3.3 COMPLETE SOLUTION TO $A\vec{x} = \vec{b}$

- solution to $A\vec{x} = \vec{b}$ takes the form $\vec{x} = \vec{x}_p + \vec{x}_n$ where $\vec{x}_p$ = one particular solution to $A\vec{x} = \vec{b}$, $\vec{x}_n$ = any solution to $A\vec{x} = \vec{0}$
  ↑ comes from null space
- This expression is obtained by

  augmented matrix $[A \ \vec{b}] \longrightarrow [R \ \vec{d}]$ ↙ RREF of A

- Two extreme cases: What if $\text{rank}\underbrace{A}_{m \times n} = n$ or m? → extreme case

COMPLETE SOLUTION TO $A\vec{x} = \vec{b}$

EXAMPLE : Find all solutions to $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

#1) row reduce the augmented matrix

° Remember RREF, all 0's above pivots

$$[A \ \ \vec{b}] = \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

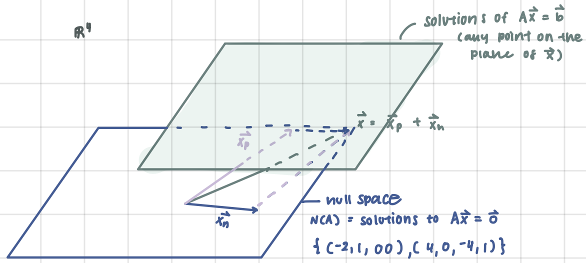
$$\xrightarrow{R_3 \cdot R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R \ \ \vec{d}]$$

#2) set pivots = free vars

$$\Rightarrow \begin{cases} x_1 + 2x_2 = -4x_4 = 7 \\ x_3 + 4x_4 = -3 \end{cases} \Rightarrow \begin{cases} x_1 = 7 - 2x \\ x_3 = -3 - 4x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 7 - 2x_2 + 4x_4 \\ x_2 \\ -3 - 4x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\vec{x}_n}$$

where $A\vec{x}_p = \vec{b}$ and $\vec{x}_n \in N(A)$ i.e. $A\vec{x}_n = \vec{0}$



$\vec{x}_p = (7, 0, 3, 0)$

$\vec{x}_n =$

EXTREAM CASES

#1 Full Column Rank Case

EXAMPLE:

$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$

a) Does $A\vec{x} = \vec{b}$ have solutions for any $\vec{b} \in \mathbb{R}^3$? If not, find a condition on $\vec{b}$ so that $A\vec{x} = \vec{b}$ has a solution.

recall $A\vec{x} = \vec{b}$ has a solution means $\vec{b} \in C(A)$

$C(A) = \text{span}\{(1,2,1), (2,5,4)\}$ = plane $\neq \mathbb{R}^3$

↳ so $A\vec{x} = \vec{b}$ does not have a solution for some $\vec{b} \in \mathbb{R}^3$

$$[A \ \ \vec{b}] = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 5 & b_2 \\ 1 & 4 & b_3 \end{bmatrix} \xrightarrow[R_3 - 1R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 2 & b_3 - b_1 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 3b_1 - 2b_2 + b_3 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 5b_1 - 2b_2 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 3b_1 - 2b_2 + b_3 \end{bmatrix}$$

b are constants in general

$A\vec{x} = \vec{b}$ has a solution when $3b_1 - 2b_2 + b_3 = 0$

b) find all solutions to $A\vec{x} = \vec{b}$ under the condition found in (a). $\vec{x}_p$? $\vec{x}_n$?

Assuming $3b_1 - 2b_2 + b_3 = 0$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \end{bmatrix}}_{\vec{x}_p} + \underbrace{\vec{0}}_{\vec{x}_n}}$$

there's only one solution

EXPLAINS THIS

c) rank of A? # of free variables?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 1 & 4 \end{bmatrix} \rightarrow \begin{array}{cc} x_1 & x_2 \\ \left[\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\right] \end{array} \Rightarrow \text{rank } A = 2$$

↳ 0 free variable → so $N(A) = \{\vec{0}\}$

FULL COL. RANK GENERAL CASE

In general:

rank $A = n$ (# of columns) means # of pivots = # of colums in A (every column has a pivot)

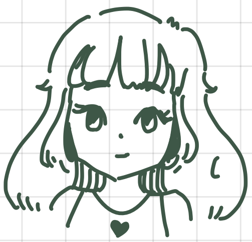
then...

rank A implies:

(i) there are no free variables, $\vec{x}_n = 0$, $N(A) = \{\vec{0}\}$

(ii) $A\vec{x} = \vec{b}$, if solveable, has only 1 solution

since $\vec{x}_p + \vec{x}_n = \vec{x}$, but $\vec{x}_n = 0$, there's no wiggle room for our solution



#2 FULL ROW RANK CASE

Rank A = m (# of rows)

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$$

a) Does $A\vec{x} = \vec{b}$ have solutions for any $\vec{b} \in \mathbb{R}^2$? IF NOT, find a condition on $\vec{b}$ so that $A\vec{x} = \vec{b}$ has a solution.

$C(A) = \text{span}\{\underbrace{(1,2),(2,4)}_{\text{scalar}},(3,7)\}$ this span is in $\mathbb{R}^2$

↳ THIS MEANS $A\vec{x} = \vec{b}$ HAS SOLUTION $\forall \vec{b} \in \mathbb{R}^2$, IN FACT:

$$\begin{bmatrix} A & \vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 7 & b_2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \end{bmatrix}$$ → NO INCONSISTENCIES

$$\xrightarrow[R_1 - 3R_2]{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 7b_1 - 3b_2 \\ 0 & 0 & 1 & b_2 - 3b_1 \end{bmatrix}$$

b) find all solutions to $A\vec{x} = \vec{b}$ under the condition found in (a). $\vec{x}_p$? $\vec{x}_n$?

$$\left.\begin{aligned} x_1 &= 7b_1 - 3b_2 - 2x_2 \\ x_3 &= b_2 - 2b_1 \end{aligned}\right\} \text{ tells us our } \vec{x} \text{ would be: } \begin{bmatrix} 7b_1 - 3b_2 - 2x_2 \\ x_2 \\ b_2 - 2b_1 \end{bmatrix}$$

x_2 is a free variable

$$= \underbrace{\begin{bmatrix} 7b_1 - 3b_2 \\ 0 \\ b_2 - 2b_1 \end{bmatrix}}_{\vec{x}_p} + x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_n}$$

becomes $\vec{x}_n$, so there are infinite solutions

c) rank of A? # of free variables?

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \\ p & & p \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \Rightarrow \text{rank } A = 2, \text{ 1 free variable}$$

FULL ROW RANK GENERAL CASE

IN GENERAL:

Rank $A = m$ means # of pivots = # of rows in A = every row has a pivot

$$A \longrightarrow R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

→ rank $A = m$ implies

(i) $A\vec{x} = \vec{b}$ has solutions for any $\vec{b} \in \mathbb{R}^m$

(ii) $C(A) = \mathbb{R}^m$