2.7 TRANSPOSE " PREMUTATIONS

- transpose of A^T of A
- Symmetric matrices : $S^T = S$
- Premutation matrices and their transpose

TRANSPOSE

WHAT IS A TRANSPOSE?

→ TRANSPOSE CAN BE DEFINED FOR EVERY MATRIX

LET A be an $n \times m$ matrix. Its transpose A^T is the $n \times m$ matrix defined by $(A^T)_{ij} = A_{ji}$

EXAMPLE :

ON A VECTOR:

$$\vec{b} = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} \quad \vec{b}^T = \begin{bmatrix} 7 & 1 & 2 \end{bmatrix}$$

3 X 1

ON A N X M :

$$A = \begin{bmatrix} 3 & -1 & 7 \\ 5 & 2 & -3 \end{bmatrix}$$

2X 3

ROWS OF OLD MATRIX BECOMES COLUMNS OF NEW MATRIX → $A^T =$

$$A^T = \begin{bmatrix} 3 & 5 \\ -1 & 2 \\ 7 & -3 \end{bmatrix}$$

3X 2

FOR A SQUARE MATRIX :

$$B = \begin{bmatrix} 4 & 7 & -2 \\ 6 & 8 & 1 \\ 5 & -9 & 3 \end{bmatrix} \xrightarrow{T} B^T = \begin{bmatrix} 4 & 6 & 5 \\ 7 & 8 & -9 \\ -2 & 1 & 3 \end{bmatrix}$$

3 X 3

APPLY THE PRINCIPLE W/ COLUMS → ROWS THIS TIME

$$\begin{array}{c} B \\ \begin{bmatrix} 4 & 7 & -2 \\ 6 & 8 & 1 \\ 5 & -9 & 3 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} B^T \\ \begin{bmatrix} 4 & 6 & 5 \\ 7 & 8 & -9 \\ -2 & 1 & 3 \end{bmatrix} \end{array}$$

ON A SQUARE, CAN ALSO SEE IT AS FLIPPING THE ENTRIES DIAGONALLY.

PROPERTIES OF TRANSPOSE

1) $(A + B)^T = A^T + B^T$

2) $(A\vec{x})^T = \vec{x}^T A^T$ order changed!

" assuming columns in A = rows in x

3) $(AB)^T = B^T A^T$ order changes too!

3) If A is invertible then $(A^T)^{-1} = (A^{-1})^T$ can take transpose of the inverse

SYMMETRIC MATRICES

A square matrix S is said to be symmetric if $S^T = S$

$$S = \begin{bmatrix} 7 & 5 \\ 5 & -3 \end{bmatrix} \rightarrow S^T = \begin{bmatrix} 7 & 5 \\ 5 & -3 \end{bmatrix}$$

ANY DIAGONAL MATRIX ARE SYMMETRICAL

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ all 0's except for diagonal

EXAMPLE T/F :

① IF A + B ARE SYMMETRIC ⇒ A + B = SYMMETRIC

IMPLIES A + B ARE SQUARE

" TRUE "

PROOF : show $(A \pm B)^T$ = SYMMETRIC

$(A \pm B)^T = A^T \pm B^T = A \pm B$ □

USING PROPERTY OF TRANSPOSE

② SUPPOSE A + B IS SYMMETRIC ⇒ A + B ARE SYMMETRIC

" FALSE "

counter example:

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{BUT } A + B = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$$

NOT SYM. IS SYM.

③ A and B are symmetric ⇒ AB is symmetric

" FALSE "

SHOW W/ CONTRADICTION :

$$\begin{array}{c} A \\ \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \end{array} \begin{array}{c} B \\ \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} \end{array} = \begin{array}{c} AB \\ \begin{bmatrix} 4 & 10 \\ 11 & 4 \end{bmatrix} \end{array}$$

NOT SYM.

$(AB)^T = B^T A^T = BA \neq AB$

IMPLIES FALSE

IT'S TRUE THO WHEN A + B ARE DIAGONAL MATRICES

LDU FACTORIZATION OF SYMMETRIC MATRIX

S is symmetric, and $S = LDU$

$$S^T = (LDU)^T = U^T D^T L^T = U^T D L^T \quad \text{so...} \quad S^T = S$$

$D^T = D$ U^T = LOWER △ L^T: UPPER △

$$\Rightarrow U^T = L, \; L^T = U$$

$$\Rightarrow S^T = S = LDU = U^T D L^T = L D L^T = U^T D U$$

EXAMPLE. FIND LDU FACTORIZATION OF S

$$S: \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & 5 \end{bmatrix} \xrightarrow[R_3 + R_1]{\ell_{31} = -1,\ \ell_{21} = 2,\ R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 6 & 4 \end{bmatrix} \xrightarrow[R_3 - 2R_2]{\ell_{32} = 2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & -8 \end{bmatrix} \xrightarrow{DU} U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$$

FLIP IT

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

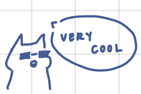
PERMUTATION MATRICES + THEIR TRANSPOSE

RECALL: (ELEMENTARY) PREMUTATION MATRICES $P_{ij} = R_i \leftrightarrow R_j$

EXAMPLE:

$P_{13} = P_{13}^T$ because $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ P_{13} + P_{13}^T ARE SYMMETRIC

we generalize this:



$$P_{ij}^T = P_{ij} \qquad P_{ij} = (P_{ij})^{-1}$$

RECALL THIS TOO!

WHAT IF WE MULTIPLY 2 DIFFERENT P'S?

let's try $(P_{23})(P_{13})$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we can just apply P_{23} to P_{13}, so swap 2nd + 3rd row of P_{13}

THIS IS, NONE OF THE P_{ij}'S BUT IT IS A MATRIX OBTAINED BY SHUFFLING THE ROWS OF I

SO WE WANT TO REDEFINE PERMUTATION MATRIX: A SQUARE MATRIX WHOSE ROWS ARE THOSE OF THE I (of same size) EACH APPEARING ONCE!

ex: I, any of P_{ij} + any products of P_{ij}s.