

# MATH 2XX3 - Advanced Calculus II

Sang Woo Park

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# 1 Introduction

In this course, we are going to study calculus using the concepts from linear algebra.

## 1.1 Vector norm

**Definition 1.1.** Euclidean norm of  $\vec{x} = (x_1, x_2, \dots, x_n)$  is given as

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{j=1}^n x_j^2}$$

**Theorem 1.1** (Properties of a norm).

1.  $\|\vec{x}\| \geq 0$  and  $\|\vec{x}\| = 0$  iff  $\vec{x} = \vec{0} = (0, 0, \dots, 0)$ .
2. For all scalars  $a \in \mathbb{R}$ ,  $\|a\vec{x}\| = |a| \cdot \|\vec{x}\|$ .
3. (Triangle inequality)  $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ .

We say that this is a property of a norm because there are other norms, which measure distance in  $\mathbb{R}^n$  in different ways!

**Example 1.1.1** (A non-pythagorean norm - The Taxi Cab Norm). Consider the following vector  $\vec{p} = (p_1, p_2) \in \mathbb{R}^2$ . The euclidean norm gives the length of the diagonal line. On the other hand,

$$\|\vec{p}\|_1 = |p_1| + |p_2|$$

gives us the total distance in a rectangular grid system.

For  $\vec{p} = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ ,  $\|\vec{p}\|_1 = \sum_{j=1}^n |p_j|$ . Note that the Taxi Cab norm is a valid norm because it satisfies all properties of a norm above. So it also gives us a valid alternative way to measure distance in  $\mathbb{R}^n$ ,  $\text{dist}(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\|$ . This way of measuring distance gives  $\mathbb{R}^n$  a different geometry.

**Definition 1.2.** Neighborhood of a point  $\vec{p}$ , or disks centered at  $\vec{p}$  is defined as

$$D_r(\vec{p}) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{p}\| < r\}$$

*Remark.* The neighborhood around  $\vec{a}$  of radius  $r$  may be written using any of the following notations:

$$D_r(\vec{a}) = B_r(\vec{a}) = B(\vec{a}, r)$$

**Definition 1.3.** Sphere is defined as

$$S_r(\vec{p}) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{p}\| = r\}$$

What neighborhood and sphere look like depends on which norm you choose. First, let's start with the familiar euclidean norm. Then, the sphere is given by

$$\begin{aligned} \|\vec{x} - \vec{p}\| &= r \\ \iff \sqrt{\sum_{j=1}^n (x_j - p_j)^2} &= r \end{aligned}$$

Then, we have

$$\sum_{j=1}^n (x_j - p_j)^2 = r^2$$

If  $n = 3$ , we have  $(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = r^2$ , usual sphere in  $\mathbb{R}^3$  with center  $\vec{p} = (p_1, p_2, p_3)$

If  $n = 2$ , we have  $(x_1 - p_1)^2 + (x_2 - p_2)^2 = r^2$ , usual circle in  $\mathbb{R}^2$  with center  $\vec{p} = (p_1, p_2)$ .

If we replace Euclidean norm by the Taxi Cab norm (for simplicity, take  $\vec{p} = \vec{0}$ ), we have

$$S_r^{\text{taxi}}(\vec{0}) = \left\{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{0}\|_1 = r \right\}$$

In other words, we have

$$\vec{x} \in S_r^{\text{taxi}}(\vec{0}) \iff \sum_{j=1}^n |x_j| = r$$

Looking at it in  $\mathbb{R}^2$ , we have  $\vec{x} = (x_1, x_2)$ . Then,  $r = |x_1| + |x_2|$ . This, in fact, is a diamond.

*Remark.* Note that  $|x_1| + |x_2| = r$  is a *circle* in  $\mathbb{R}^2$  under the Taxi Cab norm. Then, we have

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{8r}{2r} = 4$$

## 1.2 Subset

Let's introduce some properties of *subsets* in  $\mathbb{R}^n$ .  $A \subset \mathbb{R}^n$  means  $A$  is a *collection* of points  $\vec{x}$ , drawn from  $\mathbb{R}^n$ .

**Definition 1.4.** Let  $A \subset \mathbb{R}^n$ , and  $\vec{p} \in A$ . We say  $\vec{p}$  is an *interior point* of  $A$  if there exists a neighbourhood of  $\vec{p}$ , i.e. an open disk, which is entirely contained in  $A$ :

$$D_r(\vec{p}) \subset A.$$

**Example 1.2.1.**

$$A = \left\{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \neq \vec{0} \right\}$$

Take any  $\vec{p} \in A$ , so  $\vec{p} \neq \vec{0}$ . Then, let  $r = \|\vec{p} - \vec{0}\| > 0$ , and  $D_r(\vec{p}) \subset A$ , since  $\vec{0} \notin D_r(\vec{p})$ . (Notice: any smaller disk,  $D_s(\vec{p}) \subset D_r(\vec{p}) \subset A$ , where  $0 < s < r$  works to show that  $\vec{p}$  is an interior point).

So every  $\vec{p} \in A$  is an interior point to  $A$ .

**Definition 1.5.** If every  $\vec{p} \in A$  is an interior point, we call  $A$  an open set.

**Example 1.2.2.**  $A = \{\vec{x} \in \mathbb{R}^n | \vec{x} \neq \vec{0}\}$  is an open set.

**Example 1.2.3.**  $A = D_R(\vec{0})$  is an open set.

*Proof.* If  $\vec{p} = \vec{0}$ ,  $D_r(\vec{0}) \subseteq A = D_R(\vec{0})$  provided  $r \leq R$ . So  $\vec{p} = \vec{0}$  is interior to  $A$ . Consider any other  $\vec{p} \in A$ . It's evident that  $D_r(\vec{p}) \subset A = D_R(\vec{0})$  provided that  $0 \leq r \leq R - \|\vec{p}\|$ . Therefore,  $A = D_R(\vec{0})$  is an open set.  $\square$

**Example 1.2.4.** Suppose we use Taxi Cab disks instead of Euclidean disk. It does not change which points are interior to  $A$  since the diamond is inscribed in a circle. In other words,

$$D_r^{\text{taxi}}(\vec{p}) \subset D_r^{\text{Euclid}}(\vec{p})$$

**Definition 1.6.** The complement of set  $A$  is

$$A^c = \{\vec{x} | \vec{x} \notin A\}$$

**Definition 1.7.**  $\vec{b}$  is a boundary point of  $A$  if for every  $r > 0$ ,  $D_r(\vec{b})$  contains both points in  $A$  and points not in  $A$ :

$$D_r(\vec{b}) \cap A \neq \emptyset \text{ and } D_r(\vec{b}) \cap A^c \neq \emptyset$$

In the example 1.2.3, the set of all boundary points of  $A = D_R(\vec{0})$

$$\{\vec{b} | \|\vec{b}\| = R\}$$

is a sphere of radius  $R$ .

**Definition 1.8.** A set  $A$  is closed if  $A^c$  is open.

**Theorem 1.2.**  $A$  is closed if and only if  $A$  contains all its boundary points.

**Example 1.2.5.** Consider the following set:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 > 0\}$$

If  $\vec{p}_1 = (p_1, p_2)$ , where  $p_1 > 0, p_2 > 0$ , then  $\vec{p}_1$  is an interior point. Take  $r = \min\{p_1, p_2\}$ . Then,  $D_r(\vec{p}) \subset A$ . On the other hand, any  $\vec{p}$  that lies on either axes (including  $\vec{0}$ ) is a boundary point. Since there are boundary points in  $A$ ,  $A$  can't be open.

## 2 Functions

### 2.1 Limits and continuity

In this section, we will be considering vector values functions such that

$$F : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^k.$$

We can rewrite this using a matrix notation:

$$F(x_1, x_2, \dots, x_n) = \begin{bmatrix} F_1(x_1, x_2, \dots, x_n) \\ F_2(x_1, x_2, \dots, x_n) \\ \vdots \\ F_k(x_1, x_2, \dots, x_n) \end{bmatrix}$$

**Example 2.1.1.** For a  $(k \times n)$  matrix  $M$ ,

$$F(\vec{x}) = M\vec{x}$$

First, we wish to study limits. What does  $\lim_{\vec{x} \rightarrow \vec{a}} F(\vec{x}) = \vec{L}$  mean? Note that it's not enough to treat the variables  $x_1, x_2, \dots, x_n$  separately.

**Example 2.1.2.** Consider the following function:

$$F(x, y) = \frac{xy}{x^2 + 4y^2},$$

where  $(x, y) \neq (0, 0)$ . First, we can attempt to find its limit by considering them separately.

$$\lim_{x \rightarrow 0} \left( \lim_{y \rightarrow 0} F(x, y) \right) = \lim_{x \rightarrow 0} \left( \frac{0}{x^2} \right) = \lim_{x \rightarrow 0} 0 = 0$$

Similarly, we have

$$\lim_{y \rightarrow 0} \left( \lim_{x \rightarrow 0} F(x, y) \right) = 0$$

However, if  $(x, y) \rightarrow (0, 0)$  along a straight line path with  $y = mx$ , where  $m$  is constant, we have

$$F(x, mx) = \frac{mx^2}{x^2 + 4m^2x^2} = \frac{m}{1 + 4m^2}$$

In this case, we have

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=mx}} F(x, y) = \frac{m}{1 + 4m^2}$$

Therefore, the values of  $F(x, y)$  don't approach any particular value as  $(x, y) \rightarrow (0, 0)$ .

**Example 2.1.3** (Worse). Consider the following function:

$$F(x, y) = \frac{y^2}{x^4 + y^2}.$$

If we approach  $(0, 0)$  along  $y = mx$ , limit equals 1. However, if we approach along a parabola,  $y = mx^2$ , limit equals  $m^2/(1 + m^2)$ . We get different limits along different parabolas.

We showed that computing

$$\lim_{\vec{x} \rightarrow \vec{a}} = \vec{b}$$

is tricky because  $\vec{x} \rightarrow \vec{a}$  has to be more precise. It can't depend on the path or direction on which  $\vec{x}$  approaches  $\vec{a}$ , but only on *proximity*. In other words, we want  $\|F(\vec{x}) - \vec{b}\|$  to go to zero as  $\|\vec{x} - \vec{a}\|$  goes to zero.

**Definition 2.1.** We say  $\lim_{\vec{x} \rightarrow \vec{a}} F(\vec{x}) = \vec{b}$  if for any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that whenever  $0 < \|\vec{x} - \vec{a}\| < \delta$ , we have  $\|F(\vec{x}) - \vec{b}\| < \varepsilon$ . Therefore,

$$\lim_{\vec{x} \rightarrow \vec{a}} F(\vec{x}) = \vec{b} \iff \lim_{\vec{x} \rightarrow \vec{a}} \|F(\vec{x}) - \vec{b}\| = 0$$

*Remark.* Geometrically, for any given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$F(\vec{x}) \in D_\varepsilon(\vec{b}),$$

where  $\vec{x} \in D_\delta(\vec{a})$ .

Before doing examples, here's a useful observations. Take  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ . Then, we have

$$\|\vec{v}\| = \sqrt{\sum_{j=1}^n v_j^2} \geq \sqrt{v_i^2} = |v_i|$$

for each coordinate  $i = 1, 2, \dots, n$ .

**Example 2.1.4.** Show

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^2 + y^2} = 0$$

*Proof.* Note that  $F : \mathbb{R} \setminus \{\vec{0}\} \rightarrow \mathbb{R}$ ,  $b = 0$ ,  $\vec{a} = (0, 0)$ . Call

$$R = \|\vec{x} - \vec{a}\| = \|\vec{x}\| = \sqrt{x^2 + y^2}$$

Since  $F(\vec{x}) \in \mathbb{R}$ , we have

$$\begin{aligned}
\|F(\vec{x}) - \vec{b}\| &= |F(\vec{x}) - b| \\
&= \left| \frac{2x^2y}{x^2 + y^2} - 0 \right| \\
&= \frac{2|x|^2|y|}{x^2 + y^2} \\
&\leq \frac{2 \cdot R^2 \cdot R}{R^2} \\
&= 2R \\
&= 2\|\vec{x} - \vec{a}\|
\end{aligned}$$

By letting  $\|\vec{x} - \vec{a}\| = \|\vec{x}\| < \varepsilon/2$ , we get  $\|F(\vec{x}) - \vec{b}\| < \varepsilon$ . Therefore, definition is satisfied with  $\delta \leq \varepsilon/2$   $\square$

**Example 2.1.5.** Consider the following function,  $F : \mathbb{R}^3 \setminus \{\vec{0}\} \rightarrow \mathbb{R}$ :

$$\frac{3z^2 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2}.$$

Determine whether

$$\lim_{(x,y,z) \rightarrow (0,0,0)} F(x,y,z) = 2.$$

*Proof.* We have

$$\begin{aligned}
\|F(x,y,z) - \vec{b}\| &= |F(x,y,z) - 2| \\
&= \left| \frac{3z^3 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2} - 2 \right| \\
&= \frac{3|z|^3}{x^2 + 2y^2 + 3z^2} \\
&\leq \frac{3R^3}{x^2 + y^2 + z^3} \\
&= \frac{3R^3}{R^2} \\
&= 3R
\end{aligned}$$

Then,

$$\|F(x,y,z) - \vec{b}\| < 3R < \varepsilon$$

provided that

$$R = \|\vec{x} - \vec{0}\| < \delta = \frac{\varepsilon}{3}$$

$\square$

**Definition 2.2.** Consider a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  with domain  $A \subseteq \mathbb{R}^n$ . For  $\vec{a} \in A$ , we say  $F$  is continuous at  $\vec{a}$  in the domain of  $F$  iff

$$F(a) = \lim_{\vec{x} \rightarrow \vec{a}} F(\vec{a})$$

**Example 2.1.6.** Going back the example 2.1.5, if we redefine  $F$  as follows,

$$F = \begin{cases} \frac{3z^2+2x^2+4y^2+6z^2}{x^2+2y^2+3z^2} & (x, y, z) \neq (0, 0, 0) \\ 2 & (x, y, z) = (0, 0, 0) \end{cases}$$

then  $F$  is continuous at  $(0, 0, 0)$  (and in fact at all  $\vec{x} \in \mathbb{R}$ ).

If  $F$  is continuous at every  $\vec{a} \in A$ , ( $\forall \vec{x} \in A$ ), we say  $F$  is continuous on the set  $A$ . Continuity is always preserved by the usual algebraic operations: sum, product, quotient, and composition of continuous functions is continuous<sup>1</sup>.

## 2.2 Differentiability

**Definition 2.3.** For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , its derivative is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

If it exists, we say  $f$  is differentiable at  $x$ .

**Theorem 2.1.** If  $f$  is differentiable at  $x$ ,  $f(x)$  is also continuous at  $x$ .

Note that differentiable functions,  $f(x)$ , are well approximated by their tangent lines (also known as linearization). We wish to extend this idea to  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

First, we try dealing with the independent variables,  $x_1, x_2, \dots, x_n$ , one at a time by using partial derivatives. We start by introducing the standard basis in  $\mathbb{R}^n$ :

$$\begin{aligned} \vec{e}_1 &= (1, 0, 0, \dots, 0) \\ \vec{e}_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ \vec{e}_n &= (0, 0, 0, \dots, 1) \end{aligned}$$

(In  $\mathbb{R}^3$ ,  $\vec{e}_1 = \vec{i}, \vec{e}_2 = \vec{j}, \vec{e}_3 = \vec{k}$ ).

For any  $\vec{x} \in \mathbb{R}^n$ , and  $h \in \mathbb{R}$ ,  $(\vec{x} + h\vec{e}_j)$  moves from  $\vec{x}$  parallel to the  $x_j$  axis by distance  $h$ . In other words,

$$\vec{x} + h\vec{e}_j = (x_1, x_2, \dots, x_j + h, x_{j+1}, \dots, x_n).$$

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<sup>1</sup>Provided we remain in the domain of continuity of both functions and denominators aren't zero



**Definition 2.4.** *Partial derivatives of  $f(x)$  is defined as*

$$\frac{\partial f}{\partial x_j}(\vec{x}) = \lim_{h \rightarrow 0} \frac{f(\vec{x} + h\vec{e}_j) - f(\vec{x})}{h},$$

for all  $j = 1, 2, \dots, n$ .

Partial derivatives calculate the derivatives of  $f$ , treating of  $\vec{x}_j$  as the only variable, and all others treated as constants. For a vector valued function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ :

$$F(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{bmatrix},$$

we treat each component  $F_i(\vec{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  separately as a real valued function. Each has  $n$  partial derivatives, and so  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has  $(m \times n)$  partial derivatives, which form an  $(m \times n)$  matrix:

$$\left( \frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}}.$$

We call this the derivative matrix or *Jacobian matrix*,  $DF(\vec{x})$ .

**Example 2.2.1.** Consider a function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ :

$$F(\vec{x}) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^4 \end{bmatrix}.$$

Jacobian of the function is given by

$$\begin{aligned} DF(\vec{x}) &= \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 4x_2^3 \end{bmatrix} \end{aligned}$$

Do we get the same properties for  $DF(\vec{x})$  as we did for single-value calculus?

**Example 2.2.2.** Consider the following function:

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Do the partial derivatives exist at  $(0, 0)$ ?

By definition,

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{h \cdot 0}{(h^2+0^2)^2} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} = 0\end{aligned}$$

Similarly,  $\frac{\partial f}{\partial y}(0,0) = 0$  (symmetry of  $x, y$ ). Therefore,

$$Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Although partial derivatives exist,  $f$  is not continuous at  $(0,0)$ ! (For example,  $f(x, mx) \rightarrow \pm\infty$  as  $x \rightarrow 0^\pm$  for  $m \neq 0$ ).

To get reasonable information from  $Df(\vec{x})$ , we need to say more. First, let's go back to  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Note

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \iff \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} - f'(x) \right) &= 0 \\ \iff \lim_{h \rightarrow 0} \left( \frac{f(x+h) - \overbrace{[f(x) + hf'(x)]}^L}{h} \right) &= 0\end{aligned}$$

Numerator is the difference between  $f(x+h)$  and its linear approximation,  $L$  (i.e. the tangent line). So  $f$  is differentiable at  $x$  if its linear approximation gives an estimate of the values  $f(x+h)$  within an error which is small compared to  $\Delta x = h$ . More precisely, the linearization of  $f(x)$  at  $x = a$  (or the tangent line) is given by

$$L_a(x) = f(a) + f'(a)(x - a)$$

We wish to extend this idea to higher dimensions. For  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $F(\vec{x})$  has  $(m \times n)$  partial derivatives (see definition 2.4). Then, the linearization of  $F$  at  $\vec{a}$  is

$$L_{\vec{a}}(\vec{x}) = \underbrace{F(\vec{a})}_{m \times 1} + \underbrace{DF(\vec{a})}_{m \times n} \underbrace{(\vec{x} - \vec{a})}_{n \times 1}.$$

So,  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , just like  $F$ . The derivative matrix  $DF(\vec{a})$  is a *linear transformation* of  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Notice that when  $n = 2$  and  $m = 1$ , For  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ , we have

$$DF(\vec{a}) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(\vec{a}) & \frac{\partial F}{\partial x_2}(\vec{a}) \end{bmatrix},$$

a  $(1 \times 2)$  row vector and

$$\vec{x} - \vec{a} = \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \end{bmatrix},$$

so we have

$$L_{\vec{a}}(\vec{x}) = F(\vec{a}) + \frac{\partial F}{\partial x_1}(x_1 - a_1) + \frac{\partial F}{\partial x_2}(x_2 - a_2),$$

a familiar equation of the tangent plane to  $z = F(x_1, x_2)$ .

Finally, we can introduce the idea of differentiable:

**Definition 2.5** (Differentiability). *We say  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable if*

$$\lim_{\vec{x} \rightarrow \vec{a}} \frac{\|F(\vec{x}) - F(\vec{a}) - DF(\vec{a})(\vec{x} - \vec{a})\|}{\|\vec{x} - \vec{a}\|} = 0.$$

*Equivalently,*

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{\|F(\vec{x} + \vec{h}) - F(\vec{x}) - DF(\vec{x})\vec{h}\|}{\|\vec{h}\|} = 0$$

In summary,  $F$  is differentiable if  $\|F(\vec{x}) - L_{\vec{a}}(\vec{x})\|$  is small compared to  $\|\vec{x} - \vec{a}\|$ . Or,  $F(\vec{x})$  is approximated by  $L_{\vec{a}}(\vec{x})$  with an error which is much smaller than  $\|\vec{x} - \vec{a}\|$ . Note that we write  $o(\|\vec{x} - \vec{a}\|)$  “little-oh” for quantity which is small compared to  $\|\vec{x} - \vec{a}\|$ . Using this notation, differentiability can be written as

$$\|F(\vec{x}) - F(\vec{a}) - DF(\vec{a})(\vec{x} - \vec{a})\| = o(\|\vec{x} - \vec{a}\|).$$

**Example 2.2.3.** Is the following function differentiable at  $\vec{a} = \vec{0}$ ?

$$F(x_1, x_2) = \begin{cases} \frac{x_2^2 \sin x_1}{\sqrt{x_1^2 + x_2^2}}, & \vec{x} \neq \vec{0} \\ 0, & \vec{x} = \vec{0} \end{cases}$$

First, we have

$$\begin{aligned} \frac{\partial F}{\partial x_1}(\vec{0}) &= \lim_{h \rightarrow 0} \frac{F(\vec{0} + h\vec{e}_1) - F(\vec{0})}{h} \\ &= \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0 \end{aligned}$$

Similarly, we have

$$\frac{\partial F}{\partial x_2}(\vec{0}) = 0$$

So we have

$$DF(\vec{0}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

For differentiability, we have to look at:

$$\begin{aligned} &\left| \frac{x_2^2 \sin x_1}{\sqrt{x_1^2 + x_2^2}} - 0 - \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| \\ &= \frac{x_2^2 |\sin x_1|}{\sqrt{x_1^2 + x_2^2}} \end{aligned}$$

Then,

$$\begin{aligned} \frac{|F(\vec{x}) - L_{\vec{0}}(\vec{x})|}{\|\vec{x} - \vec{0}\|} &= \frac{x_2^2 |\sin x_1|}{\left(\sqrt{x_1^2 + x_2^2}\right)^2} = \frac{x_2^2 |\sin x_1|}{x_1^2 + x_2^2} \\ &\leq \frac{R^2 \cdot R}{R^2} = R = \|\vec{x} - \vec{0}\| \end{aligned}$$

By squeeze theorem, we have

$$\lim_{\vec{x} \rightarrow \vec{0}} \frac{|F(\vec{x}) - L_{\vec{0}}(\vec{x})|}{\|\vec{x} - \vec{0}\|} = 0$$

Therefore,  $F$  is differentiable at  $\vec{x} = \vec{0}$

**Example 2.2.4.** Verify that  $F$  is differentiable at  $\vec{a} = \vec{0}$ .

$$F(\vec{x}) = \begin{bmatrix} 1 + x_1 + x_2^2 \\ 2x_2 - x_1^2 \end{bmatrix}$$

First, note that

$$F(\vec{a}) = F(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We also need to compute the Jacobian at  $\vec{0}$ :

$$DF(\vec{0}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then, we get the following linearization of the function:

$$\begin{aligned} L_{\vec{0}}(\vec{x}) &= F(\vec{0}) + DF(\vec{0})(\vec{x} - \vec{0}) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 + x_1 \\ 2x_2 \end{bmatrix} \end{aligned}$$

Then, look at

$$\frac{\|F(\vec{x}) - L_{\vec{0}}(\vec{x})\|}{\|\vec{x} - \vec{0}\|} = \frac{\left\| \begin{bmatrix} x_2^2 \\ -x_1^2 \end{bmatrix} \right\|}{\|\vec{x}\|} = \frac{\sqrt{x_2^4 + x_1^4}}{\sqrt{x_1^2 + x_2^2}} \leq \frac{R^4 + 4}{R} = \sqrt{2}R = \sqrt{2}\|\vec{x} - \vec{0}\|$$

As  $\vec{x} \rightarrow \vec{0}$ ,  $\|\vec{x} - \vec{0}\| = R \rightarrow 0$ , so by the squeeze theorem, the desired limit goes to 0. Therefore,  $F$  is differentiable at  $\vec{0}$ .

**Theorem 2.2.** Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $\vec{a} \in \mathbb{R}^n$ . If there exists a disk  $D_r(\vec{a})$  in which all the partial derivatives  $\partial(F_i(\vec{x}))/\partial x_j$  exist and are continuous, then  $F$  is differentiable at  $\vec{x} = \vec{a}$ .

**Definition 2.6.** A function which satisfies Theorem 2.2 is called continuously differentiable, of  $C^1$ .

So far as our example, we calculate the partial for  $\vec{x} \neq \vec{0}$ :

$$\begin{aligned}\frac{\partial F}{\partial x_1} &= x_2^2 \left( \cos x_1 (x_1^2 + x_2^2)^{-\frac{1}{2}} + \left( -\frac{1}{2} (x_1^2 + x_2^2)^{-\frac{3}{2}} 2x_1 \right) \sin x_1 \right) \\ &= \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} [\cos x_1 (x_1^2 + x_2^2) - x_1 \sin x_1]\end{aligned}$$

which is continuous as long as  $\vec{x} \neq \vec{0}$ . We do the same for  $\frac{\partial F}{\partial x_2}$  and conclude that  $F$  is  $C^1$  at all  $\vec{x} \neq \vec{0}$ . We summarize these ideas in the figure below:

### 2.3 Chain rule

**Definition 2.7.** Suppose  $A \subseteq \mathbb{R}^n$  is open, and we have a function

$$F : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

Similarly, supposed  $B \subseteq \mathbb{R}^m$  is open, and we have a function

$$G : B \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^p.$$

Assume  $\vec{a} \in A$  and  $F(\vec{a}) = \vec{b} \in B$ . The composition

$$H(\vec{x}) = G \circ F(\vec{x}) = G(F(\vec{x}))$$

is a function  $\mathbb{R}^n \rightarrow \mathbb{R}^p$ .

**Example 2.3.1.** Consider the following linear functions:

$$\begin{cases} F(\vec{x}) = M\vec{x} & M \text{ an } (m \times n) \text{ matrix} \\ G(\vec{y}) = N\vec{y} & N \text{ an } (p \times m) \text{ matrix} \end{cases}$$

Then,

$$H(\vec{x}) = G(F(\vec{x})) = NM\vec{x}$$

is also a linear and represented by the product  $NM$ .

**Theorem 2.3.** Assume  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $\vec{x} = \vec{a}$  and  $G : \mathbb{R}^m \rightarrow \mathbb{R}^p$  is differentiable at  $\vec{b} = F(\vec{a})$ . Then,  $H = G \circ F$  is differentiable at  $\vec{x} = \vec{a}$  and

$$DH(\vec{a}) = \underbrace{DG(\vec{b})}_{DG(F(\vec{a}))} DF(\vec{a})$$

Note that all of the various forms of Chain Rule done in first year calculus can be derived directly from this general formula.

**Example 2.3.2.** Consider the following functions,  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ :

$$F(\vec{x}) = \begin{bmatrix} x_1^2 + x_2x_3 \\ x_1^2 + x_3^2 \end{bmatrix}, G(\vec{y}) = \begin{bmatrix} -y_2^3 \\ y_1 + y_2 \end{bmatrix}$$

Let  $H = G \circ F(\vec{x})$ . Find  $DH(\vec{a})$  where  $a = (1, -1, 0)$ .

First, we have

$$DF(\vec{x}) = \begin{bmatrix} 2x_1 & x_3 & x_2 \\ 2x_1 & 0 & 2x_3 \end{bmatrix}, DF(1, -1, 0) = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

Similarly, we have

$$DG(\vec{y}) = \begin{bmatrix} 0 & -3y_2^2 \\ 1 & 1 \end{bmatrix}, DG(1, 1) = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

By Chain Rule, we get

$$\begin{aligned} DH(1, -1, 0) &= DG(1, 1)DF(1, -1, 0) \\ &= \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 0 & 0 \\ 4 & 0 & -1 \end{bmatrix} \end{aligned}$$

### 3 Paths and Curves

#### 3.1 Directional derivative

**Definition 3.1.** A path is  $\vec{C} : \mathbb{R} \rightarrow \mathbb{R}^n$  is a vector-valued function of a scalar independent variable, usually,  $t$ :

$$\vec{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

$\vec{c}(t)$  can be thought of as a moving vector. It takes out a curve in  $\mathbb{R}^n$  as  $t$  increases. Basically, path is a way of describing a curve using functions. Note that this is not the only way to describe a curve.

**Example 3.1.1.** A unit circle in  $\mathbb{R}^2$  described as a path is

$$\vec{c}(t) = (\cos t, \sin t),$$

where  $t \in [0, 2\pi)$ . But we could also describe the unit circle *non-parametrically* as

$$x^2 + y^2 = 1$$

Note that the same curve can be described by different paths. Going back to unit circle, we can also write

$$\vec{b}(t) = (\sin(t^2), \cos(t^2)).$$

Using different paths can change (1) time dynamics and (2) direction of the curve. This curve has a non-constant speed and reversed orientation.

If  $\vec{c}$  is differentiable,  $D\vec{c}(t)$  is an  $(n \times 1)$  matrix. Since each component  $\vec{c}_j(t)$  is a real-valued function of only one variable, the *partial-derivative* is the usual derivative:

$$\frac{\partial c_j}{\partial t} = \frac{dc_j}{dt} = c'_j(t) = \lim_{h \rightarrow 0} \frac{c_j(t+h) - c_j(t)}{h}$$

So  $D\vec{c}(t) = \vec{c}'(t)$  is written as a column vector:

$$\begin{aligned} D\vec{c}(t) &= \begin{bmatrix} c'_1(t) \\ c'_2(t) \\ \vdots \\ c'_n(t) \end{bmatrix} \\ &= \lim_{h \rightarrow 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h} \end{aligned}$$

which is a vector which is tangent to the curve traced out at  $\vec{x} = \vec{x}(t)$ . Physically,  $\vec{c}'(t)$  is the velocity vector for motion along the path.

**Example 3.1.2** (Lines). Given two points,  $\vec{p}_1, \vec{p}_2 \in \mathbb{R}^n$ , there is a unique line connecting them. One path which represents this line is

$$\vec{c}(t) = \vec{p}_1 + t\vec{v},$$

where  $\vec{v} = \vec{p}_2 - \vec{p}_1$ . Velocity is then given by  $\vec{c}'(t) = \vec{v}$ , a constant.

**Definition 3.2.**  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a scalar-valued function.

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable,  $Df(\vec{x})$  is a  $(1 \times n)$  matrix:

$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

We use *paths*  $\vec{c}(t)$  to explore  $f(x)$  by looking at

$$h(t) = f \circ \vec{c}(t) = f(\vec{c}(t)).$$

where  $h : \mathbb{R} \rightarrow \mathbb{R}$ , By chain rule,

$$\begin{aligned} Dh(t) = h'(t) &= \underbrace{Df(\vec{c}(t))}_{1 \times n} \underbrace{D\vec{c}(t)}_{n \times 1} \\ &= Df(\vec{c}(t)) \vec{c}'(t) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix} \end{aligned}$$

We can think of this as a dot product of  $\vec{c}'(t)$  with a vector  $Df^T = \nabla f$ , the gradient vector:

$$h'(t) = \nabla f(\vec{c}(t)) \cdot \vec{c}'(t)$$

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\vec{a} \in \mathbb{R}^n$ , and we have a path  $\vec{c} : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\vec{c}(0) = \vec{a}$ . Let  $\vec{v} = \vec{c}'(0)$ . Then,  $h'(0)$  measures rate of change of  $f$  along the path as we cross through  $\vec{a}$ :

$$\begin{aligned} h'(0) &= \nabla f(\vec{c}(0)) \cdot \vec{c}'(0) \\ &= \nabla f(\vec{c}(0)) \cdot \vec{v} \end{aligned}$$

Note that we get the same value for  $h'(0)$  for *any* path  $\vec{c}(t)$  going through  $\vec{a}$  with velocity  $\vec{c}'(t) = \vec{v}$ . In other words,  $h'(0)$  says something about  $f$  at  $\vec{a}$ , and not the path  $\vec{c}(t)$ .

**Definition 3.3** (Directional derivative). *The directional derivative of  $f$  at  $\vec{a}$  in direction  $\vec{v}$  is given by*

$$D_{\vec{v}}f(\vec{a}) = Df(\vec{a})\vec{v} = \nabla f(\vec{a}) \cdot \vec{v}.$$



Now, we can make some observations. Using the Chain Rule, directional derivatives can be rewritten as

$$D_{\vec{v}}f(\vec{a}) = \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}.$$

Note the similarity to partial derivatives, where  $\vec{v} = \vec{e}_j$ .

Second, notice that  $D_{2\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot (2\vec{v}) = 2D_{\vec{v}}f(\vec{a})$ . To get the information on how fast  $f$  is changing at  $\vec{a}$ , we need to restrict to unit vectors  $\|\vec{v}\| = 1$ .

Directional derivatives also gives a geometrical interpretation of the gradient vector,  $\nabla f(\vec{a})$ . We use the Cauchy-Schwartz Inequality<sup>2</sup> to do so. By applying the Cauchy-Schwartz inequality, we get:

$$D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v} \leq \|\nabla f(\vec{a})\| \|\vec{v}\| = \|\nabla f(\vec{a})\|.$$

Therefore, we can conclude that the length of  $\|\nabla f(\vec{a})\|$  is the largest of  $D_{\vec{v}}f(\vec{a})$  among all choices of unit directions  $\vec{v}$ . In other words, the direction  $\vec{v}$  in which  $f(\vec{x})$  increases most rapidly is the direction of  $\nabla f(\vec{a})$ , i.e.

$$\vec{v} = \frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|},$$

provided that  $\nabla f(\vec{a}) \neq \vec{0}$ .

Similarly,  $-\nabla f(\vec{a})$  points in the direction of largest of  $f(\vec{x})$ , i.e.

$$\vec{v} = -\frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|},$$

gives the most negative directional derivative.

### 3.2 Parameterized curve

A path,  $\vec{c}(t)$ , is  $\{\text{continuous, differentiable, and } C^1\}$  provided that each component  $c_j(t)$ ,  $j = 1, 2, \dots, n$  are. Note that  $\{\vec{c}(t) : t \in [a, b]\}$  traces out a curve in  $\mathbb{R}^n$ , with initial endpoint,  $\vec{a}$ , and final endpoint,  $\vec{b}$ . Therefore, the path  $\vec{c}(t)$  *parameterizes* the curve drawn out.

Recall that for any function  $F : \mathbb{R}^k \rightarrow \mathbb{R}^n$ , differentiability means that tangent (i.e. linearization) makes a good approximation. For a differentiable path,  $\vec{c}'(t)$  is a tangent vector to the curve drawn out when  $\vec{c}'(t) \neq \vec{0}$ . We call  $\vec{v}(t) = \vec{c}'(t)$  the velocity vector ( $v = \|\vec{v}\| = \|\vec{c}'(t)\|$  is the speed).

Finally, we can define the unit tangent vector:

**Definition 3.4.** *Unit tangent vector is defined as*

$$\vec{T}(t) = \frac{\vec{v}}{\|\vec{v}(t)\|} = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}$$

---

<sup>2</sup>For any vectors  $\vec{u} \cdot \vec{v} \leq \|\vec{u}\| \|\vec{v}\|$ , and equality holds if and only if  $\vec{u} = t\vec{v}$  for a scalar  $t$ .

**Example 3.2.1.** Consider a path,  $\vec{c}: \mathbb{R} \rightarrow \mathbb{R}^2$ :

$$\vec{c}(t) = (\cos^3 t, \sin^3 t), t \in [-\pi, \pi].$$

This is a  $C^1$  path<sup>3</sup> whose velocity vector is given by

$$\vec{c}'(t) = (-3 \cos^2 t \sin t, 3 \sin^2 t \cos t).$$

To find the unit tangent, we have to find its speed first:

$$\begin{aligned} v &= \|(-3 \cos^2 t \sin t, 3 \sin^2 t \cos t)\| \\ &= 3 |\sin t \cos t| \|(-\cos t, \sin t)\| \\ &= 3 |\sin t \cos t| \end{aligned}$$

Then, its unit tangent is given by

$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \left( -|\cos t| \frac{\sin t}{|\sin t|}, |\sin t| \frac{\cos t}{|\cos t|} \right)$$

Note that its tangent is undefined when  $\sin t = 0$  or  $\cos t = 0$ , i.e. at multiples of  $\frac{\pi}{2}$ . Worse,  $\frac{\sin t}{|\sin t|}, \frac{\cos t}{|\cos t|}$  flip discontinuously as  $t$  crosses a multiple of  $\pi/2$  from  $-1$  to  $+1$ , or vice versa. Although the path is  $C^1$ , the curve is not smooth! When  $\vec{v}(t) = \vec{c}'(t) = 0$ , it allows the curve to have cusps.

Note that it is possible to have a nice tangent direction even when  $\vec{c}'(t) = 0$ :

**Example 3.2.2.** Consider a parameterized straight line:

$$\vec{c} = \vec{a} + \vec{w}t^3$$

Its velocity vector,  $\vec{c}'(t) = 3\vec{w}t^2$ , is equal to  $\vec{0}$  when  $t = 0$ . However, it still has a tangent direction which is parallel to  $\vec{w}$ .

**Definition 3.5.** We say a parameterized curve is smooth<sup>4</sup> (or regular) if its path is  $C^1$ , i.e. if it can be parameterized by a path  $\vec{c}(t)$ , and  $\|\vec{c}'(t)\| \neq 0$  for any  $t$ .

If  $\vec{c}(t)$  is twice-differentiable,  $\vec{c}''(t) = \vec{a}(t)$  gives us the acceleration vector.

**Theorem 3.1.**

1. If  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $\vec{c}: \mathbb{R} \rightarrow \mathbb{R}^n$ , both differentiable,

$$\begin{aligned} \frac{d}{dt}(f(t)\vec{c}(t)) &= f(t)\vec{c}'(t) + f'(t)\vec{c}(t) \\ &= \sum_{j=1}^n \frac{d}{dt}(f(t)\vec{c}_j(t))\vec{e}_j \\ &= \sum_{j=1}^n \frac{d}{dt}(f'(t)\vec{c}_j(t) + f(t)\vec{c}_j'(t))\vec{e}_j \end{aligned}$$

<sup>3</sup>In fact, it is  $C^\infty$ , differentiable to all orders!

<sup>4</sup>For a smooth curve, the unit tangent  $\vec{T}(t)$  is continuous.

2. If  $\vec{c}, \vec{d}: \mathbb{R} \rightarrow \mathbb{R}^n$  are differentiable,

$$\frac{d}{dt}(\vec{c}(t) \cdot \vec{d}(t)) = \vec{c}'(t) \cdot \vec{d}(t) + \vec{c}(t) \cdot \vec{d}'(t)$$

3. If  $\vec{c}, \vec{d}: \mathbb{R} \rightarrow \mathbb{R}^3$  are differentiable,

$$\frac{d}{dt}(\vec{c}(t) \times \vec{d}(t)) = \vec{c}(t) \times \vec{d}'(t) + \vec{c}'(t) \times \vec{d}(t),$$

where  $\vec{c} \times \vec{d} = \sum_{i,j,k=1}^3 c_i d_j \vec{e}_k \varepsilon_{ijk}$ .  $\varepsilon_{ijk}$  is defined in the footnotes<sup>5</sup>.

**Example 3.2.3.** Suppose  $\vec{c}$  is a twice differentiable path and  $\vec{a}(t) = k\vec{c}(t)$  for some constant  $k \neq 0$ . Show that  $\vec{c}(t)$  describes a motion in a fixed plane.

Define a vector

$$\vec{n} = \vec{c}(t) \times \vec{v}(t) = \vec{c}(t) \times \vec{c}'(t)$$

Notice  $\vec{n} \perp \vec{c}(t)$  and  $\vec{v}(t)$ , i.e.  $\vec{n}$  is normal to the plane.

$$\begin{aligned} \frac{d\vec{n}}{dt} &= \frac{d}{dt}(\vec{c}(t) \times \vec{c}'(t)) = \vec{c}(t) \times \underbrace{\vec{c}''(t)}_{\vec{a}(t)} + \underbrace{\vec{c}'(t) \times \vec{c}'(t)}_{\vec{0}} \\ &= \vec{c}(t) \times k\vec{c}(t) \\ &= \vec{0} \end{aligned}$$

Therefore,  $\vec{n}$  is constant in time!

So  $\vec{c}(t)$  and  $\vec{v}(t)$  are, for all  $t$ , perpendicular to the constant vector  $\vec{n}$ . Then,

$$P = \{\vec{w} \mid \vec{w} \cdot \vec{n} = 0\}$$

is the plane through  $\vec{0}$ . So  $\vec{c}(t) \in P$  for all  $t$ .

**Definition 3.6** (Arclength). *The arclength (or distance travelled along the parameterized curve) for  $a \leq t \leq b$  is*

$$\int_a^b \underbrace{\|\vec{c}'(t)\|}_{\text{speed}} dt$$

For a variable time interval, the arclength function

$$s(t) = \int_a^t \|\vec{c}'(u)\| du$$

is a distance travelled from time  $a$  to time  $t$ .

---

<sup>5</sup>  $\varepsilon_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } k = 1 \\ 1 & \text{if } (i, j, k) \text{ is positively ordered} \\ -1 & \text{if } (i, j, k) \text{ is negatively ordered} \end{cases}$

**Example 3.2.4.** Consider the following path:

$$\vec{c}(t) = (3 \cos t, 3 \sin t, 4t), \quad t \in [0, 4\pi].$$

Its velocity vector is given by

$$\vec{v}(t) = (-3 \sin t, 3 \cos t, 4).$$

It follows that its speed is exactly equal to 5. Then, we can compute the arclength:

$$s(t) = \int_0^t v(t) dt = \int_0^t 5 du = 5t$$

Therefore, total arclength is  $s(4\pi) = 20\pi$ .

**Definition 3.7.** When the path  $\vec{c}(t)$  traces out the curve with speed  $\|\vec{v}(t)\| = 1$  for all  $t$ , we say that the curve is arclength parameterized.

If a curve is arclength parameterized, arclength function becomes

$$s(t) = t$$

Then, we can use  $s$  instead of  $t$  as a parameterization in the path.

**Example 3.2.5.** In example 3.2.4, helix is not arclength parameterized but we can re-parameterize it so that it is. To do so, we need to solve for  $t = \varphi(s)$  to invert the function,  $s(t)$ .

Going back the example, we had  $s(t) = 5t$ . It follows that  $t = \frac{1}{5}s$ . Then,

$$\vec{c}(s) = \vec{c}(\varphi(s)) = \vec{c}\left(\frac{s}{5}\right) = \left(3 \cos\left(\frac{s}{5}\right), 3 \sin\left(\frac{s}{5}\right), \frac{4s}{5}\right)$$

is an arclength parameterization of the original helix, i.e.  $\|\vec{c}'(s)\| = 1, \forall s$ .

### 3.3 Geometry of curves in $\mathbb{R}^3$

Path,

$$\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = (x, y, z)(t),$$

traces out a curve, for  $t \in [a, b]$ , in space, and its velocity vector and speed are given by  $\vec{c}'(t)$  and  $\|\vec{c}'(t)\|$ , respectively. This is a smooth parameterization if  $\vec{c} \in C^1$  and  $\|\vec{c}'(t)\| \neq 0$  for any  $t \in [a, b]$ .

We introduced the arclength function,

$$s(t) = \int_a^t \|\vec{c}'(u)\| du,$$

the total distance along the curve up to time  $t$ .

We also introduced the idea of arclength parameterization, where  $s(t) = t$ . Then, since

$$\frac{ds}{dt} = \|\vec{c}'(t)\|,$$

arclength parameterization is a path that travels along the curve with unit speed,  $ds/dt = 1$ , throughout. Therefore, any path with  $\|\vec{c}'(t)\| \neq 0$  can be parameterized by arclength by inverting  $s = s(t)$  such that  $t = \varphi(s)$ . Note that we can always do this for a smooth path ( $ds/dt > 0$  so  $s(t)$  is monotonically increasing). In practice, however, you may not be able to find an explicit formula for the arclength parameterization!

**Example 3.3.1.** Consider the following path:

$$\vec{c}(t) = (t, \frac{1}{2}t^2) :$$

Since  $y = x^2/2$ , it's a parabola. Then, we observe that

$$\vec{c}'(t) = (1, t), \|\vec{c}'(t)\| = \sqrt{1+t^2} \geq 1 > 0.$$

So the path is smooth. Then, we have

$$s(t) = \int_0^t \|\vec{c}'(u)\| du = \int_0^t \sqrt{1+u^2} du = \frac{1}{2} \left( \ln \left| \sqrt{1+t^2} + t \right| + t\sqrt{1+t^2} \right).$$

Clearly, there's no way we can solve for  $t$  as a function of  $s$ . The way out of this trouble is to treat all  $\vec{c}$  as if they were parameterized by arclength and use Chain rule with  $ds/dt = \|\vec{c}'(t)\|$  to compensate.

Recall that unit tangent vector to  $\vec{c}(t)$  is

$$\vec{T}(t) = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}.$$

We wish to understand how direction of the curve changes over time:

**Definition 3.8.** *The curvature of a curve is defined as rate of change of unit tangent:*

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|.$$

By chain rule,

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

So, in the original time parameter,  $t$ ,

$$\kappa(t) = \left\| \frac{1}{\frac{ds}{dt}} \frac{d\vec{T}}{dt} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}$$

**Example 3.3.2.** Consider a circle of radius  $R > 0$  in  $xy$ -plane:

$$\vec{c}(t) = (R \sin t, R \cos t).$$

Now, we can easily find its velocity vector and speed:

$$\begin{aligned}\vec{c}'(t) &= (R \cos t, -R \sin t) \\ \|\vec{c}'(t)\| &= R\end{aligned}$$

Notice that this travels with constant speed but is not arclength parameterized.

We can also find its unit tangent:

$$\vec{T}(t) = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|} = \frac{\vec{c}'(t)}{R} = (\cos t, -\sin t)$$

Then,

$$\vec{N}(t) = \vec{T}'(t) = (-\sin t, -\cos t)$$

Again, notice that  $\vec{N}(t)$  is perpendicular to  $\vec{T}(t)$ .

Finally, we have

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{1}{R}.$$

Therefore, circle with large radius has less curvature.

**Example 3.3.3.** Consider the following helix:

$$\vec{c}(t) = (3 \cos t, 3 \sin t, 4t).$$

Following the same approach as shown in the previous example, we get

$$\begin{aligned}\vec{c}'(t) &= (-3 \sin t, 3 \cos t, 4) \\ \|\vec{c}'(t)\| &= 5 \\ \vec{T}(t) &= \left(-\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5}\right) \\ \vec{T}'(t) &= \left(-\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0\right)\end{aligned}$$

Then,

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

This curve also has a constant curvature.

**Definition 3.9** (Principal normal vector).

$$\vec{N} = \frac{\vec{T}'(s)}{\|\vec{T}'(s)\|} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Since  $\|\vec{T}(s)\| = 1$  for all  $s$ ,  $\vec{T}(s) \cdot \vec{T}(s) = \|\vec{T}(s)\|^2 = 1$ . By implicit differentiation, we have

$$\begin{aligned}\frac{d}{ds}(1) &= \frac{d}{ds}(\vec{T}(s) \cdot \vec{T}(s)) \\ 0 &= \vec{T}'(s) \cdot \vec{T}(s) + \vec{T}(s) \cdot \vec{T}'(s) \\ &= 2\vec{T}'(s) \cdot \vec{T}(s)\end{aligned}$$

Therefore,  $\vec{T}'(s) \perp \vec{T}(s)$  for all  $s$ . So as long as  $\vec{T}'(s) \neq 0$ , i.e.  $\kappa \neq 0$ , we have  $\vec{N}(s) \perp \vec{T}(s)$ . In fact,  $\vec{T}'(s) = \|\vec{T}'(s)\|\vec{N} = \kappa\vec{N}$ , so the tangent turns in the direction of  $\vec{N}$ . For motion in a line, where  $\kappa(s) = 0$  for all  $s$ ,  $\vec{N}$  cannot be defined!

$\vec{T}, \vec{N}$  determines a plane in  $\mathbb{R}^3$ , the osculating plane. The normal vector to the osculating plane is given by

$$\vec{B} = \vec{T} \times \vec{N}.$$

**Definition 3.10** (Binormal vector).  $\vec{B} = \vec{T} \times \vec{N}$

We observe that  $\vec{B} \perp \vec{T}$ ,  $\vec{B} \perp \vec{N}$ , and

$$\|\vec{B}\| = \|\vec{T}\|\|\vec{N}\|\sin\theta = 1 \cdot 1 \cdot \sin(\pi/2) = 1$$

Therefore,  $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$  is a moving *orthonormal basis* for  $\mathbb{R}^3$  at each point along the curve. This plane is also referred to as *moving frame* or *frenet frame*. Now, we introduce the following consequences:

(1). If curvature  $\kappa(s) = 0$  for all  $s$ , then the curve is a straight line.

To see this,  $\vec{T}'(s) = \kappa\vec{N}(s) = 0$  for all  $s$ . Therefore,  $\vec{T}(s) = \vec{u}$  is a constant vector and

$$\vec{r}(s) = \vec{u}s + \vec{p},$$

a line thorough  $\vec{p} = \vec{r}(0)$  with direction vector  $\vec{u}$ .

(2). When  $\kappa = 0$ ,  $\vec{N}$  and  $\vec{B}$  cannot be defined.

(3). If  $\vec{B}(s)$  is a constant vector, then  $\vec{c}(t)$  ( $\vec{r}(t)$ ) move in a fixed plane, with normal vector  $\vec{B}$ .

Now, suppose  $\vec{B}(s)$  isn't constant. First,  $\|\vec{B}(s)\| = 1$  for all  $s$ . Then,

$$1 = \|\vec{B}(s)\|^2 = \vec{B}(s) \cdot \vec{B}(s)$$

holds for all  $s$ . So we can apply implicit differentiation:

$$0 = \frac{d}{ds}(1) = \frac{d}{ds}(\vec{B} \cdot \vec{B}) = 2\vec{B}' \cdot \vec{B}.$$

Then, it follows that  $\vec{B}' \perp \vec{B}$ , for every  $s$ .

Next, since  $\vec{B}(s) \perp \vec{T}(s)$  for all  $s$ , we have  $\vec{B} \cdot \vec{T} = 0$  for all  $s$ . Then,

$$\frac{d}{ds} (\vec{B} \cdot \vec{T}) = \vec{B}'(s) \cdot \vec{T}(s) + \vec{B}(s) \cdot \vec{T}'(s) = 0.$$

Since  $\vec{T}' = \kappa \vec{N}$  and  $\vec{B} \cdot \vec{N} = 0$ , it follows that

$$\vec{B}'(s) \cdot \vec{T}(s) = 0 \iff \vec{B}'(s) \perp \vec{T}(s)$$

Since  $\{\vec{T}, \vec{N}, \vec{B}\}$  form an orthonormal basis for  $\mathbb{R}^3$ , we must have  $\vec{B}'(s)$  parallel to  $\vec{N}$ . Therefore,

$$\vec{B}'(s) = -\tau(s) \vec{N}(s)$$

for a function  $\tau(s)$  called the *torsion*. Since  $\tau = \|d\vec{B}/ds\|$ , it measures how fast the normal  $\vec{B}$  to the osculating plane is twisting.

**Definition 3.11** (Torsion).

$$\tau = \left\| \frac{d\vec{B}}{ds} \right\| = \frac{\|\vec{B}'(t)\|}{\|\vec{c}'(t)\|}$$

Putting all the information together we get *Frenet formulas*:

**Theorem 3.2** (Frenet formula).

$$\begin{cases} \frac{d\vec{T}}{ds} = \kappa \vec{N} \\ \frac{d\vec{B}}{ds} = -\tau \vec{N} \\ \frac{d\vec{N}}{ds} = -\kappa \vec{T} + \tau \vec{B} \end{cases}$$

**Example 3.3.4.** Consider the following helix:

$$\vec{c}(t) = (3 \cos t, 3 \sin t, 4t)$$

Then, we have

$$\begin{aligned} \|\vec{c}'(t)\| &= 5, \\ \vec{T}(t) &= \left( -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right), \\ \vec{T}'(t) &= \left( -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right), \\ \kappa &= \frac{3}{25}, \\ \vec{N}(t) &= (-\cos t, -\sin t, 0), \\ \vec{B}(t) &= \vec{T} \times \vec{N} = \left( \frac{4}{5} \sin t, -\frac{4}{5} \cos t, \frac{3}{5} \right), \\ \vec{B}' &= \left( \frac{4}{5} \cos t, \frac{4}{5} \sin t, 0 \right), \\ \tau &= \frac{4}{25}. \end{aligned}$$



### 3.4 Dynamics

How do these quantities relate to dynamical quantities? Given,  $\vec{c}(t)$ , a position vector along the curve,  $\vec{c}'(t) = \vec{v}(t) = \vec{T}(t) \cdot ds/dt$  is its velocity vector and  $\|\vec{c}'(t)\| = ds/dt$  is its speed.

**Definition 3.12** (Acceleration).  $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t)$

First, observe that

$$\vec{v}(t) = \vec{c}'(t) = \frac{ds}{dt} \cdot \vec{T}(t)$$

Then,

$$\begin{aligned} \vec{a}(t) &= \frac{d}{dt} \left( \frac{ds}{dt} \cdot \vec{T}(t) \right) = \frac{d^2s}{dt^2} \cdot \vec{T}(t) + \frac{ds}{dt} \cdot \vec{T}'(t) \\ &= \frac{d^2s}{dt^2} \cdot \vec{T} + \frac{ds}{dt} \cdot \left( \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \right) \end{aligned}$$

So we have

$$\vec{a}(t) = \underbrace{\frac{d^2s}{dt^2} \cdot \vec{T}}_{\text{Linear acceleration}} + \underbrace{\kappa \left( \frac{ds}{dt} \right)^2 \vec{N}}_{\text{Steering-term}}$$

By looking at the steering term, we see that acceleration to turn on a curve is proportional to the curvature and (speed)<sup>2</sup>.

**Example 3.4.1.** Consider the following path

$$\vec{c}(t) = (e^t \cos t, e^t \sin t, e^t)$$

that draws a spiral in  $xy$  direction and monotonically increases along  $z$  coordinate.

First, observe that

$$\begin{aligned} \vec{v}(t) = \vec{c}'(t) &= (-e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t, e^t) \\ \frac{ds}{dt} = \|\vec{c}'(t)\| &= \sqrt{3}e^t \end{aligned}$$

Then, we have

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|} = \frac{1}{\sqrt{3}}(-\sin t + \cos t, \cos t + \sin t, 1), \\ \vec{T}'(t) &= \frac{\vec{c}''(t)}{\|\vec{c}'(t)\|} = \frac{1}{\sqrt{3}}(-\cos t - \sin t, -\sin t + \cos t, 0). \end{aligned}$$

Since  $\|\vec{T}'(t)\| = \sqrt{2/3}$ , we can easily find the principal normal vector:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{\sqrt{2}}(-\cos t - \sin t, -\sin t + \cos t, 0)$$

Then,

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\sqrt{2}}{3}e^{-t}.$$

Furthermore,

$$\begin{aligned}\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) = \dots = \frac{1}{\sqrt{6}}(\cos t - \sin t, -\sin t - \cos t, 2) \\ \vec{B}'(t) &= \frac{1}{\sqrt{6}}(-\sin t - \cos t, -\cos t + \sin t, 0)\end{aligned}$$

Therefore, torsion of the curve is given by

$$\tau(t) = \frac{\|\vec{B}'(t)\|}{\|\vec{c}'(t)\|} = \frac{1}{3}e^{-t}$$

We can then verify formula for  $\vec{a}$  in terms of  $\vec{T}, \vec{N}, \kappa$ , (and verify that it agrees with  $\vec{a} = \vec{v}'(t)$  calculated directly).

Now, we present an alternative equation for curvature using dynamical quantities:

**Theorem 3.3.**

$$\kappa(t) = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3} = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

*Proof.* To verify it, we use the decomposition of  $\vec{a}$ :

$$\begin{aligned}\vec{v} \times \vec{a} &= \vec{v} \times \left( \frac{d^2s}{dt^2} \cdot \vec{T} + \kappa \left( \frac{ds}{dt} \right)^2 \vec{N} \right) \\ &= \frac{d^2s}{dt^2} (\vec{v} \times \vec{T}) + \kappa \left( \frac{ds}{dt} \right)^2 (\vec{v} \times \vec{N}) \\ &= \kappa \left( \frac{ds}{dt} \right)^3 (\vec{T} \times \vec{N}) \\ &= \kappa \left( \frac{ds}{dt} \right)^3 \vec{B}\end{aligned}$$

Then,  $\kappa(ds/dt)^3\|\vec{B}\| = \|\vec{v} \times \vec{a}\|$ . Since  $\vec{B}$  is a unit vector, the desired result has been achieved.  $\square$

## 4 Implicit functions

### 4.1 The Implicit Function Theorem I

Often, we have an *implicit* relationship between variables,

$$F(x_1, x_2, \dots, x_n) = 0,$$

rather than an *explicit* function relation, such as

$$x_n = f(x_1, x_2, \dots, x_{n-1}).$$

**Example 4.1.1.** Look at a familiar example in  $\mathbb{R}^2$ ,

$$x^2 + y^2 = 1.$$

This fails vertical line test ( $y \neq f(x)$ ) as well as horizontal line test ( $x \neq g(y)$ ); globally, this relation does not define a function. Locally, we can write this as a function, i.e. by restricting attention to small pieces of the curve.

First, define

$$F(x, y) = x^2 + y^2 - 1$$

If  $y_0 > 0$ ,  $x_0^2 + y_0^2 = 1$ , i.e.  $F(x_0, y_0) = 0$ , and we look at a window (or *neighborhood*) around  $(x_0, y_0)$ , which lies entirely in the upper half plane, we can solve for  $y = f(x)$ ,

$$y = \underbrace{\sqrt{1 - x^2}}_{f(x)}$$

We could calculate  $y' = f'(x)$  from the explicit formula but we can also get it via *implicit differentiation*:

$$\begin{aligned} \frac{d}{dx} (F(x, f(x))) &= \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot f'(x) \\ &= 2x + 2yf'(x), \end{aligned}$$

so  $f'(x) = -x/y$ .

For a general  $F(x, y) = 0$ , we can solve for  $f'(x)$  where its coefficient

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0,$$

where  $y$  is the variable we want to solve for. This gives the limitation on which we can solve for  $y = f(x)$  locally! For the circle example,

$$\frac{\partial F}{\partial y} = 2y.$$

When  $y = 0$ , the vertical line test fails in every neighborhood of  $(x_0, y_0) = (\pm 1, 0)$ .

In general, suppose we have a  $C^1$  function,

$$F : \mathbb{R}^{n+1} \rightarrow \mathbb{R},$$

and consider all functions of  $F(\vec{x}, y) = 0$ . In order that  $y = g(\vec{x})$ , i.e. we can solve for  $y$  as a differentiable function of  $\vec{x}$ , we do the same implicit differentiation, with the chain rule,

$$\frac{\partial}{\partial x_i} (F(x_1, x_2, \dots, x_n, f(\vec{x}))) = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial y} \frac{\partial f}{\partial x_i}$$

for each  $i = 1, 2, \dots, n$ . We can then solve for each

$$\frac{\partial f}{\partial x_i} = - \frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}},$$

provided  $\partial F / \partial y \neq 0$ . This is a sufficient condition to solve for  $y = f(\vec{x})$ .

**Theorem 4.1** (Implicit Function Theorem I). *Assume  $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is  $C^1$  in a neighborhood of  $(\vec{x}_0, y_0)$  with  $F(\vec{x}_0, y_0) = 0$ . If  $\frac{\partial F}{\partial y}(\vec{x}_0, y_0) \neq 0$ , then there exists neighborhood  $U$  of  $\vec{x}_0$  and  $V$  of  $y_0$  and a  $C^1$  function*

$$f : U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R},$$

for which  $F(\vec{x}, f(\vec{x})) = 0$  for all  $\vec{x} \in U$ . In addition,

$$Df(\vec{x}) = \frac{-1}{\frac{\partial F}{\partial y}(\vec{x}, y)} D_{\vec{x}} F(\vec{x}, y),$$

where

$$D_{\vec{x}} F(\vec{x}, y) = \left[ \frac{\partial F}{\partial x_1} \quad \frac{\partial F}{\partial x_2} \quad \cdots \quad \frac{\partial F}{\partial x_n} \right].$$

**Example 4.1.2.** Consider the following function:

$$xy + y^2 z + z^3 = 1.$$

For which parts on this surface can we write  $z = f(x, y)$ , i.e.

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}, F(x, y, z) = xy + y^2 z + z^3 - 1$$

is  $C^1$ ?

We want to solve for  $z$ , so we look at

$$\frac{\partial F}{\partial z} = y^2 + 3z^2$$

We observe that  $\partial F / \partial z = 0$  iff  $y = 0$  and  $z = 0$ . However,  $y = 0$  and  $z = 0$  is not defined on this surface. At all points on this surface,  $\partial F / \partial z \neq 0$ . So at

every  $(x_0, y_0, z_0)$  with  $F(x_0, y_0, z_0) = 0$ , we can solve for  $z = f(x, y)$  locally near  $(x_0, y_0)$ !

We can then use the implicit differentiation formula in the theorem to calculate  $Df(x, y)$ :

$$D_{(x,y)}F = [y \quad (x + 2yz)],$$

so we get

$$Df(x, y) = \frac{-D_{(x,y)}F}{\partial F/\partial z} = \left[ -\frac{y}{y^2 + 3z^2} \quad -\frac{x + 2yz}{y^2 3z^2} \right]$$

or

$$\nabla f(x, y) = \left( -\frac{y}{y^2 + 3z^2}, -\frac{x + 2yz}{y^2 3z^2} \right).$$

**Example 4.1.3.** Consider the following equation:

$$x^4 + xz^2 + z^4 = 1.$$

Show that we can solve for  $z = g(x)$  near  $(x_1, z_1) = (-1, 1)$  but not near  $(x_2, z_2) = (1, 0)$ .

*Proof.* First, let

$$F(x, z) = x^4 + xz^2 + z^4 - 1.$$

Clearly,  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^1$  for all  $(x, z) \in \mathbb{R}^2$ . Observe that

$$\frac{\partial F}{\partial z} = 2xz + 4z^3,$$

and so  $\partial F(-1, 1)/\partial z \neq 0$ .

By the Implicit Function Theorem, we can solve for  $z = g(x)$  locally near  $(x_1, z_1) = (-1, 1)$ . In addition, we can get an explicit formula for its derivative:

$$Dg(x) = g'(x) = -\frac{\partial F(x, z)/\partial x}{\partial F/\partial z} = -\frac{4x^3 + z^2}{2xz + 4z^3}$$

Finally, since  $\partial F(1, 0)/\partial z = 0$ , the Implicit Function Theorem does not apply near  $(1, 0)$ .  $\square$

**Example 4.1.4.** Consider the following equation:

$$x - z^3 = 0$$

Clearly,  $F(x, z) = x - z^3$  is  $C^1$  for all  $(x, z) \in \mathbb{R}^2$ . Note that

$$\frac{\partial F}{\partial z} = -3z^2.$$

Clearly,  $\partial F/\partial z = 0$  at  $(x, z) = (0, 0)$ . However, we can write  $z = x^{1/3}$  globally. So  $z = g(x) = x^{1/3}$  exists but isn't differentiable at  $(x_0, z_0) = (0, 0)$ .

**Example 4.1.5.** Suppose we have a system of equations with more unknowns:

$$\begin{cases} u^2 - v^2 - x^3 = 0 \\ 2uv - y^5 = 0 \end{cases}$$

Can we solve for  $(u, v)$  as functions of  $(x, y)$ ?

First, consider a  $C^1$  function,  $F : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ , that is defined as follows:

$$\begin{cases} F_1(x, y, u, v) = u^2 - v^2 - x^3 = 0 \\ F_2(x, y, u, v) = 2uv - y^5 = 0 \end{cases}$$

Following what we did before, we can assume  $(u, v) = g(x, y)$  and see when we can calculate  $Dg$ . Note that

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} F_1(x, y, u(x, y), v(x, y)) = \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v} \frac{\partial v}{\partial x} \\ 0 &= \frac{\partial}{\partial x} F_2(x, y, u(x, y), v(x, y)) = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F_2}{\partial v} \frac{\partial v}{\partial x} \end{aligned}$$

Then, we can solve for  $\partial u / \partial x$  and  $\partial v / \partial x$ . Rearranging,

$$\begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_1}{\partial x} \\ -\frac{\partial F_2}{\partial x} \end{bmatrix}$$

This can be solved if  $D_{(u,v)}F$  is invertible, i.e.  $\det [D_{(u,v)}F] \neq 0$ .

Simiarly, we can also solve for  $\partial u / \partial y$  and  $\partial v / \partial y$ . As a result, we get a different linear system to solve but with the same matrix  $[D_{(u,v)}F]$ . The second version of the Implicit Function Theorem says that this is the correct condition to solve for  $g(x)$  in this setting.

## 4.2 The Implicit Function Theorem II

Implicit differentiation allows us to look at an underdetermined system of (non-linear) equations. Given a following function,

$$\begin{aligned} F_1(x_1, \dots, x_n, u_1, \dots, u_m) &= 0 \\ F_2(x_1, \dots, x_n, u_1, \dots, u_m) &= 0 \\ &\vdots \\ F_m(x_1, \dots, x_n, u_1, \dots, u_m) &= 0 \end{aligned}$$

we want to solve for  $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$  as a function,  $\vec{u} = g(\vec{x})$ , of  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Via implicit differentiation, for the case of  $n = m = 2$ , we arrived at an appropriate condition where this is possible.

**Theorem 4.2** (Implicit Function Theorem II - General Form). *Let*

$$F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$$

*be a  $C^1$  function in a neighborhood of  $(\vec{x}_0, \vec{u}_0) \in \mathbb{R}^{n+m}$ , with  $F(\vec{x}_0, \vec{u}_0) = \vec{0}$ . If, in addition,  $D_{\vec{u}}F(\vec{x}_0, \vec{u}_0)$  is invertible, then there exists neighborhoods  $\mathcal{V}$  of  $\vec{x}_0$  and  $\mathcal{U}$  of  $\vec{u}_0$ , for which solutions of  $F(\vec{x}, \vec{u}) = \vec{0}$  lie on a  $C^1$  graph,  $\vec{u} = g(\vec{x})$ ,*

$$g : \mathcal{V} \subset \mathbb{R}^n \rightarrow \mathcal{U} \subset \mathbb{R}^m$$

**Example 4.2.1.** Consider the following set of equations:

$$\begin{cases} 2xu^2 + yv^4 = 2 \\ xy(u^2 - v^2) = 0 \end{cases}$$

Can we solve for  $(u, v) = g(x, y)$  near  $(x_0, y_0, u_0, v_0) = (1, 1, -1, -1)$ ?

Let

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \vec{x} = (x, y), \vec{u} = (u, v),$$

where  $F$  is defined as

$$F_1(\vec{x}, \vec{u}) = 2xu^2 + yv^4 - 2 = 0$$

$$F_2(\vec{x}, \vec{u}) = xy(u^2 - v^2) = 0$$

Then, we get the following Jacobian

$$\begin{aligned} D_{\vec{u}}F &= \frac{\partial(F_1, F_2)}{\partial(u, v)} \\ &= \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix} \\ &= \begin{bmatrix} 4xu & 4yv^3 \\ 2xy & -2vxy \end{bmatrix} \end{aligned}$$

Substituting the given values, we have

$$D_{\vec{u}}F(1, 1, -1, -1) = \begin{bmatrix} -4 & -4 \\ -2 & 2 \end{bmatrix}$$

Since  $\det D_{\vec{u}}F = -16 \neq 0$ , the Implicit Function Theorem does apply, and we can solve for  $\vec{u} = (u, v) = g(\vec{x}) = g(x, y)$  near  $(x_0, y_0, u_0, v_0) = (1, 1, -1, -1)$ .

*Remark.* In general, we can't get an explicit formula for  $g$ , but we can get a formula for  $Dg(x, y)$ , /ie its partial derivatives, using implicit differentiation.

**Example 4.2.2.** Consider the following set of equations:

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

Note that this example fits the form of the Implicit Function Theorem, but it's a special case. We want to invert this relation, i.e. given,  $\vec{x} = f(\vec{u})$ , we want to solve for  $\vec{u} = g(\vec{x})$ .

To get a nice theorem for this special case, we can use the framework of the Implicit Function Theorem:

$$F_1(\vec{x}, \vec{u}) = f_1(\vec{u}) - x = 0, F_2(\vec{x}, \vec{u}) = f_2(\vec{u}) - y = 0$$

Since

$$D_{\vec{u}}F(\vec{x}, \vec{u}) = Df(\vec{u}),$$

we can do this locally near a point  $(\vec{x}_0, \vec{u}_0)$  provided that

$$\det(Df(\vec{u})) \neq 0$$

Note that if we had a linear system,  $\vec{x} = M\vec{u}$ , we can solve  $\vec{u} = M^{-1}\vec{x}$  provided  $M \neq 0$ . This is why we call this derivative matrix,  $Df(\vec{x})$  the *linearization* of  $f(\vec{u})$ .

### 4.3 Inverse Function Theorem

In general, suppose we have a  $C^1$  function,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $\vec{x} = f(\vec{u})$ . How do we solve for  $\vec{u} = g(\vec{x})$ ?

First, let's In single-variable calculus, a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *one-to-one* on an interval  $[a, b]$  if and only if  $f$  is strictly monotone on  $[a, b]$ . For these functions,  $f$  has an inverse  $g = f^{-1}$ ,

$$g(f(x)) = x, \quad \forall x \in [a, b]$$

If  $f$  is differentiable on  $[a, b]$ , and  $f'(x) > 0$  on  $[a, b]$  (or  $f'(x) < 0$  on  $[a, b]$ ), then the inverse  $g(x)$  is also differentiable, and

$$g'(f(x)) = \frac{1}{f'(x)}, \quad \forall x \in [a, b]$$

If, for example,  $f'(x) > 0$  for all  $x \in \mathbb{R}$ , then it's globally invertible, i.e.  $g(f(x)) = x$  for all  $x \in \mathbb{R}$ . How do we apply this for  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $n \geq 2$ ?

**Theorem 4.3** (Inverse Function Theorem). *Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  which is  $C^1$  in a neighborhood of  $\vec{u}_0$ , with  $f(\vec{u}_0) = \vec{x}_0$ . If  $\det(Df(\vec{u}_0)) \neq 0$ , then there exist neighborhoods  $\mathcal{U}$  of  $\vec{u}_0$  and  $\mathcal{V}$  of  $\vec{x}_0$  and a  $C^1$  function  $g : \mathcal{V} \rightarrow \mathcal{U}$ , with*

$$\vec{x} = f(\vec{u}) \iff \vec{u} = g(\vec{x}),$$

*with  $\vec{u} \in \mathcal{U}$                       with  $\vec{x} \in \mathcal{V}$*

i.e. near  $\vec{x}_0$  and  $\vec{u}_0$ ,  $g$  is the inverse of  $f$ .



**Example 4.3.1.** Apply the Inverse Function Theorem to the function that was defined in the previous example:

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

Observe that

$$\det(Df(u, v)) = \det \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4u^2 + 4v^2 \neq 0$$

as long as  $(u_0, v_0) \neq (0, 0)$ . So we can invert the variables and solve for  $(u, v) = g(x, y)$ , locally near any  $(u_0, v_0) \neq (0, 0)$ .

Notice that

$$\begin{aligned} f_1(-u, -v) &= x = f_1(u, v) \\ f_2(-u, -v) &= y = f_2(u, v) \end{aligned}$$

So in any neighborhood of  $(0, 0)$  there are 2 values of  $(u, v)$  corresponding to each  $(x, y)$ . So  $f$  is not invertible near  $(u, v) = (0, 0)$ .

**Example 4.3.2.** Consider the following equations:

$$\begin{cases} x = e^y \cos v \\ y = e^u \sin v \end{cases}$$

For which  $(u, v, x, y)$  can we solve for  $u, v$  as functions of  $x, y$ ?

Call

$$f(u, v) = \begin{bmatrix} e^u \cos v \\ e^u \sin v \end{bmatrix}.$$

Then, we have

$$Df(u, v) \begin{bmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{bmatrix}$$

Then, we can compute  $\det(Df(u, v))$ , (or  $\det\left(\frac{\partial(x, y)}{\partial(u, v)}\right)$ ):

$$\det(Df(u, v)) = e^{2u}.$$

Clearly,  $\det(Df(u, v)) > 0$  for all  $u, v$ . By the Inverse Function Theorem, we can invert and solve for  $(u, v) = g(x, y)$ , near any  $(u_0, v_0)$ .

We can invert locally near any point; can we find a global inverse, i.e. a  $g$  for which  $(u, v) = g(x, y)$  for every  $(u, v) \in \mathbb{R}^2$ ? If so, then  $f$  would have to be a one-to-one function. However,

$$f(u, v + 2\pi k) = f(u, v)$$

for all  $k \in \mathbb{Z}$ . Therefore,  $f$  can't be globally inverted.

**Example 4.3.3.** Consider the following equations:

$$\begin{cases} x = f_1(u, v) = u^3 - 3uv^2 \\ y = f_2(u, v) = -v^3 + 3u^2v \end{cases}$$

Since they're polynomials,  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $C^1$ . Then, we have

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{bmatrix} 3u^2 - 3v^2 & -6uv \\ 6uv & -3v^2 + 3u^2 \end{bmatrix} \\ \det \left( \frac{\partial(x, y)}{\partial(u, v)} \right) &= (3u^2 - 3v^2) + (6uv)^2 \end{aligned}$$

Clearly,  $\det(\partial(x, y)/\partial(u, v)) = 0$  iff  $u = v = 0$ . So, Inverse Function Theorem holds for all  $(u_0, v_0) \neq (0, 0)$ , and we can solve for  $(x, y) = g(u, v)$  around any  $(u_0, v_0) \neq (0, 0)$ .

## 5 Taylor's Theorem

### 5.1 Taylor's Theorem in one dimension

Consider a one-dimensional function:

$$g : \mathbb{R} \rightarrow \mathbb{R},$$

which is  $C^{k+1}$ , i.e. it is  $(k+1)$  times continuously differentiable; i.e., each derivative,

$$\frac{d^j g}{dx^j}(x), j = 1, 2, \dots, k+1, \text{ (of order up to and including the } (k+1)^{\text{st}}),$$

exists and is a continuous function (in some interval). Then, we can approximate  $g(x)$  locally near  $x = a$  by a polynomial of degree  $k$ , *Taylor's polynomial*,  $P_k(x)$ :

$$P_k(x) = g(a) + g'(a)(x-a) + \frac{1}{2!}g''(a)(x-a)^2 + \dots + \frac{1}{k!}\frac{d^k g}{dx^k}(a)(x-a)^k$$

These are chosen to match  $g(x)$  up to the  $k^{\text{th}}$  derivative at  $x = a$ ,

$$\frac{d^j P_k}{dx^j}(a) = \frac{d^j g}{dx^j}(a), j = 0, 1, 2, \dots, k.$$

For example,  $P_1(x) = g(a) + g'(a)(x-a)$  is the tangent line. Since we know that  $g$  is differentiable,

$$\lim_{x \rightarrow a} \frac{|g(x) - P_1(x)|}{|x-a|} = 0 \text{ or } g(x) = P_1(x) + o(|x-a|)$$

**Theorem 5.1** (Taylor's Theorem). *Assume  $g : \mathbb{R} \rightarrow \mathbb{R}$  is  $C^{k+1}$  in a neighborhood  $N$  around  $x = a$ . Then, for each  $x \in N$ , there is a  $c$  between  $a$  and  $x$  for which*

$$g(x) = P_k(x) + \underbrace{\frac{1}{(k+1)!}\frac{d^{k+1}g}{dx^{k+1}}(c)(x-a)^{k+1}}_{\text{Remainder term } R_k(a,x)}$$

Since we assume  $g$  is continuous, we have

$$\lim_{x \rightarrow a} \frac{R_k(a,x)}{|x-a|^k} = 0,$$

i.e.  $R_k(a,x) = o(|x-a|^k)$ . So  $R_k(a,x)$  is small compared with each of the terms in  $P_k(x)$ .

*Remark.* Locally,  $g(x)$  is well approximated by its Taylor polynomials, but only near  $x = a$ .

**Example 5.1.1.** Notice that

$$g(x) = \cos x = 1 - \underbrace{\frac{1}{2}x^2 + 0x^3 + o(x^4)}_{P_3(x)}$$

This tells us that  $\cos x$  is quadratic near  $a = 0$ . However, this clearly doesn't work for  $x$  that is not near  $a = 0$ .

## 5.2 Taylor's Theorem in higher dimensions

Can we apply Taylor's Theorem for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , i.e., approximate a smooth function locally near  $\vec{x} = \vec{a}$  via polynomial? We can do so by restricting our attention to each line,  $\vec{x} = \vec{a} + t\vec{u}$ , through  $\vec{a}$  in direction  $\vec{u}$ .

Assume  $f \in C^3$  near  $\vec{x}_0 \in \mathbb{R}^n$ . Let's sample  $f(\vec{x})$  along a line running through  $x_0$ . Take a unit vector  $\vec{u}$ ,  $\|\vec{u}\| = 1$ , and the line,

$$\vec{l}(t) = \vec{x}_0 + t\vec{u},$$

that goes through  $\vec{x}_0$  at  $t = 0$  in the direction of  $\vec{u}$ . Then, we get

$$g(t) = f(\vec{l}(t)) = f(\vec{x}_0 + t\vec{u}),$$

so  $g : \mathbb{R} \rightarrow \mathbb{R}$ .

By chain rule, if  $f$  is  $C^3$  near  $\vec{x}_0$ , then  $g$  is  $C^3$  near  $t = 0$ . So we use Taylor's Theorem in  $g$ :

$$g(0) = f(\vec{x}_0),$$

$$g'(t) = Df(\vec{x}_0 + t\vec{u}) \cdot \vec{l}'(t) = Df(\vec{x}_0 + t\vec{u})\vec{u}.$$

So  $g'(0) = Df(\vec{x}_0)\vec{u} = \nabla f(\vec{x}_0) \cdot \vec{u}$ . Using coordinates,

$$g'(t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})u_i$$

so

$$g''(t) = \sum_{i=1}^n \underbrace{\frac{d}{dt} \left( \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u}) \right)}_{D \frac{\partial f}{\partial x_i} \cdot \vec{u}} u_i = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0 + t\vec{u})u_j u_i$$

Therefore,

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)u_j u_i$$

Now, call

$$H(\vec{x}) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) \right]_{i,j=1,\dots,n},$$

the *Hessian matrix* of  $f$  at  $\vec{x}_0$ . For  $f$ , a  $C^2$  function,  $f_{x_i x_j} = f_{x_j x_i}$ , so  $H(\vec{x}_0)$  is a symmetric matrix.

So  $g''(0) = \vec{u} \cdot H(\vec{x}_0)\vec{u}$ . Using Taylor's Theorem, for  $g$ , we get:

**Theorem 5.2** (Second order Taylor's approximation). Assume  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and is  $C^3$  in a neighborhood of  $\vec{x}_0$ . Then,

$$\underbrace{f(\vec{x})}_{g(\vec{x}_0+t\vec{u})} = \underbrace{f(\vec{x}_0)}_{g(0)} + \underbrace{Df(\vec{x}_0)(\vec{x} - \vec{x}_0)}_{g'(0)(t-0)} + \frac{1}{2!} \underbrace{(\vec{x} - \vec{x}_0) \cdot H(\vec{x}_0)(\vec{x} - \vec{x}_0)}_{g''(0)(t-0)^2} + R_2(\vec{x}_0, \vec{x}),$$

where

$$H(\vec{x}_0) = D^2 f(\vec{x}_0) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j=1,2,\dots,n}$$

is the hessian and

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \frac{R_2(\vec{x}_0, \vec{x})}{\|\vec{x} - \vec{x}_0\|^2} = 0,$$

i.e.  $R_2(\vec{x}_0, \vec{x}) = o(\|\vec{x} - \vec{x}_0\|^2)$ . Alternatively, the second order Taylor's approximation can be written as

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h} + R_2(\vec{a}, \vec{h}),$$

with

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{R_2(\vec{a}, \vec{h})}{\|\vec{h}\|^2} = 0.$$

**Example 5.2.1.** Find the second order Taylor polynomial of the following functions:

$$f(x, y) = \cos(xy^2)$$

near  $\vec{a} = (\pi, 1)$ .

First, we compute the derivatives:

$$\begin{aligned} f(\vec{a}) &= f(\pi, 1) = \cos(\pi) = 1, \\ \frac{\partial f}{\partial x} &= -y^2 \sin(xy^2), \\ \frac{\partial f}{\partial y} &= -2xy \sin(xy^2), \\ \frac{\partial^2 f}{\partial x^2} &= -y^2 \cos(xy^2) y^2, \\ \frac{\partial^2 f}{\partial x \partial y} &= -2y \sin(xy^2) - 2xy^3 \cos(xy^2), \\ \frac{\partial^2 f}{\partial y^2} &= -2x \sin(xy^2) - 2xy \cos(xy^2). \end{aligned}$$

Then, at  $\vec{a} = (\pi, 1)$ , we find that

$$\begin{aligned} Df(\vec{a}) &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ D^2 f(\vec{a}) &= \begin{bmatrix} 1 & 2\pi \\ 2\pi & 4\pi^2 \end{bmatrix} \end{aligned}$$

So, we have

$$\begin{aligned} f(\vec{a} + \vec{h}) &= -1 + \frac{1}{2} \vec{h} \cdot \begin{bmatrix} 1 & 2\pi \\ 2\pi & 4\pi^2 \end{bmatrix} \vec{h} + R_2 \\ f(\pi + h_1, 1 + h_2) &= -1 + \frac{1}{2} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \cdot \begin{bmatrix} h_1 + 2\pi h_2 \\ 2\pi h_1 + 4\pi^2 h_2 \end{bmatrix} \\ &= -1 + \frac{1}{2} (h_1^2 + 4\pi h_1 h_2 + 4\pi^2 h_2^2) + o(\|\vec{h}^2\|) \end{aligned}$$

In terms of a point  $\vec{x}$  (near  $\vec{a}$ ), we can write  $\vec{x} = \vec{a} + \vec{h}$ , so  $\vec{h} = \vec{x} - \vec{a}$ , and then

$$\cos(xy^2) = -1 + \frac{1}{2} ((x - a_1)^2 + 4\pi(x - a_1)(y - a_2) + 4\pi^2(y - a_2)^2) + R_2.$$

Advantage to the  $f(\vec{a} + \vec{h})$  form is that it makes it easier to guess the behaviour of  $f(\vec{x})$  near  $\vec{x} = \vec{a}$ .

### 5.3 Local minima/maxima

**Definition 5.1.** We say  $\vec{a}$  is a local minimum for  $f$  if there exists an open disk  $D_r(\vec{a})$  for which

$$f(\vec{a}) \leq f(\vec{x})$$

for all  $\vec{x} \in D_r(\vec{a})$ .  $\vec{a}$  is a strict local minimum if

$$f(\vec{a}) < f(\vec{x})$$

for all  $\vec{x} \neq \vec{a}, \vec{x} \in D_r(\vec{a})$ .

**Definition 5.2.** We say  $\vec{a}$  is a local maximum for  $f$  if  $\exists r > 0$  with  $f(\vec{a}) \geq f(\vec{x})$ ,  $\forall \vec{x} \in D_r(\vec{a})$ .  $\vec{a}$  is a strict local max if  $f(\vec{a}) > f(\vec{x})$ ,  $\forall \vec{x} \in D_r(\vec{a}) \setminus \{\vec{a}\}$ .

Note that if  $f$  is differentiable, we have a necessary condition for local maxima and minima.

**Theorem 5.3.** If  $f$  has a local maxima or minima at  $\vec{a}$  and is differentiable at  $\vec{a}$ , then  $Df(\vec{a}) = \vec{0}$ .

*Proof.* Again, we start by restricting to line through  $\vec{a}$ :

$$g(t) = f(\vec{a} + t\vec{u}),$$

where  $\vec{u}$  is a unit vector. If  $f$  has a local minima at  $\vec{a}$ , then

$$g(0) = f(\vec{a}) \leq f(\vec{a} + t\vec{u}) = g(t),$$

for all  $t$  with  $|t| < r$ . So  $g(t)$  has a local minima at  $t = 0$ . By a calculus theorem,  $g'(0) = 0$ . But,

$$0 = g'(0) = Df(\vec{a})\vec{u},$$

for all  $\vec{u}$ . Then, by taking  $\vec{u} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ , we get

$$\frac{\partial f}{\partial x_j}(\vec{a}) = 0,$$

for each  $j = 1, 2, 3, \dots, n$ . Therefore,  $Df(\vec{a}) = 0$ .  $\square$

**Definition 5.3.** An  $\vec{a}$  for which  $Df(\vec{a}) = 0$  is called a critical point.

**Example 5.3.1.** In the previous example,  $\vec{a} = (\pi, 1)$  was a critical point.

Now, we want to combine Taylor's Theorem and linear algebra to classify critical points as local minima, maxima, or others<sup>6</sup>. Taylor's theorem states that for  $\vec{x} = \vec{a} + \vec{h}$ , if  $\|\vec{h}\|$  is small,

$$f(\vec{x}) = f(\vec{a} + \vec{h}) = f(\vec{a}) + \underbrace{Df(\vec{h})}_0 + \underbrace{\frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h}}_{\text{quadratic form}} + \underbrace{R_2(\vec{a}, \vec{h})}_{o(\|\vec{h}\|^2)}$$

So we expect the behaviour of  $f(\vec{x})$  near  $\vec{a}$  to be determined by the quadratic term.

Notice that the Hessian matrix,  $M = D^2 f(\vec{a})$ , is a symmetric matrix. This allows us to apply the following theorem:

**Theorem 5.4** (Spectral Theorem). Assume  $M$  is a symmetric  $(n \times n)$  matrix. Then,

- All eigenvalues of  $M$  are real,  $\lambda_i \in \mathbb{R} \forall i = 1, 2, \dots, n$ .
- There is an orthonormal basis composed of eigenvectors of  $M$ ,

$$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}, M\vec{u}_i = \lambda_i \vec{u}_i, \|\vec{u}_i\| = 1, \vec{u}_i \cdot \vec{u}_j = 0 \text{ for } i \neq j$$

- In the basis of eigenvectors,  $M$  is a diagonal matrix. In other words, if we let  $U$  be the matrix whose columns are the  $\vec{u}_i$ ; then

$$MU = U\Lambda,$$

where  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ .

---

<sup>6</sup> **Cramer's Rule.** Given a system of linear equations that is represented by  $2 \times 2$  matrices,

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases},$$

solution of the system is given by

$$x = \frac{\det \begin{pmatrix} s & b \\ t & d \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}, y = \frac{\det \begin{pmatrix} a & s \\ c & t \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

*Remark.* Note that since the eigenvalues are real, they can be ordered, smallest to largest:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

However, they may not be necessarily distinct.

Written in the orthonormal basis of eigenvalues, the quadratic form,  $\vec{h} \cdot M\vec{h}$ , has an easy expression. First, we write

$$\vec{h} = \sum_{i=1}^n c_i \vec{u}_i, \quad c_i \in \mathbb{R}, \quad \forall i = 1, 2, \dots, n$$

Notice that

$$\|\vec{h}\| = \sqrt{\sum_{i=1}^n c_i^2}$$

Then, we have:

$$\begin{aligned} \vec{h} \cdot M\vec{h} &= \vec{h} \cdot \sum_{i=1}^n c_i M\vec{u}_i \\ &= \vec{h} \cdot \sum_{i=1}^n \lambda_i c_i \vec{u}_i \\ &= \sum_{i=1}^n \lambda_i c_i (\underbrace{\vec{h} \cdot \vec{u}_i}_{c_i}) \\ &= \sum_{i=1}^n \lambda_i c_i^2 \end{aligned}$$

**Theorem 5.5.** Suppose  $M$  is a symmetric matrix with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Then,

$$\lambda_1 \|\vec{h}\|^2 \leq \vec{h} \cdot M\vec{h} \leq \lambda_n \|\vec{h}\|^2$$

*Proof.* First, we have

$$\begin{aligned} \vec{h} \cdot M\vec{h} &= \sum_{i=1}^n \lambda_i c_i^2 \\ &\leq \sum_{i=1}^n \lambda_n c_i^2 \\ &= \lambda_n \sum_{i=1}^n c_i^2 = \lambda_n \|\vec{h}\|^2, \end{aligned}$$

which proves the right hand inequality. For the left hand one,

$$\vec{h} \cdot M\vec{h} \geq \sum_{i=1}^n \lambda_1 c_i^2 = \lambda_1 \|\vec{h}\|^2.$$



This proves both sides of the inequality.  $\square$

Now we apply this idea to the Hessian via Taylor's Theorem to get the following theorem:

**Theorem 5.6** (Second derivative test). *Suppose  $f$  is  $C^3$  in a neighborhood of a critical point  $\vec{a}$ . Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  be the eigenvalues of  $D^2f(\vec{a})$ . Then,*

1. *If all eigenvalues are positive, then  $\vec{a}$  is a strict local minima of  $f$ .*
2. *If all eigenvalues are negative, then  $\vec{a}$  is a strict local maxima of  $f$ .*
3. *If  $D^2f(\vec{a})$  has at least one positive and at least one negative eigenvalue, then  $\vec{a}$  is a saddle point. In other words,  $\exists r_0 > 0$  for which in  $D_r(\vec{a})$ ,  $0 < r < r_0$ , there are points with  $f(\vec{x}) > f(\vec{a})$  and points with  $f(\vec{x}) < f(\vec{a})$ .*

*Proof.* Let's verify (1). By Taylor's Theorem, with  $\vec{x} = \vec{a} + \vec{h}$ , we have

$$f(\vec{x}) = f(\vec{a} + \vec{h}) = f(\vec{a}) + \frac{1}{2} \vec{h} \cdot D^2f(\vec{a})\vec{h} + R_2(\vec{a}, \vec{h})$$

Notice that

$$\frac{1}{2} \vec{h} \cdot D^2f(\vec{a})\vec{h} \geq \frac{1}{2} \lambda_1 \|\vec{h}\|^2,$$

where  $\lambda_1$  is the smallest eigenvalue.

Now, we look at the rectangular term:  $R_2(\vec{a}, \vec{h}) = o(\|\vec{h}\|^2)$ . By taking  $\epsilon = \lambda_1/4 > 0$ , there exists  $\delta > 0$  for which

$$\frac{|R_1|}{\|\vec{h}\|^2} < \epsilon = \frac{1}{4} \lambda_1,$$

if  $0 < \|\vec{h}\| < \delta$ , i.e. if  $\vec{h} \in D_\delta(\vec{0})$ , then  $|R_2| < \frac{1}{4} \lambda_1 \|\vec{h}\|^2$ . This implies that  $R_2 > -\frac{1}{4} \lambda_1 \|\vec{h}\|^2$ .

Combining these two results with Taylor expansion, if  $\vec{x} \in D_\delta(\vec{a})$ ,  $\vec{h} \in D_\delta(\vec{0})$ , we get

$$\begin{aligned} f(\vec{x}) &\geq f(\vec{a}) + \frac{1}{2} \lambda_1 \|\vec{h}\|^2 - \frac{1}{4} \lambda_1 \|\vec{h}\|^2 \\ &\geq f(\vec{a}) + \frac{1}{4} \lambda_1 \|\vec{h}\|^2 \\ &> f(\vec{a}) \end{aligned}$$

if  $\vec{h} \neq 0$ ,  $\vec{h} \in D_\delta(\vec{0})$ , i.e.  $\vec{x} \in D_\delta(\vec{a})$ .  $\square$

*Remark.* When  $D^2f(\vec{a})$  has zero as an eigenvalue, things can get complicated. For example, if  $\lambda_i \geq 0$  for all  $i$ , you *could* still have a local minima. In this case, the behaviour would be determined by higher order terms in Taylor Series. We call this *Degenerate critical point*.

**Example 5.3.2.** Consider

$$f(x, y, z) = x^3 - 3xy + y^3 + \cos z$$

Find all critical points and classify them using the Hessian.

First, observe that

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3y \\ \frac{\partial f}{\partial y} = -3x + 3y^2 \\ \frac{\partial f}{\partial z} = -\sin z \end{cases}$$

Critical points are defined as  $\nabla f(\vec{a}) = \vec{0}$  so we get the following critical points

$$(0, 0, n\pi), (1, 1, n\pi),$$

where  $n \in \mathbb{Z}$ .

Then, we want to compute the Hessian at each point.

$$D^2 f(\vec{a}) = \begin{bmatrix} 6x & -3 & 0 \\ -3 & 6y & 0 \\ 0 & 0 & -\cos z \end{bmatrix}$$

Notice that at  $(0, 0, n\pi)$ , we get

$$\begin{aligned} D^2 f(0, 0, 2k\pi) &= \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ D^2 f(0, 0, (2k+1)\pi) &= \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

When  $n$  is even, we find that its eigenvalues are

$$\lambda = -3, -1, 3$$

so we get a saddle at  $(0, 0, 2k\pi)$ ,  $k \in \mathbb{Z}$ . Similarly, when  $n$  is odd, we find that its eigenvalues are

$$\lambda = -3, 1, 3$$

which is also a saddle. Thus, we get a saddle at  $(0, 0, n\pi)$  for all  $n \in \mathbb{Z}$ .

At  $(1, 1, n\pi)$ , we get

$$D^2 f(1, 1, n\pi) = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & (-1)^{n+1} \end{bmatrix}$$

By observation, we find that  $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is an eigenvector with  $\lambda = (-1)^{n+1}$ .

Then, the two eigenvalues are eigenvalues of  $\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ . Since its trace is 12

and determinant is 27, its characteristic equation is given by

$$\lambda^2 - 12\lambda + 27 = 0.$$

So we find that two other eigenvalues are  $\lambda = 3, 9$ . Therefore,  $(1, 1, (2k+1)\pi)$  is a local minima, and  $(1, 1, 2k\pi)$  are saddles.

**Example 5.3.3.** Consider

$$f(x, y) = x^2 + y^4$$

We find that

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix}$$

so we get only one critical point,  $(x, y) = (0, 0)$ . Notice that

$$D^2f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} \text{ so } D^2f(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

So we get  $\lambda = 2, 0$ . Since the quadratic doesn't dominate the remainder, we call this a *degenerate* case.

Still,  $f(0, 0) < f(x, y)$  for all  $(x, y) \neq (0, 0)$  so it's a minima even if the Hessian test doesn't tell us so.

**Example 5.3.4.** Consider

$$g(x, y) = x^2 - y^4$$

This has the same second order Taylor expansion as the previous example but has a different remainder,  $R_2 = -y^4$ . This is a degenerate saddle.

Notice that for the converse, eigenvalues don't have to be strictly larger or smaller than 0. In other words, if  $\vec{a}$  is a local minima, then  $\vec{a}$  is a critical point and all the eigenvalues of  $D^2f(\vec{a})$  must be greater than or equal to 0 (not necessarily strictly greater than 0).

## 6 Calculus of Variations

### 6.1 Single variable

In this section, we look at extremal problems, i.e. maxima, minima, and saddle points, where the unknown is a function (of one or several variables) which should optimize some real-valued expression.

**Example 6.1.1.** Set  $[a, b] \subset \mathbb{R}$  and choose values  $c, d \in \mathbb{R}$ . Consider all  $C^2$  functions,  $u(x)$ , joining  $P_1 = (a, c)$  and  $P_2 = (b, d)$ , i.e.  $u(a) = c$  and  $u(b) = d$ . Among all  $C^2$  curves,  $u(x)$ , connecting  $P_1$  to  $P_2$ , find the one with shortest arclength.

Let  $\vec{c}(t) = (t, u(t))$ . Then, we have

$$\|\vec{c}'(t)\| = \sqrt{1 + u'(t)^2}.$$

This allows us to compute the arclength:

$$I(u) = \int_a^b \sqrt{1 + u'(x)^2} dx.$$

Now, call  $\mathcal{A} = \{u : [a, b] \rightarrow \mathbb{R} \mid u \in C^2, u(a) = c, u(b) = d\}$ . Then,  $I : \mathcal{A} \rightarrow \mathbb{R}$  is a function of functions, or *functional*. We want to minimize  $I(u)$  over all  $u \in \mathcal{A}$ .

**Example 6.1.2.** For the same class  $\mathcal{A}$  of functions, take  $u \in \mathcal{A}$  and rotate around the  $x$  axis, creating a surface of revolution. Which  $y = u(x)$  generate the surface of least area?

This time, we let

$$I(u) = 2\pi \int_a^b u(x) \sqrt{1 + u'(x)^2} dx.$$

A function which is a local minima of this  $I(u)$  is called a *minimal surface*.

**Example 6.1.3.** Find  $y = u(x)$  for which one object sliding along the curvature from point  $P_1$  to  $P_2$  in the shortest time (with gravity).

For this problem, it is convenient to chose  $P_1 = (0, 0)$ . Then, we want to minimize

$$I(u) = \int_0^b \frac{\sqrt{1 + u'(x)^2}}{\sqrt{2g(u(x))}} dx$$

So the idea is to think of  $u$  as vectors and use analogy to  $f(\vec{a})$ . Recall that a local maxima or minima of a function  $g(t) = f(\vec{a} + t\vec{v})$  should have a critical point at  $t = 0$ :

$$0 = g'(0) = Df(\vec{x})\vec{v},$$

for all directions  $\vec{v}$ . We can apply the same idea for  $I(u)$ . We assume  $u$  is a local minima or maxima for  $I(u)$ , and let  $g(t) = I(u + tv)$ , where  $v$  is a function which creates *variation* of  $u$ .

Problem is that we minimize (or maximize) under the assumption that  $u$  connects  $P_1$  to  $P_2$ , i.e.  $(i + tv) \in a$  is required and end points cannot change. So we insist that  $v \in C^1$  and  $v(a) = v(b) = 0$ .

Now, call  $\mathcal{A}_0$  the set of all  $v(x)$  (notice that  $\mathcal{A}_0$  is a vector space). Then, if  $g(t) = I(u + tv)$  and  $I(u)$  has an extreme value, we get

$$0 = g'(0) = \left. \frac{d}{dt}(I(u + tv)) \right|_{t=0}.$$

This equation is called the *first variation* and must hold for all variations,  $v \in \mathcal{A}_0 = \{v \in C^1 | v(a) = v(b) = 0\}$ .

**Example 6.1.4.** Solve example 6.1.1.

We had  $I(u) = \int_a^b \sqrt{1 + u'(x)^2} dx$ . Then, we have

$$\begin{aligned} g'(0) &= \left. \frac{d}{dt}(I(u + tv)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left( \int_a^b \sqrt{1 + (u' + tv')^2} dx \right) \right|_{t=0} \\ &= \int_a^b \left. \frac{d}{dt} \sqrt{1 + (u' + tv')^2} \right|_{t=0} dx \\ &= \int_a^b \frac{1}{2} (1 + (u' + tv')^2)^{-1/2} \cdot 2(u' + tv') \cdot v' dx \Big|_{t=0} \\ &= \int_a^b \frac{u'(x)v'(x)}{\sqrt{1 + u'(x)^2}} dx = 0, \end{aligned}$$

for all  $v \in \mathcal{A}_0$ . So we can call  $DI(u)v = 0$  its *critical point equation*.

Now we want to express the critical point equation in terms of  $v(x)$  instead of  $v'(x)$ . To do so, we are going to integrate by parts:

$$\begin{aligned} \int_a^b \frac{u'(x)v'(x)}{\sqrt{1 + u'(x)^2}} dx &= \left[ \frac{u'}{\sqrt{1 + (u')^2}} v(x) \right]_a^b - \int_a^b \frac{d}{dx} \left( \frac{u'}{\sqrt{1 + (u')^2}} \right) v(x) dx \\ &= \int_a^b -\frac{d}{dx} \left( \frac{u'}{\sqrt{1 + (u')^2}} \right) v(x) dx \\ &= 0 \end{aligned}$$

Notice that the equation holds for all  $v \in \mathcal{A}_0$ . This implies that the derivative of  $u'/\sqrt{1 + (u')^2}$  must be identically equal to 0 on  $[a, b]$  by the following lemma:

**Lemma 6.1** (The Fundamental Lemma of the Calculus of Variations). *Assume  $h(x)$  is continuous on  $[a, b]$  and  $\int_a^b h(x)v(x)dx = 0$  for all  $v \in \mathcal{A}_0$ . Then,  $h(x) \equiv 0$  on  $[a, b]$ .*

*Proof.* Assume  $\int_a^b h(x)v(x)dx = 0$  for all  $v \in \mathcal{A}_0$ . Let  $h(x_0) \neq 0$  for some  $x_0$  and assume that  $h(x) > 0$  on an interval  $(\alpha, \beta)$ , which contains  $x_0$ .

Now, choose  $v(x)$  with  $v(x) > 0$  in  $(\alpha, \beta)$  and zero outside. Then, we have

$$h(x)v(x) = \begin{cases} > 0 & \text{in } (\alpha, \beta) \\ = 0 & \text{outside } (\alpha, \beta) \end{cases}$$

So

$$\begin{aligned} 0 &= \int_a^b h(x)v(x)dx \\ &= \int_\alpha^\beta h(x)v(x)dx > 0, \end{aligned}$$

yielding a contradiction. Therefore,  $h(x) = 0$  for all  $x \in [a, b]$ .  $\square$

Going back to the critical point equation, we get a second order ODE, which is also referred to as *the Euler-Lagrange Equation*:

$$-\frac{d}{dx} \left( \frac{u'}{\sqrt{1 + (u')^2}} \right) = 0$$

Then, we get

$$\frac{u'}{\sqrt{1 + (u')^2}} = C,$$

where  $C$  is a constant. Solving, we find that  $u'(x) = C_1 = \pm \sqrt{C^2/(1 - C^2)}$ , which yields

$$u(x) = C_1 x + C_2.$$

Therefore, we can conclude that a straight line is the path of least arc length.

**Example 6.1.5.** Solve example 6.1.2.

We were given

$$I(u) = 2\pi \int_a^b u(x) \sqrt{1 + (u'(x))^2} dx$$

Notice that this functional has the form of

$$I(u) = \int_a^b F(u', u, x) dx,$$

where  $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ . For surface area, we have  $F(p, u, x) = 2\pi u(1 + p^2)$

In general, the first variation is given by

$$\begin{aligned} 0 &= \frac{d}{dt} (I(u + tv)) \Big|_{t=0} \\ &= \frac{d}{dt} \left( \int_a^b F(u'(x) + tv'(x), u(x) + tv(x), x) dx \right) \Big|_{t=0} \\ &= \int_a^b \left( \frac{\partial F}{\partial p} v'(x) + \frac{\partial F}{\partial u} v(x) \right) dx \end{aligned}$$

where  $p = u'(x) + tv'(x)$ . Now, we apply integration by parts to obtain

$$\begin{aligned}\int_a^b \left( \frac{\partial F}{\partial p} v'(x) + \frac{\partial F}{\partial u} v(x) \right) dx &= \left[ \frac{\partial F}{\partial p} v(x) \right]_a^b - \int_a^b \left[ \frac{d}{dx} \left( \frac{\partial F}{\partial p} \right) v - \frac{\partial F}{\partial u} v \right] dx \\ &= \int_a^b \left[ \frac{d}{dx} \left( \frac{\partial F}{\partial p} \right) - \frac{\partial F}{\partial u} \right] v dx = 0\end{aligned}$$

Then, by the Fundamental Lemma of the Calculus of Variation, we get

$$\frac{d}{dx} \left( \frac{\partial F}{\partial p} (u'(x), u(x), x) \right) - \frac{\partial F}{\partial u} (u'(x), u(x), x) = 0,$$

which is the general solution to the Euler-Lagrange Equation.

Here, we also present an alternate derivation of a general solution to the Euler-Lagrange equation. Assume  $F = F(p, u)$  but not explicitly a function of  $x$ . Then, we can use the following trick. Notice that

$$\begin{aligned}\frac{d}{dx} \left[ \frac{\partial F}{\partial p} (u', u) u'(x) - F(u', u) \right] &= \frac{d}{dx} \left( \frac{\partial F}{\partial p} \right) u' + \frac{\partial F}{\partial p} u'' - \left( \frac{\partial F}{\partial p} u + \frac{\partial F}{\partial u} u' \right) \\ &= \left[ \frac{d}{dx} \left( \frac{\partial F}{\partial p} \right) - \frac{\partial F}{\partial u} \right] u' = 0\end{aligned}$$

Thus, by integrating the last equation, we get

$$\frac{\partial F}{\partial p} (u', u) u' - F(u', u) = C$$

Now, we apply this to the surface area equation. Since  $F(p, u) = 2\pi u \sqrt{1 + p^2}$ , we have

$$\begin{aligned}C &= \frac{\partial F}{\partial p} (u', u) u' - F(u', u) \\ &= 2\pi u \frac{u'}{\sqrt{1 + (u')^2}} - 2\pi u \sqrt{1 + (u')^2} \\ &= 2\pi \left( \frac{u}{\sqrt{1 + (u')^2}} \right)\end{aligned}$$

By squaring both sides, we get

$$\begin{aligned}\frac{C^2}{4\pi^2} &= \frac{u^2}{1 + (u')^2} \\ u' &= \sqrt{\frac{u^2}{C_1^2} - 1}\end{aligned}$$

Since this is a separable equation, we can solve it by substituting  $u/c_1 = \cosh \theta$ :

$$\begin{aligned}\int \frac{du}{\sqrt{u^2/c_1^2 - 1}} &= \int 1 dx \\ \int \frac{c_1 \sinh \theta}{\sinh \theta} d\theta &= x + c_2 \\ c_1 \cosh^{-1} \left( \frac{u}{c_1} \right) &= c_1 \theta = x + c_2\end{aligned}$$

Therefore, we get

$$u(x) = c_1 \cosh \left( \frac{x + c_2}{c_1} \right)$$

We call this surface of revolution a *Catenoid* – a minimal surfaces!

**Example 6.1.6** (The brachistochrone). A brachistochrone is a curve on which motion downward from  $P_1$  to  $P_2$  travels in least time.

To find the brachistochrone curve, we first write the function we want to minimize:

$$I(u) = \int_0^b \frac{\sqrt{1 + (u'(x))^2}}{\sqrt{2gu(x)}} dx,$$

for  $u \in \mathcal{A}$ . We can rewrite this as

$$F(p, u) = \frac{\sqrt{1 + p^2}}{\sqrt{2gu}}.$$

Since this is of the same special form, we can use the *integrated* form of the Euler-Lagrange equation!

$$\begin{aligned}C &= \frac{\partial F}{\partial p}(u', u, x)u' - F(u', u) \\ &= \frac{u'}{\sqrt{1 + (u')^2} \sqrt{2gu}} u' - \frac{\sqrt{1 + (u')^2}}{\sqrt{2gu}} \\ &= \frac{-1}{\sqrt{1 + (u')^2} \sqrt{2gu}}\end{aligned}$$

Now, we take the reciprocal and take the square on both sides:

$$u(1 + (u')^2) = \frac{1}{2gc^2} = k$$

We get a separable first order ODE. Unfortunately, the integral needed cannot be expressed in *closed form*. However, we can still write it as a parametric curve:

$$\begin{cases} x = \frac{k}{2}(t - \sin t) \\ y = \frac{k}{2}(1 - \cos t) \end{cases}, t \in \mathbb{R}$$



Notice that  $(\sin t, \cos t)$  describes a circle whereas  $(t, 1)$  describes a horizontal motion to the right with constant speed of 1. So we get a *cycloid*.

Notice that

$$u'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

Then,

$$\begin{aligned} u(1 + (u')^2) &= \frac{k}{2}(1 - \cos t) \left( 1 + \frac{(1 - \cos^2 t)}{(1 - \cos t)^2} \right) \\ &= k \end{aligned}$$

## 6.2 Multi-variable

We wish to expand the idea to a function of several variables.

**Definition 6.1** (Closure). *Let  $D$  be a bounded, open set in  $\mathbb{R}^n$  with smooth boundary  $\partial D$  ( $\partial D$  is a closed curve for  $D \subset \mathbb{R}^2$  and is a surface for  $D \subset \mathbb{R}^3$ ). Then,  $\bar{D} = D \cup \partial D$  is the closure of  $D$  (a closed set).*

For  $C^1$  functions  $u : \bar{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , we define a functional,

$$I(u) = \int_D F(\nabla u(\vec{x}), u(\vec{x}), \vec{x}) dx_1 \cdots dx_n$$

Then, the task is to minimize (or maximize)  $I(u)$  over some class of  $C^1$  function  $u(\vec{x})$ .

For example, all having the same given boundary values,

$$u(\vec{x}) = g(\vec{x}), \forall \vec{x} \in \partial D$$

is the boundary condition where  $g$  is a fixed, given function. We'll do as we did before: define a class of *variations*,  $v(\vec{x})$ , and look at the directional derivative of  $I$ ,

$$\left. \frac{d}{dt} I(u + tv) \right|_{t=0} = 0.$$

Let's start with an example:

**Example 6.2.1.** In  $\mathbb{R}^2$ , let the bounded, open set  $D$  represent an elastic membrane, at rest. Assume that the membrane is attached to its  $\partial D$  in the  $z = 0$  plane, like a drum skin to its rim. We then apply a loading force (i.e., a weight) to the membrane, represented by  $f(x, y)$ .

Call  $u(x, y)$  the graph of the membrane shape, under loading. According to linear elasticity theory, the equilibrium shape of the membrane minimizes the elastic energy functional,

$$I(u) = \iint_D \left[ \frac{1}{2} \|\nabla u\|^2 - f(x, y)u(x, y) \right] dx dy,$$

over all  $u \in C^1(\bar{D})$  with  $u(x, y) = 0, \forall (x, y) \in \partial D$ .

Let  $\mathcal{A}_0 = \{v \in C^1(\bar{D}) | v(\vec{x}) = 0, \forall \vec{x} \in \partial D\}$ , the class which we're minimizing but also the usual class of admissible variations. So, if  $u, v \in \mathcal{A}_0$  and  $t \in \mathbb{R}$ , then  $u + tv \in \mathcal{A}_0$ . As before, if  $u$  is extremal for  $I(u)$ , then

$$\left. \frac{d}{dt} I(u + tv) \right|_{t=0} = 0, \forall v \in \mathcal{A}_0$$

Now, let's calculate it directly (rather than develop of general formula for the  $F(\nabla u, u, \vec{x})$  case):

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \iint_D \left[ \frac{1}{2} \|\nabla u + t\nabla v\|^2 - f(u + tv) \right] dx dy \\ &= \iint_D \left. \frac{d}{dt} \right|_{t=0} \left[ \frac{1}{2} \|\nabla u\|^2 + t\nabla u \cdot \nabla v + \frac{t^2}{2} \|\nabla v\|^2 - fu - tfv \right] dx dy \\ &= \iint_D \left[ \nabla u \cdot \nabla v - fv + t\|\nabla v\|^2 \right] \Big|_{t=0} dx dy \end{aligned}$$

So we get

$$0 = \iint_D [\nabla u \cdot \nabla v - fv] dx dy, \forall v \in \mathcal{A}_0$$

Before we proceed, we introduce integration by parts for multiple integrals:

**Lemma 6.2.** Suppose  $D \subset \mathbb{R}^n$  ( $n = 2$  or  $3$ ) with smooth  $\partial D$ ,  $u \in C^1(\bar{D})$ , and  $\vec{A} : \bar{D} \rightarrow \mathbb{R}^n$  is a  $C^1$  vector field. Let  $\vec{n}$  be the unit exterior normal vector to  $\partial D$ . Then,

$$\iint_D \vec{A}(\vec{x}) \cdot \nabla u dx dy = \int_{\partial D} \underbrace{u \vec{A}(\vec{x}) \cdot \vec{n} ds}_{\text{line integral}} - \iint_D u \operatorname{div}(\vec{A}) dx dy.$$

In  $3D$ ,

$$\iiint_D \vec{A} \cdot \nabla u dx dy dz = \iint_{\partial D} \underbrace{u \vec{A}(\vec{x}) \cdot \vec{n} dS}_{\text{surface integral}} - \iiint_D u \operatorname{div}(\vec{A}) dx dy dz$$

*Proof.* This comes from combining a product rule,

$$\operatorname{div}(\vec{A}u) = \vec{A} \cdot \nabla u + \operatorname{div}(\vec{A})u$$

with the Divergence Theorem,

$$\iint_D \operatorname{div}(\vec{A}u) dx dy = \int_{\partial D} (\vec{A}u) \cdot \vec{n} ds.$$

We can then double integrate both sides of the identity and rearrange to get the integration by parts formula.  $\square$

Continuing with our example, integrate  $\iint_D \nabla u \cdot \nabla v dx dy$  by parts:

$$0 = \int_{\partial D} \underbrace{v}_{=0} \nabla u \cdot \vec{n} ds - \iint_D v \operatorname{div}(\nabla u) dx dy - \iint_D f v dx dy$$

So for all  $v \in \mathcal{A}_0$ ,  $\Delta u = \operatorname{div}(\nabla u)$  is the Laplacian:

$$0 = - \iint_D [\Delta u + f(x, y)] v dx dy$$

By FLCoV, we find that

$$\Delta u(x, y) + f(x, y) = 0$$

at each  $(x, y) \in D$ .

*Remark.* The expansion we did is a Taylor polynomial,

$$I(u + tv) = I(u) + tDI(u)v + \frac{1}{2}t^2 \iint_D \|\nabla v\|^2 dx dy,$$

so the solution  $u$  is a minimum!

## 7 Fourier Series

Fourier series was developed by using solutions to the heat equation of a special form,

$$e^{kt} \sin(wx),$$

and linear combinations of functions of this form. Fourier claimed that any function can be expressed as a series of functions of this form:

$$\sum_{n=1}^{\infty} \sin(w_n x),$$

where  $w_n$  is a constant. Fourier was right; almost any functions can be expressed as a series of the following function:

$$\sum_{n=1}^{\infty} e_n \sin(nw_0 x)$$

Notice that  $f(x) = A \sin(wx + \varphi)$  is a periodic function with period  $T = 2\pi/w$ , i.e.  $f(x + T) = f(x)$  for all  $x \in \mathbb{R}$ . Similarly, if  $k \in \mathbb{N}$ , then  $f_k(x) = A \sin(kwx + \varphi)$  is also  $T$ -periodic. So any linear combination of these  $f_k(x)$  are  $T$ -periodic and we can't represent non-periodic function by Fourier series, at least not on the whole real line  $\mathbb{R}$ . So the solution is to restrict to an interval of length  $T$ .

*Remark.* If  $f(x)$  is  $T$ -periodic, it's enough to study it on any interval of length  $T$ , i.e.  $[a, a + T]$ , for any  $a$ . By this identity, we get

$$A \sin(wx + \varphi) = \alpha \cos(wx) + \beta \sin(wx),$$

where  $A = \sqrt{\alpha^2 + \beta^2}$  and  $\alpha = A \sin \varphi, \beta = A \cos \varphi$ .

It will be convenient to use both  $\sin$  and  $\cos$  and use a symmetric interval of length  $T = 2L$  where  $x \in [-L, L]$ . We follow Fourier and look at integer multiple of the frequency,  $w = \pi/L$ , i.e.

$$\alpha_k \cos\left(\frac{k\pi x}{L}\right) + \beta_k \sin\left(\frac{k\pi x}{L}\right),$$

where  $k = 1, 2, 3, \dots$  as our basic building blocks. Using these building blocks, we want to answer 2 questions: (1) given a function  $f(x)$  on  $[-L, L]$ , can we choose the coefficients  $\alpha_k, \beta_k$  such that  $f$  is represented by the following form?

$$\alpha_0 + \sum_{k=1}^{\infty} \left( \alpha_k \cos\left(\frac{k\pi x}{L}\right) + \beta_k \sin\left(\frac{k\pi x}{L}\right) \right)$$

(2) does the series converge to  $f(x)$  and in what sense does it converge?

## 7.1 Orthogonality

Consider the vector space of all continuous,  $T$ -periodic functions:

$$f : \mathbb{R} \rightarrow \mathbb{R},$$

which we call  $C_T$ . We introduce an inner product (or scalar product) on  $C_T$ :

**Definition 7.1** (Inner product).  $\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$

We can easily verify the symmetry and linearity of the inner product. Now, we also define a norm:

**Definition 7.2** (Norm).  $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-L}^L (f(x))^2 dx}$

Using this definition of norm, we can define distance between  $f$  and  $g$  as  $\|f(x) - g(x)\|$ , which is equal to square root of area under the graph of  $(f(x) - g(x))^2$ .

This definition of norm should have the property that  $\|f\| = 0$  iff  $f(x) = 0$ . This is true by an arrangement which is similar to the FLCoU!

With this definition of inner product, we find that

$$\left\{1, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \mid k = 1, 2, 3, \dots\right\}$$

is an orthogonal family of functions. In other words,

$$\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = 0$$

for  $m \neq k$ .

## 7.2 Fourier Series

So the idea is to take a function  $f(x)$  defined on an interval of length  $T = 2L$ , typically  $(-L, L)$  and associate to it a series of trigonometric function of period  $T = 2L$ :

$$\begin{aligned} f(x) &\sim S(x) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right) \end{aligned}$$

Further, we want to derive the coefficients  $a_k$  and  $b_k$  from  $f(x)$  and test whether the series converges or not.

To find the coefficients, we use orthogonality. Define an inner product on  $C_T$ , a set of continuous functions with period  $T$ :

$$\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx.$$

Then,  $f \perp g$  when  $\langle f, g \rangle = 0$  and we find that

$$\left\{ \frac{1}{2}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \mid k = 1, 2, 3, \dots \right\}$$

is an orthogonal family on  $C_T$ . Further, we find that

$$\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \left\langle \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \begin{cases} \sqrt{L} & m = k \\ 0 & m \neq k \end{cases}$$

Recall for vectors  $u, v \in \mathbb{R}^n$ , the projection of  $v$  onto  $u$  is

$$\frac{v \cdot u}{\|u\|^2} u.$$

By analogy, we get

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(x) dx \\ a_k &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx \\ b_k &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx \end{aligned}$$

So the Fourier series of  $f(x)$  is

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right).$$

If we look at a partial sum of this series,

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n \left( a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right),$$

we find that  $S_n(x)$  is a trigonometric polynomial. Then, for large values of  $n$ , this should be a good approximation to  $f(x)$  on  $(-L, L)$ .

**Example 7.2.1.** Consider  $f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x)$  with  $L = \pi$  on  $[-\pi, \pi]$ . Then,

$$\begin{cases} a_0 = 1, a_1 = 0, a_2 = 1/2, a_{k \geq 3} = 0 \\ b_k = 0, \forall k \end{cases}$$

Any function of the form  $\cos^m x, \sin^m x$  or  $\cos^m x \sin^m x$  is a trig polynomial and it can be written as a linear combination using trigonometric identities.

**Example 7.2.2.** Consider  $f(x) = x^2$  for  $x \in (-\pi, \pi)$ . Then, we get

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(kx) dx = 0, \forall k \in \mathbb{N} \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3} \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx = \dots = \frac{4}{k^2} (-1)^k \end{aligned}$$

So

$$f(x) \sim S(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx)$$

**Example 7.2.3.** Consider  $f(x) = x^2$  where  $x \in (0, 2\pi)$ . This periodic extension is different from the one above. Especially, this extension is discontinuous at all  $2\pi m$ . So even in the best case, where the series,  $S(x)$ , converges everywhere, it's discontinuous function at  $x = 2\pi m$ , for all  $m \in \mathbb{Z}$ . Then does it converge there, and if so, to which values?

We calculate the Fourier coefficients, remembering that a  $2\pi$ -periodic function can be integrated over any interval over length  $2\pi$ , and give the same answer. So

$$a_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(kx) dx$$

In this case, we get  $a_0 = 8\pi^2/3$ ,  $a_k = 4/k^2$ ,  $b_k = -4\pi/k$ . Since  $b_k \neq 0$ , we know that this function is not symmetric.

**Example 7.2.4.** Consider  $f(x) = x$  where  $x \in (-\pi, \pi)$  and  $L = \pi$ . Then, we get

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(kx) dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin(kx) dx$$

$$= \frac{2}{k} (-1)^{k+1}$$

So as  $f(x) = x$  on  $(-\pi, \pi)$  is odd, we have a sine series:

$$f(x) \sim S(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

$S(x)$  has discontinuities,  $f(-\pi) = -\pi \neq \pi = f(\pi)$ . So what happens at the discontinuities,  $x = (2m-1)\pi$ ?

$$S(n\pi) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kn\pi) = \sum_{k=1}^{\infty} 0 = 0$$

It will be a general fact that at a *jump discontinuity*, the Fourier series,  $S(x)$ , takes the midpoint value. It does not choose the value of  $f(x)$  from either the right or the left at a jump discontinuity.

### 7.3 Convergence

Let's deal with convergence! Start with numerical series,  $\sum_{k=1}^{\infty} c_k$ . The series converges to a number  $s$  if

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n c_k = s.$$

We call  $s_n = \sum_{k=1}^n c_k$ , the  $n$ -th *partial sum* of the series. If the limit,  $\lim_{n \rightarrow \infty} S_n$ , does not exist, we say the series diverges.

**Example 7.3.1** (Geometric Series). Consider

$$\sum_{k=0}^{\infty} r^k.$$

This is exceptional, because we can explicitly calculate  $s_n$  for these,

$$s_n = \frac{1 - r^{n+1}}{1 - r}, r \neq 1.$$

So series converges if  $|r| < 1$  and diverges if  $|r| \geq 1$ . For  $r = 1$ ,  $s_n = n$  and it diverges as  $n \rightarrow \infty$ . When  $r = -1$ ,  $s_n = 1 - 1 + 1 - 1 + 1 - \dots \pm 1$ . So

$$s_n = \begin{cases} 1, & \text{if even} \\ 0, & \text{if odd} \end{cases}$$

and  $s_n$  diverges. In conclusion, geometric series converges iff  $|r| < 1$  and  $s = 1/(1 - r)$ .