2.6 ELIMINATION = FACTORIZATION

why is this useful?

→ we can use $A = LU$ to solve $A\vec{x} = \vec{b}$

IF $A \xrightarrow[\text{W/O SWAPPING}]{\text{elimination}} U$ (upper triangle form), then $A = LU$

↑ lower triangle form

LU FACTORIZATION / DECOMPOSITION

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 4 \\ 3 & 8 & -5 \end{bmatrix}$$

CAN ONLY USE ELIMINATION! (NO SWAPPING)

- see below PLU SECTION 4 SWAPS

a) reduce a to upper triangular matrix using elimination + ⓑ list all elimination matrices + multipliers for all steps

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 4 \\ 3 & 8 & -5 \end{bmatrix} \xrightarrow[E_{21}]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 3 & 8 & -5 \end{bmatrix} \xrightarrow[E_{31}]{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[E_{32}]{R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

ℓ_{21} mult: -2 $\qquad$ ℓ_{31} mult: 3 $\qquad$ ℓ_{32} mult: $1/2$

$E_{21}A$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

c) write $A = LU = (\text{lower})(\text{upper})$

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix} = E_{32}E_{31}E_{21}A$$

TO GET L, TAKE INVERSE OF E'S

→ because $U = E_{32}E_{31}E_{21}A$, → $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

$A = (E_{32}E_{31}E_{21})^{-1}U$

$A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \underbrace{E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}}_{\text{multiply:}}$$

undo the operations!

think of it as apply the operations to each other

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1/2 & 1 \end{bmatrix}$$

THESE ARE JUST THE MULTIPLIERS! CAN GET L BY PLUGGING IN MULTIPLIERS

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

LDU FACTORIZATION

FROM PREVIOUS ANSWER: $A = LU = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1/2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix}}_{U \text{ old}}$

L has the '1' diagonal, but U doesn't!

so to make it fair we make another diagonal matrix

(THIS IS BASICALLY FOR AESTHETICS ONLY)

$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1/2 & 1 \end{bmatrix}}_{\text{L STAYS THE SAME}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}}_{\text{DIAGONAL}} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{NEW U}}$

THINK OF IT AS A PUZZLE

THINK OF WHAT TO MULTIPLY BY U by to get D

PLU FACTORIZATION

EXAMPLE:

$A = \begin{bmatrix} 0 & -2 & 6 \\ 1 & 2 & 3 \\ 3 & 8 & 0 \end{bmatrix}$

FOR THIS EXAMPLE, WE CANNOT GET U USING ONLY ELIMINATION: C like a temp. error

$\xrightarrow[\text{W/O SWAPPING}]{\text{(FORWARD) ELIMINATION}} U$

WE FIX IT BY SWAPPING

$A \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 3 & 8 & 0 \end{bmatrix} = P_{12} A$

$P_{12}: \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\overset{P_{12}A}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 3 & 8 & 0 \end{bmatrix}} \xrightarrow[l_{31}=3]{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 2 & -9 \end{bmatrix} \xrightarrow[l_{32}=-1]{R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix} = U \quad \sim \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$

SO ... $P_{12}A = LU$

$A = (P_{12}^{-1})LU = \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}}$

!, RECALL: $P_{12} = P_{12}^{-1}$

$A\vec{x} = \vec{b} \Rightarrow L\underbrace{U\vec{x}}_{\vec{c}} = \vec{b} \Rightarrow \begin{cases} L\vec{c} = \vec{b} \\ U\vec{x} = \vec{c} \end{cases}$

we split $A\vec{x} = \vec{b}$ into 2 equations

if $\vec{b}$ is known, we can solve for $\vec{c}$ or if $\vec{c}$ is known, we solve for $\vec{b}$

THIS IS MUCH FASTER THAN FINDING INVERSES

THE IDEA: FIRST SOLVE $L\vec{c} = \vec{b}$ for $\vec{c}$, and then solve $U\vec{x} = \vec{c}$ for $\vec{x}$

EXAMPLE: solve $A\vec{x} = \vec{b}$

$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 4 \\ 3 & 8 & -5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -3 \\ 2 \\ -17 \end{bmatrix}$ where (DO LU) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -3 \end{bmatrix} = LU$

much FASTER!

FIRST SOLVE: $L\vec{c} = \vec{b}$ for $\vec{c}$

forward sub.

$L\vec{c} = \vec{b} \Leftrightarrow \begin{cases} c_1 = -3 \\ -2c_1 + c_2 = 2 \\ 3c_1 + \frac{1}{2}c_2 + c_3 = -17 \end{cases} \quad \begin{matrix} c_1 = -3 \\ c_2 = -4 \\ c_3 = -6 \end{matrix}$

C IS SOLVED EASILY! $\longrightarrow$ $\vec{c} = (-3, 4, -6)$

backward sub.

$U\vec{x} = \vec{c} \Leftrightarrow \begin{cases} x_1 + 2x_2 - x_3 = -3 \Rightarrow -3 + 4 + 2 = 3 \\ 4x_2 + 2x_3 = -4 \Rightarrow 4x_2 = 8 \Rightarrow x_2 = -2 \\ -3x_3 = -6 \Rightarrow x_3 = 2 \end{cases}$

$\vec{x} = (3, -2, 2)$