PB #1

• late work only accepted if email to prof. + prof. emails TA

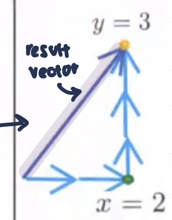
VECTORS + LINEAR COMBINATIONS

1.1 Vectors and Linear Combinations

1 $3\boldsymbol{v} + 5\boldsymbol{w}$ is a typical **linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

2 For $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ that combination is $3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+10 \\ 3+15 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$.

3 The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ goes across to $x = 2$ and up to $y = 3$ in the xy plane.



4 The combinations $c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ fill the whole xy plane. They produce every $\begin{bmatrix} x \\ y \end{bmatrix}$.

3 dimensions

5 The combinations $c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ $\in \mathbb{R}^3$ fill a **plane** in xyz space. Same plane for $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

6 But $\begin{matrix} c + 2d = 1 \\ c + 3d = 0 \\ c + 4d = 0 \end{matrix}$ has no solution because its right side $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not on that plane.

↳ later rigorous proof will show this

Suppose the vectors u, v, w are in three-dimensional space:

1. What is the picture of *all* combinations $c\boldsymbol{u}$?
2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$? $u \neq kv$
3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

CASES

u, v, w are random

1. The combinations $c\boldsymbol{u}$ fill a ***line through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$. $0 \cdot u = (0,0,0)$
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$. $c = 0,\ d = 0$, get the 0 vector
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

if $u = a \cdot v + b \cdot w$ then the vector is linear dependent
↳ cannot get a 3D shape

3Dimension $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $x_1, x_2, x_3 \in \mathbb{R}$

1 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)}\quad u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \qquad \text{(b)}\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(c)}\quad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Solution: **(a)** Let $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (3, 6, 9)$. We can observe here that $\mathbf{v} = 3\mathbf{u}$. Hence, geometrically the linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$ is a line.

$$c \cdot \vec{v} + d \cdot \vec{u}$$
$$= c \cdot (3\vec{u}) + d\vec{u} = (3c+d)\vec{u}$$

b) $u = (1,0,0)$ + $v = (0,2,3)$

here $v \neq cu$ where c is constant → the linear combo is not a line, the plane is in $\mathbb{R}^3$

c) let $u = (2,0,0)$, $v = (0,2,2)$, $w = (2,2,3)$

2 representations: $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ → 1st component, 2nd, 3rd ⋮

$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = v = w = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ ↳ all corresponding components should be the same for them to = each other!

Here there are three vectors which are not have a common multiple. Hence, they cannot form a line in R^3. In order to determine if it forms a plane, we can check if either one of the three vectors can be expressed as a combination of the remaining two.

Let,

$$v = \frac{1}{c} w - \frac{d}{c} v$$

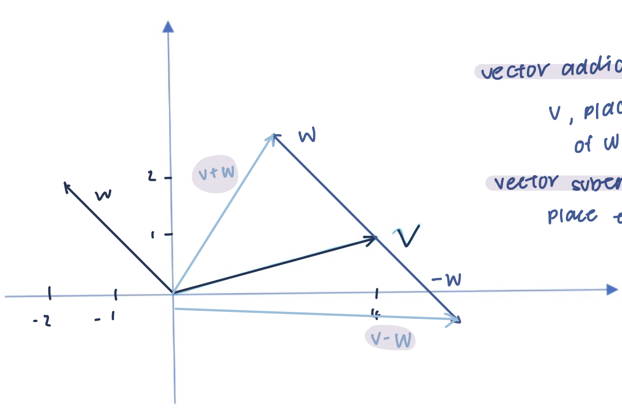
$$\mathbf{w} = c\mathbf{u} + d\mathbf{v}, \quad c, d \in R$$
$$\implies (2, 2, 3) = c(2, 0, 0) + d(0, 2, 2)$$
$$\implies (2, 2, 3) = (2c, 2d, 2d)$$
$$\implies 2 = 2c, 2 = 2d \text{ and } 3 = 2d$$
$$\implies c = 1 and d = 1 \text{ as well as } d = 3/2 \text{ which is a contradiction}$$

if c + d and c both have a solution, then $w = cu + dv$ holds → meaning there is only 2 directions, the combination will create a plane NOT $\mathbb{R}^3$

↳ c + D are not solveable!

Hence, the linear combination of the three vectors will give all of the R^3.

2 Draw $\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ in a single xy plane.



vector addition: at end of v, place the beginning of w

vector subtraction: at the end of v, place the beginning of −w

vectors + + −

method 1 → draw arrows

method 2 → find sum/difference of the components

$v+w = \begin{bmatrix} 4+(-2) \\ 1+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$v-w = \begin{bmatrix} 4-(-2) \\ 1-(2) \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

3 If $\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}5\\1\end{bmatrix}$ and $\boldsymbol{v}-\boldsymbol{w}=\begin{bmatrix}1\\5\end{bmatrix}$, compute and draw the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

add them to get 2V

Solution: We have $\mathbf{v}+\mathbf{w}=(5,1)$ and $\mathbf{v}-\mathbf{w}=(1,5)$. Adding these two vectors we get,

directly evaluate the vectors

$$2\mathbf{v}=(6,6) \implies \mathbf{v}=(3,3)$$

$(V+W)+(V-W) = 2V$

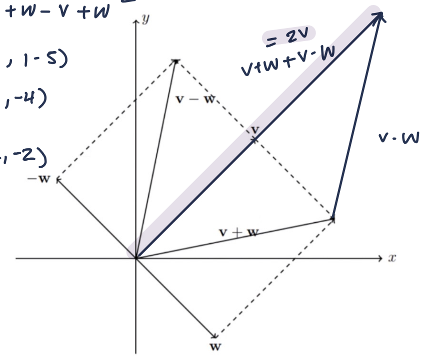
Using this information we get $\mathbf{w}=(2,-2)$

$(V+W)-(V-W) = V+W-V+W$

$= (5-1, 1-5)$

$= (4,-4)$

$W = (2,-2)$



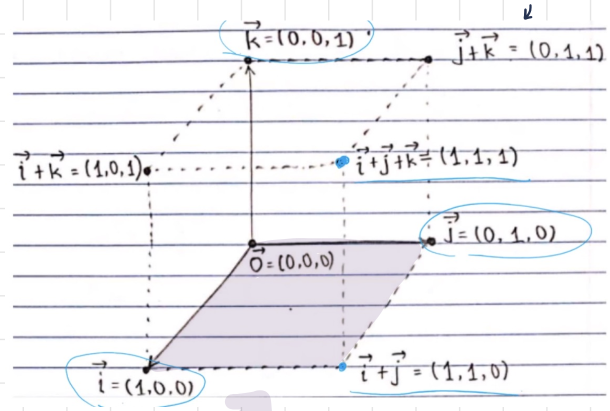
10 Which point of the cube is $\boldsymbol{i}+\boldsymbol{j}$? Which point is the vector sum of $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{j}=(0,1,0)$ and $\boldsymbol{k}=(0,0,1)$? Describe all points (x,y,z) in the cube.

$i+j = (1,0,0)+(0,1,0) = (1,1,0)$

$c\vec{i}+d\vec{j}+e\vec{k},\ 0\le c\le 1,\ 0\le d\le 1,\ 0\le e\le 1$

all in cube

what's $i+j+k$? → $(1,1,1)$



12 *Review Question.* In xyz space, where is the plane of all linear combinations of $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{i}+\boldsymbol{j}=(1,1,0)$?

Solution: The linear combination of $\vec{i}=(1,0,0)$ and $\vec{i}+\vec{j}=(1,1,0)$ is the xy plane in the xyz space.

it's a plane because all the components are linearly dependent to eachother!

11 Four corners of this unit cube are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are ______. The cube has how many edges?

↳ 4 top, 4 bottom, 4 vertical $= 12$

↳ $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$

↳ $\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}+\frac{1}{2}\vec{k} = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$

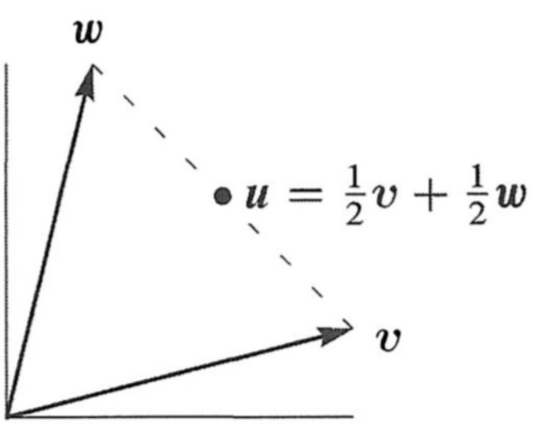
center point of the 6 faces

top: $(\frac{1}{2},\frac{1}{2},1)$ left: $(\frac{1}{2},0,\frac{1}{2})$ front: $(1,\frac{1}{2},\frac{1}{2})$

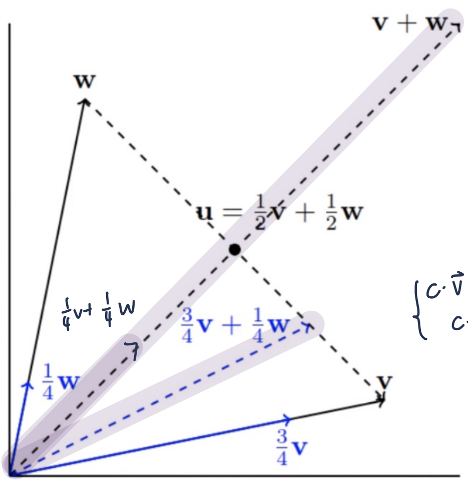
bottom: $(\frac{1}{2},\frac{1}{2},0)$ right: $(\frac{1}{2},1,\frac{1}{2})$ back: $(0,\frac{1}{2},\frac{1}{2})$

$\frac{1}{2}(\frac{1}{2}v+\frac{1}{2}w)$

15 Figure 1.5a shows $\frac{1}{2}\,\boldsymbol{v}+\frac{1}{2}\,\boldsymbol{w}$. Mark the points $\frac{3}{4}\,\boldsymbol{v}+\frac{1}{4}\,\boldsymbol{w}$ and $\frac{1}{4}\,\boldsymbol{v}+\frac{1}{4}\,\boldsymbol{w}$ and $\boldsymbol{v}+\boldsymbol{w}$.



1.5 (a)



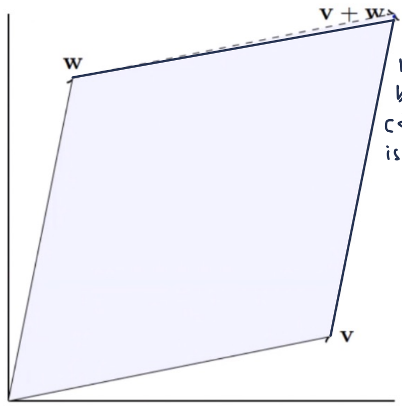
Solution to problem # 15.

$\{c\cdot\vec{v}+d\vec{w} : c+d=1\}$

can be proven

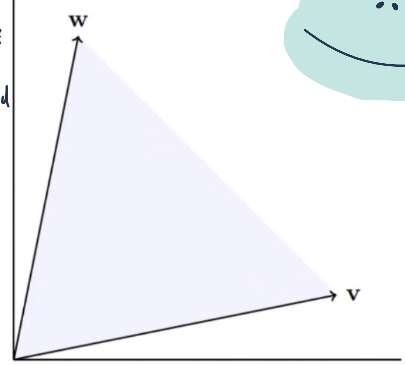
18 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

19 Restricted only by $c \ge 0$ and $d \ge 0$ draw the "cone" of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

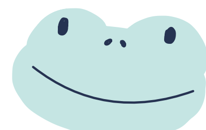


should be solid line because bounds of c+d → 1 is included

Solution to problem # 18.



Solution to problem # 19.



it doesn't mean "same direction" could be opposite direction too, so just on the same line

26 What combination $c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix}$ produces $\begin{bmatrix}14\\8\end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

Solution: We get two equations for the two components,

$$c(1,2) + d(3,1) = (14,8)$$
$$\implies c + 3d = 14 \text{ and } 2c + d = 8$$
$$\implies c = 2, d = 4.$$

Hence, $2(1,2) + 4(3,1) = (14,8)$.

work in this linear combo so

$\hookrightarrow c = 14 - 3d$

$\hookrightarrow c(14-3d) + d = 8$

$28 - 6d + d = 8$

$5d = 20$

$d = 4$

plug d in

$c = 14 - 3(4) = 2$

30 The linear combinations of $\boldsymbol{v} = (a,b)$ and $\boldsymbol{w} = (c,d)$ fill the plane unless $\vec{v} = c \cdot \vec{w}$. Find four vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ with four components each so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} + f\boldsymbol{z}$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

$\vec{v}$ = scalar multiple of $\vec{w}$

Solution:
Unless they both lie on the same line through (0, 0), i.e. they have the same direction.

One example of four vectors in 4-dimensional space whose linear combination fills
lets say up the whole space:
$\vec{u}$ = (1, 0, 0, 0)
$\vec{v}$ = (0, 1, 0, 0)
$\vec{w}$ = (0, 0, 1, 0)
$\vec{z}$ = (0, 0, 0, 1)
They are known as the standard basis.

$c\vec{u} + d\vec{v} + e\vec{w} + f\vec{z} = 0$
no solution for c, d, e, f

28 Find vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ so that $\boldsymbol{v} + \boldsymbol{w} = (4,5,6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2,5,8)$. This is a question with _____ unknown numbers, and an equal number of equations to find those numbers.

Solution:
<u>Method1:</u>
Denote v = (x1, x2, x3), w = (y1, y2, y3).
So, v + w = (x1 + y1, x2 + y2, x3 + y3), v - w = (x1 - y1, x2 - y2, x3 - y3).
4 5 6 2 5 8
According to the given equation, we have
x1 + y1 = 4,
x2 + y2 = 5,
x3 + y3 = 6,
x1 – y1 = 2,
x2 – y2 = 5,
x3 – y3 = 8.
Using the six equations to solve for the 6 unknown numbers and get:
v = (3, 5, 7), w = (1, 0, -1).

<u>Method2: (more efficient)</u>
Here v + w and v – w are given.
By adding them we get 2v = (v + w) – (v - w) = (6, 10, 14), so v = (3, 5, 7).
By taking difference we get 2w = (v + w) – (v - w) = (2, 0, -2), so w = (1, 0, -1).

there are 6 components to find, so 6 unknown numbers.

31 Write down three equations for c, d, e so that $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$. Can you somehow find c, d, e for this $\boldsymbol{b}$?

$$\boldsymbol{u} = \begin{bmatrix}2\\-1\\0\end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix}-1\\2\\-1\end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix}0\\-1\\2\end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Solution:
Each component produces an equation.
Writing $c\mathbf{u} + d\mathbf{v} + e\mathbf{w} = b$ we get the following equations,

$$c\vec{u} + d\vec{v} + e\vec{w} = c(2,-1,0) + d(-1,2,-1) + e(0,-1,2) = (1,0,0)$$
$$\implies 2c - d = 1 \quad c(2,-1,0)$$
$$-c + 2d - e = 0 \quad d(-1,2,-1)$$
$$-d + 2e = 0 \quad e(0,-1,2)$$

on RHS

The last equation tells that d = 2e.
The first equation tells that d=2c-1.

} 2nd eq. $\Rightarrow \underbrace{-\tfrac{1}{2}(d+1)}_{-c} + 2d - \underbrace{\tfrac{1}{2}d}_{e} = 0$

get unique solution for c, d, + e

So we can replace c and e in the second equation by expressions of d, and get d = ½.
Therefore, e = ¼. ~~C~~ c = ¾.