Ch.3 VECTOR SPACES & SUBSPACES

3.1 VECTOR SPACES

" this is the real college level linear algebra "

° getting comfy w/ higher dimensions

WHAT IS VECTOR SPACE?

SET OF OBJECTS (NOT NECESSARILY VECTORS PER SE) THAT POSSESS SAME ALGEBRAIC PROPERTIES AS VECTORS IN $\mathbb{R}^n$

WHAT IS SUBSPACE?

VECTOR SPACE INSIDE VECTOR SPACE

an overview

GETTING TO KNOW VECTOR SPACE

$\mathbb{R}^n$ AS A PROTOTYPE FOR VECTOR SPACE

WHAT IS $\mathbb{R}^n$ ANYWAYS?

° its a set of vectors w/ n components $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

- addition $\vec{u} + \vec{v}$ is defined $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$
- scalar mult. $c\vec{u}$ is defined $\forall c \in \mathbb{R}$ $\forall \vec{u} \in \mathbb{R}^n$
- 8 properties:

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$
2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ $\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$
3) $\exists \vec{0} \in \mathbb{R}^n$ $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ $\forall \vec{u} \in \mathbb{R}^n$
4) $\forall \vec{u} \in \mathbb{R}^n$, $\exists -\vec{u} \in \mathbb{R}^n$ such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$
5) $1\vec{u} = \vec{u}$ $\forall \vec{u} \in \mathbb{R}^n$
6) $(c_1c_2)\vec{u} = c_1(c_2\vec{u})$ $\forall c_1, c_2 \in \mathbb{R}$ $\forall \vec{u} \in \mathbb{R}^n$
7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ $\forall c \in \mathbb{R}$ $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$
8) $(c_1 + c_2)\vec{u} = c_1\vec{u} + c_2\vec{u}$ $\forall c_1, c_2 \in \mathbb{R}$ $\forall \vec{u} \in \mathbb{R}^n$

REAL DEFINITION OF VECTOR SPACE

LET V BE A SET OF OBJECTS (LIKE VECTORS/FUNCTIONS/MATRIX) FOR WHICH ADDITION + SCALAR MULTIPLICATION ARE DEFINED

" we will know what we mean by $u+v$ + cu $\forall c \in \mathbb{R}$ + $\forall u, v \in V$

V is called a vector space if all 8 properties (w/ $\mathbb{R}^n$ replaced by V) are satisfied.

EXAMPLE:

° $\mathbb{R}^n$ (it was our prototype)

° $V = M_{22}$ = set of all 2x2 matrices $= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$ W/ STANDARD ADDITION + SCALAR MULTIPLICATION

↳ we claim this set V is a vector space

↳ IT SATISFIES ALL 8 PROPERTIES, IN FACT, M_{22} IS BASICALLY THE SAME AS $\mathbb{R}^4$:

M_{22} $\mathbb{R}^4$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$

$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \longleftrightarrow (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2)$

8 PROPERTIES ✓

"THAT'S TOO EZ, LETS MAKE IT MORE ABSTRACT"

° $V = P_2$ = set of all polynomials of degree ≤ 2 $= \{ a_0 + a_1t + a_2t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

W/ STANDARD ADD. + SCALAR MULT

$p(t) = a_0 + a_1t + a_2t^2$, $q(t) = b_0 + b_1t + b_2t^2$

$p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$ →

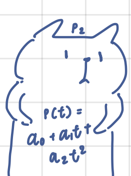
$cp(t) = (ca_0) + (ca_1) + (ca_2)t^2$

THEN P_2 can be identified with $\mathbb{R}^3$

P_2 $\mathbb{R}^3$

$a_0 + a_1t + a_2t^2$ (a_0, a_1, a_2)

SO ALL 8 PROPERTIES ARE SATISFIED → P_2 IS A VECTOR SPACE

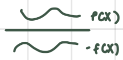


we can't come up w/ $\mathbb{R}^n$ ∘ $V = F(\mathbb{R})$ = set of all functions defined on $\mathbb{R}$ (x-axis) w/ standard addition + scalar multiplication

" doesn't identify w/ a $\mathbb{R}^n$ "

$f(x)+g(x)$ ↑ $cf(x)$ ↑

- what corresponds to $\vec{0}$ is the zero function $z(x) = 0 \quad \forall x \in \mathbb{R}$
- ⊖ exist



⋮

all 8 properties are satisfied

⇒ $F(\mathbb{R})$ is a vector space

what is this for?

this helps us see higher dimensions better!

subspaces are very useful later on too!

SUBSPACES

TRIVIAL EXAMPLE:

Given a vector space V, V itself is a subspace of V

A subspace W of a vector space V is a subset that satisfies the following.

i) $\forall \vec{w}, \vec{y} \in W \quad \vec{w} + \vec{y} \in W$

ii) $\forall c \in \mathbb{R} \quad \forall \vec{w} \in W \quad c\vec{w} \in W$

if these are satisfied, W is a subspace of V

★ must define a vector space before trying to define a subspace ★

THEOREMS

1 Any linear combo of elements from a subspace W is in W

proof:

let $\vec{w}, \vec{y} \in W$, show $c\vec{w} + d\vec{y} \in W$

$c\vec{w}, d\vec{y} \in W$ by (ii)

⇒ $c\vec{w} + d\vec{y} \in W$ by (i) ∘

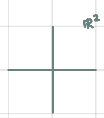
2 IF W IS A SUBSPACE OF V, then W contains $\vec{0} \in V$

IN OTHER WORDS: IF A SUBSET W of V DOES NOT CONTAIN $\vec{0}$, THEN W CANNOT BE A SUBSPACE

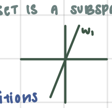
EXAMPLES

∘ DETERMINE WHETHER THE GIVEN SET IS A SUBSPACE OF $\mathbb{R}^2$ OR NOT

① $V = \mathbb{R}^2$ →



a) $W_1 = \{(x,y) \mid y = 2x\}$



we need to check the conditions

↳ $\{(x, 2x) \mid x \in \mathbb{R}\}$

(i) $\forall a, b \in \mathbb{R}$

$(a, 2a) + (b, 2b)$

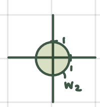
$= ((a+b), 2(a+b)) \in W$ ✓

(ii) $\forall a, c \in \mathbb{R}$

$c(a, 2a) = (ca, 2ca) \in W$, ✓ ⇒

W_1 is a subspace of $\mathbb{R}^2$

② $W_2 = \{(x,y) \mid x^2 + y^2 \leq 1\}$



(i) take $(1,0)$ + $(0,1)$, both in W_2

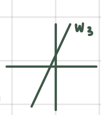
but $(1,0) + (0,1) = (1,1) \notin W_2$ ✗

(ii) $(1,0) \in W_2$ but

$2(1,0) = (2,0) \notin W_2$ ✗

violated. NOT A SUBSPACE

③ $\{(x,y) \mid y = 2x + 1\}$



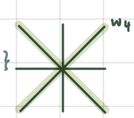
★ $\vec{0}$ IS NOT IN W_3, W_3 IS NOT A SUBSPACE!!! ★

also... (i) violates it

$(2, 5) + (3, 7)$

$(5, 13)$ — x, y ↑ not $2x+1$

④ $W_4 = \{(x,y) \mid y = \pm x\}$



NOT A SUBSPACE, VIOLATES (i)

(i) $(1,1) + (1,-1) = (2,0) \notin W_4$

NOT IN W_4! NOT SUBSPACE

⑤ $W_5 = \{(n, 0) \mid n \in \mathbb{Z}\}$

(i) $\forall n_1, n_2 \in \mathbb{Z} \quad (n_1, 0) + (n_2, 0) = ((n_1+n_2), 0)$ ✓

(ii) $\sqrt{2}(1,0) = (\sqrt{2}, 0) \notin \mathbb{Z} \notin W_5$ = VIOLATION

↑ c can be $\mathbb{R}$

NOT A SUBSPACE

→ HARDER TO VISUALIZE,
MUST WORK W/ DEFINITIONS

• GIVEN $M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

DETERMINE WHETHER THE GIVEN SET IS A SUBSET OF M_{22}

① W_1 = all 2x2 diagonal matrices

" (• ω •) " IT LOOKS LIKE A SUBSPACE

$= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| \, a, d \in \mathbb{R} \right\}$

(i) $\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & d_1+d_2 \end{bmatrix}$

STILL A DIAGONAL MATRIX $\in W_1$ ✓

(ii) $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in W_1$, $c\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $c \in \mathbb{R}$

$= \begin{bmatrix} ca & 0 \\ 0 & cd \end{bmatrix} \in W_1$ ✓

W_1 IS A SUBSPACE OF M_{22}

② W_2 = all 2x2 invertible matrices

(o⁻ ᵕ ⁻o) I THINK NOT

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, ad - bc \neq 0 \right\}$ ↳ THIS HAS NO $\vec{0}$, a, b, c, d can't be 0

$0: \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2$ → NOT A SUBSPACE X

(i) + (ii) ARE NOT SATISFIED EITHER

(i) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in W_2$

$I + (-I) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2$ X

(ii) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W_2$

$c = 0$, $c \in \mathbb{R}$

$0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W_2$ X

③ W_3 = all 2x2 symmetric matrices

$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

let's define it more abstractly

$= \{ A \in M_{22} \mid A^T = A \}$ ← use this to prove

(i) $A, B \in W_3$

$(A+B)^T = A^T + B^T$ by property

$A^T = A$, $B^T = B$ by def.

$A^T + B^T = A + B \Rightarrow A + B \in W_3$ ✓

(ii) let $A \in W_3$ and $c \in \mathbb{R}$

$(cA)^T = cA^T = cA = cA \in W_3$ ✓

W_3 is a subspace of M_{22}

• LET W be the subspace of P_2 defined as

$W = \{ \underset{a_0 + a_1 t + a_2 t^2}{p(t)} \in P_2 \mid \underset{\text{ENSURES THE } \vec{0}}{p(3) = 0} \}$

there are polynomials in P_2 that's not in W

that's the one condition so like if $f(t) = t^2 + 1 \notin W$: $f(3) = \underline{10 \neq 0}$ → IT IS NOT IN THE SUBSPACE W

↳ $g(t) = t^2 - 2t - 3 \in W$: $g(3) = 0$ ✓ IT IS IN THE SUBSPACE

(i) let $p(t), g(t) \in W$, i.e. $p(3) = q(3) = 0$

$(p+q)(3) = \underset{0}{p(3)} + \underset{0}{q(3)} = 0$ ✓

(ii) $p(t) \in W$ i.e. $p(3) = 0$

$\forall c \in \mathbb{R}$

$3(p(3)) = 3(0) = 0$ ✓

W IS A SUBSPACE OF P_2

SUBSPACE SPANNED BY VECTORS

→ LET $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^m$. Then the set of all linear combinations of them.

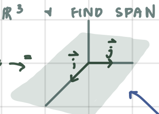
$\text{span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \{ x_1\vec{a}_1 + \ldots + x_n\vec{a}_n \mid x_1, \ldots, x_n \in \mathbb{R} \}$

is called the subspace spanned by $\vec{a}_1, \ldots, \vec{a}_n$

A SPAN IS ALWAYS THE SUBSPACE OF THAT VECTOR SPACE
IN THIS CASE, IT IS A SUBSPACE OF $\mathbb{R}^3$

EXAMPLE: Take $\vec{i} = (1,0,0)$ and $\vec{j} = (0,1,0)$ in $\mathbb{R}^3$ + FIND SPAN

= x-y plane →



$\text{span}\{\vec{i}, \vec{j}\} = \{ x\vec{i} + y\vec{j} = (x, y, 0) \mid x, y \in \mathbb{R} \}$

FINDING THE SUBSET AS ALL THE LIN. COMBOS OF $\vec{i} + \vec{j}$

SO THE SPAN OF $\{\vec{i}, \vec{j}\}$ is all the vectors on the plane (i, j)

COLUMN SPACE

Given an $m \times n$ matrix $A = [\vec{a}_1, \ldots, \vec{a}_n]$ where $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^m$

WE DEFINE THE COLUMN SPACE $C(A)$ OF THIS MATRIX AS $C(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ = all lin. combos of $\vec{a}_1, \ldots, \vec{a}_n$

$= \{ x_1\vec{a}_1 + \ldots + x_n\vec{a}_n \mid x_1, \ldots, x_n \in \mathbb{R} \} = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$

Then, according to the above example, $C(A)$ is a subspace of $\mathbb{R}^m$. also, $A\vec{x} = \vec{b}$ has a solution $\vec{x} \in \mathbb{R}^n \leftrightarrow \vec{b} = A\vec{x} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n$ for some $\vec{x} \in \mathbb{R}^n$

$\leftrightarrow \vec{b} = C(A)$

EXAMPLE:

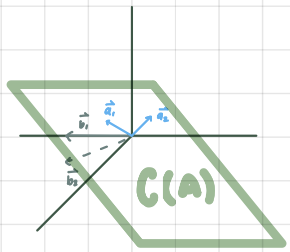
$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 2 \end{bmatrix} = \left[\vec{a}_1 , \vec{a}_2 \right]$$

a) describe the column space $C(A)$

$C(A) = \text{span}\{\vec{a}_1, \vec{a}_2\} = \{ x_1\vec{a}_1 + x_2\vec{a}_2 \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R} \}$

since $\vec{a}_2 \neq c\vec{a}_1 \quad \forall c \in \mathbb{R}$, $\vec{a}_1$ and $\vec{a}_2$ are not parallel.

Hence $C(A)$ is the plane containing both $\vec{a}_1$ + $\vec{a}_2$



b) Determine whether $\vec{b}_1 = (3, -6, 4)$ is in $C(A)$

$$A\vec{x} = \vec{b} \leftrightarrow \begin{cases} 2x + y = 3 & ① \\ -x + 4y = -6 & ② \\ 3x + 2y = 4 & ③ \end{cases}$$

$① + 2② : 9y = -9 \Rightarrow y = -1$

$③ + 3② . \ 14y = -14 \Rightarrow y = -1$

then $② \rightarrow x = 4y + 6 = 2$. so $(x, y) = (2, -1)$

This implies that $\vec{b}_1 = 2\vec{a}_1 - \vec{a}_2$, i.e., $\vec{b} \in C(A)$

c) Determine whether $\vec{b}_1 = (3, -6, 2)$ is in $C(A)$

$$A\vec{x} = \vec{b}_1 \leftrightarrow \begin{cases} 2x + y = 3 & ① \\ -x + 4y = -6 & ② \\ 3x + 2y = 2 & ③ \end{cases}$$

$① + 2② : 9y = -9 \Rightarrow y = -1$

$③ + 3② : 14y = -16 \Rightarrow y = -16/14$

INCONSISTENT $\Rightarrow A\vec{x} = \vec{b}_2$ has no sol.

$\Rightarrow \vec{b}_2$ is <u>NOT</u> a lin. comb. of $\vec{a}_1 + \vec{a}_2 \Rightarrow \vec{b}_2 \notin C(A)$