EIGEN STUFF 6.1

$A\vec{x} = \lambda \vec{x}$

eigen value (λ)

eigen vector ($\vec{x}$)

FOR A GENERAL SQUARE MATRIX

WHAT ARE THEY EXACTLY?

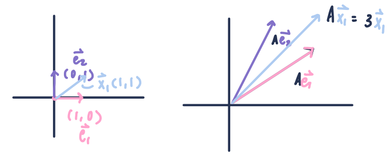
EXAMPLE -

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ defines a transformation $\vec{x} \mapsto A\vec{x}$ of any vector $\vec{x} \in \mathbb{R}^2$

$\vec{x}$ + $A\vec{x}$ are in general not on a single line : $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto A\vec{e}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto A\vec{e}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{e}_1$ + $\vec{e}_2$ transformations are different



BUT IF YOU CHOOSE A GOOD $\vec{x}$, THEN YOU WILL HAVE A VECTOR PARALLEL TO THE ORIGINAL VECTOR.

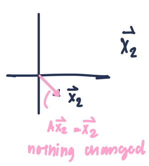
THIS IS EIGEN VECT.

PICKED SOMETHING BETWEEN $\vec{e}_1$ + $\vec{e}_2$ — $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto A\vec{x}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}_1$

eigenvalue 3

scalar multiple of $\vec{x}_1$

OR CAN DO



$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longmapsto A\vec{x}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}_2$

eigen value 1

just $\vec{x}_2$

THE ACTUAL DEFINITION:

NON-ZERO BC IF $\vec{0}$, THEN THERE'S INFINITE SOLUTIONS.
↳ A TRIVIAL SOLUTION.

LET A = $n \times n$ matrix. A scalar λ is called an eigen value of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$. , $\vec{x}$ = eigenvector of A (corresponding to λ)

SO HOW DO WE FIND THE EIGENVALUE + EIGENVECTOR?

λ IS AN EIGENVALUE OF A = THERE IS A NON-ZERO VECTOR S.T. $A\vec{x} = \lambda\vec{x}$

$= \exists \vec{x} \neq \vec{0}$ S.T. $(A - \lambda I)\vec{x} = \vec{0}$ ↳ TELLS US $\vec{x}$ LIVES IN THE NULLSPACE

LOOKS LIKE $A\vec{x} = \vec{0}$, $\vec{x}$ = NULLSPACE

$= \exists \vec{x} \neq \vec{0}$ S.T. $\vec{x} \in N(A - \lambda I)$ → CANNOT BE $\vec{0}$ ITSELF

$= N(A - \lambda I) \neq \{\vec{0}\}$ ← NULL SPACE CAN'T BE $\vec{0}$ ITSELF

= THERE IS A FREE VARIABLE.

= RANK$(A - \lambda I) < n$

= $(A - \lambda I)$ IS NOT INVERTIBLE / NON-INVERTIBLE

(bc no pivot in every row)

CAN TURN THIS INTO A SINGLE EQUATION :

$\det(A - \lambda I) = 0$ - CHARACTERISTIC EQUATION

EQUATION 4 λ!

* SINGULAR MATRIX'S DET = 0 *

APPLE TO EXAMPLE 1.

FIND EIGENVALUES + EIGENVECTORS OF $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

1) solve the characteristic equation

char. eq: $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$ } ad−bc

det.

$= (2-\lambda)^2 - 1 = 0$

$= (2-\lambda)^2 = 1$

$= (\lambda - 2)^2 = 1$

$= \lambda - 2 = \pm 1$

$= \lambda = 3, \lambda = 1$ } THE ONLY POSSIBLE EIGEN VALUES

NOW FIND EIGENVECTOR

· given by non-zero solution to $(A - \lambda I)\vec{x} = \vec{0}$

(i) $\lambda = 3$

$(A - 3I)\vec{x} = \vec{0}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\vec{x} = \vec{0}$ find nullspace by doing REF

$\hookrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = x_2$

so we are looking at:

$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{x} \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

THIS IS AN EIGENSPACE

IN THIS CASE

so almost anything can be an eigen vector of $\lambda = 3$

we pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

~ ANY NON-ZERO VECTOR IN THE EIGENSPACE CAN BE EIGENVECTOR ~

(ii) $A - 1\lambda$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$: $x_1 = -x_2$

$\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow$ EIGENSPACE = span $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

we can pick $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = eigen vector

same as in example

PROPERTIES OF EIGENVALUES

ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 3, 1$

$\det A = 3 = 3 * 1$

(i) PRODUCT OF ALL EIGENVALUES (INCLUDING REPEATED ONES) OF A = det A

sometimes you get ex: 2 + 2 then you do 2 * 2

(ii) SUM OF ALL EIGENVALUES (INCLUDING REPEATED ONES) OF A = $\underbrace{tr A}_{a_{11} + a_{22} + \ldots + a_n}$ DIAGONAL ENTRIES

ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 3, 1$

$tr A = 2 + 2 = 4 = 3 + 1$

(iii) A IS SINGULAR IFF 0 IS AN EIGENVALUE OF A

because then $\det A = 0$

$= \det(A - 0I) = 0$

$\lambda = 0$ IS A SOLUTION TO CHAR. EQ.

$= \lambda = 0$ for A

EXAMPLE 2

FIND EIGENVALUES + EIGEN VEC. OF $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

EIGENVALUE:

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & -1-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix}$ ↪ DET (UPPER TRIANGULAR) = 0

$(2-\lambda)(-1-\lambda)(3-\lambda) = 0$

↳ $\lambda = 2, -1, 3$

↑ THIS IS UPPER TRIANGULAR MATRIX

n x n matrix should have n eigen values

IF MATRIX IN UPPER △, λ'S = DIAGONAL ENTRIES

EIGEN VECTORS:

(i) $\lambda = 2$

$A - 2I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ↪ $x_2 = 0$, $x_3 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$: eigenvector for $\lambda = 2$

(ii) $\lambda = -1$

$A + I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ↪ $x_1 = -2/3\, x_2 - 1/3\, x_2$, $x_3 = 0$

$\vec{x} = \begin{bmatrix} -2/3\, x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ eigen can be → $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ for $\lambda = -1$

(iii)

$A - 3I = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 3x_3$, $x_2 = x_3$

$\vec{x} = \begin{bmatrix} 3x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ is an eig. v. for $\lambda = 3$