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$ is an invalid integral since $\frac{1}{x^2}$ is not defined when $x = 0$. However, $\int_0^1 \frac{1}{x^2}\mathrm{d}x$ is an improper integral since $\frac{1}{x^2}$ tends to infinity as $x \to 0$ and it is well-defined everywhere else in the interval of integration.

In this section, we consider only those improper integrals where the function is not defined at one end of the interval.

Key Point 8.13

We can evaluate integrals of the form $\int_a^b f(x)\mathrm{d}x$ where $f(x)$ is not defined when $x = a$ by replacing the limit a with an X and then taking the limit as $X \to a$, provided the limit exists.

Similarly, we can evaluate integrals of the form $\int_a^b f(x)\mathrm{d}x$ where $f(x)$ is not defined when $x = b$ by replacing the limit b with an X and then taking the limit as $X \to b$, provided the limit exists.

Example 8.16 Find the value of $\int_0^2 \frac{5}{x^2}\mathrm{d}x$ if it exists.

Solution. The function $f(x) = \frac{5}{x^2}$ is not defined when $x = 0$. Consider

$$\int_X^2 \frac{5}{x^2}\mathrm{d}x = \int_X^2 5x^{-2}\mathrm{d}x = [-5x^{-1}]_X^2 = \left(-\frac{5}{2}\right) - \left(-\frac{5}{X}\right) = \frac{5}{X} - \frac{5}{2}.$$

As $X \to 0$, $\frac{5}{X}$ tends to infinity, and thus $\int_0^2 \frac{5}{x^2}\mathrm{d}x$ is undefined.

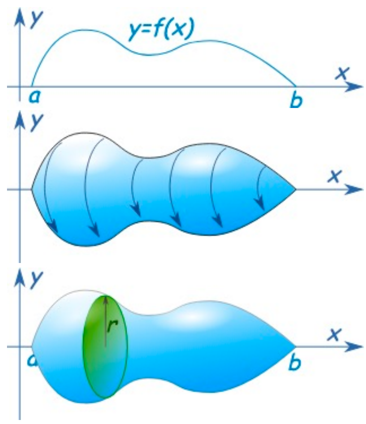
Example 8.17 Find the value of $\int_0^2 \frac{3}{\sqrt{2-x}}\mathrm{d}x$ if it exists.

Solution. The function $f(x) = \frac{3}{\sqrt{2-x}}$ is not defined when $x = 2$. Consider

$$\int_0^X \frac{3}{\sqrt{2-x}}\mathrm{d}x = \int_0^X 3(2-x)^{-1/2}\mathrm{d}x = \left[\frac{3}{-1/2}(2-x)^{1/2}\right]_0^X = [-6\sqrt{2-x}]_0^X = 6\sqrt{2} - 6\sqrt{2-X}.$$

As $X \to 2$, $6\sqrt{2-X} \to 0$, and thus $\int_0^2 \frac{3}{\sqrt{2-x}}\mathrm{d}x = 6\sqrt{2}$.

8.4 Volumes of Revolution



Consider the area bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

When this area is rotated about the x-axis through $360°$, a **solid of revolution** is formed. The volume of this solid is called a **volume of revolution**.

Similar to the area calculation, we can approximate the volume of the solid by a series of cylindrical discs of thickness Δx and radius $y = f(x)$. The volume of each cylindrical disc is $\pi y^2 \Delta x$, and thus the approximated volume is $\sum \pi y^2 \Delta x$.

As $\Delta x \to 0$, $\sum \pi y^2 \Delta x \to \int_a^b \pi y^2 \mathrm{d}x$.

Key Point 8.14

The volume V, obtained when the function $y = f(x)$ is rotated through $360°$ about the x-axis between the boundary values $x = a$ and $x = b$ is

$$V = \int_a^b \pi y^2 \mathrm{d}x.$$

Example 8.18 Find the volume obtained when the function $y = \dfrac{9}{3x+2}$ is rotated through $360°$ about the x-axis between the boundary values $x = 1$ and $x = 2$.

Solution. The volume is

$$\int_1^2 \pi y^2 \mathrm{d}x = \pi \int_1^2 \left(\frac{9}{3x+2}\right)^2 \mathrm{d}x = \pi \int_1^2 81(3x+2)^{-2} \mathrm{d}x = \pi \left[\frac{-27}{3x+2}\right]_1^2 = \frac{81}{40}\pi \text{ units}^3.$$

Similarly, we can also use definite integral to find the volume obtained when a curve is rotated about the y-axis.

Key Point 8.15

The volume V, obtained when the function $x = f(y)$ is rotated through $360°$ about the y-axis between the boundary values $y = a$ and $y = b$ is

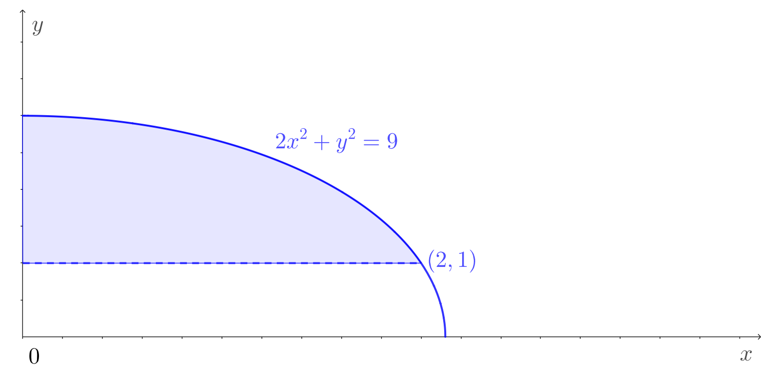
$$V = \int_a^b \pi x^2 \mathrm{d}y.$$

Example 8.19 Find the volume obtained when the curve $y = x^2$ is rotated through $360°$ about the y-axis between the boundary values $y = 2$ and $y = 5$.

Solution. The volume is

$$\int_2^5 \pi x^2 \mathrm{d}y = \pi \int_2^5 y \mathrm{d}y = \pi \left[\frac{y^2}{2}\right]_2^5 = \frac{21}{2}\pi \text{ units}^3.$$

Example 8.20 Find the volume of the solid obtained when the shaded region is rotated through 360° about the x-axis.



Solution. When the shaded region is rotated about the x-axis, a solid with a cylindrical hole is formed. The radius of the cylindrical hole is 1 unit and the length of the hole is 2 units. Thus the volume of solid is

$$\begin{aligned}\int_0^2 \pi y^2 \mathrm{d}x - \text{Volume of cylinder} &= \pi \int_0^2 (9 - 2x^2)\mathrm{d}x - \pi r^2 h \\ &= \pi \left[9x - \frac{2}{3}x^3\right]_0^2 - \pi \times 1^2 \times 2 \\ &= \pi \left(18 - \frac{16}{3}\right) - 2\pi \\ &= \frac{32\pi}{3} \text{ units}^3.\end{aligned}$$

Appendix A List of formulae (MF19)

In P1 paper, list of formulae (MF19) is given, and we may use:

- Mensuration
  - Volume of sphere $= \frac{4}{3}\pi r^3$.
  - Surface area of sphere $= 4\pi r^2$.
  - Volume of cone or pyramid $= \frac{1}{3} \times$ Base area $\times$ Height.
  - Area of curved surface of cone $= \pi r \times$ Slant height.
  - Arc length of circle $= r\theta$, where θ in radians.
  - Area of sector of circle $= \frac{1}{2}r^2\theta$, where θ in radians.
- Algebra
  - For the quadratic equation $ax^2 + bx + c = 0$: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - For an arithmetic series:
    - $u_n = a + (n-1)d$
    - $S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n(2a + (n-1)d)$.
  - For a geometric series:
    - $u_n = ar^{n-1}$
    - $S_n = \dfrac{a(1-r^n)}{1-r}$, where $r \neq 1$
    - $S_\infty = \dfrac{a}{1-r}$, where $|r| < 1$.
  - $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + b^n$, where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.
- Trigonometry
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
  - $\cos^2\theta + \sin^2\theta = 1$.
  - Principal values: $-\frac{1}{2}\pi \leq \sin^{-1}x \leq \frac{1}{2}\pi, 0 \leq \cos^{-1}x \leq \pi$ and $-\frac{1}{2}\pi < \tan^{-1}x < \frac{1}{2}\pi$.
- Differentiation
  - If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
  - If $x = f(t)$ and $y = g(t)$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \div \dfrac{\mathrm{d}x}{\mathrm{d}t}$.
- Integration
  - If $f(x) = x^n$, then $\displaystyle\int f(x)\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$, where $n \neq 1$.