3.4 INDEPENDENCE, BASIS, AND DIMENSION

OVERVIEW

TERMS FOR A SET OF VECTORS

- linear Independence: No "redundant" vectors, that is, not too many
- Spanning: Enough vectors to produce rest of the space, that is, not too few
- Basis: both of the above, that is, not too many + not too few

DIMENSIONS OF VECTOR SPACE

LINEAR INDEPENDENCE

Let V be a vector space. $\vec{a}_1, \dots, \vec{a}_n$ in V are said to be linearly independent if:

$$x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0} \text{ implies } x_1 = x_n = 0$$

(the only solution is the trivial solution when $x_1 = x_n = 0$)

- $V = \mathbb{R}^m$

WHAT LIN. INDEPENDENCE MEAN IN DIFF. TERMS:

$\vec{a}_1, \dots, \vec{a}_n$ in $\mathbb{R}^m$ are linearly independent

means $\underbrace{\begin{bmatrix} \vec{a}_1, \dots, \vec{a}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \vec{0}$ implies $x_1 = \dots = x_n = 0$

$A = m \times n$

THE NULL SPACE:

$\longleftrightarrow$ $A\vec{x} = \vec{0}$ has unique solution $\vec{x} = \vec{0}$

$\longleftrightarrow$ $N(A) = \{\vec{0}\}$

$\longleftrightarrow$ THERE ARE NO FREE VARIABLES

$\longleftrightarrow$ PIVOT IN EVERY COLUMN

$\longleftrightarrow$ rank $A = n =$ full column rank

EXAMPLE OF THIS IDEA:

WE HAVE $\vec{a}_1 = (1,0,2)$, $\vec{a}_2 = (2,1,4)$, $\vec{a}_3 = (3,2,5)$, $\vec{a}_4 = (2,3,4)$

FIND THE LARGEST POSSIBLE NUMBER OF LINEAR INDEPENDENT VECTORS AMONG THEM

intuitively the answer is probably 2-3 because it is an $\mathbb{R}^3$ space.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 4 & 5 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

PUT IN UPPER △ FORM

= 3 PIVOTS, THIS TELLS YOU:

because not full column rank

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are linearly dependent

$\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly independent and is the largest set of linearly independent vectors among them.

SPANNING

$\vec{a}_1, \dots, \vec{a}_n$ in V is said to span V if any vector in $\vec{b} \in V$ can be written as a linear combination of $\vec{a}_1, \dots, \vec{a}_n$ meaning:

$\exists x_1, \dots, x_n \in \mathbb{R}$ such that $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$

↳ definition of when $\vec{b}$ is in some vector space V

- $V = \mathbb{R}^m$ let's come up w/ more equivalence equations to span a vector space

$\vec{a}_1, \dots, \vec{a}_n$ span $\mathbb{R}^m$ $\longleftrightarrow$ $\forall \vec{b} \in \mathbb{R}^m$, $\exists x_1, \dots, x_n \in \mathbb{R}$ such that $\underbrace{x_1\vec{a}_1 + \dots + x_n\vec{a}_n}_{A\vec{x}} = \vec{b}$ → any vector in $\mathbb{R}^m$ can be written as a linear combination of the column vectors of the matrix A

$\longleftrightarrow$ $\forall \vec{b} \in \mathbb{R}^m$, $\exists \vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = b$

$\longleftrightarrow$ $C(A) = \mathbb{R}^m$

$\longleftrightarrow$ every row has a pivot.

↳ inconsistency could never happen

$\longleftrightarrow$ rank $A = m$, meaning full row rank

EXAMPLE:

$\vec{a}_1 = (1, -3, 0)$, $\vec{a}_2 = (0, 2, 4)$, $\vec{a}_3 = (3, -5, 8)$, $\vec{a}_4 = (4, -10, 5)$

a) FIND a set 3 vectors from above that spans $\mathbb{R}^3$

could drop 1 of the 4 vectors n they still span $\mathbb{R}^3$

$$[\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4] = \begin{bmatrix} 1 & 0 & 3 & 4 \\ -3 & 2 & -5 & -10 \\ 0 & 4 & 8 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ 3 PIVOTS, PIVOT IN EVERY ROW

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ indeed span $\mathbb{R}^3$, remove $\vec{a}_3$, still span $\mathbb{R}^3$

$\vec{a}_1, \vec{a}_2, \vec{a}_4$ span $\mathbb{R}^3$

b) write the vector not chosen in a) as a linear combo. of the other ones

col 3 = 3(col 1) + 2(col 2) → $\vec{a}_3 = 3\vec{a}_1 + 2\vec{a}_2 + 0\vec{a}_4$

BASIS

A set of elements in V are called a <u>Basis</u> if they are linearly independent but also span V

- $V = \mathbb{R}^m$

THESE TWO COND. MUST BE MET

$\vec{a}_1, \dots, \vec{a}_n$ form a basis for $\mathbb{R}^m$ $\leftrightarrow$ $\begin{cases} \vec{a}_1, \dots, \vec{a}_n \text{ are lin. independent} \\ \vec{a}_1, \dots, \vec{a}_n \text{ span } \mathbb{R}^m \end{cases}$

Full Column Rank + Row Rank

$A = [\vec{a}_1 \dots \vec{a}_n]$ basically means rank $A = n$ + rank $A = m$

meaning $\longleftrightarrow$ $m = n$ (A is $n \times n$) and rank $A = n$

$\longleftrightarrow$ $A \to I$, RREF$(A) = I$

$\longleftrightarrow$ A IS INVERTIBLE

EXAMPLE :

$\vec{a_1} = (1,-3,0)$, $\vec{a_2} = (0,2,4)$, $\vec{a_3} = (3,-5,8)$, $\vec{a_4} = (4,-10,5)$

FIND THE SET OF VECTORS FROM ABOVE THAT FORM A BASIS FOR $\mathbb{R}^3$

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} & \vec{a_4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ try to see which column you can drop for this to become the identity matrix

↳ get rid of $\vec{a_3}$

↳ $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow I$

DIMENSIONS

Given a vector space V, its dimension dim V = # of elements in a basis for V

ex:

$$\begin{bmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \vec{e_1}, \vec{e_2}, \vec{e_3} \text{ is a basis for } \mathbb{R}^3$$

$\dim \mathbb{R}^3 = 3$

BASIS 4 SUBSPACE

a subspace W for vector space V is itself, what's the basis for W ?

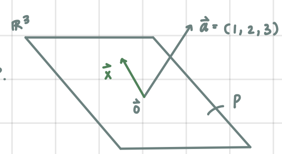
EXAMPLE :

"more abstract problem"

Let P be the plane passing through the origin and perpendicular to $\vec{a} = (1,2,3)$

Then P is a subspace of $\mathbb{R}^3$

FIND A BASIS FOR P, or FIND DIMENSIONS OF P.



1) P is a set of vectors, ⊥ to $\vec{a}$

$P = \{\vec{x} \in \mathbb{R}^3 \mid \vec{a} \cdot \vec{x} = \vec{0}\} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{A} \vec{x} = \vec{0}$

$\vec{x} = \vec{0} \Rightarrow$ meaning this is N(A)

2) $A = \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \end{array}$ $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ already in RREF

$x_1 = -2x_2 - 3x_3$

$$\vec{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{s_1}} + x_3 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{s_2}}$$

3) $x_2 \vec{s_1} + x_3 \vec{s_2} = 0$ implies $x_2 = x_3 = 0 \Rightarrow \vec{s_1} + \vec{s_2}$ are linearly independent.

$\forall \vec{x} \in P$ is a linear combo of $\vec{s_1} + \vec{s_2} \Rightarrow \vec{s_1} + \vec{s_2}$ span P

$\vec{s_1} + \vec{s_2}$ form a basis for P $\Rightarrow \dim P = 2$