MATH 2C03 - Differntial Equations

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Course Outline

- Website: https://ms.mcmaster.ca/lovric/2C3.html
- Textbook: *Elementary Differential Equations with Boundary-Value Problems*
- Course pack must be bought.
- Assignments are online assignments.

Contents

1 Introduction

1.1 Differential equations

Definition 1.1. *Differential equation is an equation where the unknown object is a function or a set of functions, and which involve their derivaties.*

Definition 1.2. An ordinary differential equation (ODE) *is a differential equation involving one variable:* $f'(x)$

Example 1.1.1 (ODE)**.** The following differential equation represents exponential growth:

$$P'(t) = kP(t), k > 0$$

Definition 1.3. *If a function,* $f(x, y, z, \dots)$*, involves more than one variable, we must use partial derivatives, e.g.* $\partial f/\partial x, \partial^2 f/\partial x \partial y, \dots$*. This is called* a partial differential equation (PDE)*.*

Example 1.1.2 (PDE)**.** Let $c(x, t)$ represent the concentration at a location that is x units away from the source at time t. Then, we have

$$c_t(x, t) = Ac_{xx}(x, t)$$

Remark. Note that example 1.1.1 can be rearranged as follows:

$$\frac{P'(t)}{P(t)} = k$$

$P'(t)/P(t)$ represents the *relative rate of change*.

More often, k is not a constant. For example, we can incorporate seasonal variation to the exponential growth model.

Example 1.1.3.

$$\frac{P'(t)}{P(t)} = k \sin(at)$$

Definition 1.4. *If a model does not involve any chance effecs, it's* a deterministic model*. Otherwise, it's* a stochastic model.

Example 1.1.4 (Deterministic model)**.** Example 1.1.1 is a deterministic model.

Example 1.1.5 (Stochastic model)**.** Going back to example 1.1.1, we may define k as follows:

$$k = \begin{cases} 0.6 & 35\%\text{chance} \\ 0.5 & 65\%\text{chance} \end{cases}$$

This is a stochastic model.

1.2 Ordinary differential equations

All ordinary differential equations (ODEs) contain the following:

- An independent variable
- Unknown function
- Derivative of the function

In other words, all ordinary equations have the following form:

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

Example 1.2.1. If $F(s,t) = s^2 + 2t - 4$, then

$$F(x, y) = x^2 + 2y - 4.$$

Note that this function is defined *implicitly.*

Example 1.2.2. If $F(s,t,u) = e^{su} - t^2$, then

$$F(x, y, y') = e^{xy'} = y^2.$$

This is a first order ODE.

Example 1.2.3. If $F(s,t,u,v,w) = w^2 - s + 4u$, then

$$F(x, y, y', y'', y''') = (y''')^2 - xy + 4y' = 0$$

This is a third order ODE.

Definition 1.5 (Order of an ODE)**.** *Order of an ODE is determined by the highest non-zero derivative. If the highest non-zero derivative is the n-th derivative, the ODE also has an order of n.*

Remark. We categorize differential equations because each category requires a different methods to find its solution.

Example 1.2.4. Consider the following differential equation:

$$y'' - 3x^2y' + xy - 7e^x = 0$$

Since the highest non-zero derivative is the second derivative, this is a second order ODE. Note that this equation has the following form:

$$F(x, y, y', y'') = 0$$

We can rearrange this equation by solving for the highest derivative:

$$y'' = 3x^2y' - xy + 6e^x$$

We can also decide to put all y terms on LHS:

$$y'' - 3x^2y' + xy = 7e^x$$

Here, $7e^x$ is referred to as a *homogeneous term* since it does not contain y.

Definition 1.6 (Homogeneous ODE)**.** *If the homogeneous term of an ODE is equal to* 0*, the ODE is called* homogeneous*.*

Definition 1.7 (Linear ODE)**.** *Let $y^{(n)}(x)$ be the n-th derivative of $y(x)$. If an ODE can be written in the following form,*

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = p(x),$$

it is a linear*.*

Example 1.2.5. Consider the following equations:

1. $y''' - 3x^2y'' + 4y' = 76\sin x$
2. $y'' - 3x^2\sqrt{y} = 76\sin x$
3. $4y'' - 3y' + 4y = 6$
4. $y^{(4)} - 3y''y' - 4x^3y = 0$

Only equation 1 and 3 are linear.

Equation 1

$$\begin{cases} a_3(x) = 1 \\ a_2(x) = -3x^2 \\ a_1(x) = 4 \\ a_0(x) = 0 \end{cases}$$

Equation 2

$\sqrt{y}$ is not linear.

Equation 43

$$\begin{cases} a_3(x) = 4 \\ a_2(x) = -3 \\ a_0(x) = 4 \\ p(x) = 6 \end{cases}$$

Equation 4

$y''y'$ is not linear.

2 First order ODEs

2.1 Initial Value Problems

Commonly, first order ordinary equations are written as

$$y' = G(x, y)$$

Example 2.1.1. $y' = 3x^3y^4 + 7\ln x$.

Definition 2.1 (Pure time ODE)**.** $y' = f(x)$

Definition 2.2 (Autonomous ODE)**.** $y' = f(y)$

Example 2.1.2 (Pure time ODE)**.** Consider the following ODE:

$$y' = 3x^2 - e^x$$

Since this is a pure time ODE, we can solve this ODE by integrating both sides with respect to the time variable x.

$$\begin{aligned} y &= \int (3x^2 - e^x)dx \\ &= x^3 - e^x + C \end{aligned}$$

Note that this solution gives a *family* of curves (we get a different curve for each value of C). Note that changing c shifts the curve vertically. To identify one solution from the family (i.e. to find the exact value of C), we need to know *initial condition.* If $y(0) = 4$, we have $C = 5$.

Definition 2.3 (General solution)**.** *Integrating the ordinary differential eqau-tion gives us a solution with integration constant. This solution is called a* general solution*.*

Definition 2.4 (Particular solution)**.** *Once we are given the initial condition, we can determine the exact value of the integration constant. Solution obtained by solving the* Initial Value Problem *is called a* particular solution*.*

Example 2.1.3. Consider the following ODE:

$$y'' = 4x^2,$$

where $y(0) = 3$ and $y'(0) = -7$. Since both values are given at $x = 0$, they are called *initial conditions.* We can also be asked to find the particular solution where $y(0) = 2$ and $y(4) = -3$. Since they occur at different x values, they are called *boundary conditions.*

When we're given the initial conditions, we can find the particular solution by integrating the ODE once:

$$y' = \frac{4}{3}x^3 + C.$$

However, when we're given the boundary conditions, we must integrate the ODE twice:

$$y = \frac{1}{3}x^4 + Cx + D$$

Definition 2.5 (General IVP)**.** *Consider the following ODE:*

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

IVP requires us to find the particular solution of the ODE given the initial conditions: $y(x_0) = y_0, y(x_1) = y_1, \ldots y^{(n-1)}(x_0) = y_{n-1}$*. Note that the solution of an IVP, BVP is a function, and it can be algebraic, geometric, numeric, and/or qualitative.*

Example 2.1.4. Show that

$$y = 1 - x + 4x \ln x$$

is a solution of the following IVP:

$$\begin{aligned} x^2y'' - xy' + y &= 1 \\ y(1) &= 0 \\ y'(1) &= 3 \end{aligned}$$

Proof. First, we start by differentiating y:

$$\begin{aligned} y' &= -1 + 4\ln x + 4x\frac{1}{x} = 4\ln x + 3 \\ y'' &= \frac{4}{x} \end{aligned}$$

Then, by substitution, we have

$$\begin{aligned} &x^2y'' - xy' + y \\ &= x^2\left(\frac{4}{x}\right) - x(4\ln x + 3) + 4\ln x + 3 \\ &= 1 \end{aligned}$$

It is also easy to verify that $y(1) = 0$ and $y(1) = 3$. □

Example 2.1.5. Show that $x = a\sin 2t, a \in \mathbb{R}$ is a solution of

$$\begin{aligned} &x''(t) + 4x(t) = 0 \\ &x(0) = 0 \\ &x(\pi) = 0 \end{aligned}$$

Proof. Note that this is a homogeneous ODE with constant coefficients. Also, since we are given two values at different points, we have a boundary condition problem.

First, we start by taking the derivatives:

$$\begin{aligned} x' &= 2a\cos 2t \\ x'' &= -4a\sin 2t \end{aligned}$$

Then, we have

$$x''(t) + 4x(t) = -4a\sin 2t + 4\left(a\sin 2t\right) = 0$$

Finally, it is easy to check that the boundary conditions also hold. Therefore, $x = a\sin 2t$ is a solution of a given system. $\square$

Example 2.1.6. Show that $e^{x^2y} = y - x$ is a solution of

$$\left(x^2e^{x^2y} - 1\right)y' = -1 - 2xye^{x^2y}$$

Proof. Since we are given an implicit function in this problem, we must work with implicit differentiation:

$$\begin{aligned} \frac{d}{dx}\left(e^{x^2y}\right) &= \frac{d}{dx}(y - x) \\ e^{x^2y}\left(\left(x^2\right)'y + x^2y'\right) &= y' - 1 \\ e^{x^2y}\left(2xy + x^2y'\right) &= y' - 1 \\ \left(x^2e^{x^2y} - 1\right)y' &= -1 - 2xye^{x^2y} \end{aligned}$$

$\square$

Example 2.1.7. Show that $f(x) = \int_0^x te^{-t^2}dt$ satisfies the IVP

$$\begin{aligned} &f''(x) + 2xf'(x) - e^{-x^2} = 0 \\ &f(0) = 0 \\ &f'(0) = 0 \end{aligned}$$

Proof. Recall fundamental theorem of calculus:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Using FTA, we have

$$\begin{aligned} &f(x) = \int_0^x te^{-t^2}dt \\ &f'(x) = xe^{-x^2} \\ &f''(x) = e^{-x^2} + xe^{-x^2}(-2x) \end{aligned}$$

Then, it is easy to verify that f satisfies the given IVP. $\square$

Remark (Chain rule in FTC). Note the following:

$$\left(\int_0^{\sin x} te^{-t^2} dt\right)' = \sin x e^{-(\sin x)^2} \cos x.$$

Example 2.1.8. Show that

$$y = Ce^{-\int P(x)dx}$$

is a solution of $y' + P(x)y = 0$.

Proof. Note that $y' + P(x)y$ is a first order, linear, and non-constant coefficient ODE. We start the proof by differentiating the equation:

$$\begin{aligned} y' &= C\left(e^{-\int P(x)dx}\right)' \\ &= C\left(e^{-\int P(x)dx}\right)\left(-\int P(x)dx\right)' \\ &= Ce^{-\int P(x)dx}(-P(x)) \end{aligned}$$

Thus, $y' + P(x)y = 0$. $\square$

Example 2.1.9. Show that $f(x) = \sum\limits_{n=0}^{\infty} \frac{2^n}{n!}x^n$ is a solution of IVP:

$$\begin{aligned} &f'(x) - 2f(x) = 0, \\ &f(0) = 1. \end{aligned}$$

Proof. Before we begin the proof, we note that $0! = 1$ and $0^0 = 1$. The latter result follows from the l'Hospital rule:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = \cdots = 1,$$

Also, note that we can differentiate and integrate power series (term-wise), and radius of convergence does not change. If $x \in \mathbb{R}, R = \infty$.

First, we observe that the series can be expanded as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = 1 + 2x + 2x^2 + \frac{8}{6}x^3 + \dots$$

By taking the derivative, we have

$$f'(x) = \sum_{n=1}^{\infty} \frac{2^n}{n!}nx^{n-1} = 2\sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)!}x^{n-1} = 2\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$$

Therefore, we have $f'(x) - 2f(x) = 0$. It is easy to confirm that $f(0) = 1$. Note that this problem can also be solved by using the fact that $f(x) = e^{2x}$. $\square$

Example 2.1.10. Intensity, $I(t)$, decreases linearly with depth, d, and intensity at surface is I_0. This problem can be written into the following IVP:

$$I'(d) = -K$$
$$I(0) = I_0$$

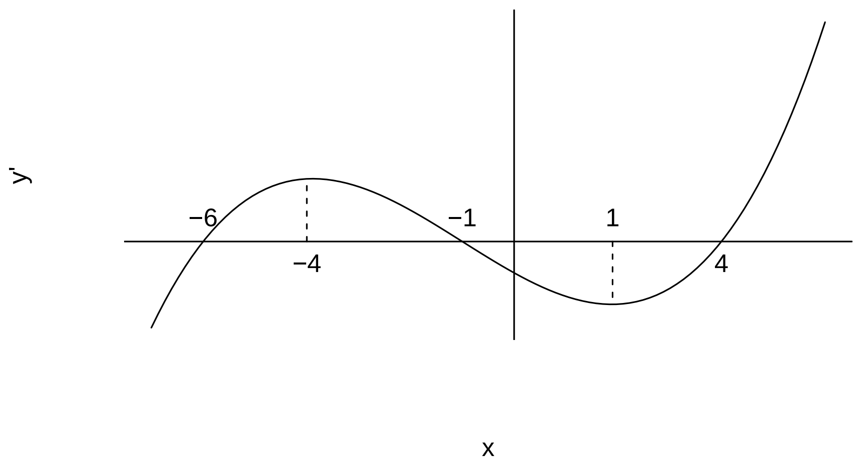
Example 2.1.11. Consider the following:

$$my'' = mg - Dy'$$

What are the units of D if y is in metres, m is in kilograms, and t is in time?

Note that y' has units of m/s and y'' has units of m/s^2. Since my'' has units of $kg \cdot \frac{m}{s^2}$, D should have units of kg/s.

Example 2.1.12. We can think about the geometric aspect of a curve:



First, we can look at its derivatives:

-6 -1 4

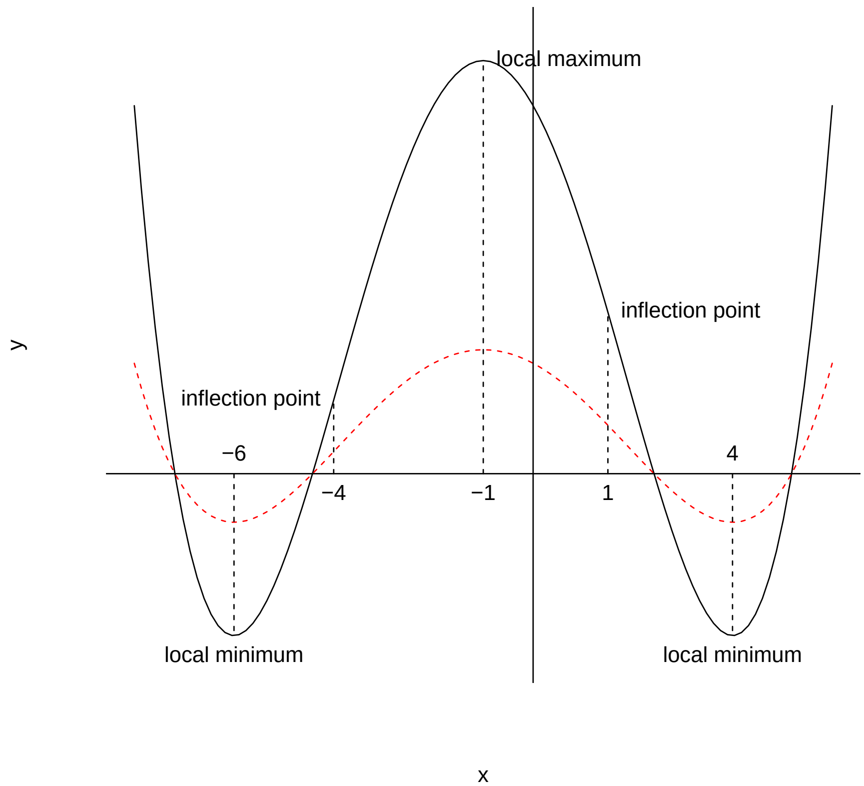
y'	-	+	-	+
y	decreasing	increasing	decreasing	increasing

When $y' = 0$ we can a local maximum/minimum. We can also look at its second derivatives. Note than $y'' = 0$ occurs where the curve has a horizontal tangent.

-4 1

y'	+	-	+
y'	concave up	concave down	concave up

From these observations, we can attempt to draw the curve of y.



This is called a *qualitative analysis*.

Example 2.1.13. Perform a qualitative analysis on the following ODE:

$$P'(t) = KP(t),$$

where $P(t) \geq 0$ and $K > 0$.

First, we observe that $P'(t) > 0$. This indicates that $P(t)$ is a monotonically increasing function. Now, we can also look at the second derivative:

$$P''(t) = KP'(t) = K\left(KP(t)\right) = K^2P(t) > 0$$

Therefore, this function is always concave up.