3.5 DIMENSIONS OF THE 4 SUBSPACES

FOR EACH $m \times n$ MATRIX A, THERE ARE FOUR ASSOCIATED SUBSPACES:

(i) Column Space $C(A)$

(ii) Nullspace $N(A)$

(iii) Row space $C(A^T)$

(iv) Left Nullspace $N(A^T)$

WHAT ARE BASES AND DIMENSIONS OF THESE SUBSPACES?

THEY'RE ALL SUBSPACES OF SOMETHING

EXAMPLE:

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 & 6 \\ 2 & 8 & 3 & 7 & 9 \\ 1 & 4 & 2 & 5 & 3 \end{bmatrix}$$

a) what vector space is each of the 4 subspaces a subspace of?

$C(A)$ is a subspace of $\mathbb{R}^3$ because $A = 3 \times 5$

$C(A^T)$ is a subspace of $\mathbb{R}^5$ because $A^T = 5 \times 3$

$N(A)$ is a subspace of $\mathbb{R}^5$: A has 5 columns, $N(A)$ needs to be an element in $\mathbb{R}^5$

$N(A^T)$ is a subspace of $\mathbb{R}^3$ because A^T has 3 columns, $N(A)$ needs to be an element in $\mathbb{R}^3$

b) what are the basis n dimensions of the 4 subspaces?

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 & 6 \\ 2 & 8 & 3 & 7 & 9 \\ 1 & 4 & 2 & 5 & 3 \end{bmatrix} \xrightarrow{\text{ERO'S}} \overset{\text{RREF:}}{\begin{bmatrix} 1 & 4 & 0 & -1 & 9 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} = R$$

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4 \ \vec{a}_5] \qquad [\vec{r}_1 \ \vec{r}_2 \ \vec{r}_3 \ \vec{r}_4 \ \vec{r}_5]$$

C(A)

By def. $\vec{a}_1, \ldots, \vec{a}_5$ span $C(A)$

BUT THEY ARE NOT LINEARLY INDEPENDENT, SO WE NEED TO PICK A LIN. INDEPENDENT SUBSET.

notice: $\vec{r}_2 = 4\vec{r}_1$, $\vec{r}_4 = -\vec{r}_1 + 3\vec{r}_3$, $\vec{r}_5 = 9\vec{r}_1 - 3\vec{r}_3$ $\xrightarrow{\text{meaning}}$ (replace r w/ a) $\vec{a}_2 = 4\vec{a}_1$, $\vec{a}_4 = -\vec{a}_1 + 3\vec{a}_3$, $\vec{a}_5 = 9\vec{a}_1 - 3\vec{a}_3$

so, $\vec{a}_1 + \vec{a}_3$ are linearly independent bc $\vec{r}_1 + \vec{r}_3$ are pivots

$\therefore$ $\vec{a}_1 + \vec{a}_3$ span $C(A)$ + also are lin. independent.

↳ $\vec{a}_1 + \vec{a}_3$ are the basis for $C(A)$

↳ FOUND 2! so DIMENSION OF $C(A) = 2$ (SAME AS # OF PIVOTS)

BUT! $C(A) \neq C(A^T)$

N(A)

$$R\vec{x} = \vec{0} \Rightarrow \begin{cases} x_1 = -4x_2 + x_4 - 9x_5 \\ x_3 = -3x_4 + 3x_5 \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} -4x_2 + x_4 - 9x_5 \\ x_2 \\ -3x_4 + 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{s}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{s}_2} + x_5 \underbrace{\begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{s}_3}$$

WE WROTE THE FREE VECTORS W/ THE PIVOTS

IF WE SET THIS $= 0$ ⟶ this tells us x_2, x_4, x_5 must be 0

THE SPECIAL SOLUTIONS $\vec{s}_1, \vec{s}_2, \vec{s}_3$ form a basis for $N(A)$

→ DIMENSION OF $N(A)$ IS 3 = # OF FREE VARS

C(A^T)

A → ERO'S → R: THIS TELLS US EACH ROW OF R IS A LIN. COMBO. OF ROWS OF A

R → ERO'S → A: THIS TELLS US EACH ROW OF A IS A LIN. COMBO. OF ROWS OF R

ROW SPACE OF A = ROW SPACE OF R ($C(A^T)$)

ROW SPACE DOES NOT CHANGE BY ERO'S!

→ $(1,4,0,-1,9) + (0,0,1,3,-3)$ FORM A BASIS FOR ROW SPACE OF R. HENCE FORMING ROW SPACE FOR ROW A AS WELL

↳ DIMENSION OF $C(A^T) = 2$ (FROM # OF PIVOTS)

because we know

$$\underbrace{C_1(\text{ROW}_1) + C_2(\text{ROW}_2)}_{\text{MUST BE LIN. INDEPENDENT}} + C_3(\text{ROW}_3) = 0$$

$\vec{0}$ = LIN. DEPENDENT BECAUSE C_3 CAN BE ANYTHING

N(A^T)

COULD FIND IT BY FINDING RREF OF A^T, BUT THAT SOUNDS LIKE A LOT OF WORK.

XD SO WE DONT DO IT!

SHORT-CUT:

$\vec{y} \in N(A) \longleftrightarrow A^T\vec{y} = \vec{0}$

$\longleftrightarrow \vec{y}^T A = \vec{0}$ → $\vec{y}$ to left of A: WHY THIS IS "LEFT NULL SPACE"

$\vec{y}^T [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4 \ \vec{a}_5] = [0\ 0\ 0\ 0\ 0]$ → $\vec{y}^T \vec{a}_i = 0, \ i = 1, \ldots, 5$ → $\vec{y}^T \vec{a}_1 = 0$ and $\vec{y}^T \vec{a}_3 = 0$ → $\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_3^T \end{bmatrix} \vec{y} = \vec{0}$

BECAUSE $\vec{a}_2, \vec{a}_4, \vec{a}_5$ are automatically 0

NOW WE NEED TO LOOK AT THAT SMALLER MATRIX:

$$\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \overset{\text{RREF:}}{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}} \rightarrow \begin{cases} y_1 = y_3 \\ y_2 = -y_3 \end{cases} \rightarrow \vec{y} = \begin{bmatrix} y_3 \\ -y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

→ $(1,-1,1)$ FORMS THE BASIS FOR $N(A^T)$

↳ DIMENSION = 3

SUMMARY 4 GENERAL CASES

GIVEN AN $M \times N$ MATRIX A

$A \xrightarrow{\text{EROs}} R$ (RREF) w/ r PIVOTS

$\rightarrow \text{rank} A = r$

i $C(A)$ subspace of $\mathbb{R}^m$

↳ BASIS = PIVOT COLUMNS OF A (NOT OF R)

↳ DIM C(A) = # OF PIVOTS = rank A

ii $N(A)$ subspace of $\mathbb{R}^n$

↳ BASIS = SPECIAL SOLUTIONS TO $A\vec{x} = \vec{0}$ or $R\vec{x} = \vec{0}$

↳ dim N(A) = # OF FREE VARIABLES / # OF SPECIAL SOLUTIONS = # col. − # PIVOTS $(n - r)$

= # of columns of A − # of pivots (rankA)

iii $C(A^T)$ subspace of $\mathbb{R}^n$

↳ BASIS = PIVOT ROWS OF R

↳ $\underbrace{\text{DIM } C(A^T)}_{\text{rank } A^T}$ = # of PIVOTS = rank A = r

IMPLIES rank A = rank A^T

iv $N(A^T)$ subspace of $\mathbb{R}^m$

↳ BASIS = SPECIAL SOLUTIONS TO $\begin{bmatrix} \text{BASIS VECTORS OF } C(A) \\ \text{PLACED AS ROWS} \end{bmatrix} \vec{y} = \vec{0}$

↳ DIM $N(A^T)$ = (# of columns of A^T) − (# of pivots in A^T)

$= m - \underset{\text{rank } A}{\text{rank } A^T} = m - r$

FUNDALMENTAL THEOREM OF LIN. ALG.

dim C(A) + dim N(A) = N # OF ROWS

dim $C(A^T)$ + dim $N(A^T)$ = M # OF COLUMNS