1.3 MATRICES

- ANNOTATIONS

MATRICES

MATRIX - VECTOR MULTIPLICATION

LINEAR EQUATIONS + INVERSE MATRICES

LINEAR INDEPENDENCE + DEPENDENCE

MATRICES

$$A = \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}^{n \text{ columns}} \Big\} m \text{ rows}$$

ex.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 5 \end{bmatrix}$$

3 x 2 MATRIX

- 2 column vectors → $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ + $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$ in $\mathbb{R}^3$
- 3 row vectors → $[2\ 1]$, $[-1\ 4]$, + $[3\ 5]$ in $\mathbb{R}^2$

m x n matrix = array of numbers w/ m rows + n columns

MATRIX - VECTOR MULTIPLICATION

A $m \times n$ matrix, $\vec{x} \in \mathbb{R}^n \Rightarrow A\vec{x} \in \mathbb{R}^m$

EX.

• concept: $A\vec{x}$ is a linear combo. of columns of A w/ $x_1 \ldots x_n$ as coefficients

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 5 \end{bmatrix} \quad 3 \times 2 \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \rightarrow A\vec{x} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \\ 3 & 5 \end{bmatrix}$$

TWO WAYS!

i) more important / conceptual view

$$= \underbrace{x_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}}_{\text{lin. combo of columns of } A} = \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 4x_2 \\ 3x_1 + 5x_2 \end{bmatrix}$$

this is basically what we did last week to see if the 3rd vector ($\vec{z}$) in $\{\vec{x}, \vec{y}, \vec{z}\}$ will form a plane or be in $\mathbb{R}^3$

ii) DOT PRODUCT

$$= \begin{bmatrix} (2,1) \cdot (x_1, x_2) \\ (-1,4) \cdot (x_1, x_2) \\ (3,5) \cdot (x_1, x_2) \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 4x_2 \\ 3x_1 + 5x_2 \end{bmatrix}$$

★ NOTE: $A\vec{x}$ is not defined if A is $m \times n$ + $x \in \mathbb{R}^k$ where $k \neq n$

★ NOTE: (ii) is easier for computations, (i) is more important

LINEAR EQUATIONS + INVERSE MATRIX

1) ~~suppose that we are solving~~ Given a system of linear equations:

$$\begin{cases} x_1 = b_1 \\ -x_1 + x_2 = b_2 \\ -x_2 + x_3 = b_3 \end{cases} \xrightarrow{\text{solve for } (x_1, x_2, x_3)} \begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_1 + b_2 + b_3 \end{cases}$$

pretend they are known
known

$$\updownarrow \qquad\qquad \updownarrow$$

$$\begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$

$$\updownarrow \qquad\qquad \updownarrow$$

easier to remember

$\vec{b} = A\vec{x}$

$A\vec{x} = \vec{b}$

work backwards to find A

+ return $\vec{b}$ = 3 rows

figure out what A is

$\vec{x}$ is in $\mathbb{R}^3$ so 3 columns

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

inverse matrix

$\vec{x} = A^{-1}\vec{b}$: A^{-1} size would be 3×3 bc the size of $\vec{x}$ → row mult. vector $\vec{b}$ → columns

$\vec{x}$ + $\vec{b}$ determines the size of A + A^{-1}

A^{-1}

because when you solve for $\vec{x}$ you get $\vec{x} = \frac{b}{A}$

$= A^{-1}b$

although it is inverse, it's not as simple as taking the reciprocal!

2) w/ $\vec{x} = (x_1, x_2, x_3)$ + $\vec{b} = (b_1, b_2, b_3)$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$\vec{b} = A\vec{x}$

$\vec{x} = A^{-1}\vec{b}$

↑ # of rows ↑ # of columns

INVERSE MATRIX!

A^{-1} = inverse of A

SPECIAL CASE:

particularly, if $\vec{b} = 0$, then

based on this special case

$A\vec{x} = \vec{0}$ $\xrightarrow{\text{solve for } \vec{x}}$ $\vec{x} = A^{-1}\vec{0} = \vec{0}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

↳ $\vec{x} = \vec{0}$ = only solution

inverse does not exist for all matrices (even if not all entries are 0)

consider this matrix:

$$\begin{bmatrix} x_1 - x_3 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \leftrightarrow C\vec{x} = \vec{b} \text{ w/ } C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

CLAIM: C^{-1} does not exist

→ consider the special case w/ $\vec{b} = \vec{0}$

$$\begin{bmatrix} x_1 - x_3 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{cases} x_1 = x_3 \\ x_1 = x_2 \\ x_2 = x_3 \end{cases} \leftrightarrow x_1 = x_2 = x_3$$

so $\vec{x} = a(1,1,1)$ $\forall a \in \mathbb{R}$ is a solution to $C\vec{x} = \vec{0}$

↳ meaning $C\vec{x} = \vec{0}$ has ∞ MANY solutions

BUT if C^{-1} did exist, then $\underbrace{C\vec{x} = \vec{0} \to \vec{x} = C^{-1}\vec{0} = \vec{0}}$ → what we should've gotten if C^{-1} exists

meaning $\vec{x} = \vec{0}$ is the only solution = CONTRADICTION! So C^{-1} does not exist FOR THIS MATRIX

- if you try to solve for C^{-1} w/ $\vec{b} = \vec{0}$
  - + if does only has $\vec{x} = \vec{0}$ as a solution it HAS AN INVERSE
  - - if it has other solutions! THEN IT DOES NOT HAVE AN INVERSE

★ HE SAID: MOST IMPORTANT!

LINEAR INDEPENDENCE + DEPENDENCE

TO THINK ABOUT THIS:

think of this as a challenge:

you need to find x_1, x_2, x_3 that can make $\vec{x} \cdot \vec{a}$ $(x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3) = \vec{0}$

- there's an easy solution tho! set $x_1 = x_2 = x_3 = 0$! → trivial solution

★ So! the real question is... are there any other solutions?

a) if you can find x_1, x_2, x_3 NOT ALL $= 0$, then they are linearly dependent

b) if all $x_{1,2,3} = 0$ is the only solution, then they are linearly independent

Vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ in $\mathbb{R}^3$ are said to be linearly independent if the only solution $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ to $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}$ is $\vec{x} = \vec{0}$, meaning $x_1 = x_2 = x_3 = 0$

otherwise, if the only solution $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ to $x_1\vec{a}_1 + x_2\vec{a}_2 + x\vec{a}_3 = \vec{0}$ exists $\exists\, \vec{x} \neq \vec{0}$ such that $x_1\vec{a}_1 + x_2\vec{a}_2 + x\vec{a}_3 = \vec{0}$ holds, "there exists" then it is linearly dependent

DIFFERENT WAY OF THINKING WITH MATRIX

Setting $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$ 3×3 matrix,

if the only solution $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ to $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0} \Leftrightarrow A\vec{x} = \vec{0}$

So. $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly independent if only solutions to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

dependent if $A\vec{x} = \vec{0}$ has solutions other than $\vec{x} = \vec{0}$

EX: (A + C from ↑)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

a) are $\vec{a}_1, \vec{a}_2, \vec{a}_3$ linear independent / dependent?

- we saw the only solution to $A\vec{x} = \vec{0}$ so $\vec{x} = 0$
  ↳ linearly independent

b) are $\vec{c}_1, \vec{c}_2, \vec{c}_3$ linear independent / dependent?

↳ we saw that $C\vec{x} = \vec{0}$ had many solutions where $\vec{x} \neq \vec{0}$

$\vec{c}_1, \vec{c}_2, \vec{c}_3$ are linearly dependent

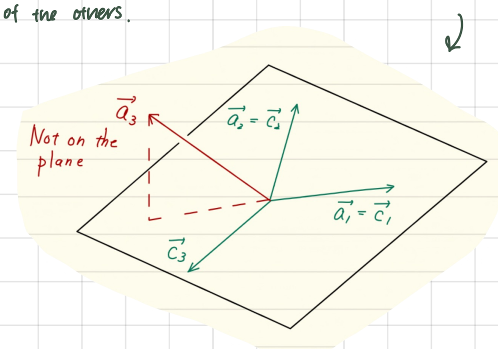
c) write $\vec{c}_3$ as a linear combination of $\vec{c}_1 + \vec{c}_2$

since $\vec{x} = (1,1,1)$ is a solution to $C\vec{x} = \vec{0} \to x_1\vec{c}_1 + x_2\vec{c}_2 + x_3\vec{c}_3 = 0$

$\vec{c}_1 + \vec{c}_2 + \vec{c}_3 = \vec{0} \Rightarrow \vec{c}_3 = -\vec{c}_1 - \vec{c}_2$

there can be many more solutions though

NOTE: It's impossible to write any of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ as a linear combination of the others.



IN GENERAL: $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are linear independent they are on the same plane

dependent if they are not on the same plane