Calculus note

微积分 备注

ElegantLaTeX Program

Victory won't come to us unless we go to it.

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目　录

第 1 章 极限

Example 1.1: (2017/10/20) 求极限 $\lim_{x\to\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right)$

Proof:

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\begin{aligned}
\text{原式} &\xlongequal{\text{和差化积}} -2\lim_{x\to\infty}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}\\
&\xlongequal{\text{有理化}} -2\lim_{x\to\infty}\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}\\
&\xlongequal{\sin x\sim x} -2\lim_{x\to\infty}\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}\\
&\xlongequal{\text{有界乘无穷小量}} 0
\end{aligned}$$

□

Exercise 1.1: 计算

$$\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n}\sin(\frac{2k}{2n})}{\sum_{k=1}^{n}\sin(\frac{2k-1}{2n})}\right)^{n}=e^{2\cot(1/2)}$$

Proof: Using what is inside the parentheses:

$$\sum_{k=1}^{n}\sin(k/n)=\frac{1}{2i}\sum_{k=1}^{n}\left(e^{\frac{ki}{n}}-e^{\frac{-ki}{n}}\right)$$

$$\sum_{k=1}^{n}\sin(\frac{2k-1}{2n})=\frac{1}{2i}\sum_{k=1}^{n}\left(e^{\frac{(2k-1)i}{2n}}-e^{\frac{-(2k-1)i}{2n}}\right)$$

So, we get:

$$\frac{\frac{1}{2i}\sum_{k=1}^{n}\left(e^{\frac{ki}{n}}-e^{\frac{-ki}{n}}\right)}{\frac{1}{2i}\sum_{k=1}^{n}\left(e^{\frac{(2k-1)i}{2n}}-e^{\frac{-(2k-1)i}{2n}}\right)}$$

Now, factor a little:

$$\frac{\sum_{k=1}^{n}\left(e^{i/n}\right)^{k}-\sum_{k=1}^{n}\left(e^{-i/n}\right)^{k}}{e^{-i/2n}\sum_{k=1}^{n}\left(e^{i/n}\right)^{k}-e^{i/2n}\sum_{k=1}^{n}\left(e^{-i/2n}\right)^{k}}$$

These are partial geometric series. They should simplify down.

The top left one evaluates to

$$\frac{(e^{i}-1)e^{i/n}}{e^{i/n}-1}$$

The top right one:

$$\frac{(e^{i}-1)e^{-i}}{e^{i/n}-1}$$

The bottom left:

$$\frac{e^{-i/2n}(e^i-1)e^{i/n}}{e^{i/n}-1}$$

The bottom right:

$$\frac{e^{i/2n}(e^i-1)e^{-i}}{e^{i/n}-1}$$

Putting these altogether, it should whittle down to some trig functions involving sin and/or cos. But, I have not finished yet.

It looks encouraging though.

EDIT:

Well, I spent some time trying to hammer down the results of the sums above.

They become:

$$\frac{\sin(1/n+1)-\sin(1/n)-\sin(1)}{\sin(1/2n+1)+\sin(1/2n-1)-2\sin(1/2n)}$$

this is equivalent to:

$$\frac{\cos(1/2n)\sin(1/2)+\sin(1/2n)\cos(1/2)}{\sin(1/2)}$$

Using the product-to-sum formulas on the numerator, it whittles down to: $=\dfrac{\sin(\frac{n+1}{2n})}{\sin(1/2)}$

But, I admit I left tech do most of the work and played around with some trial and error.

So, we finally get:

$$\lim_{n\to\infty}\left(\frac{\sin(\frac{n+1}{2n})}{\sin(1/2)}\right)^n$$

Now, since this is a limit and there is an 'e' in the required solution, I figured I would make the sub $n=1/k$ in order to get something that resembles the 'e' limit.

$$\lim_{k\to 0}\left(\frac{\sin(\frac{k+1}{2})}{\sin(1/2)}\right)^{1/k}$$

So, take logs:

$$\lim_{k\to 0}\frac{1}{k}[\log(\sin(\frac{k+1}{2})-\log(\sin(1/2))]$$

Using L'Hopital and taking this limit:

$$\lim_{k\to 0}\frac{1}{2}\cot(\frac{k+1}{2})$$

results in

$$1/2\cot(1/2)$$

Now, e:

$$e^{1/2\cot(1/2)}=e^{\frac{1}{2\tan(1/2)}}$$

□

Example 1.2: (2017/10/8) 求极限

$$\lim_{x\to 0^+}\frac{\sqrt{1-e^{-2x}}-\sqrt{1+2x-\cos x}}{\sqrt{x^3}}$$

☞ Proof:

$$\text{原式} \xlongequal{\text{有理化}} \lim_{x\to0^+} \frac{-e^{-2x}-2x+\cos x}{\sqrt{x^3}\Big(\underbrace{\sqrt{1-e^{-2x}}}_{\sim\sqrt{2x}}+\underbrace{\sqrt{1+2x-\cos x}}_{\sim\sqrt{2x}}\Big)}$$

$$\xlongequal{\text{等价无穷小}} \lim_{x\to0^+} \frac{\cos x-e^{-2x}-2x}{2\sqrt{2}x^2}$$

$$\xlongequal{\text{洛必达}} \lim_{x\to0^+} \frac{-\sin x+2e^{-2x}-2}{4\sqrt{2}x}$$

$$\begin{cases} \xlongequal{\text{等价无穷小}} \lim\limits_{x\to0^+} \dfrac{-x+(-4x)}{4\sqrt{2}x} = -\dfrac{5}{4\sqrt{2}} \\ \xlongequal{\text{洛必达}} \lim\limits_{x\to0^+} \dfrac{-\cos x-4e^{-2x}}{4\sqrt{2}} = -\dfrac{5}{4\sqrt{2}} \end{cases}$$

$$\text{原式} \xlongequal{\text{有理化}} \lim_{x\to0^+} \frac{-e^{-2x}-2x+\cos x}{\sqrt{x^3}\Big(\underbrace{\sqrt{1-e^{-2x}}}_{\sim\sqrt{2x}}+\underbrace{\sqrt{1+2x-\cos x}}_{\sim\sqrt{2x}}\Big)}$$

$$\xlongequal{\text{等价无穷小}} \lim_{x\to0^+} \frac{\cos x-e^{-2x}-2x}{2\sqrt{2}x^2}$$

$$\xlongequal{\text{泰勒展开}} \lim_{x\to0^+} \frac{\left(1-\frac{1}{2!}x^2+o(x^2)\right)-\left(1+(-2x)+\frac{1}{2!}(-2x)^2+o(x^2)\right)-2x}{2\sqrt{2}x^2}$$

$$= \lim_{x\to0^+} \frac{\left(-\frac{1}{2!}-\frac{1}{2!}(-2)^2\right)x^2}{2\sqrt{2}x^2} = -\frac{5}{4\sqrt{2}}$$

□

Exercise 1.2: 求极限

$$\lim_{x\to0} x^6\left(\frac{1}{\sin^8 x}-\frac{1}{x^8}\right)$$

✎ Solution

$$\begin{aligned}
\lim_{x\to0} x^6\left(\frac{1}{\sin^8 x}-\frac{1}{x^8}\right) &= \lim_{x\to0} \frac{x^8-\sin^8 x}{x^2\sin^8 x} \\
&= \lim_{x\to0} \frac{(x^4-\sin^4 x)(x^4+\sin^4 x)}{x^{10}} \\
&=2\lim_{x\to0} \frac{x^4-\sin^4 x}{x^6} \\
&=2\lim_{x\to0} \frac{(x^2-\sin^2 x)(x^2+\sin^2 x)}{x^6} \\
&=4\lim_{x\to0} \frac{x^2-\sin^2 x}{x^4} \\
&=4\lim_{x\to0} \frac{(x-\sin x)(x+\sin x)}{x^4} \\
&=8\lim_{x\to0} \frac{x-\sin x}{x^3} \\
&=8\lim_{x\to0} \frac{1-\cos x}{3x^2} = \frac{8}{3}\lim_{x\to0} \frac{\frac{1}{2}x^2}{x^2}
\end{aligned}$$

$$=\frac{4}{3}$$

◀

Exercise 1.3: 设函数 $f(x)=x^{\frac{1}{x}}, x>1$

① 证明：$\forall x>1$, 恒有 $1<f(x)<1+e^{\frac{1}{e}}\cdot\frac{\ln x}{x}$

② 计算：$\lim\limits_{n\to\infty}\frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}}{n}$

③ 设数列 $I_n=\sum\limits_{k=1}^{n^2}\frac{1+2^{\frac{1}{2^k}}+3^{\frac{1}{3^k}}+\cdots+n^{\frac{1}{n^k}}}{n^2+k^2}$, 求 $\lim\limits_{n\to\infty}I_n$

Solution ① $f(x)=x^{\frac{1}{x}}=e^{\frac{\ln x}{x}}, x>1$ 并注意到 $\frac{\ln x}{x}>0\ (x>1)$ 故 $f(x)>e^0=1$

由于 e^x 在 $\left[0,\frac{\ln x}{x}\right]$ 可导，由拉格朗日中值定理有

$$e^{\frac{\ln x}{x}}-e^0=\frac{\ln x}{x}e^{\xi}\quad \xi\in\left(0,\frac{\ln x}{x}\right)$$

令 $g(x)=\frac{\ln x}{x}$ 则 $g'(x)=\frac{1-\ln x}{x^2}$ 故 $g(x)$ 在 $(1,e)\uparrow$ 在 $(e,+\infty)\downarrow$ 因此 $g_{max}(x)=\frac{1}{e}$ 故

$$f(x)=e^{\frac{\ln x}{x}}=1+e^{\xi}\frac{\ln x}{x}<1+\frac{\ln x}{x}e^{\frac{\ln x}{x}}<1+\frac{\ln x}{x}e^{\frac{1}{e}}$$

②: 由 ① 知

$$1\leqslant\lim_{n\to\infty}\frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}}{n}\leqslant\lim_{n\to\infty}\left(1+\frac{1}{n}\sum_{i=1}^{n}\frac{\ln i}{i}e^{\frac{1}{e}}\right)$$

其中

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\ln i}{i}e^{\frac{1}{e}}<e^{\frac{1}{e}}\frac{\ln n}{n}\sum_{i=1}^{n}\frac{1}{i}<e^{\frac{1}{e}}\frac{\ln n(\ln n+1)}{n}\to 0\quad(n\to\infty)$$

故由夹逼准则知

$$\lim_{n\to\infty}\frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}}{n}=1$$

用到不等式

$$\ln n<\sum_{i=1}^{n}\frac{1}{i}<\ln n+1$$

③: 由 ② 知

$$\left(1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}\right)=\sum_{i=1}^{n}i^{\frac{1}{i}}\sim n+o(n)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}\leqslant\lim_{n\to\infty}I_n\leqslant\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{\sum\limits_{i=1}^{n}i^{\frac{1}{i}}}{n^2+k^2}$$

下面计算极限 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n^2}\frac{n}{n^2+k^2}$

一方面

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}&=\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k-1}^{k}\frac{n}{n^2+k^2}\mathrm{d}x\\&\leqslant\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k-1}^{k}\frac{n}{n^2+x^2}\mathrm{d}x=\int_0^{n^2}\frac{n}{n^2+x^2}\mathrm{d}x=\frac{\pi}{2}\end{aligned}$$

另一方面

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}&=\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k}^{k+1}\frac{n}{n^2+k^2}\mathrm{d}x\\&\geqslant\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k}^{k+1}\frac{n}{n^2+x^2}\mathrm{d}x=\int_1^{n^2+1}\frac{n}{n^2+x^2}\mathrm{d}x=\frac{\pi}{2}\end{aligned}$$

故由夹逼准则知

$$\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}=\frac{\pi}{2}$$

因此

$$\lim_{n\to\infty}I_n=\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{1+2^{\frac{1}{2^k}}+3^{\frac{1}{3^k}}+\cdots+n^{\frac{1}{n^k}}}{n^2+k^2}=\frac{\pi}{2}$$

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Exercise 1.4: 求极限

$$\lim_{x\to0}\frac{\tan\tan x-\sin\sin x}{\tan x-\sin x}$$

Solution

$$\begin{aligned}\lim_{x\to0}\frac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}&=\lim_{x\to0}\frac{\tan(\tan x)-\sin(\tan x)}{\tan x-\sin x}+\lim_{x\to0}\frac{\sin(\tan x)-\sin(\sin x)}{\tan x-\sin x}\\&=\lim_{x\to0}\frac{\frac{1}{2}\tan^3x+o(\tan^3x)}{\frac{1}{2}x^3+o(x^3)}+\lim_{x\to0}\frac{2\cos\frac{\tan x+\sin x}{2}\sin\frac{\tan x-\sin x}{2}}{\tan x-\sin x}\\&=1+1=2\end{aligned}$$

◀

Solution

$$\begin{aligned}\lim_{x\to0}\frac{\tan\tan x-\sin\sin x}{\tan x-\sin x}&=\lim_{x\to0}\frac{\tan\tan x-\tan\sin x+\text{tansin}x-\sin\sin x}{\tan x-\sin x}\\&=\lim_{x\to0}\frac{\tan\tan x-\tan\sin x}{\tan x-\sin x}+\lim_{x\to0}\frac{\text{tansin}x-\sin\sin x}{\tan x-\sin x}\\&=\lim_{x\to0}\frac{\tan\tan x-\tan\sin x}{\tan x-\sin x}+\lim_{x\to0}\frac{\text{tansin}x\,(1-\text{cossin}x)}{\tan x\,(1-\cos x)}\end{aligned}$$

$$= \underset{\varepsilon\in(\sin x,\tan x)}{(\tan\varepsilon)'} + \lim_{x\to 0}\frac{x\times\frac{1}{2}x^2}{x\times\frac{1}{2}x^2} = \frac{1}{\cos^2\varepsilon}+1 = 1+1 = 2$$

◀

Exercise 1.5: 求极限

$$\lim_{n\to\infty} n\left[\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n\right]$$

Solution(小灰灰)

$$\begin{aligned}
I &= \lim_{n\to\infty} n\left[\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n\right] = \lim_{n\to\infty} n\left[\sum_{k=0}^{\infty}e^{-k}-\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^n\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}-\left(1-\frac{k}{n}\right)^n+\sum_{k=n}^{\infty}e^{-k}\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(1-\left(1-\frac{k}{n}\right)^n e^k\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(-\ln\left(1-\frac{k}{n}\right)^n-k\right)+O\left(\left(-\ln\left(1-\frac{k}{n}\right)^n-k\right)^2\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(n\left(\frac{k}{n}+\frac{k^2}{2n^2}+o\left(\frac{k^3}{3n^3}\right)\right)-k\right)+o\left(\left(-\ln\left(1-\frac{k}{n}\right)^n-k\right)^2\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(\frac{k^2}{2n}+O\left(\frac{k^3}{n^2}\right)\right)+O\left(\left(\frac{k^2}{2n}\right)^2\right)\right]\\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}e^{-k}\frac{k^2}{2}+\frac{1}{n}o\left(\sum_{k=0}^{n-1}e^{-k}k^3\right)+\frac{1}{4n}o\left(\sum_{k=0}^{n-1}e^{-k}k^4\right)\\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}e^{-k}\frac{k^2}{2} = \sum_{k=0}^{\infty}e^{-k}\frac{k^2}{2} = S = \sum_{k=1}^{\infty}e^{-k+1}\frac{(k-1)^2}{2}\\
&= eS-\sum_{k=1}^{\infty}e^{-k+1}\frac{2k-1}{2} = \frac{1}{2e-2}\sum_{k=1}^{\infty}e^{-k+1}(2k-1)\\
&= \frac{1}{2e-2}\sum_{k=0}^{\infty}e^{-k}(2k+1) = \frac{1}{2e-2}+e^{-1}S+\frac{1}{2e-2}\sum_{k=1}^{\infty}2e^{-k}\\
&= \frac{1}{1-e^{-1}}\frac{1}{2e-2}\left(1+\sum_{k=1}^{\infty}2e^{-k}\right) = \frac{e^{-1}(e^{-1}+1)}{2(1-e^{-1})^3}\\
&= \frac{e(e^2+1)}{2(e-1)^3}
\end{aligned}$$

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第 2 章 导数

Example 2.1: 设函数 $f(x)$ 可导, $F(x)=f(x)(1+|\sin x|)$, $F(x)$ 在 $x=0$ 处可导, 求 $f(0)$

Solution $f(x)$ 可导知 $f(x)$ 连续, 于是可知 $\lim\limits_{x\to 0} f(x)=f(0)$

$$\begin{aligned}
F'_{-}(0) &= \lim_{x\to 0^-}\frac{F(x)-F(0)}{x}=\lim_{x\to 0^-}\frac{f(x)(1+|\sin x|)-f(0)}{x}\\
&=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}+\lim_{x\to 0}f(x)\lim_{x\to 0^-}\frac{|\sin x|}{x}\\
&=f'(0)+f(0)\lim_{x\to 0^-}\frac{-\sin x}{x}=f'(0)-f(0)
\end{aligned}$$

$$\begin{aligned}
F'_{+}(0) &= \lim_{x\to 0^+}\frac{F(x)-F(0)}{x}=\lim_{x\to 0^+}\frac{f(x)(1+|\sin x|)-f(0)}{x}\\
&=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}+\lim_{x\to 0}f(x)\lim_{x\to 0^+}\frac{|\sin x|}{x}\\
&=f'(0)+f(0)\lim_{x\to 0^+}\frac{\sin x}{x}=f'(0)+f(0)
\end{aligned}$$

$F(x)$ 在 $x=0$ 处可导 $\Longrightarrow F'(0)=F'_{-}(0)=F'_{+}(0)\Longrightarrow f(0)=0$ ◀

Exercise 2.1: 设 $f(x)$ 在 $x=0$ 处连续, 且 $\lim\limits_{x\to 0}\dfrac{f(2x)-f(x)}{x}$ 存在. 求证 $f'(0)$ 存在.

Solution: 令 $A=\lim\limits_{x\to 0}\dfrac{f(2x)-f(x)}{x}$, 下证 $f'(0)=A$.
事实上, 对任意 $\varepsilon>0$, 根据所给条件, 存在 $\delta>0$, 使得

$$\left|\frac{f(2x)-f(x)}{x}-A\right|<\varepsilon, \forall 0<|x|<\delta,$$

即

$$Ax-\varepsilon|x|<f(2x)-f(x)<Ax+\varepsilon|x|, \forall 0<|x|<\delta.$$

从而

$$\frac{Ax}{2^i}-\frac{\varepsilon|x|}{2^i}<f\left(\frac{x}{2^{i-1}}\right)-f\left(\frac{x}{2^i}\right)<\frac{Ax}{2^i}+\frac{\varepsilon|x|}{2^i}, \forall 0<|x|<\delta, i\in\mathbb{N}.$$

因此对任意 $0<|x|<\delta$ 和 $n\in\mathbb{N}$,

$$\left(1-\frac{1}{2^n}\right)(Ax-\varepsilon|x|)<f(x)-f\left(\frac{x}{2^n}\right)<\left(1-\frac{1}{2^n}\right)(Ax+\varepsilon|x|).$$

令 $n\to\infty$, 利用 f 在 $x=0$ 处的连续性, 得到

$$Ax-\varepsilon|x|\leqslant f(x)-f(0)\leqslant Ax+\varepsilon|x|, \forall 0<|x|<\delta.$$

即
$$\left|\frac{f(x)-f(0)}{x}-A\right| \leqslant \varepsilon, \forall 0<|x|<\delta,$$
这就证明了 $f'(0)$ 存在, 且 $f'(0)=A$. □

Exercise 2.2: 已知 $y=x^2e^{2x}$, 求 $y^{(20)}$

✎ Solution 设 $u=e^{2x}$, $v=x^2$, 则
$$u^{(k)}=2^ke^{2x}\ (k=1,2,\cdots,20)$$
$$v'=2x,\ v''=2,\ v^{(k)}=0\ (k=3,4,\cdots,20)$$
代入莱布尼茨公式, 得
$$\begin{aligned} y^{(20)} &= (x^2e^{2x})^{(20)} \\ &= 2^{20}e^{2x}\cdot x^2+20\cdot 2^{19}e^{2x}\cdot 2x+\frac{20\cdot 19}{2!}2^{18}e^{2x}\cdot 2 \\ &= 2^{20}e^{2x}(x^2+20x+95) \end{aligned}$$
莱布尼茨公式
$$(uv)^n=\sum_{k=0}^{n}C_n^ku^{(n-k)}v^{(k)}$$
◀

第 3 章 积分

3.1 不定积分

Exercise 3.1: 求不定积分

$$\int \frac{x^2}{\sqrt{1+x+x^2}}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1+x+x^2}}\,\mathrm{d}x &\xlongequal{x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan t} \int \left(\frac{\sqrt{3}}{2}\tan t-\frac{1}{2}\right)^2 \sec t\,\mathrm{d}t \\
&= \frac{3}{4}\int \tan^2 t\sec t\,\mathrm{d}t - \frac{\sqrt{3}}{2}\int \tan t\sec t\,\mathrm{d}t + \frac{1}{4}\int \sec t\,\mathrm{d}t \\
&= \frac{3}{4}\int \sec t(\sec^2 t-1)\,\mathrm{d}t - \frac{\sqrt{3}}{2}\sec t + \frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{4}\int \sec^3 t\,\mathrm{d}t - \frac{3}{4}\int \sec t\,\mathrm{d}t - \frac{\sqrt{3}}{2}\sec t + \frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{4}\sec t\tan t - \frac{3}{4}\int \tan^2 t\sec t\,\mathrm{d}t - \frac{3}{4}\int \sec t\,\mathrm{d}t - \frac{\sqrt{3}}{2}\sec t + \frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{8}\sec t\tan t - \frac{\sqrt{3}}{2}\sec t - \frac{1}{8}\ln|\sec t+\tan t| + C \\
&= \frac{1}{4}(2x-3)\sqrt{x^2+x+1} - \frac{1}{8}\ln|2\sqrt{x^2+x+1}+2x+1| + C
\end{aligned}$$

Exercise 3.2: 计算积分

$$\int \frac{1}{x\sqrt{x^2-2x-3}}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2-2x-3}}\,\mathrm{d}x &= \int \frac{1}{x\sqrt{(x-1)^2-4}}\,\mathrm{d}x \\
&\xlongequal{x-1=2\sec t} \int \frac{2\tan t\sec t}{2(2\sec t+1)\tan t}\,\mathrm{d}t = \int \frac{1}{2+\cos t}\,\mathrm{d}t \\
&= \int \frac{2-\cos t}{4-\cos^2 t}\,\mathrm{d}t \\
&= 2\int \frac{1}{4\sin^2 t+3\cos^2 t}\,\mathrm{d}t - \int \frac{\cos t}{3+\sin^2 t}\,\mathrm{d}t \\
&= \int \frac{1}{(2\tan t)^2+3}\,\mathrm{d}(2\tan t) - \int \frac{1}{3+\sin^2 t}\,\mathrm{d}(\sin t)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{3}} \arctan \frac{2\tan t}{\sqrt{3}} - \frac{1}{2\sqrt{3}} \arctan \frac{\sin t}{\sqrt{3}} + C \\
&= \frac{1}{\sqrt{3}} \arctan \frac{\frac{2\tan t}{\sqrt{3}} - \frac{\sin t}{\sqrt{3}}}{1 + \frac{2\tan t}{\sqrt{3}} \times \frac{\sin t}{\sqrt{3}}} + C \\
&= -\frac{1}{\sqrt{3}} \arctan \frac{x+3}{\sqrt{3}\sqrt{x^2-2x-3}} + C
\end{aligned}
$$

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Exercise 3.3: 求不定积分: $\int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x$

Solution 注意到

$$
\begin{aligned}
\frac{1}{\sin^6 x + \cos^6 x} &= \frac{\sin^2 x + \cos^2 x}{\sin^4 x(1-\cos^2 x) + \cos^4 x(1-\sin^2 x)} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^4 x\cos^2 x - \cos^4 x\sin^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x}
\end{aligned}
$$

故

$$
\begin{aligned}
\int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x &= \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x} \mathrm{d}x \\
&= \int \frac{\tan^2 x + 1}{\tan^4 x - \tan^2 x + 1} \mathrm{d}(\tan x) \\
&\xlongequal{t=\tan x} \int \frac{t^2+1}{t^4-t^2+1} \mathrm{d}t \\
&= \int \frac{1}{\left(t-\frac{1}{t}\right)^2+1} \mathrm{d}\left(t - \frac{1}{t}\right) \\
&= \arctan\left(t - \frac{1}{t}\right) + C \\
&= -\arctan(2\cot x) + C
\end{aligned}
$$

◀

Exercise 3.4: 求不定积分: $\int \frac{1}{\sin x + \cos x} \mathrm{d}x$

Solution

$$
\begin{aligned}
\int \frac{1}{\sin x + \cos x} \mathrm{d}x &= \int \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x} \mathrm{d}x \\
&= \int \frac{1}{1-2\sin^2 x} \mathrm{d}(\sin x) + \int \frac{1}{2\cos^2 x - 1} \mathrm{d}(\cos x) \\
&= -\frac{1}{\sqrt{2}} \int \frac{1}{2\sin^2 x - 1} \mathrm{d}\left(\sqrt{2}\sin x\right) + \frac{1}{\sqrt{2}} \int \frac{1}{2\cos^2 x - 1} \mathrm{d}\left(\sqrt{2}\cos x\right)
\end{aligned}
$$

$$= -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$$

◀

Solution

$$\begin{aligned}\int\frac{1}{\sin x+\cos x}\,\mathrm{d}x &= \int\frac{1}{\cos^2\left(\frac{1}{2}x\right)-\sin^2\left(\frac{1}{2}x\right)+2\cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}x\right)}\,\mathrm{d}x\\ &= 2\int\frac{1}{-\left(\tan\left(\frac{1}{2}x\right)-1\right)^2+2}\,\mathrm{d}\big(\tan\left(\tfrac{1}{2}x\right)-1\big)\\ &= -\frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{1}{2}x\right)-1-\sqrt{2}}{\tan\left(\frac{1}{2}x\right)-1+\sqrt{2}}\right|+C\end{aligned}$$

◀

Solution

$$\begin{aligned}\int\frac{1}{\sin x+\cos x}\,\mathrm{d}x &= \int\frac{1}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}x\\ &= \frac{1}{\sqrt{2}}\int\frac{1}{\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}\left(x+\frac{\pi}{4}\right)\\ &= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x+\frac{\pi}{4}}{2}\right)\right|+C\\ &= \frac{1}{\sqrt{2}}\ln\left|\csc\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right|+C\end{aligned}$$

◀

Exercise 3.5: 求不定积分

$$\int x^2\sqrt{x^2+1}\,\mathrm{d}x$$

Solution

$$\begin{aligned}I &= \int x^2\sqrt{x^2+1}\,\mathrm{d}x = \int x\sqrt{x^4+x^2}\,\mathrm{d}x = \frac{1}{2}\int\sqrt{x^4+x^2}\,dx^2\\ &= \frac{1}{2}\int\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx^2 = \frac{1}{2}\int\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,d\left(x^2+\frac{1}{2}\right)\\ &= \frac{1}{2}\left(x^2+\frac{1}{2}\right)\sqrt{x^4+x^2}-\frac{1}{2}\int\frac{\left(x^2+\frac{1}{2}\right)^2}{\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,d\left(x^2+\frac{1}{2}\right)\\ &= \frac{1}{2}\left(x^2+\frac{1}{2}\right)\sqrt{x^4+x^2}-\frac{1}{2}\int\frac{\left(x^2+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}{\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,d\left(x^2+\frac{1}{2}\right)\\ &= \frac{1}{2}\left(x^2+\frac{1}{2}\right)\sqrt{x^4+x^2}-\frac{1}{2}\int\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,d\left(x^2+\frac{1}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}\int \frac{1}{\sqrt{\left(x^2+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}d\left(x^2+\frac{1}{2}\right)\\
&=\frac{1}{2}\left(x^2+\frac{1}{2}\right)\sqrt{x^4+x^2}-I-\frac{1}{8}\ln\left(x^2+\frac{1}{2}+\sqrt{x^4+x^2}\right)\\
\Rightarrow I &=\frac{1}{4}\left(x^2+\frac{1}{2}\right)\sqrt{x^4+x^2}-\frac{1}{16}\ln\left(x^2+\frac{1}{2}+\sqrt{x^4+x^2}\right)+c_1\\
&=\frac{1}{8}x\left(2x^2-1\right)\sqrt{x^2+1}-\frac{1}{16}\ln\left(x+\sqrt{x^2+1}\right)^2+c\left(c=c_1+\frac{\ln 2}{16}\right)\\
&=\frac{1}{8}x\left(2x^2-1\right)\sqrt{x^2+1}-\frac{1}{8}\ln\left(x+\sqrt{x^2+1}\right)+c
\end{aligned}
$$

◀

Exercise 3.6: 求不定积分: $\displaystyle\int \frac{16x+11}{\left(x^2+2x+2\right)^2}\,\mathrm{d}x$

Solution

$$
\begin{aligned}
\int \frac{16x+11}{\left(x^2+2x+2\right)^2}\mathrm{d}x &= 8\int \frac{2x+2}{\left(x^2+2x+2\right)^2}\mathrm{d}x-\int \frac{5}{\left(x^2+2x+2\right)^2}\mathrm{d}x\\
&=8\int \frac{1}{\left(x^2+2x+2\right)^2}d\left(x^2+2x+2\right)-\underbrace{5\int \frac{1}{\left(\left(x+1\right)^2+1\right)^2}\mathrm{d}x}_{x+1=\tan t}\\
&=-\frac{8}{x^2+2x+2}-5\int \frac{\sec^2 t}{\left(\tan^2 t+1\right)^2}\mathrm{d}t\\
&=-\frac{8}{x^2+2x+2}-5\int \cos^2 t\,\mathrm{d}t\\
&=-\frac{8}{x^2+2x+2}-5\int \frac{1+\cos 2t}{2}\mathrm{d}t\\
&=-\frac{8}{x^2+2x+2}-\frac{5}{2}\int \mathrm{d}t-\frac{5}{4}\int \cos 2t\,d\left(2t\right)\\
&=-\frac{8}{x^2+2x+2}-\frac{5}{2}t-\frac{5}{4}\sin 2t+C\\
&=-\frac{5x+21}{2\left(x^2+2x+2\right)}-\frac{5}{2}\arctan\left(x+1\right)+C
\end{aligned}
$$

◀

Exercise 3.7: 求不定积分: $\displaystyle I=\int \frac{16x+11}{\left(x^2+2x+2\right)^2}\,\mathrm{d}x$

Solution

$$
\begin{aligned}
I &= 8\int \frac{2x+2}{\left(x^2+2x+2\right)^2}\mathrm{d}x-\int \frac{5}{\left(x^2+2x+2\right)^2}\mathrm{d}x\\
&=8\int \frac{1}{\left(x^2+2x+2\right)^2}d\left(x^2+2x+2\right)-5\int \frac{1}{\left(x^2+2x+2\right)^2}\mathrm{d}x\\
&=-\frac{8}{x^2+2x+2}-5\int \frac{1+\left(x+1\right)^2-\left(x+1\right)^2}{\left(x^2+2x+2\right)^2}\mathrm{d}x\\
&=-\frac{8}{x^2+2x+2}-5\int \frac{1}{\left(x+1\right)^2+1}d\left(x+1\right)-\frac{5}{2}\int \left(x+1\right)d\left(\frac{1}{x^2+2x+2}\right)
\end{aligned}
$$

$$= -\frac{8}{x^2+2x+2} - 5\arctan(x+1) - \frac{5x+5}{2(x^2+2x+2)} + \frac{5}{2}\int \frac{1}{(x+1)^2+1} d(x+1)$$

$$= -\frac{5x+21}{2(x^2+2x+2)} - \frac{5}{2}\arctan(x+1) + C$$

◀

Exercise 3.8: 求不定积分

$$\int \sqrt[3]{\frac{1+\sin x}{1-\sin x}}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
I &= \int \sqrt[3]{\frac{1+\sin x}{1-\sin x}}\mathrm{d}x \\
&\xlongequal{x=2\theta} 2\int \sqrt[3]{\frac{1+\sin 2\theta}{1-\sin 2\theta}}\mathrm{d}\theta = 2\int \sqrt[3]{\left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)^2}\mathrm{d}\theta \\
&= 2\int \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^{\frac{3}{2}}\mathrm{d}\theta \\
&\xlongequal{\phi=\frac{\pi}{4}+\theta} 2\int \left[\tan\left(\frac{\pi}{4}+\theta\right)\right]^{\frac{3}{2}}\mathrm{d}\theta \\
&= 2\int \tan^{\frac{3}{2}}\phi\,\mathrm{d}\phi \\
&\xlongequal{\sqrt{\tan\phi}=t} 4\int \frac{t^4}{1+t^4}\mathrm{d}t = 4t - 4\int \frac{1}{1+t^4}\mathrm{d}t \\
&= 4t - 2\int \frac{(t^2+1)-(t^2-1)}{1+t^4}\mathrm{d}t \\
&= 4t - 2\int \frac{t^2+1}{1+t^4}\mathrm{d}t + 2\int \frac{t^2-1}{1+t^4}\mathrm{d}t \\
&= 4t - 2\int \frac{1}{\left(t-\frac{1}{t}\right)^2+2}\mathrm{d}\left(t-\frac{1}{t}\right) + 2\int \frac{1}{\left(t+\frac{1}{t}\right)^2-2}\mathrm{d}\left(t+\frac{1}{t}\right) \\
&= 4t - \frac{1}{\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right| + C \\
&= 4\sqrt{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)} - \frac{1}{\sqrt{2}}\arctan\frac{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)-1}{\sqrt{2}} \\
&\quad - \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)+\sqrt{2}\sqrt{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)}+1}{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)-\sqrt{2}\sqrt{\tan\left(\frac{1}{2}x-\frac{\pi}{4}\right)}+1}\right| + C
\end{aligned}$$

◀

Exercise 3.9: 求不定积分

$$\int \frac{1}{1+\sqrt{\tan x}}\,\mathrm{d}x$$

Solution

$$I = \int \frac{1}{1+\sqrt{\tan x}}\,\mathrm{d}x \xlongequal{\sqrt{\tan x}=t} \int \frac{2t}{(1+t)(1+t^4)}$$

$$\frac{2t}{(1+t^4)(1+t)}=\frac{At^3+Bt^2+Ct+D}{1+t^4}+\frac{E}{1+t}$$
$$=\frac{(A+E)t^4+(A+B)t^3+(B+C)t^2+(C+D)t+(D+E)}{(1+t^4)(1+t)}$$

$$\begin{cases}A+E=0\\A+B=0\\B+C=0\\C+D=2\\D+E=0\end{cases}\Longrightarrow\begin{cases}A=1\\B=-1\\C=1\\D=1\\E=-1\end{cases}\Longrightarrow\frac{2t}{(1+t^4)(1+t)}=\frac{t^3-t^2+t+1}{1+t^4}-\frac{1}{1+t}$$

$$\begin{aligned}I&=\int\frac{t^3-t^2+t+1}{1+t^4}\,\mathrm{d}t-\int\frac{1}{1+t}\,\mathrm{d}t\\&=\frac{1}{4}\int\frac{1}{1+t^4}d(t^4)+\frac{1}{2}\int\frac{1}{1+t^4}d(t^2)+\int\frac{\frac{1}{t^2}-1}{\frac{1}{t^2}+t^2}\,\mathrm{d}t-\ln|1+t|\\&=\frac{1}{4}\ln\left(1+t^4\right)+\frac{1}{2}\arctan t^2-\ln|1+t|+\int\frac{1}{\left(t+\frac{1}{t}\right)^2-2}d\left(t+\frac{1}{t}\right)\\&=\frac{1}{4}\ln\left(1+\tan^2x\right)+\frac{1}{2}\arctan\tan x-\ln\left|1+\sqrt{\tan x}\right|-\frac{\sqrt{2}}{4}\ln\left|\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right|+c\end{aligned}$$

◀

3.1.1 非初等表达

Exercise 3.10: 求不定积分

$$\int\frac{\arctan x}{x}\,\mathrm{d}x$$

Solution 设 $f(x)=\arctan x$ 则

$$f'(x)=\frac{1}{1+x^2}=\frac{1}{(1-ix)(1+ix)}=\frac{1}{2}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right)$$

利用幂级数展开 $f'(x)$, 首先我们知道 $\frac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n, x\in(-1,1)$

因此

$$f'(x)=\frac{1}{(1-ix)(1+ix)}=\frac{1}{2}\left(\sum_{n=0}^{\infty}(ix)^n+\sum_{n=0}^{\infty}(-ix)^n\right)$$

对两边积分有:

$$\int_0^x f'(x)\,\mathrm{d}x=\int_0^x\frac{1}{2}\left(\sum_{n=0}^{\infty}(ix)^n+\sum_{n=0}^{\infty}(-ix)^n\right)\mathrm{d}x$$

$$= -\frac{1}{2}i\sum_{n=0}^{\infty}\frac{(ix)^{n+1}}{n+1} + \frac{1}{2}i\sum_{n=0}^{\infty}\frac{(-ix)^{n+1}}{n+1}$$
$$= -\frac{1}{2}i\sum_{n=1}^{\infty}\frac{(ix)^{n}}{n} + \frac{1}{2}i\sum_{n=1}^{\infty}\frac{(-ix)^{n}}{n}$$

所以:

$$f(x) = \arctan x = \frac{1}{2}i\sum_{n=1}^{\infty}\frac{(-ix)^{n}}{n} - \frac{1}{2}i\sum_{n=1}^{\infty}\frac{(ix)^{n}}{n}$$

所以

$$\begin{aligned}\int\frac{\arctan x}{x}\,\mathrm{d}x &= \frac{1}{2}\int\sum_{n=1}^{\infty}\frac{(-ix)^{n-1}}{n}\,\mathrm{d}x - \frac{1}{2}i\int\sum_{n=1}^{\infty}\frac{(ix)^{n-1}}{n}\,\mathrm{d}x\\ &= \frac{1}{2}i\sum_{n=1}^{\infty}\frac{(-ix)^{n}}{n^2} - \frac{1}{2}i\sum_{n=1}^{\infty}\frac{(ix)^{n}}{n^2} + c\\ &= \frac{1}{2}i\left(\mathrm{Li}_2(-ix) - \mathrm{Li}_2(ix)\right) + c\end{aligned}$$

◀

Exercise 3.11: 求不定积分

$$\int x \tan x \,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int x \tan x \,\mathrm{d}x &= \int x \times \frac{\frac{e^{ix}-e^{-ix}}{2i}}{\frac{e^{ix}+e^{-ix}}{2}} \,\mathrm{d}x = -\int ix\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} \,\mathrm{d}x \\
&= -\int ix\frac{e^{2ix}-1}{e^{2ix}+1} \,\mathrm{d}x = -\int ix \,\mathrm{d}x + 2i\int \frac{x}{e^{2ix}+1} \,\mathrm{d}x \\
&\xlongequal{e^{2ix}=t} -\frac{1}{2}ix^2 + 2i\int \frac{\frac{1}{2i}\ln t}{t+1}\frac{1}{2it} \,\mathrm{d}t = -\frac{1}{2}ix^2 - \frac{1}{2}i\int \frac{\ln t}{(t+1)\,t} \,\mathrm{d}t \\
&= -\frac{1}{2}ix^2 - \frac{1}{2}i\left(\int \frac{\ln t}{t} \,\mathrm{d}t - \int \frac{\ln t}{t+1} \,\mathrm{d}t\right) \\
&= -\frac{1}{2}ix^2 - \frac{1}{2}i\left(\frac{1}{2}\ln^2 t - \ln t \ln(t+1) + \int \frac{\ln(1+t)}{t} \,\mathrm{d}t\right) \\
&= -\frac{1}{2}ix^2 - \frac{1}{2}i\left(\frac{1}{2}\ln^2 t - \ln t \ln(t+1) + \int \sum_{k=1}^{\infty} \frac{(-t)^{k-1}}{k} \,\mathrm{d}t\right) \\
&= -\frac{1}{2}ix^2 - \frac{1}{2}i\left(\frac{1}{2}\ln^2 t - \ln t \ln(t+1) - \sum_{k=1}^{\infty} \frac{(-t)^{k}}{k^2}\right) + c \\
&= -\frac{1}{2}ix^2 - \frac{1}{2}i\left(\frac{1}{2}\ln^2 t - \ln t \ln(t+1) - \mathrm{Li}_2\left(-t\right)\right) + c \\
&= \frac{1}{2}ix^2 + x\ln(e^{2ix}+1) + \frac{1}{2}i\,\mathrm{Li}_2\left(-e^{2ix}\right) + c
\end{aligned}$$

◀

Exercise 3.12: 求不定积分

$$\int \frac{xe^x}{1+e^x} \,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{xe^x}{1+e^x} \,\mathrm{d}x \xlongequal{t=e^x} \int \frac{\ln t}{1+t} \,\mathrm{d}t \\
&= \ln t \ln(1+t) - \int \frac{\ln(1+t)}{t} \,\mathrm{d}t \\
&= \ln t \ln(1+t) - \int \sum_{n=1}^{\infty} \frac{(-t)^{n-1}}{n} \,\mathrm{d}t \\
&= \ln t \ln(1+t) - \sum_{n=1}^{\infty} \int \frac{(-t)^{n-1}}{n} \,\mathrm{d}t \\
&= \ln t \ln(1+t) + \sum_{n=1}^{\infty} \frac{(-t)^{n}}{n^2} \,\mathrm{d}t + c \\
&= \mathrm{Li}_2(-t) + \ln t \ln(t+1) + c \\
&= \mathrm{Li}_2(-e^x) + x\ln(e^x+1) + c
\end{aligned}$$

Exercise 3.13: 求不定积分

$$\int \frac{x}{\tan x}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{x}{\tan x}\,\mathrm{d}x &= \int \frac{x\cos x}{\sin x}\,\mathrm{d}x = \int \frac{x\times\frac{e^{ix}+e^{-ix}}{2}}{\frac{e^{ix}-e^{-ix}}{2i}}\,\mathrm{d}x\\
&= \int xi\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}\,\mathrm{d}x = \int xi\frac{\left(e^{ix}-e^{-ix}+2e^{-ix}\right)}{e^{ix}-e^{-ix}}\,\mathrm{d}x\\
&= \int ix\,\mathrm{d}x + 2\int \frac{ie^{-ix}x}{e^{ix}-e^{-ix}}\,\mathrm{d}x = \frac{1}{2}ix^2 + 2\int\frac{ix}{e^{2ix}-1}\,\mathrm{d}x\\
&= \frac{1}{2}ix^2 - 2\int\frac{ix}{1-e^{2ix}}\,\mathrm{d}x\\
&\xlongequal{e^{2ix}=t} \frac{ix^2}{2} - 2\int\frac{i\times\frac{1}{2i}\ln t}{1-t}\times\left(\frac{1}{2it}\right)\mathrm{d}t\\
&= \frac{ix^2}{2} + \frac{i}{2}\int\frac{\ln t}{t\,(1-t)}\,\mathrm{d}t = \frac{1}{2}ix^2 + \frac{i}{2}\left(\int\frac{\ln t}{t}\,\mathrm{d}t + \int\frac{\ln t}{1-t}\,\mathrm{d}t\right)\\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\int \ln t\,d\ln t - \ln t\ln(1-t) + \int\frac{\ln(1-t)}{t}\,\mathrm{d}t\right)\\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \frac{1}{n}\sum_{n=1}^{\infty}\int t^{n-1}\,\mathrm{d}t\right)\\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \sum_{n=1}^{\infty}\frac{t^n}{n^2}\right) + c\\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \mathrm{Li}_2(t)\right) + c\\
&= x\ln\left(1-e^{2ix}\right) - \frac{1}{2}i\left(x^2 + \mathrm{Li}_2\left(e^{2ix}\right)\right) + c
\end{aligned}$$

Exercise 3.14: 计算不定积分

$$\int \frac{\tan x}{1+x^2}\,\mathrm{d}x$$

Solution:

$$\begin{aligned}
\int \frac{\tan x}{1+x^2}\,\mathrm{d}x &= \int \tan x\sum_{n=1}^{\infty}(-x^2)^n\,\mathrm{d}x\\
&= \sum_{n=1}^{\infty}(-1)^n\int x^{2n}\tan x\,\mathrm{d}x\\
&= \sum_{n=1}^{\infty}(-1)^n\int x^{2n}\sum_{k=1}^{\infty}\frac{B_{2k}(-4)^k(1-4^k)}{(2k)!}x^{2k-1}\,\mathrm{d}x
\end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^n B_{2k}(-4)^k(1-4^k)}{2(n+k)(2k)!}x^{2n+2k}+C$$

□

Exercise 3.15: 求不定积分

$$\int\sqrt{x+\frac{1}{x}}\,\mathrm{d}x \quad \|x\| < 1$$

Solution

$$\begin{aligned}
\int\sqrt{x+\frac{1}{x}}\,\mathrm{d}x &= \int\frac{\sqrt{x^2+1}}{\sqrt{x}}\,\mathrm{d}x \quad \|x\| < 1 \\
&= 2\int\frac{\sqrt{(\sqrt{x})^4+1}}{2\sqrt{x}}\,\mathrm{d}x = 2\int\sqrt{(\sqrt{x})^4+1}\,\mathrm{d}x \\
&\xlongequal{\sqrt{x}=u} 2\int\sqrt{u^4+1}\,\mathrm{d}u \quad \|u\| < 1 \\
&= 2\sum_{n=0}^{\infty}C_n^{1/2}\int u^{4n}\,\mathrm{d}x = 2\sum_{n=0}^{\infty}C_n^{1/2}\frac{u^{4n+1}}{4n+1}+C \\
&= 2\sum_{n=0}^{\infty}C_n^{1/2}\frac{x^{2n+\frac{1}{2}}}{4n+1}+C
\end{aligned}$$

◀

Example 3.1: 求不定积分

$$\int\frac{1}{\ln x-1}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int\frac{1}{\ln x-1}\,\mathrm{d}x &\xlongequal[\mathrm{d}x=e^v\,\mathrm{d}v]{\ln x=v} \int\frac{e^v}{v-1}\,\mathrm{d}v \\
&= -\int\frac{e^v}{1-v}\,\mathrm{d}v = -\int\left(\sum_{p=0}^{\infty}\frac{v^p}{p!}\right)\mathrm{d}v\left(\sum_{n=0}^{\infty}v^n\right) \\
&= -\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{p!}\int v^{n+p}\,\mathrm{d}v \\
&= -\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{p!}\cdot\frac{v^{n+p+1}}{n+p+1}+C \\
&= -\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{p!}\cdot\frac{(\ln x)^{n+p+1}}{n+p+1}+C
\end{aligned}$$

◀

3.2 定积分

3.2.1 定积分定义

Definition 3.1 定积分

设函数 $f(x)$ 在 $[a,b]$ 上有界, 在 $[a,b]$ 中任意插入若干个分点

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

把区间 $[a,b]$ 分为若 n 个小区间

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$$

各个小区间长度依次为

$$\Delta x_1 = x_1 - x_0, \Delta x_2 = x_2 - x_1, \cdots, \Delta x_n = x_n - x_{n-1}$$

在小区间 $[x_{i-1}, x_i]$ 上任取一点 $\xi_i\,(x_{i-1} \leqslant \xi \leqslant x_i)$, 作函数值 $f(\xi_i)$ 与小区间长度 Δx_i 的乘积 $f(\xi_i)\Delta x_i\ (i = 1, 2, \cdots, n)$, 并作出和

$$S = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

记 $\lambda = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$, 如果当 $\lambda \to 0$ 时, 这个和的极限存在, 且与闭区间 $[a,b]$ 的分法无关及点 ξ_i 的取法无关, 那么称这个极限 I 为函数 $f(x)$ 在 $[a,b]$ 上的定积分 (简称积分), 记作 $\displaystyle\int_a^b f(x)\,\mathrm{d}x$, 即

$$\int_a^b f(x)\,\mathrm{d}x = I = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

其中 $f(x)$ 叫做被积函数, $f(x)\,\mathrm{d}x$ 叫做被积表达式, x 叫做积分变量, a 叫做积分下限, b 叫做积分上限, $[a,b]$ 叫做积分区间

Exercise 3.16: 求极限

$$I = \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{n(n+1)(n+2)\cdots(2n-1)}$$

Solution

$$I = \exp\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\ln\left(1+\frac{i}{n}\right)\right) = \exp\left(\int_0^1 \ln(1+x)\,\mathrm{d}x\right) = \frac{4}{e}$$

◀

Exercise 3.17: 求极限

$$I=\lim_{n\to\infty}\left(\frac{1}{\sqrt{1^2+n^2}}+\frac{1}{\sqrt{2^2+n^2}}+\frac{1}{\sqrt{3^2+n^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right)$$

Solution

$$\begin{aligned}I&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{i^2+n^2}}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(\frac{i}{n})^2+1}}\\&=\int_0^1\frac{1}{\sqrt{x^2+1}}\mathrm{d}x=\left[\ln(x+\sqrt{x^2+1})\right]_0^1\\&=\ln(1+\sqrt{2})\end{aligned}$$

◀

Exercise 3.18: 求极限

$$\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$

Solution

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}&=\lim_{n\to\infty}\exp\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)=\exp\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)\\&=\exp\lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\cdots+\ln\frac{n}{n}\right)\\&=\exp\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\frac{i}{n}=\exp\int_0^1\ln x\,\mathrm{d}x\\&=\exp\left\{\left[x\ln x\right]_0^1-\int_0^1\mathrm{d}x\right\}=\frac{1}{e}\end{aligned}$$

◀

Exercise 3.19: 求极限

$$\lim_{n\to\infty}\left(\frac{1}{1^2+n^2}+\frac{2}{2^2+n^2}+\cdots+\frac{n}{n^2+n^2}\right)$$

Solution

$$\begin{aligned}I&=\lim_{n\to\infty}\left(\frac{1}{1^2+n^2}+\frac{2}{2^2+n^2}+\cdots+\frac{n}{n^2+n^2}\right)\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{i^2+n^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{\left(\frac{i}{n}\right)^2+1}\\&=\int_0^1\frac{x}{1+x^2}\mathrm{d}x=\frac{1}{2}\int_0^1\frac{1}{1+x^2}\mathrm{d}(1+x^2)\\&=\left[\frac{1}{2}\ln(1+x^2)\right]_0^1=\frac{1}{2}\ln 2\end{aligned}$$

◀

$$
\begin{aligned}
\int_0^1 x^4 \ln^4 x \, \mathrm{d}x &\xlongequal{-\ln x = t} \int_{+\infty}^0 e^{-4t}(-t)^4 \times (-e^{-t}) \, \mathrm{d}t \\
&= \int_0^{+\infty} e^{-5t} t^4 \, \mathrm{d}t \\
&\xlongequal{5t=u} \frac{1}{5^5} \int_0^{+\infty} e^{-u} u^4 \, \mathrm{d}u \\
&= \frac{1}{5^5}\Gamma(5) = \frac{4!}{5^5} = \frac{24}{3125}
\end{aligned}
$$

Example 3.2: 设 $f(x)$ 和 $g(x)$ 在 $[a,b]$ 上有定义, 且满足下面的条件:

1. $\forall n > N$, $f(x)e^{n\cdot g(x)} \in \mathbb{R}[a,b]$;
2. $g(x)$ 在 $[a,b]$ 上只有一个最大值点 $x = \xi \in (a,b)$, 且在任何不包含 ξ 的闭区间 $[\alpha,\beta]$ 中, 有 $\sup\limits_{x\in[\alpha,\beta]} g(x) < g(\xi)$; 中有
3. 在 ξ 的某领域内, $g''(x) < 0$ 且连续;
4. $f(\xi) \neq 0$, $f(x)$ 在 $x = \xi$ 连续.

则当 $n \to \infty$ 时, 有

$$
\int_a^b f(x)e^{n\cdot g(x)} \, \mathrm{d}x \sim f(\xi)\sqrt{-\frac{2\pi}{ng''(\xi)}} e^{ng(\xi)}
$$

Proof:

$$
\Psi(x) = f(x), h(x) = g(x)
$$

由连续性, 对 $\forall \varepsilon > 0$，取 $\delta > 0$ 使得 $\xi \pm \delta \in (a,b)$ 且当 $|x-\xi| < \delta$ 时, 有

$$
f(\xi) - \varepsilon < f(x) < f(\xi) + \varepsilon
$$

$$
g''(\xi) - \varepsilon < g''(x) < g''(\xi) + \varepsilon < 0
$$

于是

$$
\begin{aligned}
\int_a^b f(x)e^{n(h(x)-h(\xi))} \, \mathrm{d}x = &\int_{\xi-\delta}^{\xi+\delta} f(x)e^{n(h(x)-h(\xi))} \, \mathrm{d}x \\
&+ \left(\int_a^{\xi-\delta} + \int_{\xi+\delta}^b\right) f(x)e^{n(h(x)-h(\xi))} \, \mathrm{d}x
\end{aligned}
$$

因为 $g(x)$ 在 $x = \xi$ 点唯一达到有效最大值, 所以

$$
\alpha = \sup_{|x-\xi|\geqslant b} \big(g(x) - g(\xi)\big) < 0
$$

于是

$$\left|\left(\int_a^{\xi-\delta}+\int_{\xi+\delta}^b\right)f(x)e^{n(g(x)-g(\xi))}\,\mathrm{d}x\right|\leqslant e^{n\alpha}\int_\alpha^b|f(x)|\,\mathrm{d}x=O(\theta^n),\ n\to\infty$$

其中 $0<\theta<e^\alpha<1$, 所以有

$$\int_\alpha^b f(x)e^{n(g(x)-g(\xi))}\,\mathrm{d}x=\int_{\xi-\delta}^{\xi+\delta}f(x)e^{n(g(x)-g(\xi))}\,\mathrm{d}x+O(\theta^n)$$

注意到 $g'(\xi)=0$, 所以由 Tayor 展开和积分第一中值定理有

$$\begin{aligned}\int_{\xi-\delta}^{\xi+\delta}f(x)e^{n(g(x)-g(\xi))}\,\mathrm{d}x&=\int_{\xi-\delta}^{\xi+\delta}f(x)e^{\frac{1}{2}n(x-\xi)^2g''(\xi')}\,\mathrm{d}x\\&=\mu\int_{\xi-\delta}^{\xi+\delta}e^{\frac{1}{2}n(x-\xi)^2g''(\xi')}\,\mathrm{d}x\end{aligned}$$

其中 $\xi'=\xi'(x)\in(\xi-\delta,\xi+\delta)$, 而

$$f(\xi)-\varepsilon<f(x)<f(\xi)+\varepsilon$$

于是有

$$\begin{aligned}&(f(\xi)-\varepsilon)\int_{\xi-\delta}^{\xi+\delta}f(x)e^{\frac{1}{2}n(x-\xi)^2(g''(\xi)-\varepsilon)}\,\mathrm{d}x\\\leqslant&\mu\int_{\xi-\delta}^{\xi+\delta}f(x)e^{\frac{1}{2}n(x-\xi)^2g''(\xi')}\,\mathrm{d}x\\\leqslant&(f(\xi)+\varepsilon)\int_{\xi-\delta}^{\xi+\delta}f(x)e^{\frac{1}{2}n(x-\xi)^2(g''(\xi)+\varepsilon)}\,\mathrm{d}x\end{aligned}$$

利用当 $n\to\infty$ 时, 有

$$(f(\xi)\pm\varepsilon)\int_{\xi-\delta}^{\xi+\delta}f(x)e^{\frac{1}{2}n(x-\xi)^2(g''(\xi)\mp\varepsilon)}\,\mathrm{d}x\sim(f(\xi)\mp\varepsilon)\sqrt{\frac{-2\pi}{n(g''(\xi)\mp\varepsilon)}}$$

又 $\theta\in(0,1)$ 是仅依赖于 $\varepsilon>0$ 而与 n 无关的, 且

$$\theta^n=O\left(\frac{1}{n}\right),\ n\to\infty$$

所以先令 $n\to\infty$, 再令 $\varepsilon\to0$ 即得

$$\int_\alpha^b f(x)\left[\frac{f(x)}{f(\xi)}\right]^n\mathrm{d}x\sim f(x)\sqrt{\frac{-2\pi}{ng''(\xi)}}$$

或

$$\int_\alpha^b f(x)\left[\frac{f(x)}{f(\xi)}\right]^n\mathrm{d}x\sim f(x)\sqrt{\frac{-2\pi}{ng''(\xi)}}$$

注意到因为所以有从而得到徐利治的渐近分析方法及应用的 43-45 页 □

3.3 重积分

Exercise 3.20: 计算积分

$$\int_0^2\int_0^4 (6-x-y)\,\mathrm{d}x\,\mathrm{d}y$$

Solution

$$\begin{aligned}
&\int_0^2\int_0^4 (6-x-y)\,\mathrm{d}x\,\mathrm{d}y\\
=&\int_0^2\left[6x-\frac{1}{2}x^2-xy\right]_0^4\mathrm{d}y\\
=&\int_0^2(16-4y)\,\mathrm{d}y\\
=&\left[16y-2y^2\right]_0^2=24
\end{aligned}$$

◀

Exercise 3.21: 计算积分

$$\int_0^1\mathrm{d}y\int_y^1\left(\frac{e^{x^2}}{x}-e^{y^2}\right)\mathrm{d}x$$

Solution

$$\begin{aligned}
I&=\int_0^1\mathrm{d}y\int_y^1\left(\frac{e^{x^2}}{x}-e^{y^2}\right)\mathrm{d}x\\
&=\int_0^1\mathrm{d}y\int_y^1\frac{e^{x^2}}{x}\mathrm{d}x-\int_0^1\mathrm{d}y\int_y^1 e^{y^2}\mathrm{d}x\\
&=\int_0^1\mathrm{d}x\int_0^x\frac{e^{x^2}}{x}\mathrm{d}y-\int_0^1\mathrm{d}y\int_y^1 e^{y^2}\mathrm{d}x\\
&=\int_0^1 e^{x^2}\,\mathrm{d}x-\int_0^1(1-y)e^{y^2}\,\mathrm{d}y\\
&=\int_0^1 ye^{y^2}\,\mathrm{d}y\\
&=\left[\frac{1}{2}e^{y^2}\right]_0^1=\frac{e-1}{2}
\end{aligned}$$

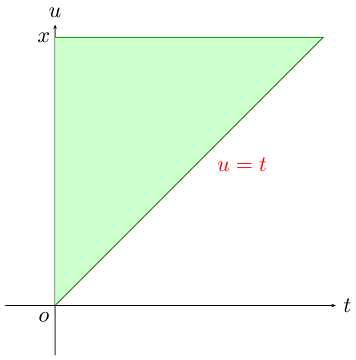
◀

Exercise 3.22: 设 $\varphi(x)=\int_0^x e^{-t^2}\,\mathrm{d}t$ 证明:

$$I=\int_0^{+\infty}\left[\frac{\sqrt{\pi}}{2}-\varphi(t)\right]\mathrm{d}t$$

Solution

$$\int_0^{+\infty}\left[\frac{\sqrt{\pi}}{2}-\varphi(t)\right]\mathrm{d}t=\lim_{x\to+\infty}\int_0^{+\infty}\left(\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u$$



$$
\begin{aligned}
&= \lim_{x\to+\infty}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t - \int_0^{x} e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u\\
&= \lim_{x\to+\infty}\int_0^{+\infty}\left(\int_x^{+\infty} e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u\\
&= \lim_{x\to+\infty}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t\int_0^{t}\mathrm{d}u\\
&= \frac{1}{2}
\end{aligned}
$$

◀

Exercise 3.23: 求极限

$$\lim_{y\to+\infty}\left(\frac{\sqrt{\pi}}{2}y - \int_0^y \mathrm{d}x\int_0^x e^{-u^2}\,\mathrm{d}u\right)$$

Solution:

$$
\begin{aligned}
&\lim_{y\to+\infty}\left(\frac{\sqrt{\pi}}{2}y - \int_0^y \mathrm{d}x\int_0^x e^{-u^2}\,\mathrm{d}u\right)\\
=&\lim_{y\to+\infty}\left(\frac{\sqrt{\pi}}{2}y - \int_0^y e^{-u^2}(y-u)\,\mathrm{d}u\right)\\
=&\lim_{y\to+\infty}\left(\frac{\sqrt{\pi}}{2}y - y\int_0^y e^{-u^2}\,\mathrm{d}u + \int_0^y ue^{-u^2}\,\mathrm{d}u\right)\\
=&\lim_{y\to+\infty}\left(y\left(\frac{\sqrt{\pi}}{2} - \int_0^y e^{-u^2}\,\mathrm{d}u\right) + \frac{1-e^{-y^2}}{2}\right)\\
=&\lim_{y\to+\infty} y\left(\frac{\sqrt{\pi}}{2} - \int_0^y e^{-u^2}\,\mathrm{d}u\right) + \lim_{y\to+\infty}\frac{1-e^{-y^2}}{2}\\
=&0 + \frac{1-0}{2} = \frac{1}{2}
\end{aligned}
$$

□

Example 3.3: 求 $\iint_D \operatorname{sgn}(xy-1)\,\mathrm{d}x\,\mathrm{d}y$, 其中 $D = \{(x,y)|0\leqslant x\leqslant 2, 0\leqslant y\leqslant 2\}$

Solution

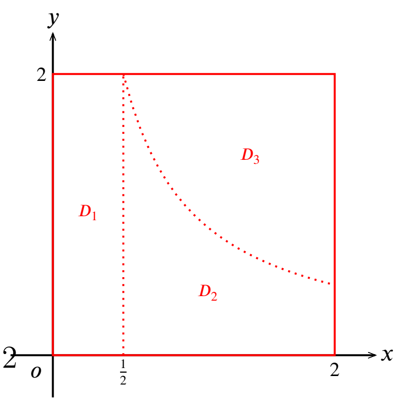
设 $D_1 = \left\{(x,y)\middle|0\leqslant x\leqslant \frac{1}{2}, 0\leqslant y\leqslant 2\right\}$

$D_2 = \left\{(x,y)\middle|\frac{1}{2}\leqslant x\leqslant 2, 0\leqslant y\leqslant \frac{1}{x}\right\}$

$D_3 = \left\{(x,y)\middle|\frac{1}{2}\leqslant x\leqslant 2, \frac{1}{x}\leqslant y\leqslant 2\right\}$

$\iint_{D_1\cup D_2}\mathrm{d}\,\mathrm{d}y = 2\times\frac{1}{2} + \int_{\frac{1}{2}}^{2}\frac{1}{x}\,\mathrm{d}x = 1 + 2\ln 2$

$\iint_{D_3}\mathrm{d}\,\mathrm{d}y = \left(2-\frac{1}{2}\right)\times 2 - \iint_{D_2}\mathrm{d}\,\mathrm{d}y = 3 - 2\ln 2$



$\iint_D \operatorname{sgn}(xy-1)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_3}\mathrm{d}\,\mathrm{d}y - \iint_{D_1\cup D_2}\mathrm{d}\,\mathrm{d}y = 2 - 4\ln 2$

◀

Exercise 3.24: 证明

$$I = \int_0^1\mathrm{d}x\int_0^1 (xy)^{xy}\,\mathrm{d}y = \int_0^1 y^y\,\mathrm{d}y = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

✎ Solution 令 $xy=t$，我们有 (注意 $xy=0$ 时给定 $xy^{xy}=1$)

$$\begin{aligned}I&=\int_0^1\mathrm{d}y\int_0^1(xy)^{xy}\,\mathrm{d}x=\int_0^1\frac{\mathrm{d}y}{y}\int_0^1(xy)^{xy}d(xy)\\&=\int_0^1\frac{\mathrm{d}y}{y}\int_0^y t^t\,\mathrm{d}t=\int_0^1\left(\int_0^y t^t\,\mathrm{d}t\right)d\ln y\\&=\ln y\cdot\int_0^y t^t\,\mathrm{d}t\Big|_0^1-\int_0^1 y^y\ln y\,\mathrm{d}y=-\int_0^1 y^y\ln y\,\mathrm{d}y\end{aligned}$$

注意到

$$\int_0^1 y^y(1+\ln y)\,\mathrm{d}y=\int_0^1 d(y^y)=\left[y^y\right]_0^1=\lim_{x\to1^-}y^y-\lim_{x\to0^+}y^y=1-1=0$$

故

$$I=\int_0^1\mathrm{d}x\int_0^1(xy)^{xy}\,\mathrm{d}y=\int_0^1 y^y(1+\ln y)\,\mathrm{d}y-\int_0^1 y^y\ln y\,\mathrm{d}y=\int_0^1 y^y\,\mathrm{d}y$$

进一步

$$\int_0^1 y^y\,\mathrm{d}y=\int_0^1 e^{y\ln y}\,\mathrm{d}x=\int_0^1\sum_{n=0}^{\infty}\frac{(y\ln y)^n}{n!}\,\mathrm{d}y=\sum_{n=0}^{\infty}\int_0^1\frac{(y\ln y)^n}{n!}\,\mathrm{d}y$$

因为

$$\begin{aligned}\int_0^1(y\ln y)^n\,\mathrm{d}y&=\int_0^1\frac{\ln^n y}{n+1}d\,y^{n+1}\\&=\left[\frac{y^{n+1}}{n+1}\ln^n y\right]_0^1-\int_0^1\frac{n}{n+1}y^n\ln^{n-1}y\,\mathrm{d}y\\&=-\frac{n}{(n+1)^2}\int_0^1\ln^{n-1}y\,d\,y^{n+1}\\&=\left[-\frac{n}{(n+1)^2}y^{n+1}\ln^{n-1}y\right]_0^1+\int_0^1\frac{n\,(n-1)}{(n+1)^2}y^n\ln^{n-2}y\,\mathrm{d}y\\&=\cdots=\frac{(-1)^n n!}{(n+1)^{n+1}}\end{aligned}$$

所以

$$\int_0^1 y^y\,\mathrm{d}y=\sum_{n=0}^{\infty}\int_0^1\frac{(y\ln y)^n}{n!}\,\mathrm{d}y=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}\approx0.783430\cdots$$

所以

$$I=\int_0^1\mathrm{d}x\int_0^1(xy)^{xy}\,\mathrm{d}y=\int_0^1 y^y\,\mathrm{d}y=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

◀

Exercise 3.25: 计算积分 $\iint\limits_D (x+y)\mathrm{d}x\mathrm{d}y$ 其中 D 是由 $x^2+y^2 \leqslant 2$ 和 $x^2+y^2 \geqslant 2x$ 所围成的区域

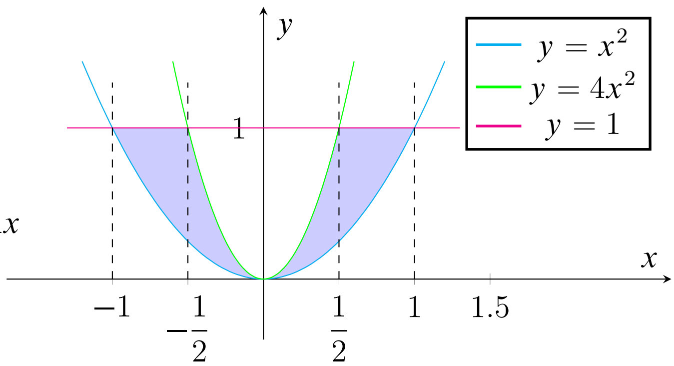
Solution

$$\begin{aligned}
\iint\limits_D (x+y)\mathrm{d}x\mathrm{d}y &= \iint\limits_D x\mathrm{d}x\mathrm{d}y + \iint\limits_D y\mathrm{d}x\mathrm{d}y = 2\iint\limits_{D_1} x\mathrm{d}x\mathrm{d}y + 0 \\
&= 2\iint\limits_{D_{11}} x\mathrm{d}x\mathrm{d}y + 2\iint\limits_{D_{12}} x\mathrm{d}x\mathrm{d}y \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} \rho^2\cos\theta\,\mathrm{d}\rho + 2\int_0^1 \mathrm{d}x \int_{\sqrt{2x-x^2}}^{\sqrt{2-x^2}} x\,\mathrm{d}y \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \left[\frac{1}{3}\rho^3\cos\theta\right]_0^{\sqrt{2}} \mathrm{d}\theta + 2\int_0^1 x\sqrt{2-x^2}\mathrm{d}x - 2\int_0^1 x\sqrt{2x-x^2}\mathrm{d}x \\
&= \frac{4\sqrt{2}}{3}\int_{\frac{\pi}{2}}^{\pi} \cos\theta\,\mathrm{d}\theta + \left[-\frac{2}{3}\sqrt{(2-x^2)^3}\right]_0^1 \\
&\quad + \int_0^1 (2-2x)\sqrt{2x-x^2}\mathrm{d}x - 2\int_0^1 \sqrt{1-(x-1)^2}\mathrm{d}x \\
&= \frac{4\sqrt{2}}{3}\Big[\sin\theta\Big]_{\frac{\pi}{2}}^{\pi} - \frac{2}{3} + \frac{4\sqrt{2}}{3} + \left[\frac{2}{3}\sqrt{(2x-x^2)^3}\right]_0^1 - 2\times\frac{\pi}{4} \\
&= -\frac{\pi}{2}
\end{aligned}$$

◀

Exercise 3.26: 计算积分 $\iint\limits_D (x+y)\mathrm{d}\sigma$ 其中 D 是由 $y=x^2$，$y=4x^2$，$y=1$ 所围成

Solution 区域 D 如图



$$\begin{aligned}
\iint\limits_D (x+y)\,\mathrm{d}\sigma &= \iint\limits_D x\,\mathrm{d}\sigma + \iint\limits_D y\,\mathrm{d}\sigma \\
&= 0 + 2\int_0^1 \mathrm{d}y \int_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} y\,\mathrm{d}x \\
&= 2\int_0^1 \Big[xy\Big]_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^1 y^{\frac{3}{2}}\mathrm{d}y = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}
\end{aligned}$$

◀

Solution

$$\begin{aligned}
\iint\limits_D (x+y)\,\mathrm{d}\sigma &= \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{-\frac{\sqrt{y}}{2}} (x+y)\,\mathrm{d}x + \int_0^1 \mathrm{d}y \int_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} (x+y)\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_{-\sqrt{y}}^{-\frac{\sqrt{y}}{2}} \mathrm{d}y + \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} \mathrm{d}y
\end{aligned}$$

$$= \int_0^1 \left(\frac{1}{2}y^{\frac{3}{2}} - \frac{3}{8}y\right) \mathrm{d}y + \int_0^1 \left(\frac{1}{2}y^{\frac{3}{2}} + \frac{3}{8}y\right) \mathrm{d}y$$
$$= \int_0^1 y^{\frac{3}{2}} \mathrm{d}y = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}$$

◀

Exercise 3.27: **计算二重积分** $\displaystyle\iint\limits_{x^2+y^2\leqslant R^2} e^x \cos y \,\mathrm{d}x\,\mathrm{d}y$.

✎ Solution 令

$$f(r) = \int_0^{2\pi} e^{r\cos\theta} \cos(r\sin\theta) d\theta, 0 \leqslant r \leqslant R,$$

则施行极坐标变换后得到

$$\iint\limits_{x^2+y^2\leqslant R^2} e^x \cos y \,\mathrm{d}x\,\mathrm{d}y = \int_0^R r f(r) dr.$$

先证

$$f(r) \equiv f(0) = 2\pi.$$

事实上,

$$f'(r) = \int_0^{2\pi} e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]d\theta$$
$$= \frac{1}{r}\left(e^{r\cos\theta}\sin(r\sin\theta)\right)\Big|_0^{2\pi} \equiv 0.$$

故 $f(r) \equiv f(0) = 2\pi$. 因此

$$\iint\limits_{x^2+y^2\leqslant R^2} e^x \cos y \,\mathrm{d}x\,\mathrm{d}y = \int_0^R r f(r) dr = \int_0^R 2\pi r dr = \pi R^2.$$

◀

✎ Solution 由于

$$\iint\limits_{x^2+y^2\leqslant R^2} e^x \sin y \,\mathrm{d}x\,\mathrm{d}y = \int_{-R}^R e^x \,\mathrm{d}x \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sin y \,\mathrm{d}y = 0,$$

故

$$\begin{aligned}\iint\limits_{x^2+y^2\leqslant R^2} e^x\cos y\,\mathrm{d}x\,\mathrm{d}y &= \iint\limits_{x^2+y^2\leqslant R^2} e^x(\cos y+i\sin y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \iint\limits_{x^2+y^2\leqslant R^2} e^{x+iy}\,\mathrm{d}x\,\mathrm{d}y\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\iint\limits_{x^2+y^2\leqslant R^2}(x+iy)^n\,\mathrm{d}x\,\mathrm{d}y\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^R r^{n+1}dr\int_0^{2\pi}e^{in\theta}d\theta\\ &= \int_0^R 2\pi r dr\\ &= \pi R^2,\end{aligned}$$

其中用到极坐标变换, 函数项级数一致收敛从而可以逐项积分和如下事实:

$$\int_0^{2\pi}e^{in\theta}d\theta=\int_0^{2\pi}(\cos n\theta+i\sin n\theta)d\theta=\begin{cases}2\pi, & n=0,\\ 0, & n\geqslant 1.\end{cases}$$

◀

3.4 特殊函数

Theorem 3.1

设 $\varphi(n)$ 是欧拉函数，且 f 是连续函数，求证

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\varphi(k)=\frac{6}{\pi^2}\int_0^1 xf(x)\,\mathrm{d}x$$

实际上这个结果利用

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}\varphi(k)=\frac{3}{\pi^2}\text{ 和 }\lim_{n\to+\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)a_k=A\int_0^1 ax^{a-1}f(x)\,\mathrm{d}x$$

其中 $A=\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{a_k}{n}, a_i>0, a>0$ 拼一起

♣

Exercise 3.28: 求极限 $\lim\limits_{s\to 0^+}\zeta(s)$

Solution

$$
\begin{aligned}
\zeta(0) &= \lim_{s\to 0^+}\zeta(s)\\
&= \lim_{s\to 0^+} 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)\\
&= \lim_{s\to 0^+} 2^s\pi^{s-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi s}{2}\right)^{2n+1}\right)\Gamma(1-s)\left(\frac{1}{(1-s)-1}+1-(1-s)\int_1^\infty\frac{x-[x]}{x^{(1-s)+1}}\,\mathrm{d}x\right)\\
&= \lim_{s\to 0^+} 2^s\pi^{s-1}\left(\frac{\pi s}{2}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi s}{2}\right)^{2n}\right)\Gamma(1-s)\left(\frac{1}{-s}+1-(1-s)\int_1^\infty\frac{x-[x]}{x^{2-s}}\,\mathrm{d}x\right)\\
&= \lim_{s\to 0^+} 2^s\pi^{s-1}\left(\frac{\pi}{2}\right)\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi s}{2}\right)^{2n}\right)\Gamma(1-s)s\left(\frac{-1}{s}+1-(1-s)\int_1^\infty\frac{x-[x]}{x^{2-s}}\,\mathrm{d}x\right)\\
&= \lim_{s\to 0^+} 2^{s-1}\pi^{s}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi s}{2}\right)^{2n}\right)\Gamma(1-s)\left(-1+s-s(1-s)\int_1^\infty\frac{x-[x]}{x^{2-s}}\,\mathrm{d}x\right)\\
&= \left(\lim_{s\to 0^+} 2^{s-1}\pi^{s}\Gamma(1-s)\left(-1+s-s(1-s)\int_1^\infty\frac{x-[x]}{x^{2-s}}\,\mathrm{d}x\right)\right)\left(\lim_{s\to 0^+} 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi s}{2}\right)^{2n}\right)\\
&= \left(2^{0-1}\pi^{0}\Gamma(1-0)\left(-1+0-0(1-0)\int_1^\infty\frac{x-[x]}{x^{2-0}}\,\mathrm{d}x\right)\right)\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi\cdot 0}{2}\right)^{2n}\right)\\
&= \left(\frac{1}{2}\cdot 1\cdot\Gamma(1)\cdot(-1+0-0)\right)\left(1+\sum_{n=1}^{\infty}0\right)\\
&= \frac{-1}{2}
\end{aligned}
$$

◀

Example 3.4: 计算积分: $\int_0^1 \ln(1-x)\ln x\ln(1+x)\mathrm{d}x$

Solution

$$
\begin{aligned}
I &= \int_0^1 \ln(1-x)\ln x\ln(1+x)\,\mathrm{d}x\\
&= \int_0^1 \ln(1-x)\ln(1+x)d(x\ln x-x+1)\\
&= \int_0^1 (x\ln x-x+1)\left[\frac{\ln(1+x)}{1-x}-\frac{\ln(1-x)}{1+x}\right]\mathrm{d}x\\
&=2\int_0^1 (x\ln x-x+1)\left[\sum_{n=0}^{\infty}(H_{2n+1}-H_n)x^{2n+1}\right]\mathrm{d}x \qquad (H_0=0)\\
&=2\sum_{n=0}^{\infty}(H_{2n+1}-H_n)\int_0^1 (x\ln x-x+1)x^{2n+1}\,\mathrm{d}x\\
&=2\sum_{n=0}^{\infty}\frac{H_{2n+1}-H_n}{(2n+3)(2n+2)}-2\sum_{n=0}^{\infty}\frac{H_{2n+1}-H_n}{(2n+3)^2}\\
&=\frac{\pi^2}{6}-\ln^2 2-2+2\ln 2-2\left[\frac{7\zeta(3)}{16}+2-\ln 2-\frac{\pi^2}{8}-\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}\right]
\end{aligned}
$$

$$=\frac{5\pi^2}{12}-\ln^2 2-6+4\ln 2+\frac{21\zeta(3)}{8}-\frac{\pi^2\ln 2}{2}$$

◀

Note:

$$2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^m}=m\zeta(m+1)-\sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1)$$

Exercise 3.29: 计算积分

$$I=\int_0^1\ln(1+x)\ln(1-x)\mathrm{d}x$$

Solution 因为

$$\ln(1+x)\ln(1-x)=\sum_{n=1}^{\infty}\frac{H_n-H_{2n}-\frac{1}{2n}}{n}x^{2n}$$

所以

$$\int_0^1\ln(1+x)\ln(1-x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{H_n-H_{2n}}{n(2n+1)}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}$$

Since

$$\begin{aligned}
I&=\int_0^1\ln(2-x)\ln x\,\mathrm{d}x\\
&=-\int_0^1 x[\frac{\ln(2-x)}{x}-\frac{\ln x}{2-x}]\,\mathrm{d}x\\
&=1-2\ln 2+\int_0^1\frac{x\ln x}{2-x}\mathrm{d}x\\
&=1-2\ln 2+2\int_0^{\frac{1}{2}}\frac{(2x)\ln(2x)}{2-2x}\mathrm{d}x\\
&=1-3\ln 2+2\ln^2 2+2\int_0^{\frac{1}{2}}\frac{x\ln x}{1-x}\mathrm{d}x\\
&=1-3\ln 2+2\ln^2 2+2\sum_{k=0}^{\infty}\int_0^{\frac{1}{2}}x^{k+1}\ln x\,\mathrm{d}x\\
&=1-3\ln 2+2\ln^2 2-\sum_{k=0}^{\infty}\frac{\ln 2}{(k+2)2^{k+1}}+\frac{1}{(k+2)^2 2^{k+1}}\mathrm{d}x\\
&=1-3\ln 2+2\ln^2 2-\ln 2[2\ln 2-1]-\frac{\pi^2}{6}+\ln^2 2+1 \qquad \text{The value of } \mathrm{Li}_2(\frac{1}{2})\\
&=2-\frac{\pi^2}{6}-2\ln 2+\ln^2 2
\end{aligned}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}=\frac{\pi^2}{6}+4\ln 2-4$$

所以

$$\sum_{n=1}^{\infty}\frac{H_{2n}-H_n}{n(2n+1)}=\frac{\pi^2}{12}-\ln^2 2$$

◀

Exercise 3.30: 计算积分:

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\,\mathrm{d}x$$

Solution 方法 1

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\,\mathrm{d}x &= \int_0^1 \ln(1-x)\ln x\, d\,\mathrm{Li}_2(x) \\
&= \int_0^1 \mathrm{Li}_2(x)\frac{\ln(1-x)}{x}\,\mathrm{d}x - \int_0^1 \mathrm{Li}_2(x)\frac{\ln x}{1-x}\,\mathrm{d}x \\
&= -\frac{1}{2}\mathrm{Li}_2^2(1) - \int_0^1 \frac{\ln x}{1-x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,\mathrm{d}x \\
&= -\frac{1}{2}\mathrm{Li}_2^2(1) + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=n+1}^{\infty}\frac{1}{k^2} \\
&= -\frac{\pi^4}{72} + \frac{\pi^4}{120} = -\frac{\pi^4}{180}
\end{aligned}$$

方法 2

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)\ln x}{x}\,\mathrm{d}x &= \frac{1}{2}\ln^2 x\ln^2(1-x)\Big|_0^1 + \int_0^1 \frac{\ln^2 x\ln(1-x)}{1-x}\,\mathrm{d}x \\
&= \int_0^1 \sum_{k=1}^{\infty}(-1)^{2k-1}H_k x^k\ln^2 x\,\mathrm{d}x = \sum_{k=1}^{\infty}(-1)^{2k-1}H_k\int_0^1 x^k\ln^2 x\,\mathrm{d}x \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\frac{H_k}{(k+1)^3} \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_{k+1}}{(k+1)^3} - \frac{1}{(k+1)^4}\right] \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_k}{k^3} - \frac{1}{k^4}\right] \\
&= -\frac{\pi^4}{36} + \frac{\pi^4}{45} = -\frac{\pi^4}{180}
\end{aligned}$$

◀

Exercise 3.31: 计算积分

$$\int_0^1 \frac{\ln x}{x^2-x-1}\,\mathrm{d}x$$

Solution 由方程 $x^2-x-1=0$ 的两个根，为了简单起见，我们记

$r_1=\varphi=\dfrac{1+\sqrt{5}}{2}, r_2=\dfrac{1-\sqrt{5}}{2}=1-\varphi,$

且 $r_1-r_2=\sqrt{5}, \varphi^2=\varphi+1, \dfrac{\varphi-1}{\varphi}=\dfrac{1}{\varphi^2}$ 则有

$$\begin{aligned}
I&=\int_0^1\frac{\ln x}{x^2-x-1}\,\mathrm{d}x=\frac{1}{r_1-r_2}\int_0^1\ln x\left(\frac{1}{x-\varphi}-\frac{1}{x-(1-\varphi)}\right)\mathrm{d}x\\
&=\frac{1}{\sqrt{5}}\int_0^1\ln x\left(\frac{1}{x-\varphi}-\frac{\varphi}{\varphi x+1}\right)\mathrm{d}x=\frac{1}{\sqrt{5}}\int_0^1\frac{\ln x}{x-\varphi}\,\mathrm{d}x-\frac{\varphi}{\sqrt{5}}\int_0^1\frac{\ln y}{\varphi y+1}\,\mathrm{d}y\\
&=\frac{1}{\sqrt{5}}\int_0^{\frac{1}{\varphi}}\frac{\ln\varphi u}{u-1}\,\mathrm{d}u-\frac{1}{\sqrt{5}}\int_0^{\varphi}\frac{\ln\frac{u}{\varphi}}{u+1}\,\mathrm{d}u\\
&=\frac{\ln\varphi}{\sqrt{5}}\left(\int_0^{\frac{1}{\varphi}}\frac{1}{u-1}\,\mathrm{d}u+\int_0^{\varphi}\frac{1}{u+1}\,\mathrm{d}u\right)-\frac{1}{\sqrt{5}}\left(\int_0^{\varphi}\frac{\ln u}{1+u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=\frac{\ln\varphi}{\sqrt{5}}\ln\frac{\varphi^2-1}{\varphi}-\frac{1}{\sqrt{5}}\left(\int_0^{\varphi}\frac{\ln u}{1+u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=-\frac{1}{\sqrt{5}}\left(\int_0^{\varphi}\frac{\ln u}{1+u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)=-\frac{1}{\sqrt{5}}\left(\int_1^{1+\varphi}\frac{\ln(u-1)}{u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=-\frac{1}{\sqrt{5}}\left(\int_1^{\varphi^2}\frac{\ln(u-1)}{u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=-\frac{1}{\sqrt{5}}\left(\int_{\frac{1}{\varphi^2}}^1\frac{\ln(1-u)-\ln u}{u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=-\frac{1}{\sqrt{5}}\left(\int_0^{\frac{1}{\varphi}}\frac{\ln u-\ln(1-u)}{u}\,\mathrm{d}u+\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u\right)\\
&=-\frac{2}{\sqrt{5}}\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u+\frac{1}{\sqrt{5}}\int_0^{\frac{1}{\varphi}}\frac{\ln(1-u)}{1-u}\,\mathrm{d}u\\
&=-\frac{2}{\sqrt{5}}\int_0^{\frac{1}{\varphi}}\frac{\ln u}{1-u}\,\mathrm{d}u-\frac{2}{\sqrt{5}}\ln^2\varphi=-\frac{2}{\sqrt{5}}\int_{\frac{1}{\varphi^2}}^1\frac{\ln(1-u)}{u}\,\mathrm{d}u-\frac{2}{\sqrt{5}}\ln^2\varphi\\
&=-\frac{2}{\sqrt{5}}\left(\int_0^1\frac{\ln(1-u)}{u}\,\mathrm{d}u-\int_0^{\frac{1}{\varphi^2}}\frac{\ln(1-u)}{u}\,\mathrm{d}u\right)-\frac{2}{\sqrt{5}}\ln^2\varphi\\
&=\frac{\pi^2}{3\sqrt{5}}-\frac{2}{\sqrt{5}}\mathrm{Li}_2\left(\frac{1}{\varphi^2}\right)-\frac{2}{\sqrt{5}}\ln^2\varphi\\
&=\frac{\pi^2}{3\sqrt{5}}-\frac{2}{\sqrt{5}}\left(\frac{\pi^2}{15}-\ln^2\varphi\right)-\frac{2}{\sqrt{5}}\ln^2\varphi=\frac{\pi^2}{5\sqrt{5}}
\end{aligned}$$

最后一步用了 dilogarithm 函数的性质，详细可以参见 http://mathworld.wolfram.com/Dilogarithm.html ◀

Example 3.5: 求定积分

$$\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\,\mathrm{d}x$$

Solution(by Renascence_5)

$$\begin{aligned}
&\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\,\mathrm{d}x \\
=&\int_0^1 \frac{\ln^2(1-y^2)}{1+\mathrm{i}y}\mathrm{i}\,\mathrm{d}y-\int_0^{\frac{\pi}{2}} \frac{\ln^2(1+e^{\mathrm{i}2\theta})}{1+e^{\mathrm{i}\theta}}\,\mathrm{d}\theta \\
=&\underbrace{\int_0^1 \frac{y\ln^2(1-y^2)}{1+y^2}\,\mathrm{d}y}_{I_1}+\underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \tan\left(\frac{\theta}{2}\right)\ln^2(2\cos\theta)\,\mathrm{d}\theta}_{I_2}+\underbrace{\int_0^{\frac{\pi}{2}} \theta\ln(2\cos\theta)\,\mathrm{d}\theta}_{I_3}-\underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \theta^2\tan\left(\frac{\theta}{2}\right)\mathrm{d}\theta}_{I_4}
\end{aligned}$$

Evaluation of I_1:

$$\begin{aligned}
I_1&=\frac{1}{2}\int_0^1 \frac{\ln^2(1-y)}{1+y}\,\mathrm{d}y=\frac{1}{4}\int_0^1 \frac{\ln^2 y}{1-y/2}\,\mathrm{d}y \\
&=\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 y^2\ln^2 y\,\mathrm{d}y=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)^3} \\
&=\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln 2+\frac{1}{6}\ln^3 2
\end{aligned}$$

Evaluation of I_2:

$$\begin{aligned}
I_2&=\int_0^1 \frac{x}{1+x^2}\ln^2\left(2\frac{1-x^2}{1+x^2}\right)\mathrm{d}x \\
&=\frac{1}{2}\int_0^1 \frac{1}{1+x}\ln^2\left(2\frac{1-x}{1+x}\right)\mathrm{d}x \\
&=\frac{1}{2}\int_0^1 \frac{\ln^2 x}{1+x}\,\mathrm{d}x+\ln 2\int_0^1 \frac{\ln x}{1+x}\,\mathrm{d}x+\frac{1}{2}\ln^2 2\int_0^1 \frac{1}{1+x}\,\mathrm{d}x \\
&=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln^2 x\,\mathrm{d}x+\ln 2\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^n\ln x\,\mathrm{d}x+\frac{1}{2}\ln^3 2 \\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}-\ln 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}+\frac{1}{2}\ln^3 2 \\
&=\frac{3}{4}\zeta(3)-\frac{\pi^2}{12}\ln 2+\frac{1}{2}\ln^3 2
\end{aligned}$$

Evaluation of I_3:

$$\begin{aligned}
I_3&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}}\theta\cos(2n\theta)\,\mathrm{d}\theta \\
&=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} \\
&=-\frac{7}{16}\zeta(3)
\end{aligned}$$

Evaluation of I_4:

$$\begin{aligned}
I_4 &= \left[\theta^2 \ln\left(\cos\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \theta \ln\left(\cos\frac{\theta}{2}\right) \mathrm{d}\theta \\
&= -\frac{\pi^2}{8}\ln 2 + 2\ln 2\int_0^{\frac{\pi}{2}} \theta\, \mathrm{d}\theta - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}} \theta\cos(n\theta)\,\mathrm{d}\theta \\
&= \frac{\pi^2}{8}\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(n\pi/2)}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(n\pi/2)}{n^2} \\
&= \frac{21}{16}\zeta(3) - \pi\mathrm{G} + \frac{\pi^2}{8}\ln 2
\end{aligned}$$

Result:

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\,\mathrm{d}x &= \left(\frac{7}{8}+\frac{3}{4}-\frac{7}{16}+\frac{21}{16}\right)\zeta(3) - \pi\mathrm{G} + \left(-\frac{\pi^2}{12}-\frac{\pi^2}{12}+\frac{\pi^2}{8}\right)\ln 2 + \left(\frac{1}{6}+\frac{1}{2}\right)\ln^3 2 \\
&= \frac{5}{2}\zeta(3) - \pi\mathrm{G} - \frac{\pi^2}{24}\ln 2 + \frac{2}{3}\ln^3 2
\end{aligned}$$

◀

3.5 积分不等式

Example 3.6: 证明

$$\int_0^{\frac{\pi}{2}} \left(e^{\sin x} - e^{-\cos x}\right) \mathrm{d}x \geqslant 2$$

Proof:

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \cdots$$

$$e^{-\sin x} = 1 - \sin x + \frac{\sin^2 x}{2!} + \cdots$$

所以

$$e^{\sin x} - e^{-\sin x} = 2\sin x + 2\cdot\frac{\sin^3 x}{3!} + \cdots \geqslant 2\sin x$$

因此

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \left(e^{\sin x} - e^{-\cos x}\right) \mathrm{d}x &= \int_0^{\frac{\pi}{2}} e^{\sin x}\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} e^{-\cos x}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} e^{\sin x}\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} e^{-\sin x}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \left(e^{\sin x} - e^{-\sin x}\right) \mathrm{d}x \\
&\geqslant \int_0^{\frac{\pi}{2}} 2\sin x = 2
\end{aligned}$$

□

Theorem 3.2

若 $D=\{(x,y)|\varphi_1(x)\leqslant y\leqslant\varphi_2(x),a\leqslant x\leqslant b\}$, $\varphi_1(x),\varphi_2(x)$ 连续, 则 D 的面积为

$$S_D=\int_a^b[\varphi_2(x)-\varphi_1(x)]\,\mathrm{d}x$$

Theorem 3.3

若 $D=\{(x,y)|\psi_1(y)\leqslant x\leqslant\psi_2(y),c\leqslant y\leqslant d\}$, $\psi_1(y),\psi_2(y)$ 连续, 则 D 的面积为

$$S_D=\int_c^d[\psi_2(y)-\psi_1(y)]\,\mathrm{d}y$$

第 4 章 级数

Example 4.1: 求 $\sum_{n=0}^{\infty}\frac{(x-2)^n}{(n+1)3^n}$ 的收敛半径与收敛域

Solution 考虑比值审敛法

$$\lim_{n\to 0}\left|\frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}}\right| = \lim_{n\to 0}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| = \left|\frac{x-2}{3}\right|$$

则当 $\left|\frac{x-2}{3}\right| < 1$ 时，即 $-1 < x < 5$ 时，级数收敛；

当 $\left|\frac{x-2}{3}\right| > 1$ 时，即 $-1 > x$ 或 $x < 5$ 时，级数发散；收敛半径 $R = 3$

且当 $x = 1$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$，由莱布尼茨判别法知收敛；

当 $x = 5$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{1}{n+1}$，显然发散；因此原级数的收敛域为 $[-1, 5)$ ◀

Example 4.2: 求 $\sum_{n=0}^{\infty}\frac{(x-3)^n}{n^2+1}$ 的收敛半径与收敛域

Solution 考虑比值审敛法

$$\lim_{n\to 0}\left|\frac{\frac{(x-3)^{n+1}}{(n+1)^2+1}}{\frac{(x-3)^n}{n^2+1}}\right| = |x-3|$$

则当 $|x-3| < 1$ 时，即 $2 < x < 4$ 时，级数收敛；

当 $|x-3| > 1$ 时，即 $2 > x$ 或 $x < 4$ 时，级数发散；收敛半径 $R = 1$

且当 $x = 2$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$，收敛；

当 $x = 4$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$，收敛；因此原级数的收敛域为 $[2, 4]$ ◀

Example 4.3: 求 $\sum_{n=0}^{\infty}\frac{x^n}{n^2+1}$ 的收敛半径与收敛域

✎ Solution 因为

$$\rho=\lim_{n\to 0}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to 0}\left|\frac{\dfrac{1}{(n+1)^2+1}}{\dfrac{1}{n^2+1}}\right|=1$$

所以收敛半径 $R=\frac{1}{\rho}=1$

且当 $x=-1$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$，收敛；

当 $x=1$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$，收敛；因此原级数的收敛域为 $[-1,1]$ ◀

Example 4.4: 求 $\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}$ 的收敛域与和函数

✎ Solution 先求收敛域，考虑比值审敛法

$$\lim_{n\to 0}\left|\frac{\dfrac{1}{2(n+1)+1}x^{2(n+1)+1}}{\dfrac{1}{2n+1}x^{2n+1}}\right|=\lim_{n\to 0}\left|\frac{2n+1}{2n+3}x^2\right|=|x|^2$$

则当 $|x|^2<1$ 时，即 $-1<x<1$ 时，级数收敛；

当 $|x|^2>1$ 时，即 $-1>x$ 或 $x<1$ 时，级数发散；收敛半径 $R=1$

且当 $x=-1$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$，由莱布尼茨判别法知收敛；

当 $x=1$ 时，级数成为 $\sum_{n=0}^{\infty}\frac{1}{2n+1}$，显然发散；因此原级数的收敛域为 $[-1,1)$

再求和函数，设和函数为 $s(x)$，即

$$s(x)=\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1},\quad x\in[-1,1)$$

则

$$s'(x)=\sum_{n=0}^{\infty}x^{2n}=\frac{1}{1-x^2},\quad x\in[-1,1)$$

上式对 0 到 x 积分

$$s(x)=s(x)-\underbrace{s(0)}_{s(0)=0}=\int_0^x s'(x)\,\mathrm{d}x=\int_0^x\frac{1}{1-x^2}\,\mathrm{d}x=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|,\quad x\in[-1,1)$$

◀

Exercise 4.1: 求幂级数 $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n\cdot 3^n}$ 的收敛域与和函数 $s(x)$.

Solution 令 $t=x-1$, 上述级数变为 $\sum_{n=1}^{\infty}\frac{t^n}{n\cdot 3^n}$, 因为 $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{3}$,
所以，收敛半径 $R=3$ 收敛区间 $|t|<3$, 即 $-2<x<4$.
当 $x=4$ 时，级数变为 $\sum_{n=1}^{\infty}\frac{1}{n}$, 这级数发散，
当 $x=-2$ 时，级数变为 $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, 这级数收敛, 原级数的收敛域为 $[-2,4)$.
设 $s(x)=\sum_{n=1}^{\infty}\frac{(x-1)^n}{n\cdot 3^n}$, 则

$$s'(x)=\sum_{n=1}^{\infty}\left(\frac{(x-1)^n}{n\cdot 3^n}\right)'=\sum_{n=1}^{\infty}\frac{(x-1)^{n-1}}{3^n}=\frac{\frac{1}{3}}{1-\frac{x-1}{3}}=\frac{1}{4-x}$$

又 $s(1)=0$, 故

$$\begin{aligned}s(x)&=s(1)+\int_1^x s'(t)\,\mathrm{d}t=0+\int_1^x\frac{1}{4-t}\,\mathrm{d}t=-\ln(4-t)\Big|_1^x\\&=\ln 3-\ln(4-x),-2\leqslant x<4.\end{aligned}$$

◀

第 5 章 不等式

Exercise 5.1: **证明：**

$$\begin{vmatrix} \int_0^1 x^2\,\mathrm{d}x & \int_0^1 x^3\,\mathrm{d}x & \int_0^1 x^4\,\mathrm{d}x & \int_0^1 xe^x\,\mathrm{d}x \\ \int_0^1 x^3\,\mathrm{d}x & \int_0^1 x^4\,\mathrm{d}x & \int_0^1 x^5\,\mathrm{d}x & \int_0^1 x^2e^x\,\mathrm{d}x \\ \int_0^1 x^4\,\mathrm{d}x & \int_0^1 x^5\,\mathrm{d}x & \int_0^1 x^6\,\mathrm{d}x & \int_0^1 x^3e^x\,\mathrm{d}x \\ \int_0^1 xe^x\,\mathrm{d}x & \int_0^1 x^2e^x\,\mathrm{d}x & \int_0^1 x^3e^x\,\mathrm{d}x & \int_0^1 e^{2x}\,\mathrm{d}x \end{vmatrix} < \frac{e^2-1}{210}$$

Solution ◀

Exercise 5.2: **证明：**

$$\begin{vmatrix} \int_{-1}^1 x^2\,\mathrm{d}x & \int_{-1}^1 (x^3+2x^3\sin x)\,\mathrm{d}x & \int_{-1}^1 (x^4+2x^4\sin^2 x)\,\mathrm{d}x \\ \int_{-1}^1 (x^3-2x^3\sin x)\,\mathrm{d}x & \int_{-1}^1 x^4\,\mathrm{d}x & \int_{-1}^1 (x^5+2x^5\sin^3 x)\,\mathrm{d}x \\ \int_{-1}^1 (x^4-2x^4\sin^2 x)\,\mathrm{d}x & \int_{-1}^1 (x^5-2x^5\sin^3 x)\,\mathrm{d}x & \int_{-1}^1 x^6\,\mathrm{d}x \end{vmatrix} > \frac{32}{2625}$$

Solution ◀

第 6 章 中值定理

Exercise 6.1: 设 $f(x)$ 在 $[0,1]$ 上可微, $f(0)=0, f(1)=1$. 三个正数 $\lambda_1,\lambda_2,\lambda_3$ 的和为 1, 证明: $(0,1)$ 内存在三个不同数 ξ_1,ξ_2,ξ_3, 使得

$$\frac{\lambda_1}{f'(\xi_1)}+\frac{\lambda_2}{f'(\xi_2)}+\frac{\lambda_3}{f'(\xi_3)}=1$$

Proof: 设 $0<x_1<x_2<1$, 对 $f(x)$ 在区间 $[0,x_1],[x_1,x_2],[x_2,1]$ 上分别使用拉格朗日中值定理可得

$$f'(\xi_1)=\frac{f(x_1)-f(0)}{x_1-0}=\frac{f(x_1)}{x_1},\quad \xi_1\in(0,x_1)\Longrightarrow\frac{\lambda_1}{f'(\xi_1)}=\frac{\lambda_1x_1}{f(x_1)}$$

$$f'(\xi_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1},\quad \xi_1\in(x_1,x_2)\Longrightarrow\frac{\lambda_2}{f'(\xi_2)}=\frac{\lambda_2(x_2-x_1)}{f(x_2)-f(x_1)}$$

$$f'(\xi_3)=\frac{f(1)-f(x_2)}{1-x_2}=\frac{1-f(x_2)}{1-x_2},\quad \xi_1\in(x_2,1)\Longrightarrow\frac{\lambda_3}{f'(\xi_3)}=\frac{\lambda_3(1-x_2)}{1-f(x_2)}$$

欲使

$$\frac{\lambda_1}{f'(\xi_1)}+\frac{\lambda_2}{f'(\xi_2)}+\frac{\lambda_3}{f'(\xi_3)}=1$$

只需

$$f(x_1)=\lambda_1, f(x_2)=\lambda_2-\lambda_1$$

又 $f(0)=0, f(1)=1$, 由连续函数的介值定理知,
存在 $x_1\in(0,1)$, 使得 $f(x_1)=\lambda_1$ 和 存在 $x_2\in(0,1)$, 使得 $f(x_1)=\lambda_2-\lambda_1$ □

Exercise 6.2: 设 $f(x)$ 在 $[0,1]$ 上可导且 $f(0)=0, f(1)=1$. 且 $f(x)$ 在 $[0,1]$ 上严格递增

证明: $(0,1)$ 内存在 $\xi_i\in(0,1)$ $(1\leqslant i\leqslant n)$, 使得

$$\frac{1}{f'(\xi_1)}+\cdots+\frac{1}{f'(\xi_n)}=n$$

Proof: 设 $\xi_i\in(0,1)$, 对 $f(x)$ 在区间 $[0,x_1],[x_2,x_3],\cdots,[x_{n-1},1]$ 上分别使用拉格朗日中值定理可得

$$f'(\xi_1)=\frac{f(x_1)-f(0)}{x_1-0}=\frac{f(x_1)}{x_1},\quad \xi_1\in(0,x_1)\Longrightarrow\frac{1}{f'(\xi_1)}=\frac{x_1}{f(x_1)}$$

$$f'(\xi_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1},\quad \xi_1\in(x_1,x_2)\Longrightarrow\frac{1}{f'(\xi_2)}=\frac{x_2-x_1}{f(x_2)-f(x_1)}$$

$$\vdots$$

$$f'(\xi_n)=\frac{f(1)-f(x_{n-1})}{1-x_{n-1}}=\frac{1-f(x_{n-1})}{1-x_{n-1}},\quad \xi_n\in(x_{n-1},1)\Longrightarrow\frac{1}{f'(\xi_n)}=\frac{1-x_{n-1}}{1-f(x_{n-1})}$$

欲使

$$\frac{1}{f'(\xi_1)}+\cdots+\frac{1}{f'(\xi_2)}=n$$

只需

$$f(x_1)=\frac{1}{n}, f(x_2)=\frac{2}{n},\cdots,f(x_{n-1})=\frac{n-1}{n}$$

又 $f(0)=0, f(1)=1$, 由连续函数的介值定理, 存在 $x_k\in(0,1)$, $k\in[1,n-1]$, 使得 $f(x_k)=\frac{k}{n}$
证毕 □

Exercise 6.3: 设 $f(x)$ 在 $[a,b]$ 上具有二阶导数, 且 $f'(a)=f'(b)=0$
证明: $\exists\,\xi\in(a,b)$, 使

$$\left|f''(\xi)\right|\geqslant\frac{4}{(b-a)^2}\left|f(b)-f(a)\right|$$

✎ **Solution** 将 $f\left(\dfrac{a+b}{2}\right)$ 分别在 a 和点 b 展开成泰勒公式, 并考虑到 $f'(a)=f'(b)=0$, 有

$$f\left(\frac{a+b}{2}\right)=f(a)+\frac{1}{2}f''(\xi_1)\left(\frac{b-a}{2}\right)^2,\quad a<\xi_1<\frac{a+b}{2}\tag{6.1}$$

$$f\left(\frac{a+b}{2}\right)=f(b)+\frac{1}{2}f''(\xi_2)\left(\frac{b-a}{2}\right)^2,\quad \frac{a+b}{2}<\xi_1<b\tag{6.2}$$

由 (6.2) − (6.1), 得

$$f(b)-f(a)+\frac{1}{8}\big[f''(\xi_2)-f''(\xi_1)\big](b-a)^2=0$$

故

$$\frac{4\big|f(b)-f(a)\big|}{(b-a)^2}\leqslant\frac{1}{2}\left(\left|f''(\xi_1)\right|+\left|f''(\xi_2)\right|\right)\leqslant f''(\xi)$$

其中 $f''(\xi)=\max\left\{\left|f''(\xi_1)\right|,\left|f''(\xi_2)\right|\right\}$ ◀

第 7 章 特殊函数

Definition 7.1 菲涅尔积分函数

Fresnel Integrals

$$\mathrm{C}\left(x\right)=\int_0^x\cos\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t$$

$$\mathrm{S}\left(x\right)=\int_0^x\sin\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t$$

Example 7.1: 计算积分: $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{x}\sin x\,\mathrm{d}x$

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\sqrt{x}\sin x\,\mathrm{d}x &\xlongequal[\mathrm{d}x=\pi t\,\mathrm{d}t]{\sqrt{x}=\sqrt{\frac{\pi}{2}}t}\pi\sqrt{\frac{\pi}{2}}\int_0^1 t^2\sin\left(\frac{1}{2}\pi t^2\right)\mathrm{d}t\\
&=\sqrt{\frac{\pi}{2}}\int_0^1 t\,d\left(-\cos\left(\frac{\pi}{2}t^2\right)\right)\\
&=\left[-\sqrt{\frac{\pi}{2}}t\cos\left(\frac{\pi}{2}t^2\right)\right]_0^1+\sqrt{\frac{\pi}{2}}\int_0^1\cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t\\
&=\sqrt{\frac{\pi}{2}}C\left(1\right)\approx 0.977451
\end{aligned}$$

◀

$$C\left(x\right)=\int_0^x\cos\left(\frac{\pi t^2}{2}\right)\mathrm{d}t\xlongequal[\mathrm{d}u=\sqrt{\frac{\pi}{2}}\,\mathrm{d}t]{\frac{\pi t^2}{2}=u^2}\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{\frac{\pi}{2}}x}\cos u^2\,\mathrm{d}u$$

$$\Longrightarrow\int_x^0\cos x^2\,\mathrm{d}x=-\int_0^x\cos x^2\,\mathrm{d}x=-\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}x\right)$$

Definition 7.2 三角积分函数

1. Sine Integrals

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$$

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,\mathrm{d}t$$

2. Cosine Integrals

$$\mathrm{Ci}(x) = \int_0^x \frac{\cos t}{t}\,\mathrm{d}t$$

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,\mathrm{d}t$$

$$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,\mathrm{d}t$$

Example 7.2: 求不定积分

$$\int \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x$$

Solution

$$\begin{aligned}
\int \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x &= -\int \sin^2 x\,\mathrm{d}\left(\frac{1}{x}\right) \\
&= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{x}\,\mathrm{d}x \\
&= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{2x}\,\mathrm{d}2x \\
&= -\frac{\sin^2 x}{x} + \mathrm{Si}(2x) + c
\end{aligned}$$

Exercise 7.1: 求不定积分

$$\int \cos\frac{1}{x}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \cos\frac{1}{x}\,\mathrm{d}x \xlongequal{x=\frac{1}{t}} -\int \frac{\cos t}{t^2}\,\mathrm{d}t &= \int \cos t\, d\frac{1}{t} \\
&= \frac{\cos t}{t} - \int \frac{\sin t}{t}\,\mathrm{d}t \\
&= \frac{\cos t}{t} - \mathrm{Si}\,(t) + c \\
&= x\cos\frac{1}{x} - \mathrm{Si}\left(\frac{1}{x}\right) + c
\end{aligned}$$

Exercise 7.2: 求不定积分

$$\int \sin x \log x \, \mathrm{d}x$$

Solution

$$\begin{aligned}\int \sin x \log x \, \mathrm{d}x &= -\int \log x d \cos x \\ &= -\log x \cos x + \int \frac{\cos x}{x} \, \mathrm{d}x \\ &= -\log x \cos x + \mathrm{Ci}(x) + c\end{aligned}$$

Definition 7.3 双曲积分函数

1. 双曲正弦积分

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh x}{x}$$

2. 双曲余弦积分

$$\mathrm{Chi}(x) = \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t} \, \mathrm{d}t = \mathrm{chi}(x)$$

Theorem 7.1

几个关于二重对数函数的等式

(1) $\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)$

(2) $\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-x^{-1}) = -\frac{1}{2}(\ln x)^2$

(3) $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{1}{6}\pi^2 - (\ln x)\ln(1-x)$

(4) $\mathrm{Li}_2(-x) + \mathrm{Li}_2(1-x) + \frac{1}{2}\mathrm{Li}_2(1-x^2) = -\frac{1}{12}\pi^2 - (\ln x)\ln(x+1)$

Exercise 7.3: 求

$$\int_0^\pi \frac{x^2}{1+\sin^2 x} \, \mathrm{d}x.$$

Solution 令 $t = x - \frac{\pi}{2}$，我们有

$$J = \int_0^\pi \frac{x^2}{1+\sin^2 x} \, \mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(t+\frac{\pi}{2}\right)^2}{1+\cos^2 t} \, \mathrm{d}t$$

$$
\begin{aligned}
&= 2\int_0^{\frac{\pi}{2}} \frac{t^2}{1+\cos^2 t}\,\mathrm{d}t + \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 t}\,\mathrm{d}t \\
&= 2\int_0^{\frac{\pi}{2}} \frac{t^2}{1+\cos^2 t}\,\mathrm{d}t + \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 t+2\cos^2 t}\,\mathrm{d}t \\
&= 2\int_0^{\frac{\pi}{2}} \frac{t^2}{1+\cos^2 t}\,\mathrm{d}t + \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t+2} d\left(\tan t\right) \\
&= \frac{\sqrt{2}}{24}\pi^3 + \frac{\sqrt{2}}{2}\pi \mathrm{Li}_2\left(3-2\sqrt{2}\right) + \frac{\sqrt{2}}{8}\pi^3 \\
&= \frac{\sqrt{2}}{2}\pi \mathrm{Li}_2\left(3-2\sqrt{2}\right) + \frac{\sqrt{2}}{6}\pi^3.
\end{aligned}
$$

◀

Exercise 7.4: 计算积分:

$$I = \int_0^{\pi} \frac{x\cos x}{1+\sin^2 x}dx$$

✎ Solution(tian_275461)

$$I = \int_0^{\pi} xd\arctan\left(\sin x\right) = -\int_0^{\pi}\arctan\left(\sin x\right)dx = -2\int_0^{\frac{\pi}{2}}\arctan\left(\sin x\right)dx$$

注意到

$$\arctan\left(\frac{\sin x}{+\infty}\right) - \arctan\left(\frac{\sin x}{1}\right) = -\int_1^{+\infty}\frac{\sin x}{y^2+\sin^2 x}dy$$

故

$$
\begin{aligned}
I &= -2\int_0^{\frac{\pi}{2}}\int_1^{+\infty}\frac{\sin x}{y^2+\sin^2 x}dydx \\
&= -2\int_1^{+\infty}\int_0^{\frac{\pi}{2}}\frac{\sin x}{y^2+\sin^2 x}dxdy \\
&= -2\int_1^{+\infty}\int_0^{1}\frac{1}{y^2+1-t^2}dtdy \qquad (t=\cos x) \\
&= -\int_1^{+\infty}\frac{1}{\sqrt{y^2+1}}\ln\left(\frac{\sqrt{y^2+1}+1}{\sqrt{y^2+1}-1}\right)dy \\
&= -\int_{\mathrm{arcsinh}1}^{+\infty}\ln\left(\frac{\cosh z+1}{\cosh z-1}\right)dz \qquad (y=\sinh z) \\
&= 2\int_{\mathrm{arcsinh}1}^{+\infty}\ln\left(\frac{1-e^{-z}}{1+e^{-z}}\right)dz \\
&= 2\int_0^{\sqrt{2}-1}\frac{\ln\left(1-t\right)-\ln(1+t)}{t}dt \qquad (t=e^{-z}) \\
&= 2\mathrm{Li}_2(1-\sqrt{2}) - 2\mathrm{Li}_2(\sqrt{2}-1)
\end{aligned}
$$

套用多重对数函数的性质 (7.1)(7.2)(7.3)

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\frac{1}{2}\ln^2 x \tag{7.1}$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{1}{6}\pi^2-\ln x\cdot\ln(1-x) \tag{7.2}$$

$$\mathrm{Li}_2(-x)-\mathrm{Li}_2(1-x)+\frac{1}{2}\mathrm{Li}_2(1-x^2)=-\frac{1}{12}\pi^2-\ln x\cdot\ln(x+1) \tag{7.3}$$

故

$$I=\int_0^\pi\frac{x\cos x}{1+\sin^2 x}dx=\ln^2(\sqrt{2}+1)-\frac{\pi^2}{4}$$

◀

Exercise 7.5: Prove that

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}=-\gamma+\log 2$$

γ is the well known Euler's constant.

Proof:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+1}&=\sum_{k=2}^{\infty}\frac{\zeta(2k-1)-1}{k}\\
\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{km^{2k-1}}&=\sum_{m=2}^{\infty}\sum_{k=2}^{\infty}\frac{m}{km^{2k}}\quad\text{since all terms are positive}\\
&=\sum_{m=2}^{\infty}\left(-m\ln\left(1-\frac{1}{m^2}\right)-\frac{1}{m}\right)=\sum_{m=2}^{\infty}\left(m\ln\left(\frac{m^2}{m^2-1}\right)-\frac{1}{m}\right)\\
&=\sum_{m=2}^{\infty}\left(m\Big(\ln(m^2)-\ln(m^2-1)\Big)-\frac{1}{m}\right)\\
&=\lim_{M\to\infty}\sum_{m=2}^{M}\left(2m\ln(m)-m\ln(m+1)-m\ln(m-1)-\frac{1}{m}\right)\\
&=\lim_{M\to\infty}\Big(\ln 2+(M+1)\ln(M)-M\ln(M+1)-H_M+1\Big)\\
&=\lim_{M\to\infty}\left(\ln 2-H_M+\ln(M)+1-M\ln\left(1+\frac{1}{M}\right)\right)\\
&=\lim_{M\to\infty}\left(\ln 2-H_M+\ln(M)+1-M\left(\frac{1}{M}+\mathcal{O}(M^{-2})\right)\right)\\
&=\ln 2-\gamma
\end{aligned}$$

□

Exercise 7.6: 证明:

$$\int_0^{2\pi}e^{\sin x}\sin x\,\mathrm{d}x=\int_0^{2\pi}e^{\sin x}\cos^2 x\,\mathrm{d}x=2\pi I_1(1)$$

Solution

$$\int_0^{2\pi}e^{\sin x}\sin x\,\mathrm{d}x=\int_0^{2\pi}e^{\sin x}\,\mathrm{d}(-\cos x)$$

$$
\begin{aligned}
&= \Big[- e^{\sin x}\cos x \Big]_0^{2\pi} + \int_0^{2\pi} e^{\sin x}\cos^2 x \,\mathrm{d}x \\
&= 0 + \int_0^{2\pi} e^{\sin x}\cos^2 x \,\mathrm{d}x \\
&= \int_0^{2\pi} e^{\sin x}\cos^2 x \,\mathrm{d}x
\end{aligned}
$$

又因为

$$
I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta
$$

$$
\begin{aligned}
\int_0^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x &= \underbrace{\int_0^{\frac{\pi}{2}} e^{\sin x}\sin x\,\mathrm{d}x}_{u=\frac{\pi}{2}+x} + \underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\sin x}\sin x\,\mathrm{d}x}_{t=x-\frac{\pi}{2}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x}_{t=x-\frac{\pi}{2}} \\
&= -\int_{\frac{\pi}{2}}^{\pi} e^{-\cos u}\cos u\,\mathrm{d}u + \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} e^{\cos t}\cos t\,\mathrm{d}t}_{v=t-\pi} \\
&= -\int_{\frac{\pi}{2}}^{\pi} e^{-\cos t}\cos t\,\mathrm{d}t + \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \int_0^{\frac{\pi}{2}} e^{-\cos v}\cos v\,\mathrm{d}v \\
&= \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \underbrace{\int_0^{\pi} e^{-\cos t}\cos t\,\mathrm{d}t}_{x=\pi-t} \\
&= \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \int_{\pi}^{0} e^{\cos x}\cos x\,\mathrm{d}x \\
&= 2\int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t \\
&= 2\pi I_1(1) \approx 3.551
\end{aligned}
$$

◀

$$
I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,\mathrm{d}\theta
$$

$$
\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x &= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x + \overbrace{\int_{\pi}^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x}^{t=2\pi-x} \\
&= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x - \int_{\pi}^{0} e^{\cos t}\cos t\,\mathrm{d}t \\
&= 2\int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x \\
&= 2\pi I_1(1) \approx 3.551
\end{aligned}
$$

Definition 7.4 贝塞尔函数

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta$$

Exercise 7.7: 计算积分:

$$\int_0^1 \frac{e^{-x^2}}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Solution 因为

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta$$

$$\begin{aligned}
\int_0^1 \frac{e^{-x^2}}{\sqrt{1-x^2}}\,\mathrm{d}x &\xlongequal{令x=\cos t} -\int_{\frac{\pi}{2}}^{0} e^{-\cos^2 t}\,\mathrm{d}t \\
&\xlongequal{令u=\frac{\pi}{2}-t} \int_0^{\frac{\pi}{2}} e^{-\sin^2 u}\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} e^{-\frac{1-\cos 2u}{2}}\,\mathrm{d}u = \frac{1}{\sqrt{e}}\int_0^{\frac{\pi}{2}} e^{\frac{1}{2}\cos 2u}\,\mathrm{d}u \\
&\xlongequal{令\theta=2u} \frac{1}{2\sqrt{e}}\int_0^{\pi} e^{\frac{1}{2}\cos\theta}\,d\theta \\
&= \frac{\pi I_0\left(\frac{1}{2}\right)}{2\sqrt{e}}
\end{aligned}$$

Definition 7.5 β 函数

定义

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$$

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,\mathrm{d}t \quad (\mathrm{Re}\,x > 0)$$

Theorem 7.2

$$(s)_n = s(s+1)\cdots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)} \quad (n \geq 1)$$

$$(s)_0 = 1,\ (s)_1 = s$$