WK 9 - 1

LINEAR ALGEBRA REVIEW

$$\begin{cases} 2x_1 - 3x_2 = 1 \\ x_1 + 2x_2 = -1 \\ 3x_1 - x_2 = 0 \end{cases} \rightarrow \begin{bmatrix} 2 & -3 \\ 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix} \quad x\ \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1/7 \\ 0 & 1 & -3/7 \\ 0 & 0 & 0 \end{bmatrix}$$

THIS IS A UNIQUE SOLUTION

use calculator

STEADY STATE DISTRIBUTION

it's like when you are making weather forecasts then you realize even after a long time, the distribution stays the same every day

(actual definition) if the distribution of a Markov chain stays the same for large n values, then the distribution is called the steady-state distribution, and is denoted

as $\pi = \underbrace{(\pi_1, \pi_2, \ldots, \pi_n)}_{\text{all the states}}$ AS IN A VECTOR

STATES

HOW WE KNOW IF THERE IS STEADY STATE → The distribution at times n is $P_0 P^n$ and the distribution at time $n+1$ is $P_0 P^{n+1}$

If the two are equal, then $P_0 P^n = P_0 P^{n+1} \Rightarrow \pi = \pi P$

USE THIS TO SOLVE + FIND STEADY-STATE DIST.

If a steady-state distribution exists, then $\lim_{n\to\infty} P^n = \begin{pmatrix} \pi_1 & \cdots & \pi_n \\ \vdots & \ddots & \vdots \\ \pi_1 & \cdots & \pi_n \end{pmatrix}$

some markov chains will have steady state after a while

BEFORE WE CAN FIND STEADY STATE, WE NEED TO KNOW REGULAR + IRREGULAR MARKOV CHAINS

REGULAR MARKOV CHAINS

$P_{ij}^{(n)} > 0$ for some n + all i + j

so you multiply the matrix n times + none of the numbers are a negative number or 0

(the probabilities)

★ All HAVE STEADY-STATE DISTRIBUTION

IRREGULAR MARKOV CHAINS

no matter what n is, there are negative probabilities

★ MAY / MAY NOT HAVE STEADY STATE DISTRIBUTION

FIND STEADY STATE BY

solving this system of linear equations:

$$\begin{cases} \pi P = \pi \\ \sum \pi_i = 1 \end{cases}$$

π = ALL THE STATES

ADD ALL THE PROBABILITY DISTRIBUTIONS TO GET 1

MORE LIN. ALG.

$$\begin{cases} 2x_1 - 3x_2 = 1 \\ x_1 + 2x_2 = -1 \end{cases} \rightarrow \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow AX = B \xrightarrow{\text{TO SOLVE:}} X = A^{-1}B = \begin{bmatrix} -1/7 \\ -3/7 \end{bmatrix}$$

THERE SHOULD ALWAYS ONLY HAVE ONE UNIQUE SOLUTION

SOLVE W/ MATRIX:

AN EXAMPLE W/ π_1, π_2:

$$[\pi_1 \ \pi_2] P = [\pi_1 \ \pi_2] \rightarrow \overset{\text{TRANSPOSE}}{P^T \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = I \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}} \rightarrow \overset{P^T - I}{(P^T - I)\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = 0} \qquad \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = 1$$

CAN ADD IDENTITY M HERE

ADD THIS AS BOTTOM ROW

IN MATRIX FORMAT: $A = \begin{bmatrix} P^T - I & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 1 \ \ldots \ 1 & 1 \end{bmatrix}$, $\text{rref}(A)$ = solution

PADDING

all 0

all 1's

FIND ROW ECHELON FORM

NOW LET'S USE ^ TO SEE WHAT'S EXAMPLE 1'S STEADY STATE

EXAMPLE 1:

IN SOME TOWN, EACH DAY IS EITHER SUNNY OR RAINY. A SUNNY DAY IS FOLLOWED BY ANOTHER SUNNY DAY W/ PROBIBILITY 0.7, WHEREAS A RAINING DAY IS FOLLOWED BY SUNNY DAY W/ A PROBABILITY OF 0.4

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \rightarrow P^T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -0.3 & 0.4 \\ 0.3 & -0.4 \end{bmatrix} \rightarrow A = \begin{bmatrix} -0.3 & 0.4 & 0 \\ 0.3 & -0.4 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 4/7 \\ 0 & 1 & 3/7 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow [\pi_1 \ \pi_2] = [4/7 \ 3/7]$$

ALWAYS WORKS

THIS MEANS, TODAY IF YOU MAKE A FORCAST FOR ANY DAY IN THE FUTURE → π_1 (SUNNY) WILL ALWAYS BE 4/7, π_2 (RAINY) WILL ALWAYS BE 3/7

ALTERNATIVE WAY TO SOLVE:

do $P^n = \begin{bmatrix} \pi_1 & \cdots & \pi_n \\ \vdots & \ddots & \\ \pi_1 & \cdots & \pi_n \end{bmatrix}$ FOR A LARGE n value

· find P^n for a large n to get steady-state in matrix

ex: $P^{100} = \begin{array}{cc} \pi_1 & \pi_2 \end{array} \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix}$

BUT! (MAY NOT ALWAYS WORK)

EXAMPLES FOR STEADY STATE :

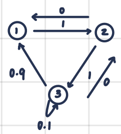
EXAMPLE 3 :

Show a markov chain with the following transition probability matrix is regular.

Find the steady-state distribution.

means when get to state 1, will remain at state 1

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0 & 0.1 \end{bmatrix} \quad P^T = \begin{bmatrix} 0 & 0 & 0.9 \\ 1 & 0 & 0 \\ 0 & 1 & 0.1 \end{bmatrix} \rightarrow P^T - I = \begin{bmatrix} -1 & 0 & 0.9 \\ 1 & -1 & 0 \\ 0 & 1 & -.9 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 & 0.9 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -0.9 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

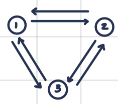


THIS IS A REGULAR M.C. *HOW DID WE KNOW THAT?*

· can try to raise to power 5 n see if all numbers become positive, n in this case, they do.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & .321428 \\ 0 & 1 & 0 & 0 & .321428 \\ 0 & 0 & 1 & 0 & .357428 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WOULD BECOME



so any state can go to any state

$\pi = (.321428, .321428, .357428)$

EXAMPLE 4 :

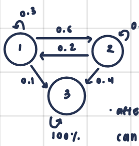
Show a markov chain with the following transition probability matrix is irregular.

steady-state distribution exists.

$$P = \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.2 & 0.4 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\pi = (0, 0, 1)$

let's try drawing a graph :



· after getting to 3, you stay at 3 forever, can never go back to 1 or 2

EXAMPLE 5:

Show a Markov chain with the following transition probability matrix is irregular. It is periodic. Steady distribution does not exist.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

PERIODIC MEANS :

$P^2 = P^4 = P^6 \ldots$

$P^1 = P^3 = P^5 \ldots$

$\lim_{n \to \infty} P^n = \text{DNE}$

→ NOT steady because it's always changing