2.2 IDEA OF ELIMINATION

WK3 · 1

ELIMINATION

- 1st step of solving linear equations systematically

★ PIVOTS : KEY PLAYER

THE IDEA:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \xrightarrow{\text{CHANGE TO C, CUT IN HALF}} \begin{cases} c_{11}x + c_{12}y + c_{13}z = d_1 \\ c_{22}y + c_{23}z = d_2 \\ c_{33}z = d_3 \end{cases}$$

PIVOTS

$\rightarrow d_1 - c_{12}\left(d_2 - c_{23}\left(\frac{d_3}{c_{33}}\right)\right) / c_{22} - c_{13}\left(d_3 / c_{33}\right) = x$

$\rightarrow \left(d_2 - c_{23}\left(\frac{d_3}{c_{33}}\right)\right) / c_{22} = y$

$\rightarrow d_3 / c_{33} = z$

DO BACKWARDS SUBSITUTION!

~SOLVING TWO EQUATIONS FOR TWO UNKNOWNS~

EXAMPLE 1: SUCCESSFUL PIVOT

PIVOT

$\begin{cases} 2x + 4y = 14 & ① \\ 3x + 5y = 19 & ② \end{cases}$

↳ WANT TO ELIMINATE THIS to get upper triangle

→ GOAL: USE ① TO ELIMINATE x from ② to get upper triangle form

→ PIVOT: first non-zero coefficient in the row (① in this case) that does the elimination

MULTIPLIER: $\frac{\text{COEFFICIENT TO ELIMINATE}}{\text{PIVOT}}$ ← in this case the 3 in 3x

↳ PIVOT = 2 ↳ MULTIPLIER = $\frac{3}{2}$

∘ equation ① will act on ② so that x in ② goes away

TO ELIMINATE:

STEP 1) DO ② − 3/2 ∗ ① (ELIM, PIVOT) (meaning "subtract $\frac{3}{2}$ ∗ ① from ②")

✿ subtract the multiplier times the pivot from the equation we want to eliminate the coefficient from.

✿ FOR TWO EQUATIONS, IT'S ALMOST ALWAYS ② − multiplier ①

unless if it is like

$\begin{cases} 7y = 21 \\ x + 3y = 4 \end{cases}$ → because in this case you would just swap to get the upper triangle form

$$\begin{array}{r} 3x + 5y = 19 \\ -\frac{3}{\cancel{2}} * (\cancel{2}x + \cancel{4}y = 14) \\ \hline \end{array} \rightarrow \begin{array}{r} 3x + 5y = 19 \\ -(3x + 6y = 21) \\ \hline -y = -2 \end{array} \rightarrow y = 2$$

STEP 2) leave ① and make lin. equation w/ the new ②

becomes pivot

$\begin{cases} 2x + 4y = 14 \rightarrow \text{Plug in } y \rightarrow 2x + 4(2) = 14 \rightarrow 2x = 6 \rightarrow x = 3 \\ -y = -2 \rightarrow y = 2 \end{cases}$

BACKWARDS SUBSITUTION

there is a pivot in every row!

eliminate by plugging in

DONE + SUCCESSFUL! WE KNOW BC UPPER TRIANGLE FORM W/ PIVOT IN EVERY ROW

meaning THERE'S ONLY ONE UNIQUE SOLUTION. $(x, y) = (3, 2)$ → THE ROW + COLUMN PICTURE: SIMILAR TO FIRST EX. IN THE LAST SECTION (2.1)

EXAMPLE 2: FAILED PIVOT

TO ELIMINATE:

$\begin{cases} 2x + 4y = 14 & ① \\ 3x + 6y = 21 & ② \end{cases}$

Find pivot: 2

find multiplier: $\frac{3}{2}$

STEP 1)

$$\begin{array}{r} \cancel{3x} + \cancel{6y} = \cancel{21} \\ -\frac{3}{2} * (\cancel{2}x + \cancel{4}y = \cancel{14}) \\ \hline 0 = 0 \end{array} \rightarrow \begin{cases} 2x + 4y = 14 \\ 0 = 0 \end{cases}$$

CAN'T MAKE A PIVOT!

→ means there are infinitely many solutions

= permanent failure: no pivot in second row

x + y could be anything ...

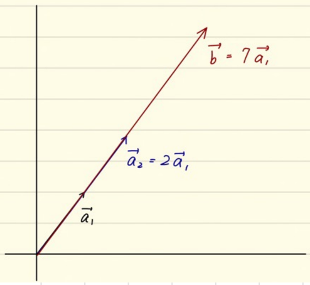
(i) Column Picture

$\vec{a}_1 = (2, 3)$, $\vec{a}_2 = (4, 6) = 2\vec{a}_1$

$\vec{b} = (14, 21) = 7\vec{a}_1$ $\vec{b} = x\vec{a}_1 + y\vec{a}_2$

∞ many way of writing $\vec{b}$ as

a lin. comb. of $\vec{a}_1$ and $\vec{a}_2$



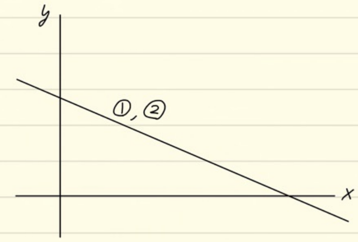
(ii) Row Picture

$2x + 4y = 14 \Leftrightarrow x + 2y = 7$

$\Leftrightarrow 3x + 6y = 21$

So ① and ② are the same line.

Every pt on the line is a soln.



EXAMPLE 3:

$$\begin{cases} 2x + 4y = 14 & ① \\ 3x + 6y = 20 & ② \end{cases}$$

PIVOT : 2

multiplier : $\frac{3}{2}$

STEP 1) $② - \frac{3}{2} * ①$

$$\cancel{3}x + \cancel{6}y = 20$$
$$-\frac{3}{2}(\cancel{2}x + \cancel{4}y = 14)$$
$$0 = -1$$

→ INCONSISTENT

STEP 2) LIST IT OUT

$$\begin{cases} 2x + 4y = 14 \\ \quad 0 = -1 \end{cases}$$

= NO SOLUTION

= permanent failure, no pivot in second row

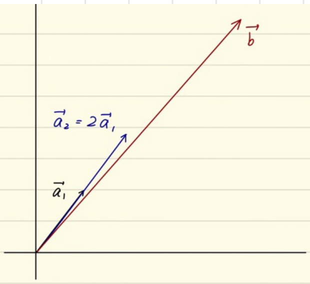
(i) Column Picture

$\vec{a}_1 = (2, 3)$, $\vec{a}_2 = (4, 6) = 2\vec{a}_1$

$\vec{b} = (14, 20) \neq c\,\vec{a}_1 \quad \forall c \in \mathbb{R}$

Impossible to write $\vec{b}$ as a lin.

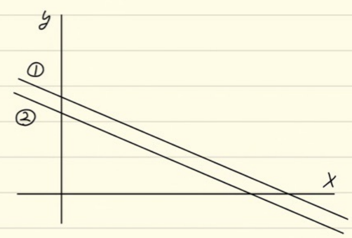
comb. of $\vec{a}_1$ and $\vec{a}_2$.



(ii) Row Picture

① and ② are parallel and

do not intersect.



" SOLVING 4 THREE EQUATIONS + THREE UNKNOWNS "

EXAMPLE 1: solving the linear system by reducing it to upper triangle form and using backwards subsitution

$$\begin{cases} 2x - 2y - 3z = -10 & ① \\ x - y + z = 5 & ② \\ 3x - 2y + z = 9 & ③ \end{cases}$$

CAN USE ② as 1st pivot, switching order is ok

→

do this because we want to use 1 as a pivot

$$\begin{cases} x - y + z = 5 & ② \\ 2x - 2y - 3z = -10 & ① \\ 3x - 2y + z = 9 & ③ \end{cases}$$

multiplier $\frac{2}{1}$ multiplier : $\frac{3}{1}$

DO $① - 2 * ②$ and $③ - 3 * ②$:

do two eliminations at the same time

STEP 1:

$$\begin{cases} x - y + z = 5 & ② \\ 2x - 2y - 3z = -10 & ① \\ 3x - 2y + z = 9 & ③ \end{cases}$$

→

$$2x - 2y - 3z = -10$$
$$-2(\cancel{x} - \cancel{y} + z = 5)$$
$$-5z = -20$$

BECOMES ①'

STEP 2:

$$\cancel{3x} - 2y + z = 9$$
$$-3(\cancel{x} - y + z = 5)$$
$$y - 2z = -6$$

BECOMES ③'

→

STEP 3:

$$\begin{cases} x - y + z = 5 \\ \quad -5z = 20 \\ \quad y - 2z = -6 \end{cases}$$

NO PIVOT TO ELIMINATE Y BELOW!

➡ TEMPORARY FAILURE: CAN BE FIXED BY SWAPPING ①' AND ③'

STEP 4: SWAP!

$$\begin{cases} x - y + z = 5 \\ \quad y - 2z = -6 \\ \quad -5z = -20 \end{cases}$$

STEP 5: BACKWARDS SUBSTITUTION

→ $x = 5 + 2 - 4 = 3$

→ $y = -6 + 2(4) = 2$

→ $z = 4$

SUCCESS!

PIVOT IN EVERY ROW

EXAMPLE 2 : HAVING TEMPORARY + PERMANENT FAILURE

$$\begin{cases} x - 3y + 2z = 2 & (1) \\ -2x + ay + 5z = 5 & (2) \\ 3x - 8y + 4z = 5 & (3) \end{cases}$$

FOR WHAT VALUE(S) OF $a \in \mathbb{R}$ does the elimination fail **temporarily**?

DOING $\overset{\text{ELIM}}{(2)} + 2 * \overset{\text{PIVOT}}{(1)}$ + $\overset{\text{ELIM}}{(3)} - 3 * \overset{\text{PIVOT}}{(1)}$

$$\begin{array}{r} -2x + ay + 5z = 5 \\ +2(\cancel{x} - 3y + 2z = 2) \\ \hline (a-6)y + 9z = 9 \\ (2)' \end{array} \qquad \begin{array}{r} \cancel{3x} - 8y + 4z = 5 \\ -3(\cancel{x} - 3y + 2z = 2) \\ \hline 2y - 2z = -1 \\ (3)' \end{array}$$

$$\rightarrow \begin{cases} x - 3y + 2z = 2 & (1) \\ (a-6)y + 9z = 9 & (2)' \\ y - 2z = -1 & (3)' \end{cases}$$

IF $a = 6$:

$$\begin{cases} x - 3y + 2z = 2 & (1) \\ \cancel{0y} + 9z = 9 & (2)' \\ y - 2z = -1 & (3)' \end{cases}$$

NEED FLIP

TEMPORARILY FAIL IF $a = 6$.

FOR WHAT VALUE(S) OF $a \in \mathbb{R}$ does the elimination fail **permanently**?

$$\begin{cases} x - 3y + 2z = 2 & (1) \\ y - 2z = -1 & (3)' \\ (a-6)y + 9z = 9 & (2)' \end{cases} \rightarrow$$

eliminate one more time

$$(2)' - (a-6) \times (3)' \rightarrow \begin{cases} x - 3y - 2z = 2 \\ y - 2z = -1 \\ (9 + 2a - 12)z = 9 + a - 6 \rightarrow (2a-3)z = a+3 \end{cases}$$

$z(2a-3) = a+3$ so PERMANENT FAILURE IF $2a = 3$, $a = 2/3$ because then it becomes $0 = \frac{3}{2} + 3$ which is INCONSISTENT

2.3 ELIMINATION W/ MATRICES

WK3-2

- Elimination from previous section $\leftrightarrow$ (by using) Multiplication of "Elementary Matrix" E_{ji}
- Swapping equations $\leftrightarrow$ (by using) multiplication of "Premutation Matrix" P_{ij}

ELIMINATION N ELEMENTARY MATRIX

EXAMPLE →

1) LET $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and suppose that $v_2 = 3v_1$ (we assume), ex: $\vec{v} = \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}$

2) CONSIDER THE ELIMINATION OF V_2 BY USING V_1, ex: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightarrow \begin{bmatrix} v_1 \\ v_2 - 3v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$

3) we write this process as (*) $\underline{R_2 - 3R}$ (row) meaning "subtract 3* Row 1 from Row 2"

4) we need to find the matrix E_{21} that does the operation (*) by multiplication, like:

- we are looking for matrix E_{21} so that we can eliminate Row 2
- Basically working backwards so that

$E_{21}\vec{v} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \xrightarrow{(*)} \begin{bmatrix} v_1 \\ v_2 - v_3 \\ v_3 \end{bmatrix} = E_{21}\vec{v} \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

INFACT:

$$E_{21}\begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$$

Easier way to find E_{21}:

E_{21} is what we get when we apply (*) to the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(*)} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ELIMINATION OF MATRIX ELEMENTS

CAN WE DO THE SAME ON MATRICES?

SUPPOSE:

STEP 1)

$$A = \left[\vec{a}_1, \vec{a}_2, \vec{a}_3\right] = \begin{bmatrix} 3 & 2 & 1 \\ 9 & 4 & 0 \\ -6 & 8 & 8 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -2 & -3 \\ -6 & 8 & 8 \end{bmatrix} = E_{21}A$$

think of A as 3 column vectors combined

got R_2 by

$$\begin{array}{r} 9 \quad 4 \quad 0 \\ -\ 9 \quad 6 \quad 3 \\ \hline 0 \ -2 \ -3 \end{array}$$

WHAT DOES THIS MEAN?

$E_{21}A$ means → $E_{21}A = [E_{21}\vec{a}_1, E_{21}\vec{a}_2, E_{21}\vec{a}_3]$

soo... same as $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ would do.

STEP 2) reduce more:

$$E_{21}A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -2 & -3 \\ -6 & 8 & 8 \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 12 & 10 \end{bmatrix} = E_{31}(E_{21}A)$$

WHATS THIS?

E_{31} is: (apply same operation to the identity matrix)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E_{31}$$

STEP 3) further reduce:

$$E_{31}E_{21}A = \begin{bmatrix}3&2&1\\0&-2&-3\\0&12&10\end{bmatrix} \xrightarrow{R_3+6R_2} \begin{bmatrix}3&2&1\\0&-2&-3\\0&0&-8\end{bmatrix} = E_{32}E_{31}E_{21}A$$

WHAT'S THIS?

E_{32}?

$$I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3+6R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&6&1\end{bmatrix} = E_{32}$$

PREMUTATION MATRICES

EXAMPLE: FIND THE MATRIX P_{23} SO THAT

$$A = \begin{bmatrix}2&4&3\\0&0&7\\0&-1&5\end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{\text{flip}} \begin{bmatrix}2&4&3\\0&-1&5\\0&0&7\end{bmatrix} = P_{23}A$$

WHAT'S THIS

P_{23} =

$$I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = P_{23}$$

APPLICATION TO LINEAR EQUATIONS

EXAMPLE (SAME AS PREVIOUS SECTION)

$$\begin{cases}2x - 2y - 3z = -10\\ x - y + z = 5\\ 3x - 2y + z = 9\end{cases} \longleftrightarrow \underbrace{\begin{bmatrix}2&-2&-3\\1&-1&1\\3&-2&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}-10\\5\\9\end{bmatrix}}_{\vec{b}}$$

REDUCE $A\vec{x} = \vec{b}$ to $U\vec{x} = \vec{c}$ where U is an upper triangular matrix

write down the elementary / premutation matrix for each operation used

STEP 1) START W/ THE <u>AUGMENTED MATRIX</u>

switch R_1 & R_2 to get 1 as a pivot

$$\begin{bmatrix}A & \vec{b}\end{bmatrix} = \left[\begin{array}{ccc|c}2&-2&-3&-10\\1&-1&1&5\\3&-2&1&9\end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c}1&-1&1&5\\2&-2&-3&-10\\3&-2&1&9\end{array}\right] \text{ w/ } P_{12} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$= P_{12}\begin{bmatrix}A & \vec{b}\end{bmatrix} = \begin{bmatrix}P_{12}A & P_{12}\vec{b}\end{bmatrix}$$

DON'T HAVE TO WRITE X, Y, Z

JUST PUT A & $\vec{b}$ next to each other

↳ don't need to keep track of X, Y, Z

STEP 2)

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 0 & -5 & -20 \\ 3 & -2 & 1 & 9 \end{bmatrix} = E_{21} P_{12} [A \ \vec{b}] \quad w/ \ E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

STEP 3)

$$\xrightarrow{R_2 \longleftrightarrow R_3,} \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 0 & -5 & -20 \\ 0 & 1 & -2 & -6 \end{bmatrix} = E_{31} E_{21} P_{12} [A \ \vec{b}] \quad w/ \ E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

STEP 4)

$$\xrightarrow{R_3 - 3R_1,} \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & -5 & -20 \end{bmatrix} = P_{23} E_{31} E_{21} P_{12} [A \ \vec{b}] \quad w/ \ P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

SO THEN WE GET: $U\vec{x} = \vec{c}$ w/

$$U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 5 \\ -6 \\ -20 \end{bmatrix}$$