2.4 RULES FOR MATRIX OPERATIONS

MATRIX ALGEBRA

PROPERTIES OF MATRIX ALGEBRA

Notation : $(A)_{ij}$ = (i,j) is the entry at the (ith row, jth column) of A

SCALAR MULTIPLICATION

for $c \in \mathbb{R}$, $m \times n$ matrix A

cA = new matrix w/ entries $(cA)_{ij} = c\,(A)_{ij}$

EXAMPLE :

$$A = \begin{bmatrix} 3 & 1/2 \\ 7 & -8 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 6 & 1 \\ 14 & -16 \end{bmatrix}$$

ADDITION + SUBTRACTION

If A and B are matrices of the same size (same # of rows and columns) then $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$

EXAMPLE :

$$A = \begin{bmatrix} 8 & 7 \\ -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 \\ 9 & 2 \end{bmatrix}$$

$$\to A + B = \begin{bmatrix} 4 & 10 \\ 4 & 8 \end{bmatrix}$$

$$\to A - B = \begin{bmatrix} 12 & 4 \\ -14 & 4 \end{bmatrix}$$

BUT IF . . .

$$C = \begin{bmatrix} 1 & -2 \\ 0 & 5 \\ 3 & 7 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & -1 & 3 \\ 7 & 0 & 4 \end{bmatrix}$$ then $C \pm D$ are undefined

ALSO !

zero matrix 0 whose entries are all zero come in any size and $A + 0 = 0 + A = A$ for any matrix A and the zero matrix of the same size.

MATRIX MULTIPLICATION

$(AB)_{ij} \neq (A)_{ij}(B)_{ij}$

AB is defined if the # of columns of A = # of rows of B

A is $m \times r$ and B is $r \times n$

AB is then the $m \times n$ matrix defined as:

i) writing $B = [\vec{b}_1 \dots \vec{b}_n]$ w/ $\vec{b}_i \in \mathbb{R}^r$ $i = 1, \dots, n$,

$$AB = [A\vec{b}_1 \dots A\vec{b}_n]$$

NOTE! $A\vec{b}_i \in \mathbb{R}^m$ so AB is $m \times n$

ii) writing

$A = m \text{ rows} \left\{ \underbrace{\begin{bmatrix} a_{i1} \dots a_{ir} \end{bmatrix}}_{r \text{ columns}} \right.$ $\quad B = r \text{ rows} \left\{ \underbrace{\begin{bmatrix} b_{1j} \\ \vdots \\ b_{rj} \end{bmatrix}}_{n \text{ columns}} \right.$

$(AB)_{ij}$ = (ith row of A) (jth column of B) $= a_{i1}b_{1j} + \dots + a_{ir}b_{rj}$

EXAMPLE 1.

$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 4 \\ -1 & 2 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

$\underset{m \quad r}{2 \times 3}$ $\quad \underset{r \quad n}{3 \times 3}$

matches

AB will be 2×3

$(AB)_{11} = (1,4,2) \cdot (3,-1,1) = 3 - 4 + 2 = 1$

$(AB)_{12} = (1,4,2) \cdot (0,2,1) = 0 + 8 + 2 = 10$

$(AB)_{13} = (1,4,2) \cdot (4,0,-2) = 4 + 0 - 4 = 0$

$(AB)_{21} = (3,1,5) \cdot (3,-1,1) = 9 - 1 + 5 = 13$

$(AB)_{22} = (3,1,5) \cdot (0,2,1) = 0 + 2 + 5 = 7$

$(AB)_{23} = (3,1,5) \cdot (4,0,-2) = 12 + 0 - 10 = 2$

$\rightarrow AB = \begin{bmatrix} 1 & 10 & 0 \\ 13 & 7 & 2 \end{bmatrix}$

* BA is NOT defined because

$\underline{3 \times 3}$ $\underline{2 \times 3}$ don't match

EXAMPLE 2:

$A = [2 \quad 3 \quad -1]$, $B = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$ FIND AB and BA if defined

$AB = \underset{1 \times 3}{[2 \quad 3 \quad -1]} \underset{3 \times 1}{\begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}} = 2 - 12 - 5 = \underset{1 \times 1}{-15}$

$BA = \underset{3 \times 1}{\begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}} \underset{1 \times 3}{[2 \quad 3 \quad -1]} = \underset{3 \times 3}{\begin{bmatrix} 2 & 3 & -1 \\ -8 & -12 & 4 \\ 10 & 15 & -5 \end{bmatrix}}$

IDENTITY MATRIX

identity matrix I or I_n = $n \times n$ matrix whose diagonal entries are all 1 and other entries are all 0

$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc...

for any matrix A:

$A I_n = I_m A = A$

MATRIX POWERS

If A is an $n \times n$ (square) matrix, then $A_k = \underbrace{A \dots A}_{k \text{ copies}}$

EXAMPLE:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find A^k for any k $\rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

if k is even, $A^k = \underbrace{A^2}_{I} \cdots \underbrace{A^2}_{I} = I$

if k is odd, $A^k = \underbrace{A^2}_{I} \cdots \underbrace{A^2}_{I} \cdot A = A$

LAWS FOR MATRIX ALGEBRA

For any matrices A, B, C of the same size:

$A + B = B + A$

$d(A+B) \quad \forall d \in \mathbb{R} = dA + dB$

↑
scalar

$(A+B) + C = A + (B+C)$

Assuming all products make sense,

$(A+B)C = AC + BC$ $\quad A(B+C) = AB + AC$

$ABC = (AB)C = A(BC)$

FUNNY THINGS W/ THE LAWS

T F : $AB = BA$ for <u>any</u> $n \times n$ matrices A & B

$$\begin{array}{c} A \\ \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \end{array} \begin{array}{c} B \\ \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix} \end{array} \rightarrow AB = \begin{bmatrix} 3 & \\ & \end{bmatrix} \quad BA = \begin{bmatrix} 38 & \\ & \end{bmatrix}$$

→ already see $3 \neq 38$

if AB is defined, doesn't mean BA is defined

T F · If $AB = AC$ and $A \neq 0$ then $B = C$

false bc → even if $A \neq 0$, $AB = AC$ but $B \neq C$

We know in scalar algebra it is true!

let's say

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -7 \\ 3 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -7 \\ 8 & 5 \end{bmatrix}$$

$B \neq C$ but $AB = AC$

$$AB = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 4 & -7 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 8 & -14 \\ 12 & -21 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 4 & -7 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -14 \\ 12 & -21 \end{bmatrix}$$

T F : If $AB = 0$ then $A = 0$ or $B = 0$

false bc → $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 10 \\ 3 & 5 \end{bmatrix}$ ⇒ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

so $AB = 0$ but $A \neq 0$ and $B \neq 0$