2.5 INVERSE MATRICES

PROPER DEFINITION OF INVERSE MATRIX

HOW TO FIND INVERSE MATRIX

→ THE NON-PROPER DEFINITION WAS

$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

(if $\vec{x} \neq \vec{0}$ then the inverse does exist)

WHAT IT IS

GIVEN AN N X N (square) MATRIX, THERE IS AN INVERSE MATRIX THAT SATISFIES

$$A^{-1}A = I \quad \text{AND} \quad AA^{-1} = I$$

← ONLY DEFINED FOR SQUARE MATRICES!

- CANNOT HAVE INVERSE IF NOT SQUARE
- OTHERE ARE SQUARE MATRICES WHO DO NOT HAVE INVERSES

RECALL: IN 1.3 WE SAID THERE WERE SQUARE MATRICES WHOSE INDEX DOES NOT EXIST!

↳ SO! A is invertible / non-singular if A^{-1} exists

but non-invertible / singular if A^{-1} does not exist

• HOW TO FIND INVERSES •

(usually a tedious process, but this special case is ez!)

FINDING INVERSE I : SPECIAL CASE

a 2 X 2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

INVERTIBLE IFF:

the $\det A \; (ad - bc) \neq 0$

"determinant"

EXAMPLE:

✓ $A = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad \det A = (2\cdot 3) - (4\cdot 2) = -2$ ✓

↳ A = INVERTIBLE

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -3/2 & 2 \\ 1 & -1 \end{bmatrix}}$$

X $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \det B = (1\cdot 4) - (2\cdot 2) = 0$ x $\quad$ B IS SINGULAR

ANOTHER SPECIAL CASE : INVERSE OF ELEMENTRY MATRIX

FIND THE INVERSE OF:

a)

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_{21} = R_2 + 2R_1$ (*)

$E_{21}A$ = what you get by applying (*) to A.

· NOW WE ARE LOOKING FOR E_{21}^{-1}

↳ $E_{21}E_{21}^{-1} = I$

$E_{21}^{-1}(E_{21}A) = A$

· E_{21}^{-1} the operation that undos the (*)

★ SO, WHAT OPERATION UNDOES THE (*)?

SO YOU DO $R_2 - 2R_1$ WHICH $= E_{21}^{-1}$

↑ UNDOS (*: $R_2 + 2R_1$)

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

APPLY THE SAME PRINCIPLE IN a)

$$P_{23}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

UNDO THE SWAP

P_{23}^{-1} corresponds to the operation that undoes $R_2 \leftrightarrow R_3$

IS ACTUALLY JUST SWAPPING AGAIN. APPLY $R_2 \leftrightarrow R_3$ TO I.

SO P_{23}^{-1} IS JUST P_{23}

INVERSE = ORIGINAL MATRIX

other special cases:

① $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \pm I \qquad a^2 = 1$

② $A = I$

$A^2 = I$

INVERSE OF PRODUCT

SUPPOSE A & B ARE BOTH $n \times n$ + ARE BOTH INVERTIBLE, WE WANT $(AB)^{-1}$

↳ HOW IS THIS RELATED TO A^{-1} + B^{-1}?

IF A + B ARE BOTH INVERTIBLE $n \times n$ matrices, THEN AB, and $(AB)^{-1} = B^{-1}A^{-1}$ ← ORDER MATTERS

PROOF:

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{A^{-1}A}_{I}B = \underbrace{B^{-1}B}_{I} = I \checkmark$$

$$(AB)(B^{-1}A^{-1}) = A\underbrace{BB^{-1}}_{I}A^{-1} = \underbrace{AA^{-1}}_{I} = I \checkmark$$

means they are candidates for inverses for each other

THEREFORE $B^{-1}A^{-1}$ = INVERSE OF AB

↳ $(AB)^{-1} = B^{-1}A^{-1}$ ◻ ← end of proof

FINDING INVERSE II : GENERAL CASE

— GAUSS - JORDAN ELIMINATION —

WHAT IF A IS NOT ANY OF THE SPECIAL CASES?

CONSIDER THE 3 TYPES OF ELEMENTARY ROW OPERATIONS (EROS)

1) ADDING / SUBTRACTING A CONSTANT * ROW TO ANOTHER ROW → $R_i \pm cR_j$
2) SWAPPING ROWS : $R_i \leftrightarrow R_j$
3) MULTIPLY A ROW BY A NON-ZERO # : cR_i

CONSIDER APPLYING THESE EROS TO A → SUPPOSE THAT k EROS REDUCE A TO I

$$A \xrightarrow{ERO} E_1A \xrightarrow{ERO} E_2E_1A \longrightarrow \dots \longrightarrow \underbrace{E_k \dots E_2E_1}A = I$$

$$\downarrow$$

$$E_k \dots E_2E_1 = A^{-1}$$

WE USE A SHORT CUT TO GET $E_k \dots E_2E_1$ ↪

APPLY THE OPERATIONS TO THE AUGMENTED MATRIX $[A \; I]$

EXAMPLE: $[A \;\; I] \longrightarrow$ GOAL $[I \;\; A^{-1}]$

LOGIC: $[A \;\; I] \xrightarrow{ERO} E_1[A \;\; I] \xrightarrow{ERO} [E_2E_1A \quad E_2E_1]$
$= [E_1A \quad E_1]$

$\xrightarrow{\text{MORE EROS}} \dots [\underbrace{E_k \dots E_2E_1}_{A^{-1}}A \quad \underbrace{E_k \dots E_2E_1}_{A^{-1}}] = [I \;\; A^{-1}]$

a) FIND INVERSE OF $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$[A \;\; I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{E_1} \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{E_2} \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 - R_2]{E_3} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] = [I \;\; A^{-1}] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

write the inverse as a multiple of the elementary matrices:

$$[\underbrace{E_3E_2E_1A}_{I} \quad \underbrace{E_3E_2E_1}_{A^{-1}}]$$

$$A^{-1} = E_3E_2E_1$$ ← IN THAT ORDER

b) find the EROs

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

WHAT'S A IN TERMS OF THE ELEMENTRY MATRICES:

$$A = (A^{-1})^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$$