1.1 VECTORS + LINEAR COMBINATIONS wk1

WHAT IS LIN ALG?

→ Doing alg. w/ linear equations with a bunch of variables

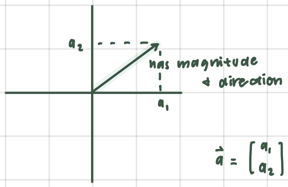
→ There are also geometric ideas

$\vec{0} = (0,0,0)$ zero vector

WHAT IS A VECTOR?

VECTORS IN $\mathbb{R}^2$

$\mathbb{R}^2$ = set of all points on the plane
= set of all vectors on plane



$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is the arrow from origin → (a_1, a_2)

- if $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ we can identify the vector w/ (a_1, a_2)
  ↓
  $= \vec{a} = (a_1, a_2)$

  notations of vector

  $\langle\ \rangle$ → don't use

- $\vec{v} \in \mathbb{R}^2$ = "v is a vector on the plane"

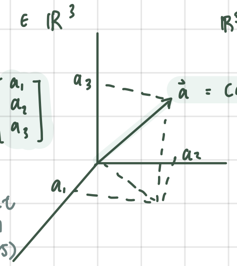
VECTORS IN $\mathbb{R}^3$

Vector $\vec{a} \in \mathbb{R}^3$

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

more strict (actual vectors)

$\vec{a} = (a_1, a_2, a_3)$ easier to write



WHAT IS A SCALAR?

a real number (as opposed to vectors)

$\mathbb{R}$ = set of real numbers

$c \in \mathbb{R}$

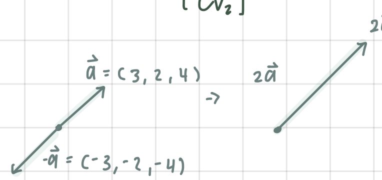
c is in set $\mathbb{R}$ = c is a $\mathbb{R}$

SCALAR MULTIPLICATION

STRETCHES / SHRINKS / REFLECTS

Given $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $c \in \mathbb{R}$

$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$



★ LINEAR COMBINATION

GIVEN $\vec{v} = (v_1, v_2)$ + $\vec{w} = (w_1, w_2)$ in $\mathbb{R}^2$ (similar in $\mathbb{R}^3$)

a <u>linear combination</u> of $\vec{v}$ + $\vec{w}$
=
a vector written as
$c\vec{v} + d\vec{w}$ w/ $c + d \in \mathbb{R}$

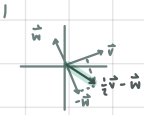
EX: $\vec{v} = (4,2)$, $\vec{w} = (-1,3)$

a) find lin. combo of $\frac{1}{2}\vec{v} - \vec{w}$ + draw out $\vec{v} + \vec{w}$

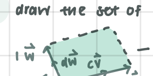
$c = \frac{1}{2}$ $d = -1$

$\frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$



b) draw the set of lin. combos $\{c\vec{v} + d\vec{w} \mid 0 \le c \le 1,\ 0 \le d \le 1\}$



move along whole arrow

— all points in this parallelogram

c) write $\vec{b} = (7,21)$ as a lin. combo of $\vec{v}$ + $\vec{w}$

$\begin{bmatrix} 7 \\ 21 \end{bmatrix} = c\vec{v} + d\vec{w}$
$= c\begin{bmatrix} 4 \\ 2 \end{bmatrix} + d\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

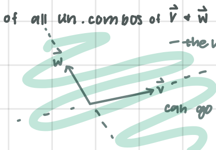
$\begin{cases} 4c - d = 7 \\ 2c + 3d = 21 \end{cases}$ $c = 3$, $d = 5$

$\vec{b} = 3\vec{v} + 5\vec{w}$

d) what does the set of all lin. combos of $\vec{v}$ + $\vec{w}$ look like?

$\{c\vec{v} + d\vec{w} \mid c, d \in \mathbb{R}\}$ $= \mathbb{R}^2$

because $\vec{v}$ + $\vec{w}$ are not ∥, any vector in $\mathbb{R}^2$ can be written as $c\vec{v} + d\vec{w}$ for some $c, d \in \mathbb{R}$



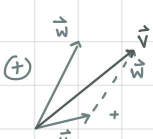
— the whole $\mathbb{R}^2$ plane

can go forever...

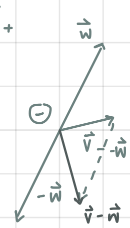
VECTOR ADDITION + SUBTRACTION

$\vec{v} = (v_1, v_2)$, $\vec{w} = (w_1, w_2)$ in $\mathbb{R}^2$

$\vec{v} \pm \vec{w} = (v_1 \pm w_1,\ v_2 \pm w_2)$



put $\vec{w}$ or $\vec{v}$ on top of each other



$\vec{v} = (1, 2, 3)$, $\vec{w} = (2, 0, 4)$, $\vec{x} = (-3, 2, 1)$, $\vec{y} = (-3, 2, 2)$

y just x but moved a bit.

Determine whether all linear combos of the given set of vectors in $\mathbb{R}^3$ is a line, plane, or all of $\mathbb{R}^3$

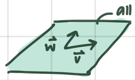
a) $\{\vec{v}, \vec{w}\}$

looking at all lin. combos of $\vec{v} + \vec{w}$

$= \{c\vec{v} + d\vec{w} \mid c, d \in \mathbb{R}\}$

→ we think it looks like a plane, but we need to check if they are parallel

$\vec{v} \neq c\vec{w}$ for $c \in \mathbb{R}$, so $\vec{v} + \vec{w}$ are not parallel



all lin. combos of $\vec{v} + \vec{w}$ = plane

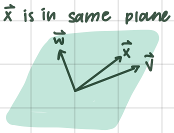
b) $\{\vec{v}, \vec{w}, \vec{x}\}$

all lin. combos of $\vec{v} + \vec{w} + \vec{x}$ =

$\{\underbrace{c\vec{v} + d\vec{w}}_{\text{just last question}} + \underbrace{e\vec{x}}_{\text{what does this add?}} \mid c, d, e \in \mathbb{R}\}$

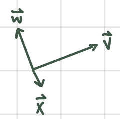
THERE ARE TWO SCENARIOS:

1) if $\vec{x}$ is in same plane of $\vec{v} + \vec{w}$



will be plane

2) if $\vec{x}$ is NOT in same plane



(it sticks out)

it will be $\mathbb{R}^3$

HOW DO WE CHECK?

$c\vec{v} + d\vec{w} = \vec{x}$

check if $\vec{v} + \vec{w}$ is a linear combo of $\vec{x}$

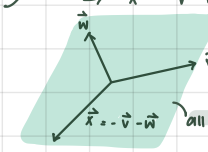
if true on plane, false off plane

$$c\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{cases} c + 2d = -3 \rightarrow -1 + 2d = -3 \rightarrow d = -1 \checkmark \\ -2c = 2 \rightarrow c = -1 \checkmark \\ 3c - 4d = 1 \rightarrow -3 - 4d = -1 \checkmark \end{cases}$$

$(c, d) = (-1, -1)$

$\Rightarrow \vec{x} = -\vec{v} - \vec{w}$



adding x doesn't add anything

all linear combos of $\{\vec{v}, \vec{w}, \vec{x}\}$ = plane

c) $\{\vec{v}, \vec{w}, \vec{y}\}$: a $\mathbb{R}^3$ example

↑ will be off the plane

is $\vec{y}$ on the plane defined by $\vec{v} + \vec{w}$?

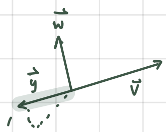
$$c\vec{v} + d\vec{w} = \vec{y} = \begin{cases} c + 2d = -3 \rightarrow d = -1 \\ -2c = 2 \rightarrow c = -1 \\ 3c - 4d = 2 \rightarrow d = -\frac{5}{4} \end{cases}$$

X INCONSISTENT

17:18

NO $c, d \in \mathbb{R}$ such that $\vec{y} = c\vec{v} + d\vec{w}$

$\vec{y}$ is not on the plane defined by $\vec{v} + \vec{w}$



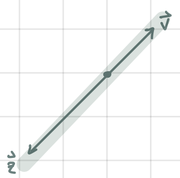
→ all linear combos of $\{\vec{v}, \vec{w}, \vec{y}\}$ = all of $\mathbb{R}^3$

d) 4th scenario : $\vec{z}$

$\vec{z} = -3\vec{v}$

$c\vec{v} + d\vec{v} = (c - 3d)\vec{v}$ } so the lin. combo is actually a line.

like this



1.2 LENGTHS + DOT PRODUCT

WK1

LENGTHS OF VEC. + UNIT VECTORS

DOT PRODUCTS : ∡ BETWEEN VECTORS

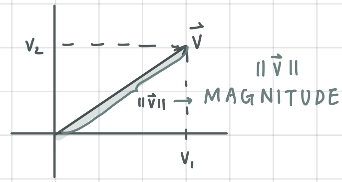
+

IMPORTANT INEQUALITIES

LENGTHS + UNIT VECTORS

Given $\vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, its length (or norm) is defined as:

$$\vec{v} = \begin{cases} \sqrt{v_1 + v_2} & \text{if } \vec{v} \in \mathbb{R}^2 \\ \sqrt{v_1 + v_2 + v_3} & \text{if } \vec{v} \in \mathbb{R}^3 \end{cases}$$



A VECTOR $\vec{U}$ IS A UNIT VECTOR IF ITS LENGTH IS ONE : $\|\vec{U}\| = 1$

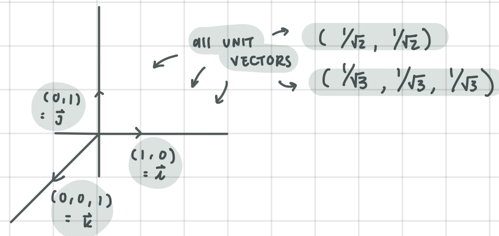
ex find a unit vector in the same direction as $\vec{v} = (1, -3, 2)$

① divide $\vec{v}$ by its length

$\vec{v} = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$

② get UNIT VECTOR

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{14}}(1, -3, 2)$



$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

DOT PRODUCT

Given two vectors $\vec{v}$ + $\vec{w}$, their dot product is defined as

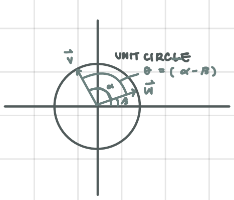
$$\vec{v} \bullet \vec{w} = \begin{cases} v_1 w_1 + v_2 w_2 & \text{if } \vec{v}, \vec{w} \in \mathbb{R}^2 \\ v_1 w_1 + v_2 w_2 + v_3 w_3 & \text{if } \vec{v}, \vec{w} \in \mathbb{R}^3 \end{cases}$$

particularly, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

we use dot product to find the ∡ between 2 vectors, let's prove this

CLAIM : we can calculate the ∡ between $\vec{v}$ + $\vec{w}$ using $\vec{v} \cdot \vec{w}$

EX: let $\vec{v}$ + $\vec{w}$ be UNIT VECTORS IN $\mathbb{R}^2$, show that $\vec{v} \cdot \vec{w} = \cos\theta$ where θ is ∡ is between $\vec{v}$ + $\vec{w}$ + $0 \le \theta \le \pi$



$\vec{v} = (\cos\alpha, \sin\alpha)$

$\vec{w} = (\cos\beta, \sin\beta)$

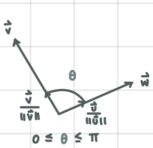
FORMULA IF THEY ARE UNIT VECTORS

$\vec{v} \cdot \vec{w}$ = $\cos\alpha \cos\beta + \sin\alpha \sin\beta$

= $\cos(\alpha - \beta)$

= $\cos\theta$

EX: WHAT IF $\vec{v}$ + $\vec{w}$ are not unit vectors ?



we see that

$\frac{\vec{v}}{\|\vec{v}\|} \bullet \frac{\vec{w}}{\|\vec{w}\|} = \cos\theta$ bc of last ex

$\rightarrow \vec{v} \bullet \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

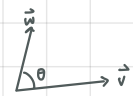
$\nearrow$ 0 vector

ASSUMING $\vec{v}, \vec{w} \neq \vec{0}$ $\qquad \|\vec{v}\|, \|\vec{w}\| > 0$

$$\vec{v} \cdot \vec{w} = \underbrace{\|\vec{v}\| \, \|\vec{w}\|}_{>0} \cos\theta \quad \begin{cases} >0 \Rightarrow \cos\theta > 0 & \text{means } 0 \le \theta < \frac{\pi}{2} \\ =0 \Rightarrow \cos\theta = 0 & \text{means } \theta = \frac{\pi}{2} \\ <0 \Rightarrow \cos\theta < 0 & \text{means } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

can figure out type of angle with just the lengths

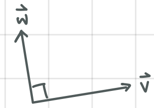
MORE EXAMPLES :



$\vec{v} \cdot \vec{w} > 0$

$0 \le \theta < \frac{\pi}{2}$

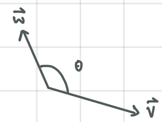
acute



$\vec{v} \cdot \vec{w} = 0$

$\theta = \pi/2$

right



$\vec{v} \cdot \vec{w} < 0$

$\frac{\pi}{2} < \theta \le \pi$

obtuse

VECTORS IN $\mathbb{R}^3$ EXAMPLE

$\vec{u} = (2, 2, -1)$, $\vec{v} = (2, -1, 2)$, $\vec{w} = (1, 1, 1)$

a) $\measuredangle$ between $\vec{u}$ + $\vec{v}$?

we need to solve for θ

1) calculate the dot product

$\vec{u} \cdot \vec{v} = 2 \cdot 2 + -1 \cdot 2 + 2 \cdot -1 = 4 - 2 - 2 = 0$

0 means $\theta = \frac{\pi}{2}$

b) $\measuredangle$ between $\vec{v}$ + $\vec{w}$?

1) calculate dot product

$\vec{v} \cdot \vec{w} = 2 - 1 + 2 = 3$, positive $= 0 \le \theta < \frac{\pi}{2}$

2) calculate lengths

$\|\vec{v}\| = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

$\|\vec{u}\| = \sqrt{1 + 1 + 1} = \sqrt{3}$

3) plug into formula $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{u}\| \cdot \cos\theta$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{u}\|} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

4) solve for θ

$\theta = \cos\left(\frac{1}{\sqrt{3}}\right)^{-1}$

SCHWARZ INEQUALITY

recall $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta \Leftrightarrow \cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}$, what's the minimum + maximum of $\cos\theta$?

we know $-1 \le \cos\theta \le 1$

so . . .

$$-1 \le \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} \le 1$$

$\Updownarrow$

$$\|\vec{v}\| \, \|\vec{w}\| \le \vec{v} \cdot \vec{w} \le \|\vec{v}\| \, \|\vec{w}\|$$

$\Updownarrow$

$$|\vec{v} \cdot \vec{w}| \le \|\vec{v}\| \, \|\vec{w}\|$$

THIS APPLIES EX :

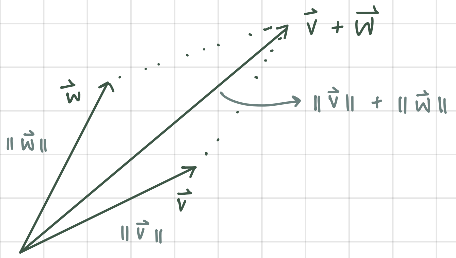
in Physics $\rightarrow$ $\vec{F}$ = force, $\vec{x}$ = displacement, $\vec{F} \cdot \vec{x}$ = work

if $\|\vec{F}\| = 4$, $\|\vec{x}\| = 3$

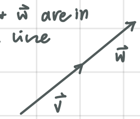
$\underline{\text{work}} = \underline{|\vec{F} \cdot \vec{x}|} \le \|\vec{F}\| \, \|\vec{x}\| = 12$

work in terms of the dot product is always less than the lengths multiplied together

TRIANGLE INEQUALITY



$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$ would only ever equal if $\vec{v}$ + $\vec{w}$ are in a line !!



EXAMPLE: $\|\vec{v}\| = 2$, $\|\vec{w}\| = 5$

$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$

$\le 2 + 5$

≤ 7