3.2 THE NULL SPACE

- Null space $N(A)$: Another subspace associated with matrix A
- $N(A)$: all solution to $A\vec{x} = \vec{0}$ ⇒ complete solution to $A\vec{x} = \vec{b}$
  - → compare: $C(A)$ ⇒ existence of solution to $A\vec{x} = \vec{b}$ (last section)
- Reduced Row Echelon Form (RREF) of A → N(A), → Rank of A = BEEFED UP VERSION OF UPPER TRIANGULAR FORM

NULL SPACE N(A)

Given an m x n matrix A,

$$N(A) = \text{all solutions } \vec{x} \in \mathbb{R}^n \text{ to } A\vec{x} = \vec{0}$$
$$= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

EXAMPLE: FIND THE NULL SPACE OF THE GIVEN MATRIX:

a) n x n invertible matrix A

If A is invertible, $A\vec{x} = \vec{0} \rightarrow \vec{x} = A^{-1}\vec{0} = \vec{0} \Rightarrow N(A) = \{\vec{0}\}$

set of $\vec{0}$ the point origin

if A is invertible $\underbrace{A^{-1}A}_{I}\vec{x} = \vec{0}$, $\vec{x}$ has to be $\vec{0}$

↳ there's only 1 solution

b) $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

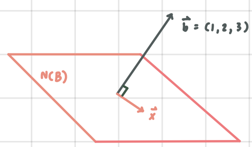
$B\vec{x} = 0 = (1,2,3)\cdot\vec{x} = \vec{0} \leftrightarrow \vec{x} \perp \underbrace{(1,2,3)}_{\vec{b}}$

$N(B) = \{\vec{x} \in \mathbb{R}^3 \mid B\vec{x} = \vec{0}\}$ } find all $\vec{b}$ that's perpendicular to $\vec{b}$

$= \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \perp \vec{b}\}$ = a plane

= plane orthogonal to $\vec{b}$ passing thru the origin

$\vec{b}\cdot\vec{x} = \vec{0} \leftrightarrow x_1 + 2x_2 + 3x_3 = 0$ Eq. for the plane.



c) $C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

$C\vec{x} = \vec{0} \leftrightarrow \begin{cases} x_1 + 3x_3 = 0 \leftrightarrow x_1 = -3x_3 \\ x_2 - 2x_3 = 0 \leftrightarrow x_2 = 2x_3 \end{cases}$

→ HAS INFINITE SOLUTIONS!

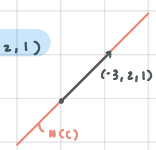
all terms are interms of other terms

So $\vec{x} = (x_1, x_2, x_3) = x_3(-3, 2, 1) \quad \forall x_3 \in \mathbb{R}$

$N(C) = \{x_3(-3,2,1) \in \mathbb{R}^3 \mid x_3 \in \mathbb{R}\}$

$= \text{span}\{(-3,2,1)\}$

= line passing thru origin and $(-3,2,1)$



THEOREM: Show that, for any m x n matrix A, N(A) is a subspace of $\mathbb{R}^n$

(i) Let $\vec{x}, \vec{y}$ be arbitrary elements in N(A), i.e., $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$

then $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$, i.e., $\vec{x} + \vec{y} \in N(A)$ ✓

(ii) Let $\vec{x}$ be an arbitrary element in N(A), i.e., $A\vec{x} = \vec{0}$. Then $\forall c \in \mathbb{R}$

$A(c\vec{x}) = cA\vec{x} = \vec{0}$, i.e., $c\vec{x} \in N(A)$. ✓

Therefore, N(A) is a subspace of $\mathbb{R}^n$

→ N(A) = SUBSPACE OF R^n

→ C(A) = SUBSPACE OF R^m

REDUCED ROW ECHELON FORM

EXAMPLE: FIND N(A) FOR $A = \begin{bmatrix}1&-3&2\\2&-4&0\\3&-5&-2\end{bmatrix}$

make all pivots 1, want 0 above every pivot

$$A=\begin{bmatrix}1&-3&2\\2&-4&0\\3&-5&-2\end{bmatrix}\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{bmatrix}1&-3&2\\0&2&-4\\0&4&-8\end{bmatrix}\xrightarrow{R_3-2R_2}\begin{bmatrix}1&-3&2\\0&2&4\\0&0&0\end{bmatrix}\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&-3&2\\0&1&-2\\0&0&0\end{bmatrix}\xrightarrow{R_1+3R_2}\begin{array}{ccc}x_1&x_2&x_3\\ \hline \end{array}\begin{bmatrix}1&0&-4\\0&1&-2\\0&0&0\end{bmatrix}=R$$

when you have R, you basically can get N(A) right away →

$A\vec{x}=\vec{0}$ = $R\vec{x}=\vec{0}$

This implies that $A\vec{x}=\vec{0}$ reduces to

$$R\vec{x}=\vec{0} \leftrightarrow \begin{cases}x_1-4x_3=0\\x_2-2x_2=0\end{cases} \leftrightarrow \begin{cases}x_1=4x_3\\x_2=2x_3\end{cases}$$

so $\vec{x} = x_3(4,2,1)$ ⇒ implies $N(A) = \text{span}\{(4,2,1)\}$

is a free variable: can freely move around

GENERAL SOLUTION

So the solutions to $A\vec{x}=\vec{0}$ are $\vec{x} = x_3(4,2,1)$, i.e., $N(A) = \text{span}\{(4,2,1)\}$

The above R is called the RREF of A

↳ In general, after reducing A to upper-triangular U, row-reduce it further by:

1) turning all pivots into 1's
2) producing 0's above all pivots
3) collecting all rows entirely of zeros at the bottom to obtain the RREF of A

e.g. $A \longrightarrow$

$$\begin{array}{ccccc}x_1&x_2&x_3&x_4&x_5\end{array}$$
$$\begin{bmatrix}1&0&*&0&*\\0&1&*&0&*\\0&0&0&1&*\\0&0&0&0&0\end{bmatrix}=R$$

x_1, x_2, x_4 = pivot variables

x_3, x_5 = free variables

We define rank A = # of pivots.

Then # of free vars = n − rank A.

in this case rank = 3

free = 2

EXAMPLE

$$A=\begin{bmatrix}1&2&1&0\\2&4&4&8\\4&8&6&8\end{bmatrix}$$

RANK = # OF PIVOTS so...

n − pivots = # OF FREE VARIABLES

a) Find RREF and rank A

$$\begin{bmatrix}1&2&1&0\\2&4&4&8\\4&8&6&8\end{bmatrix}\xrightarrow[R_3-4R_1]{R_2-2R_1}\begin{bmatrix}1&2&1&0\\0&0&2&8\\0&0&2&8\end{bmatrix}\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&2&1&0\\0&0&1&4\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{array}{cccc}x_1&x_2&x_3&x_4\end{array}\begin{bmatrix}1&2&0&-4\\0&0&1&4\\0&0&0&0\end{bmatrix}=R$$

→ rank A = # of pivot = 2

b) FIND N(A)

$$R\vec{x}=\vec{0} \leftrightarrow \begin{cases}x_1 = -2x_2+4x_4\\x_3 = -4x_4\end{cases}$$

free vars

$$\vec{x}=\begin{array}{c}\begin{bmatrix}-2x_2+4x_4\\x_2\\-4x_4\\x_4\end{bmatrix}\end{array}\begin{array}{c}x_1\\x_2\\x_3\\x_4\end{array} = x_2\underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{\vec{s}_1}+x_4\underbrace{\begin{bmatrix}4\\0\\-4\\1\end{bmatrix}}_{\vec{s}_2}$$

write $\vec{x}$ in terms of a lin. combo of $x_2 + x_4$, the "free variables"

⇒ $N(A) = \text{SPAN}\{\vec{s}_1, \vec{s}_2\}$

$\vec{s}_1$ and $\vec{s}_2$ are called special solutions of $A\vec{x}=\vec{0}$

Notice that the dimension of N(A) = # of free vars.

It is in $\mathbb{R}^4$