STATS 3U03

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Course Outline

- Textbook: Inroduction to stochastic processes
- Requirement: 5 assignments, 2 tests, and 1 final
- Test 1: Friday, February 10th
- Test 2: Friday, March 17th

Contents

1 Introduction

1.1 Review

Definition 1.1 (Independent random variables)**.** *X and Y are independent iff for any $a, b \in \mathbb{R}$, $P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$*

1.2 Stochastic processes

Definition 1.2 (Stochastic process)**.** *Let T be a subset of $[0, +\infty]$. For each $t \in T$, let X_t be a random variable. Then, the collection of $\{X_t : t \in T\}$ is called a stochastic process. Simply put, a stochastic process is just a family of random variables.*

Example 1.2.1. Let $T = \{0\}$. Then, $\{X_0\}$ is a stochastic process.

Example 1.2.2. Let $T = \{1, 2, 3, \ldots, m\}$ be a set of finite natural numbers. Then, $\{X_1, X_2, X_3, \ldots, m\}$ is a stochastic process.

Example 1.2.3. Let $T = \{0, 1, 2, \ldots\}$ be a set of all non-negative integers. Then, $\{X_1, X_2, X_3, \ldots\}$ is a stochastic process.

Example 1.2.4. Let $T = [0, +\infty)$ be a set of all non-negative real numbers. Then, $\{X_t : t \geq 0\}$ is a stochastic process.

Definition 1.3 (Time index)**.** *Let T be time index. If $T = \{0, 1, 2, \ldots\}$, then the time is discrete. If $T = [0, \infty)$, then time is continuous.*

Definition 1.4 (State Space)**.** *State space, S, is the space space where the random variable takes the values.*

Given a sample space, S, and time index $t \in T$, we can define $X_t(w) \in S$, to describe a stochastic process. Here, $\{X_t : t \in T\}$ describes the dependence relation.

We can further categorize a stochastic process by considering the following two cases: countable and uncountable state space. Time index can also be categorized as follows: discrete and continuous time. Note that each stochastic process must belong to one of the four categories.

Remark. Every stochastic process can be described by the following three factors:

1. Time index
2. State space
3. Dependence relation

Example 1.2.5. Let $S = \{0, 1\}$ and $T = \{0, 1, 2, \ldots\}$. Given,

$$X_n = \begin{cases} 1 & \text{with probability of } 1/2 \\ 0 & \text{with probability of } 1/2 \end{cases}$$

$\{X_0, X_1, X_2, \ldots\}$ is a stochastic process and is often noted as Bernoulli trials.

2 Markov chains (Discrete time Markov chains)

We will only be dealing with discrete time Markov chains in chapter 1 and 2. In other words, $T = \{0, 1, 2, \dots\}$. It follows that the state space, S, will be at most countable. Finally, Markov describes the dependence relation: $X_0, X_1, X_2, \dots$.

In example 1.2.5, every trial of the Bernoulli trials was independent. On the other hand, in a Markov model, X_{n+1} depends on X_n but not on any past stats, $X_1, X_2, \dots, X_{n-1}$.

2.1 Markov property

Definition 2.1. Markov property *can be expressed as follows:*

$$
\begin{aligned}
&P\left(X_{n+1} = x_{n+1} | X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = x_n\right) \\
&= P\left(X_{n+1} = x_{n+1} | X_n = x_n\right)
\end{aligned}
$$

$P\left(X_{n+1} = y | X_n = x\right)$ is noted as the transition probability and it describes the one step transition from x to y starting at time n. If

$$P\left(X_{n+1} = y | X_n = x\right) = P\left(X_1 = y | X_0 = x\right),$$

then the Markov chain is called to have stationary transition, or homogeneous.

Definition 2.2. *Let $\{X_n : n = 0, 1, 2, \dots\}$ be a homogeneous Markov chain. Then,*

$$P_{xy} = P\left(X_1 = y | X_0 = x\right) = P\left(X_{n+1} = y | X_n = x\right),$$

is the one-step transition probability.

Definition 2.3. *Following the definition 2.2, we can now define* one-step transition matrix*:*

$$\mathbb{P} = \left(P_{xy}\right)_{x,y \in S}$$

Remark. Given, X_0, $\pi_0(x) = P(X_0 = x)$ is called the *initial distribution.*

Given a Markov chain, we wish to answer the following fundamental questions:

1. Distribution of X_n for any $n \geq 1$.
2. Join distribution of $X_{n_1}, \dots, X_{x_k}$ for any $1 \leq n_1 < n_2 < \cdots < n_k \quad k, \geq 2$.
3. Long time behaviour, i.e.

$$\lim_{n \to \infty} P(X_n = x)$$

Example 2.1.1. We have the following Markov chain: $\{X_n : n = 0, 1, 2, \dots\}$ where $S = \{0, 1\}$. For this model, its initial distribution can be described as follows:

$$\begin{cases} \pi_0(0) = P(X_0 = 0) = a \\ \pi_0(1) = 1 - a \end{cases}$$

Transition probabilities can be written in a similar fashion:

$$\begin{array}{ll} P(X_1 = 1|X_0 = 0) = p, & P(X_1 = 0|X_0 = 0) = 1 - p \\ P(X_1 = 0|X_0 = 1) = q, & P(X_1 = 1|X_0 = 1) = 1 - q \end{array}$$

where $0 \leq p, q \leq 1$. For this Markov chain, we can consider the following three cases:

Case 1. $p = q = 0$**.**

This case is trivial.

Case 2. $p = q = 1$**.**

This case is also trivial.

Case 3. $0 \leq p + q \leq 2$**.**

$$\begin{aligned} P(X_{n+1} = 0) &= P(X_{n+1} = 0 \cap X_n = 0) + P(X_{n+1} = 0 \cap X_n = 1) \\ &= P(X_n = 0)P(X_{n+1} = 0|X_n = 0) + P(X_n = 1)P(X_{n+1} = 0|X_n = 1) \\ &= P(X_n = 0)(1 - p) + P(X_n = 1)q \\ &= P(X_n = 0)(1 - p) + (1 - P(X_n = 0))q \\ &= (1 - p - q)P(X_n = 0) + q \end{aligned}$$

We can further expand this as follows:

$$\begin{aligned} P(X_{n+1} = 0) &= (1 - p - q)P(X_n = 0) + q \\ &= (1 - p - q)\big[(1 - p - q)P(X_{n-1} = 0) + q\big] + q \\ &= (1 - p - q)^n P(X_0 = 0) + q \sum_{j=0}^{n-1} (1 - p - q)^j \end{aligned}$$

Note that

$$\sum_{j=0}^{n-1} (1 - p - q)^j \quad = \frac{(1 - p - q)^n - 1}{(1 - p - q) - 1}$$

Therefore, we have

$$\begin{aligned} P(X_{n+1} = 0) &= (1-p-q)^n a + q\frac{(1-p-q)^n - 1}{(1-p-q) - 1} \\ &= (1-p-q)^n a - \frac{q}{p+q}\big((1-p-q)^n - 1\big) \end{aligned}$$

For this Markov chain, we find that

$$\lim_{n\to\infty} P(X_n = 0) = \frac{q}{p+q}$$