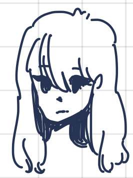
1.6 PARTIAL ORDERS
WK 3-1

LET'S LOOK AT **TOTAL ORDER** first:

ex: $\leq$ over the set $\{1,2,3,4,5\}$ — aRb iff $a \leq b$ are total order

1 2 3 4 5

the relation:
(1,1) (1,2) (1,3) (1,4) (1,5)
(2,2) (2,3) (2,4) (2,5)
(3,3) (3,4) (3,5)
(4,4) (4,5)
(5,5)



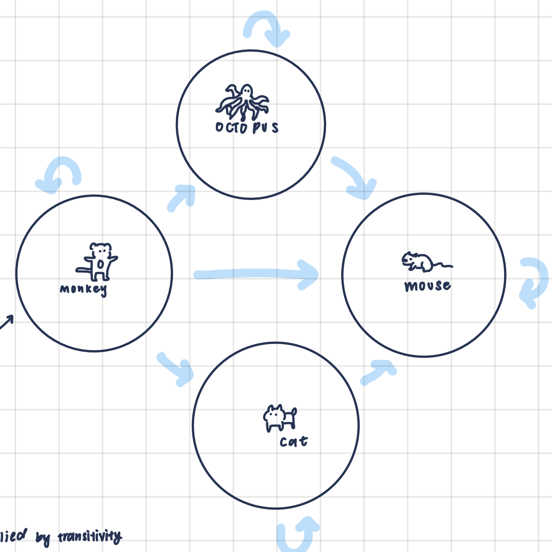
Total order:
- this is reflexive
- this is transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)
- this is anti-symmetric (whenever $a \leq b$ + $b \leq a$, $a = b$)
- for any $a + b$, either $a \leq b$ or $b \leq a$ (inclusive or) ← relax this for partial order: we are defining an order, but some pairs may not be ordered relative to each other

PARTIAL ORDER:

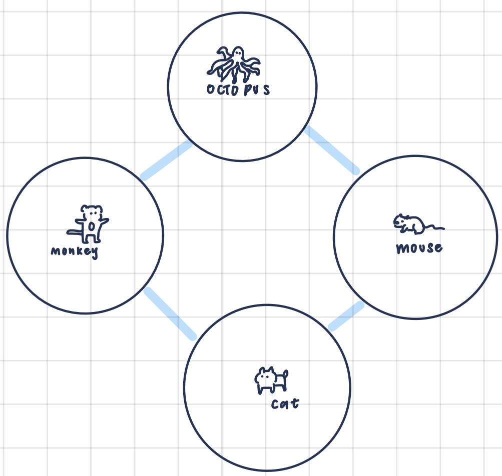
required items:
- reflexive
- transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)
- anti-symmetric (whenever $a \leq b$ + $b \leq a$, $a = b$)

↱ monkey ≤ octopus + octopus ≤ mouse then monkey ≤ mouse!

specifically $\preceq$ PARTIAL ORDER SYMBOL



HASSE DIAGRAM
of the partial order diagram
↳ an easier way to draw partial orders

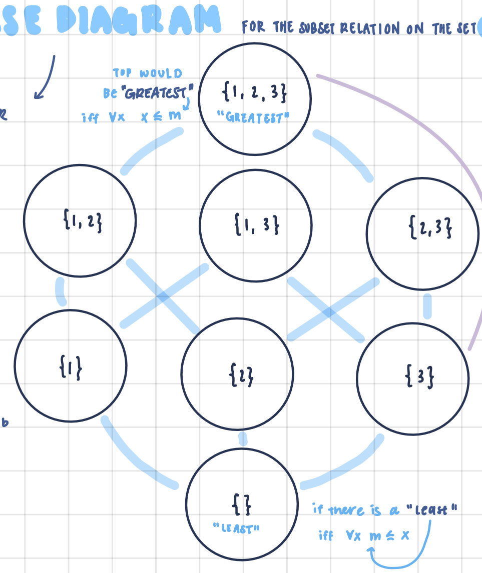


- all arrows point up
- don't draw arrows implied by transitivity
- don't draw arrows implied by reflexivity

HASSE DIAGRAM
FOR THE SUBSET RELATION ON THE SET $\mathcal{P}(\{1,2,3\})$ (POWER SET)
$\{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ = all subsets

THIS IS A PARTIAL ORDER BECAUSE NOT ALL ITEMS CAN BE COMPARED TO EACHOTHER!

TOP WOULD BE "GREATEST" iff $\forall x$ $x \leq m$



if there is a "least" iff $\forall x$ $m \leq x$

→ no lines go $\{2\} \leftrightarrow \{1,3\}$
because $\{2\} \leq \{1,3\}$ and $\{1,3\} \leq \{2\}$ ARE FALSE

this line exists but we don't draw it in since it's implied by transitivity

BEING COMPARABLE:
iff $a \preceq b$ or $b \preceq a$
for example $\{1\}$ + $\{2\}$ are NOT COMPARABLE BUT $\{\}$ + $\{3\}$ are COMPARABLE since $\{\} \preceq \{3\}$

○ it's:
- reflexive $a \leq a$
- transitive if $a \leq b$, $b \leq c$, then $a \leq c$
- anti-symmetric if $a \leq b$ + $b \leq a$ then $a = b$

HOW DO WE KNOW A PARTIAL ORDER NEVER HAS MULTIPLE GREATEST ELEMENTS?

Proof by contradiction.

we assume there are 2 greatest elements A + B

since A is greatest, B ≤ A

since B is greatest, A ≤ B

BUT! A ≠ B , violates anti-symmetry

So we also can't have multiple least elements!

* MUST BE UNIQUE

↳ MAXIMAL ELEMENT:

elements that only relate to themselves

in ^ example: {1, 2, 3}

if there is greatest + least,

they will be maximal & minimal

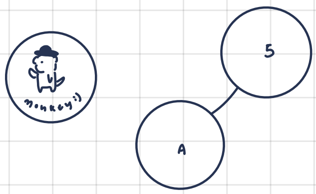
can have more than one tho

↳ MINIMAL ELEMENT:

elements that nothing else relates to

in ^ example: {}

AN ELEMENT THAT IS MINIMAL + MAXIMAL:

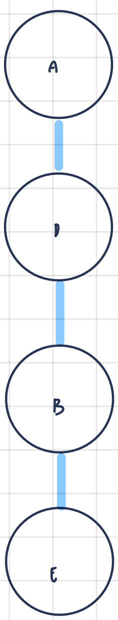


THIS IS A PARTIAL ORDER:

- GREATEST: none
- MAXIMAL: 5, Monkey
- LEAST: none
- MINIMAL: A , Monkey

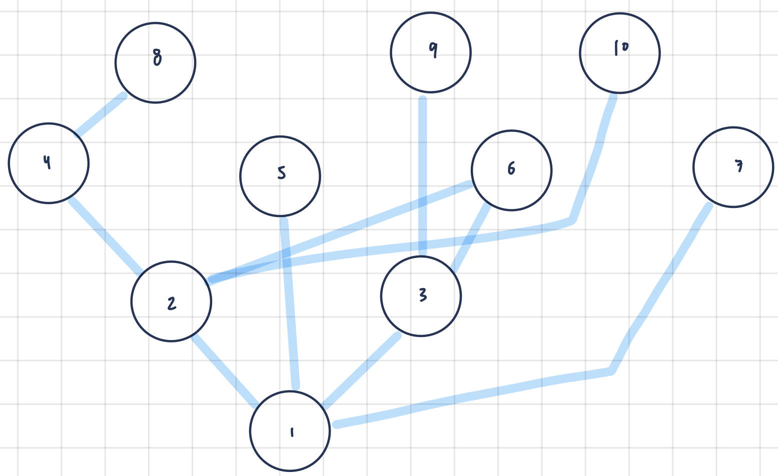
Both min + max!

HASSE DIAGRAM ↳ TOTAL ORDER



since every element is comparable, we got a chain!

↳ every total order is a partial order



MORE EXAMPLE:

- GREATEST: none
- MAXIMAL: 6, 7, 8, 9, 10
- LEAST: 1
- MINIMAL: 1

relate to nothing

if more than 1 maximal, no greatest element

if there is greatest, only 1 maximal

an ∞ set there may be no maximals at all

SIDE NOTE: CIRCULAR →

if aRb + bRc , then cRa

(a,b) (b,c) (c,a) combined w/ transitivity

↳ violates anti-symmetry, can't be partial order.