Wk9-1

SOLVING (CERTAIN) RECURRENCES

CH 8.2

WE ARE LOOKING FOR A TECHNIQUE TO SOLVE RECURRENCES OF THE FORM $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

↳ CAN EXTEND TO MORE GENERAL CASES TOO

Based on this, can we know what a_2 is?

$a_n = -4a_{n-1} - 3_{n-2}$

no, not enuff info.

"we need base cases"

↳ $a_0 = 0$, $a_1 = 1$ } based on this $a_2 = -4(1) - 3 * 0 = -4$

CAN WE COME UP W/ A CLOSED FORM SOLUTION TO THIS RECURRENCE?

we want a solution in the form $a_n = r^n$ for some value r.

(NOT GENERAL, BUT WE WILL CONSTRUCT A GENERAL SOLUTION)

$r^n = -4(r^{n-1}) - 3(r^{n-2})$ ← subing in proposed solution for a_x

$r^n + 4r^{n-1} + 3r^{n-2} = 0$ ← move all to left side

$r^2 + 4r + 3 = 0$ ← divide by r^{n-2}

$(r+1)(r+3) = 0$ ↳ roots $= -1, -3$

so what happens when we plug in $a_n = r^n$

↳ $a_n = -1^n$ $a_n = -3^n$

n	0	1	2	3	4	5
$(-1)^n$	1	-1	1	-1	1	-1

we sub this into the original recurrence.

IF IT ALL WORKS, THEN THIS IS A SOLUTION

↳ SO THIS SEQUENCE IS A SOLUTION TO OUR ORIGINAL RECURRENCE

n	0	1	2	3
$(-3)^n$	1	-3	9	-27

$-4 * -3 - 3 * 1 = 9$ ✓

THINK ABOUT IT BACKWARDS:

SUPPOSE $(r+1)(r+3) = 0$ IS TRUE

go from there ↑ up.

→ if $(r+1)(r+3) = 0$, then $r^n = -4r^{n-1} - 3r^{n-2}$

THIS IS EXACTLY THE CONDITION NEEDED TO SATISFY THE ORIGINAL RECURRENCE

NOW WE WANT MORE SOLUTIONS

$(-1)^n$	1	-1	1	-1	1	-1
=	3	-3	3	-3	3	-3

} multiply all by 3

does it still satisfy the recurrence?

↳ becomes $3(a_n) = -4(3 * a_{n-1}) - 3(3 * a_{n-2})$ = same thing

so we can multiply any value of the sequence by a $c \in \mathbb{R}$ + it would still satisfy the recurrence

= ∞ MANY SOLUTIONS BUT WAIT THERE'S MORE!

$(-1)^n$	1	-1	1	-1	1	-1
$(-3)^n$	1	-3	9	-27		
=	2	-4	10	-28		

} add together

$-4(10) - 3(-4) = -28$ ✓ THIS ALSO SATISFIES THE RECURRENCE

$\alpha_1 (-1)^n$

$\alpha_2 (-3)^n$

$\boxed{\alpha_1 (-1)^n + \alpha_2 (-3)^n}$

} ALL SATISFY THE RECURRENCE FOR ANY $\alpha_1 + \alpha_2 \in \mathbb{R}$

GENERAL FORM TO SOLUTIONS OF RECURRENCE

NOW... CAN WE FIND A CLOSED FORM SOLUTION SPECIFIC TO THESE INITIAL CONDITIONS?

$$\begin{cases} a_0 = 0 \qquad a_1 = 1 \\ \text{based on this} \quad a_2 = -4 * 1 - 3 * 0 = -4 \end{cases}$$

start w/ the general form of solutions :

$$a_n = \alpha_1 (-1)^n + \alpha_2 (-3)^n \qquad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

plug in initial conditions : $n = 0$ + $n = 1$

$$\left.\begin{aligned} a_n &= \alpha_1 (-1)^0 + \alpha_2 (-3)^0 \xrightarrow{\text{SIMPLIFY}} 0 = \alpha_1 + \alpha_2 \\ a_n &= \alpha_1 (-1)^1 + \alpha_2 (-3)^1 \rightarrow 1 = \alpha_1 (-1) + \alpha_2 (-3) \end{aligned}\right\} \text{2 equations w/ 2 unknowns}$$

solve for α_1 + α_2 :

add them!

$$\begin{aligned} 0 &= \alpha_1 + \alpha_2 \\ + \quad 1 &= \alpha_1 (-1) + \alpha_2 (-3) \\ \hline 1 &= -2\alpha_2 \end{aligned}$$

$\alpha_2 = -\frac{1}{2}$ $\qquad$ $\alpha_1 = \frac{1}{2}$

NOTE: THIS DEMONSTRATES THAT WE CAN SOLVE FOR OTHER α_1 + α_2's TO SATISFY OTHER INITIAL CONDITIONS

substitute into general solution:

$$a_n = \frac{1}{2}(-1)^n - \frac{1}{2}(-3)^n$$

plug in to check

RECAP OF PROCESS

WE WANT TO SOLVE A RECURRENCE OF THE FORM :

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

① FIND CHARACTERISTIC EQUATION :

$$r^2 - c_1 r - c_2 = 0$$ ← ALWAYS THE CHARACTERISTIC EQUATION

② FIND THE ROOTS (r_1, r_2) of the CE

↳ FACTOR / QUADRATIC FORMULA

③ WRITE GENERAL FORM OF THE SOLUTION

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for } \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

④ substitute the initial conditions into the general form to get a system of 2 equations w/ 2 unknowns

⑤ SOLVE FOR α_1 + α_2

⑥ SUBSTITUTE THOSE INTO THE GENERAL FORM SOLUTION

(r_1)^n is a solution to the recurrence
(maybe the wrong soluition, but one of them)

Multiply it by whatever, and you still have a solution.
(Call α_1 the value we multiply (r_1)^n by)