ch.2 : SOLVING LINEAR EQUATIONS

2.1 VECTORS + LINEAR EQUATIONS

there are two different geometric pictures of linear equations. $A\vec{x} = \vec{b}$:

★ i column picture ↙ more important

ii row picture

depending on how we interpret $A\vec{x} = \vec{b}$

- TWO LINEAR EQUATIONS FOR TWO UNKNOWNS -

i column picture (★ we like this)

ii row picture

EX:

$$\begin{cases} 3x + 4y = 2 \\ 2x + 3y = 7 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases} \rightarrow \underset{A}{\begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}} \rightarrow x \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{a}_1} + y \underbrace{\begin{bmatrix} -4 \\ 3 \end{bmatrix}}_{\vec{a}_2} = \underbrace{\begin{bmatrix} 2 \\ 7 \end{bmatrix}}_{\vec{b}} + \begin{matrix} 3x - 4y = 2 \\ 2x + 3y = 7 \end{matrix}$$

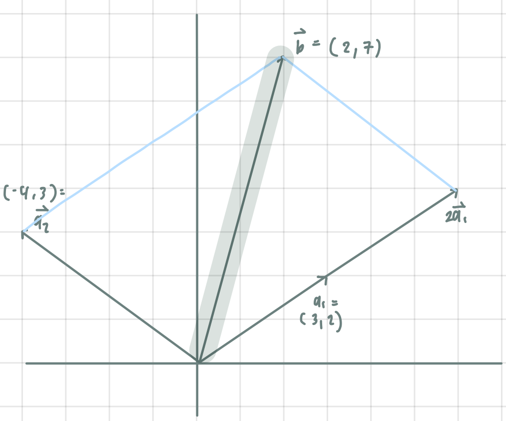
(i) (ii)

↳ to solve it the column way:

find x + y so that $\vec{b}$ is the linear combination $x\vec{a}_1 + y\vec{a}_2$ of the column vectors $\vec{a}_1 + \vec{a}_2$ of A

since $(x, y) = (2, 1)$

$2\vec{a}_1 + \vec{a}_2 = \vec{b}$



↳ to solve it the rows way

$3x - 4y = 2$ ①
$2x + 3y = 7$ ②
each defines a line

would be dot product

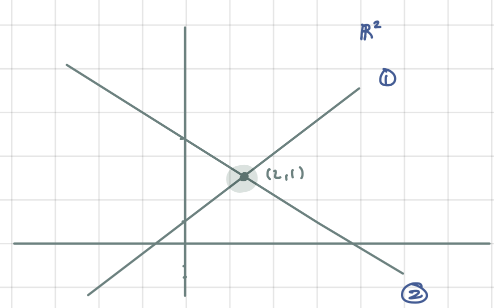
↓

$y = \frac{3}{4}x + \frac{1}{2}$ ①

$y = \frac{7}{3} - \frac{2}{3}x$ ②

↓

they intersect at (2, 1), so that is the solution



-THREE LINEAR EQUATIONS 4 THREE UNKNOWNS-

EX:

$$\begin{cases} 2x - y - 2z = -2 \\ x + z = 3 \\ 3x - 2y + z = -1 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 4 \\ z = 1 \end{cases} \rightarrow \underbrace{\begin{bmatrix} 2 & -1 & -2 \\ 1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}}_{\vec{b}}$$

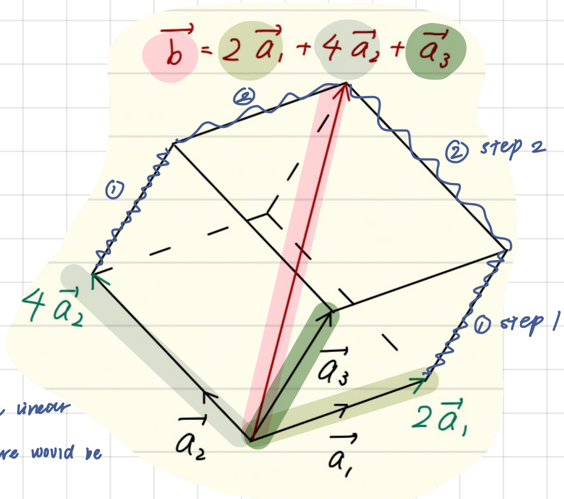
i column picture

$$x \underbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}_{\vec{a}_1} + y \underbrace{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}_{\vec{a}_2} + z \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\vec{a}_3} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

↓

solving the original set of eq. → find x, y, z such that $x\vec{a}_1 + y\vec{a}_2 + z\vec{a}_3 = \vec{b}$

solution = $(x, y, z) = (2, 4, 1) \rightarrow 2\vec{a}_1 + 4\vec{a}_2 + \vec{a}_3 = \vec{b}$



- if $\vec{b}$ cannot be a linear combination, there would be no solutions.

ii rows picture

$2x - y - 2z = -2$ ①

$x + z = 3$ ②

$3x - 2y + z = -1$ ③

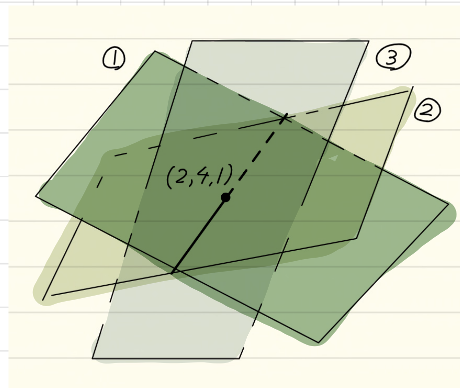
↓

make each of ① ② ③ into a plane in $\mathbb{R}^3$

+ the intersection of $(x, y, z) = (2, 4, 1)$ is the solution

↓

To form $\vec{b}$:



Matrix - vector multiplication + transformation

Given $m \times n$ matrix A, one can think of the matrix-vector mult. $A\vec{x}$ as a transformation from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ w/

input: $\vec{x} \in \mathbb{R}^n$ output: $A\vec{x} \in \mathbb{R}^m$

EX:

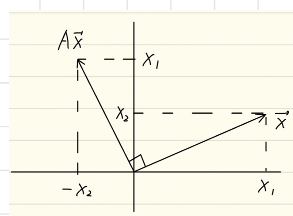
INPUT:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ transforms } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ into } \rightarrow A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \rightarrow$$

← multiply →

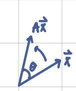
$$x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

o you multiply A to any $\vec{x}$, so $\vec{x}$ is input



$\vec{x} \cdot (A\vec{x}) = -x_1x_2 + x_1x_2 = 0 \rightarrow \vec{x} \perp A\vec{x}$

because it is 0, we know it's $\vec{x} \perp A\vec{x}$, it is a 90° rotation



if we have $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ would be a rotation by θ (counterclockwise)

← 2D ROTATION MATRIX

EX: a) Find the matrix I that multiplies every $\vec{x} \in \mathbb{R}^3$ and produces $\vec{x}$ itself as a result, meaning that $I\vec{x} = \vec{x}$

(I is called the identity matrix)

$$I\vec{x} = \begin{bmatrix} \text{1st row} \\ \text{2nd row} \\ \text{3rd row} \end{bmatrix} \vec{x} = \begin{bmatrix} (\text{1st row}) \cdot \vec{x} \\ (\text{2nd row}) \cdot \vec{x} \\ (\text{3rd row}) \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

what we need

1st row = $(1, 0, 0)$

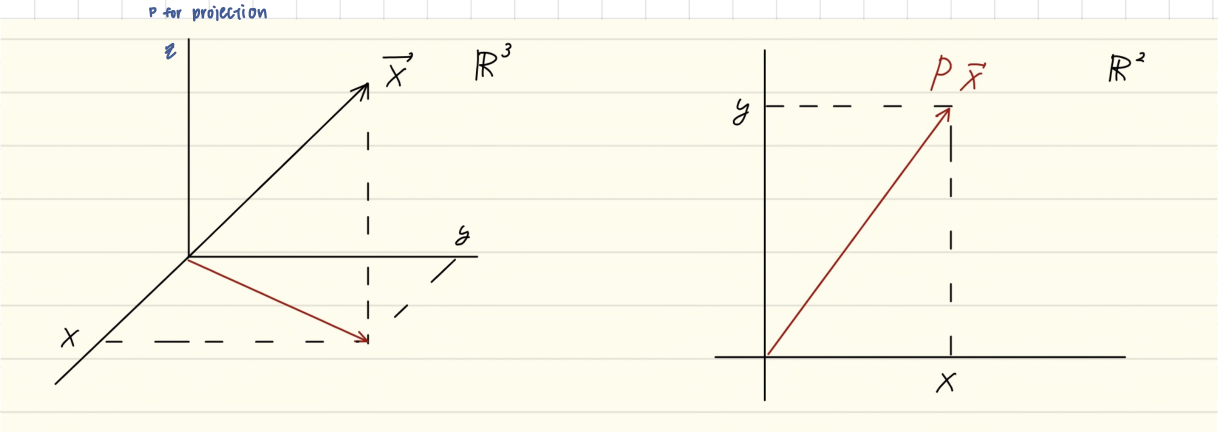
2nd row = $(0, 1, 0)$ $\rightarrow$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ called the identity matrix

3rd row = $(0, 0, 1)$

THIS IS THE MATRIX VERSION OF THE NUMBER 1

b) Find the matrix P that multiplies every $\vec{x} = (x, y, z) \in \mathbb{R}^3$ + produces the projection of $\vec{x}$ onto the x-y plane $\mathbb{R}^2$

P for projection



$\mathbb{R}^3 \rightarrow \mathbb{R}^2$; $\vec{x} = (x, y, z) \rightarrow P\vec{x} = (x, y)$. P is 2×3

implied because P cannot have 3 rows

$$P\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$P \rightarrow$ 2×3

would not have z row

if you thought of the target space as $\mathbb{R}^3$ it could still work, P would just be $\longrightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$