MATH 2XX3 - Advanced Calculus II

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March 30, 2017

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1 Introduction

In this course, we are going to study calculus using the concepts from linear algebra.

1.1 Vector norm

Definition 1.1. Euclidean norm *of $\vec{x} = (x_1, x_2, \ldots, x_n)$ is given as*

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

Theorem 1.1 (Properties of a norm)**.**

1. *$\|\vec{x}\| \geq 0$ and $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0} = (0, 0, \ldots, 0)$.*

2. *For all scalars $a \in \mathbb{R}$, $\|a\vec{x}\| = |a| \cdot \|\vec{x}\|$.*

3. *(Triangle inequality) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.*

We say that this is a property of a norm *because there are other norms, which measure* distance *in $\mathbb{R}^n$ in different ways!*

Example 1.1.1 (A non-pythagorian norm - *The Taxi Cab Norm*)**.** Consider the following vector $\vec{p} = (p_1, p_2) \in \mathbb{R}^2$. The euclidean norm gives the length of the diagonal line. On the other hand,

$$\|\vec{p}\|_1 = |p_1| + |p_2|$$

gives us the total distance in a rectangular grid system.

For $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, $\|\vec{p}\|_1 = \sum_{j=1}^{n} |p_j|$. Note that the Taxi Cab norm is a valid norm because it satisfies all properties of a norm above. So it also gives us a valid alternative way to measure *distance* in $\mathbb{R}^n$, $\text{dist}(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\|$. This way of measuring distance gives $\mathbb{R}^n$ a *different geometry*.

Definition 1.2. Neighborhood *of a point $\vec{p}$, or disks centered at $\vec{p}$ is defined as*

$$D_r(\vec{p}) = \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, \|\vec{x} - \vec{p}\| < r \right\}$$

Remark. The neighborhood around $\vec{a}$ of radius r may be written using any of the following notations:

$$D_r(\vec{a}) = B_r(\vec{a}) = B(\vec{a}, r)$$

Definition 1.3. *Sphere is defined as*

$$S_r(\vec{p}) = \left\{ \vec{x} \in \mathbb{R}^n \,\middle|\, \|\vec{x} - \vec{p}\| = r \right\}$$

What neighboorhood and sphere look like depends on which norm you choose. First, let's start with the familiar euclidean norm. Then, the sphere is given by

$$\|\vec{x} - \vec{p}\| = r$$

$$\Longleftrightarrow \sqrt{\sum_{j=1}^{n}(x_j - p_j)^2} = r$$

Then, we have

$$\sum_{j=1}^{n}(x_j - p_j)^2 = r^2$$

If $n = 3$, we have $(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = r^2$, usual sphere in $\mathbb{R}^3$ with center $\vec{p} = (p_1, p_2, p_3)$

If $n = 2$, we have $(x_1 - p_1)^2 + (x_2 - p_2)^2 = r^2$, usual circle in $\mathbb{R}^n$ with center $\vec{p} = (p_1, p_2)$.

If we replace Euclidean norm by the Taxi Cab norm (for simplicity, take $\vec{p} = \vec{0}$), we have

$$S_r^{\text{taxi}}(\vec{0}) = \left\{ \vec{x} \in \mathbb{R}^n \,\big|\, \|\vec{x} - \vec{0}\|_1 = r \right\}$$

In other words, we have

$$\vec{x} \in S_r^{\text{taxi}}(\vec{0}) \iff \sum_{j=1}^{n}|x_j| = r$$

Looking at it in $\mathbb{R}^2$, we have $\vec{x} = (x_1, x_2)$. Then, $r = |x_1| + |x_2|$. This, in fact, is a diamond.

Remark. Note that $|x_1| + |x_2| = r$ is a *circle* in $\mathbb{R}^2$ under the Taxi Cab norm. Then, we have

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{8r}{2r} = 4$$

1.2   Subset

Let's introduce some properties of *subsets* in $\mathbb{R}^n$. $A \subset \mathbb{R}^n$ means A is a *collection* of points $\vec{x}$, drawn from $\mathbb{R}^n$.

Definition 1.4. *Let $A \subset \mathbb{R}^n$, and $\vec{p} \in A$. We say $\vec{p}$ is an interior point of A if there exists a neighbourhood of $\vec{p}$, i.e. an open disk disk, which is entirely contained in A:*

$$D_r(\vec{p}) \subset A.$$

Example 1.2.1.

$$A = \left\{ \vec{x} \in \mathbb{R}^n \,|\, \vec{x} \neq \vec{0} \right\}$$

Take any $\vec{p} \in A$, so $\vec{p} \neq \vec{0}$. Then, let $r = \|\vec{p} - \vec{0}\| > 0$, and $D_r(\vec{p}) \subset A$, since $\vec{0} \notin D_r(\vec{p})$. (Notice: any smaller disk, $D_s(\vec{p}) \subset D_r(\vec{p}) \subset A$, where $0 < s < r$ works to show that $\vec{p}$ is an interior point).

So every $\vec{p} \in A$ is an interior point to A.

Definition 1.5. *If every $\vec{p} \in A$ is an interior point, we cal A an* open *set.*

Example 1.2.2. $A = \left\{ \vec{x} \in \mathbb{R}^n | \vec{x} \neq \vec{0} \right\}$ is an open set.

Example 1.2.3. $A = D_R(\vec{0})$ is an open set.

Proof. If $\vec{p} = \vec{0}$, $D_r(\vec{0}) \subseteq A = D_R(\vec{0})$ provided $r \leq R$. So $\vec{p} = \vec{0}$ is interior to A. Consider any other $\vec{p} \in A$. It's evident that $D_r(\vec{p}) \subset A = D_R(\vec{0})$ provided that $0 \leq r \leq R - \|\vec{p}\|$. Therefore, $A = D_R(\vec{0})$ is an open set. $\square$

Example 1.2.4. Suppose we use Taxi Cab disks instead of Euclidean disk. It does not change which points are interior to A since the diamond is inscribed in a circle. In other words,

$$D_r^{\text{taxi}}(\vec{p}) \subset D_r^{\text{Euclid}}(\vec{p})$$

Definition 1.6. *The complement of set A is*

$$A^c = \{ \vec{x} | \vec{x} \notin A \}$$

Definition 1.7. *$\vec{b}$ is a* boundary point *of A if for every $r > 0$, $D_r(\vec{b})$ contains both points in A and points not in A:*

$$D_r(\vec{b}) \cap A \neq \emptyset \text{ and } D_r(\vec{b}) \cap A^c \neq \emptyset$$

In the example 1.2.3, the set of all boundary points of $A = D_R(\vec{0})$

$$\left\{ \vec{b} \, | \|\vec{b}\| = R \right\}$$

is a sphere of radius R.

Definition 1.8. *A set A is* closed *if A^c is open.*

Theorem 1.2. *A is cloased if and only if A contains all its boundary points.*

Example 1.2.5. Consider the following set:

$$A = \{ (x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 > 0 \}$$

If $\vec{p_1} = (p_1, p_2)$, where $p_1 > 0, p_2 > 0$, then $\vec{p_1}$ is an interior point. Take $r = \min\{p_1, p_2\}$. Then, $D_r(\vec{p}) \subset A$. On the other hand, any $\vec{p}$ that lies on either axes (including $\vec{0}$) is a boundary point. Since there are boundary points in A, A can't be open.

2 Functions

2.1 Limits and continuity

In this section, we will be considering vector values functions such that

$$F : A \subseteq \mathbb{R}^n \to \mathbb{R}^k.$$

We can rewrite this using a matrix notation:

$$F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} F_1(x_1, x_2, \ldots, x_n) \\ F_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ F_k(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Example 2.1.1. For a $(k \times n)$ matrix M,

$$F(\vec{x}) = M\vec{x}$$

First, we wish to study limits. What does $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{L}$ mean? Note that it's not enough to treat the variables $x_1, x_2, \ldots x_n$ separately.

Example 2.1.2. Consider the following function:

$$F(x, y) = \frac{xy}{x^2 + 4y^2},$$

where $(x, y) \neq (0, 0)$. First, we can attempt to find its limit by considering them separately.

$$\lim_{x \to 0} \left(\lim_{y \to 0} F(x, y) \right) = \lim_{x \to 0} \left(\frac{0}{x^2} \right) = \lim_{x \to 0} = 0$$

Similarly, we have

$$\lim_{y \to 0} \left(\lim_{x \to 0} F(x, y) \right) = 0$$

However, if $(x, y) \to (0, 0)$ along a straight line path with $y = mx$, where m is constant, we have

$$F(x, mx) = \frac{mx^2}{x^2 + 4m^2 x^2} = \frac{m}{1 + 4m^2}$$

In this case, we have

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = mx}} F(x, y) = \frac{m}{1 + 4m^2}$$

Therefore, the values of $F(x, y)$ don't approach any particular value as $(x, y) \to (0, 0)$.

Example 2.1.3 (Worse). Consider the following function:

$$F(x, y) = \frac{y^2}{x^4 + y^2}.$$

If we approach $(0, 0)$ along $y = mx$, limit equals 1. However, if we approach along a parabola, $y = mx^2$, limit equals $m^2/(1 + m^2)$. We get different limits alond different parabolas.

We showed that computing

$$\lim_{\vec{x} \to \vec{a}} = \vec{b}$$

is tricky because $\vec{x} \to \vec{a}$ has to be more precise. It can't depend on the path or direction on which $\vec{x}$ approaches $\vec{a}$, but only on *proximity*. In other words, we want $\|F(\vec{x}) - \vec{b}\|$ to go to zero as $\|\vec{x} - \vec{a}\|$ goes to zero.

Definition 2.1. *We say $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{b}$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < \|\vec{x} - \vec{a}\| < \delta$, we have $\|F(x) - \vec{b}\| < \varepsilon$. Therefore,*

$$\lim_{\vec{x} \to \vec{a}} F(x) = \vec{b} \iff \lim_{\vec{x} \to \vec{a}} \|F(\vec{x}) - \vec{b}\| = 0$$

Remark. Geometrically, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(\vec{x}) \in D_\varepsilon(\vec{b}),$$

where $\vec{x} \in D_\delta(\vec{a})$.

Before doing examples, here's a useful observations. Take $\vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. Then, we have

$$\|\vec{v}\| = \sqrt{\sum_{j=1}^n v_j^2} \geq \sqrt{v_i^2} = |v_i|$$

for each coordinate $i = 1, 2, \ldots, n$.

Example 2.1.4. Show

$$\lim_{(x,y) \to (0,0)} \frac{2x^2 y}{x^2 + y^2} = 0$$

Proof. Note that $F : \mathbb{R} \setminus \{\vec{0}\} \to \mathbb{R}$, $b = 0$, $\vec{a} = (0, 0)$. Call

$$R = \|\vec{x} - \vec{a}\| = \|\vec{x}\| = \sqrt{x^2 + y^2}$$

Since $F(\vec{x}) \in \mathbb{R}$, we have

$$\|F(\vec{x}) - \vec{b}\| = |F(\vec{x}) - b|$$
$$= \left| \frac{2x^2 y}{x^2 + y^2} - 0 \right|$$
$$= \frac{2|x|^2|y|}{x^2 + y^2}$$
$$\leq \frac{2 \cdot R^2 \cdot R}{R^2}$$
$$= 2R$$
$$= 2\|\vec{x} - \vec{a}\|$$

By letting $\|\vec{x} - \vec{a}\| = \|\vec{x}\| < \varepsilon/2$, we get $\|F(\vec{x}) = \vec{b}\| < \varepsilon$. Therefore, definition is satisfied with $\delta \leq \varepsilon/2$ $\qquad \square$

Example 2.1.5. Consider the following function, $F : \mathbb{R}^3 \setminus \{\vec{0}\} \to \mathbb{R}$:

$$\frac{3z^2 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2}.$$

Determine whether

$$\lim_{(x,y,z) \to (0,0,0)} F(x,y,z) = 2.$$

Proof. We have

$$\|F(x,y,z) - \vec{b}\| = |F(x,y,z) - 2|$$
$$= \left| \frac{3z^3 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2} - 2 \right|$$
$$= \frac{3|z|^3}{x^2 + 2y^2 + 3z^2}$$
$$\leq \frac{3R^3}{x^2 + y^2 + z^3}$$
$$= \frac{3R^3}{R^2}$$
$$= 3R$$

Then,

$$\|F(x,y,z) - \vec{b}\| < 3R < \varepsilon$$

provided that

$$R = \|\vec{x} - \vec{0}\| < \delta = \frac{\varepsilon}{3}$$

$\qquad \square$

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Definition 2.2. *Consider a function* $F : \mathbb{R}^n \to \mathbb{R}^k$ *with domain* $A \subseteq \mathbb{R}^n$. *For* $\vec{a} \in A$, *we say* F *is* continuous *at* $\vec{a}$ *in the domain of* F *iff*

$$F(a) = \lim_{\vec{x} \to \vec{a}} F(\vec{a})$$

Example 2.1.6. Going back the example 2.1.5, if we redefine F as follows,

$$F = \begin{cases} \frac{3z^2 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2} & (x, y, z) \neq (0, 0, 0) \\ 2 & (x, y, z) = (0, 0, 0) \end{cases}$$

then F is continuous at $(0, 0, 0)$ (and in fact at all $\vec{x} \in \mathbb{R}$).

If F is continuous at every $\vec{a} \in A$, ($\forall \vec{x} \in A$), we say F is continuous on the set A. Continuity is always preserved by the usual algebraic operations: sum. product, quotient, and composition of continuous functions is continuous [1] .

2.2 Differentiability

Definition 2.3. *For a function* $f : \mathbb{R} \to \mathbb{R}$, *its derivative is defined as*

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If it exists, we say f *is* differentiable *at* x.

Theorem 2.1. *If* f *is differentiable at* x, $f(x)$ *is also continuous at* x.

Note that differentiable functions, $f(x)$, are well approximated by theire tangent lines (also known as linearization). We wish to extend this idea to $F : \mathbb{R}^n \to \mathbb{R}^m$.

First, we try dealing with the independent variables, $x_1, x_2, \ldots, x_n$, one at a time by using partial derivatives. We start by introducing the standard basis in $\mathbb{R}^n$:

$$\vec{e_1} = (1, 0, 0, \ldots, 0)$$
$$\vec{e_2} = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\vec{e_n} = (0, 0, 0, \ldots, 1)$$

(In $\mathbb{R}^3$, $\vec{e_1} = \vec{i}, \vec{e_2} = \vec{j}, \vec{e_3} = \vec{k}$).

For any $\vec{x} \in \mathbb{R}^n$, and $h \in \mathbb{R}$, $(\vec{x} + h\vec{e_j})$ moves from $\vec{x}$ parallel to the x_j axis by distance h. In other words,

$$\vec{x} + h\vec{e_j} = (x_1, x_2, \ldots, x_j + h, x_{j+1}, \ldots, x_n).$$

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[1] Provided we remain in the domain of continuity of both functions and denominators aren't zero

Definition 2.4. *Partial derivatives of $f(x)$ is defined as*

$$\frac{\partial f}{\partial x_j}(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_j) - f(\vec{x})}{h},$$

for all $j = 1, 2, \ldots, n$.

Partial derivatives calculate the derivatives of f, treating of $\vec{x}_j$ as the only variable, and all others treated as constants. For a vector valued function $F : \mathbb{R}^n \to \mathbb{R}^m$:

$$F(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{bmatrix},$$

we treat each component $F_i(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ separately as a real valued function. Each has n partial derivatives, and so $F : \mathbb{R}^n \to \mathbb{R}^m$ has $(m \times n)$ partial derivatives, which form an $(m \times n)$ matrix:

$$\left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}}.$$

We call this the derivative matrix or *Jacobian matrix*, $DF(\vec{x})$.

Example 2.2.1. Consider a function $F : \mathbb{R}^2 \to \mathbb{R}^3$:

$$F(\vec{x}) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^4 \end{bmatrix}.$$

Jacobian of the function is given by

$$DF(\vec{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 4x_2^3 \end{bmatrix}$$

Do we get the same properties for $DF(\vec{x})$ as we did for single-value calculus?

Example 2.2.2. Consider the following function:

$$f(x, y) = \begin{cases} \frac{xy}{(x^2+y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Do the partial derivatives exist at $(0, 0)$?

By definition,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h \cdot 0}{(h^2+0^2)^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

Similarly, $\frac{\partial f}{\partial y}(0,0) = 0$ (symmetry of x, y). Therefore,

$$Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Although partial derivatives exist, f is not cotinuous at $(0,0)$! (For example, $f(x, mx) \to \pm\infty$ as $x \to 0^\pm$ for $m \neq 0$).

To get reasonable information from $Df(\vec{x})$, we need to say more. First, let's go back to $f : \mathbb{R} \to \mathbb{R}$. Note

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\iff \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) = 0$$

$$\iff \lim_{h \to 0} \left(\frac{f(x+h) - \overbrace{[f(x) + hf'(x)]}^{L}}{h} \right) = 0$$

Numerator is the difference between $f(x+h)$ and its linear approximation, L (i.e. the tangent line). So f is differentiable at x if its linear approximation gives an estimate of the values $f(x+h)$ within an error which is small compared to $\Delta x = h$. More precisely, the linearization of $f(x)$ at $x = a$ (or the tangent line) is given by

$$L_a(x) = f(a) + f'(a)(x - a)$$

We wish to extend this idea to higher dimensions. For $F : \mathbb{R}^n \to \mathbb{R}^m$, $F(\vec{x})$ has $(m \times n)$ partial derivates (see definition 2.4). Then, the linearization of F at $\vec{a}$ is

$$L_{\vec{a}}(\vec{x}) = \underbrace{F(\vec{a})}_{m \times 1} + \underbrace{DF(\vec{a})}_{m \times n} \underbrace{(\vec{x} - \vec{a})}_{n \times 1}.$$

So, $L : \mathbb{R}^n \to \mathbb{R}^m$, just like F. The derivative matrix $DF(\vec{a})$ is a *linear transformation* of $\mathbb{R}^n \to \mathbb{R}^m$.

Notice that when $n = 2$ and $m = 1$, For $F : \mathbb{R}^2 \to \mathbb{R}$, we have

$$DF(\vec{a}) = \begin{bmatrix} \frac{\partial F}{\partial x_1}(\vec{a}) & \frac{\partial F}{\partial x_2}(\vec{a}) \end{bmatrix},$$

a (1×2) row vector and

$$\vec{x} - \vec{a} = \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \end{bmatrix},$$

so we have
$$L_{\vec{a}}(\vec{x}) = F(\vec{a}) + \frac{\partial F}{\partial x_1}(x_1 - a_1) + \frac{\partial F}{\partial x_2}(x_2 - a_2),$$
a familiar equation of the tangent plane to $z = F(x_1, x_2)$.

Finally, we can introduce the idea of differentiable:

Definition 2.5 (Differentiability). *We say $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if*
$$\lim_{\vec{x} \to \vec{a}} \frac{\|F(\vec{x}) - F(\vec{a}) - DF(\vec{a})(\vec{x} - \vec{a})\|}{\|\vec{x} - \vec{a}\|} = 0.$$

Equivalently,
$$\lim_{\vec{h} \to \vec{0}} \frac{\|F(\vec{x} + \vec{h}) - F(\vec{x}) - DF(\vec{x})\vec{h}\|}{\|\vec{h}\|} = 0$$

In summary, F is differentiable if $\|F(\vec{x}) - L_{\vec{a}}(\vec{x})\|$ is small compared to $\|\vec{x} - \vec{a}\|$. Or, $F(\vec{x})$ is approximated by $L_{\vec{a}}(\vec{x})$ with and error which is much smaller than $\|\vec{x} - \vec{a}\|$. Note that we write $o\left(\|\vec{x} - \vec{a}\|\right)$ "little-oh" for quantity which is small compred to $\|\vec{x} - \vec{a}\|$. Using this notation, differentiability can be written as
$$\|F(\vec{x}) - F\left(\vec{a} - Df(\vec{a})(\vec{x} - \vec{a})\right)\| = o(\|\vec{x} - \vec{a}\|).$$

Example 2.2.3. Is the following function differentiable at $\vec{a} = \vec{0}$?
$$F(x_1, x_2) = \begin{cases} \frac{x_2^2 \sin x_1}{\sqrt{x_1^2 + x_2^2}}, & \vec{x} \neq \vec{0} \\ 0, & \vec{x} = \vec{0} \end{cases}$$

First, we have
$$\frac{\partial F}{\partial x_1}(\vec{0}) = \lim_{h \to 0} \frac{F(\vec{0} + h\vec{e}_1) - F(\vec{0})}{h}$$
$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Similarly, we have
$$\frac{\partial F}{\partial x_2}(\vec{0}) = 0$$

So we have
$$DF(\vec{0}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

For differentiability, we have to look at:
$$\left| \frac{x_2^2 \sin x_1}{\sqrt{x_1^2 + x_2^2}} - 0 - \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|$$
$$= \frac{x_2^2 |\sin x_1|}{\sqrt{x_1^2 + x_2^2}}$$

Then,

$$\frac{|F(\vec{x}) - L_{\vec{0}}(\vec{x})|}{\vec{x} - \vec{0}} = \frac{x_2^2|\sin x_1|}{\left(\sqrt{x_1^2 + x_2^2}\right)^2} = \frac{x_2^2|\sin x_1|}{x_1^2 + x_2^2}$$

$$\leq \frac{R^2 \cdot R}{R^2} = R = \|\vec{x} - \vec{0}\|$$

By squeeze theorem, we have

$$\lim_{\vec{x} \to \vec{0}} \frac{|F(\vec{x}) - L_{\vec{0}}(\vec{x})|}{\|\vec{x} - \vec{0}\|} = 0$$

Therefore, F is differentiable at $\vec{x} = \vec{0}$

Example 2.2.4. Verify that F is differentiable at $\vec{a} = \vec{0}$.

$$F(\vec{x}) = \begin{bmatrix} 1 + x_1 + x_2^2 \\ 2x_2 - x_1^2 \end{bmatrix}$$

First, note that

$$F(\vec{a}) = F(\vec{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We also need to compute the Jacobian at $\vec{0}$:

$$DF(\vec{0}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then, we get the following linearization of the function:

$$L_{\vec{0}}(\vec{x}) = F(\vec{0}) + DF(\vec{x})(\vec{x} - \vec{0})$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + x_1 \\ 2x_2 \end{bmatrix}$$

Then, look at

$$\frac{\|F(\vec{x}) - L_{\vec{0}}(\vec{x})\|}{\|\vec{x} - \vec{0}\|} = \frac{\left\| \begin{bmatrix} x_2^2 \\ -x_1^2 \end{bmatrix} \right\|}{\|\vec{x}\|} = \frac{\sqrt{x_2^4 + x_1^4}}{\sqrt{x_1^2 + x_2^2}} \leq \frac{R^4+4}{R} = \sqrt{2}R = \sqrt{2}\|\vec{x} - \vec{0}\|$$

As $\vec{x} \to \vec{0}$, $\|\vec{x} - \vec{0}\| = R \to 0$, so by the squeeze theorem, the desired limit goes to 0. Therefore, F is differentiable at $\vec{0}$.

Theorem 2.2. *Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$, and $\vec{a} \in \mathbb{R}^n$. If there exists a disk $D_r(\vec{a})$ in which all the partial derivatives $\partial(F_i(\vec{x}))/\partial x_j$ exist and are continuous, then F is differentiable at $\vec{x} = \vec{a}$.*

Definition 2.6. *A function which satisfies Theorem 2.2 is called* continuously differentiable, *of C^1.*

So far as our example, we calculate the partial for $\vec{x} \neq \vec{0}$:

$$\frac{\partial F}{\partial x_1} = x_2^2 \left(\cos x_1 \left(x_1^2 + x_2^2 \right)^{-\frac{1}{2}} + \left(-\frac{1}{2} \left(x_1^2 + x_2^2 \right)^{-\frac{3}{2}} 2x_1 \right) \sin x_1 \right)$$

$$= \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} \left[\cos x_1 \left(x_1^2 + x_2^2 \right) - x_1 \sin x_1 \right]$$

which is continuous as long as $\vec{x} \neq \vec{0}$. We do the same for $\frac{\partial F}{\partial x_2}$ and conclude that F is C^1 at all $\vec{x} \neq \vec{0}$. We summarize these ideas in the figure below:

2.3 Chain rule

Definition 2.7. *Suppose $A \subseteq \mathbb{R}^n$ is open, and we have a function*

$$F : A \subseteq \mathbb{R}^n \to \mathbb{R}^m.$$

Similarly, supposed $B \subseteq \mathbb{R}^m$ is open, and we have a function

$$G : B \subseteq \mathbb{R}^m \to \mathbb{R}^p.$$

Assume $\vec{a} \in A$ and $F(\vec{a}) = \vec{b} = B$. The composition

$$H(\vec{x}) = G \circ F(\vec{x}) = G\left(F(\vec{x}) \right)$$

is a function $\mathbb{R}^n \to \mathbb{R}^p$.

Example 2.3.1. Consider the following linear functions:

$$\begin{cases} F(\vec{x}) = M\vec{x} & M \text{ an } (m \times n) \text{ matrix} \\ G(\vec{y}) = N\vec{y} & N \text{ an } (p \times m) \text{ matrix} \end{cases}$$

Then,

$$H(\vec{x}) = G\left(F(\vec{x}) \right) = NM\vec{x}$$

is also a linear and represented by the product NM.

Theorem 2.3. *Assume $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x} = \vec{a}$ and $G : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $\vec{b} = F(\vec{a})$. Then, $H = G \circ F$ is differentiable at $\vec{x} = \vec{a}$ and*

$$DH(\vec{a}) = \underbrace{DG(\vec{b})}_{DG(F(\vec{a}))} DF(\vec{a})$$

Note that all of the various forms of Chain Rule done in first year calculus can be derived directly from this general formula.

Example 2.3.2. Consider the following functions, $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$:

$$F(\vec{x}) = \begin{bmatrix} x_1^2 + x_2 x_3 \\ x_1^2 + x_3^2 \end{bmatrix}, G(\vec{y}) = \begin{bmatrix} -y_2^3 \\ y_1 + y_2 \end{bmatrix}$$

Let $H = G \circ F(\vec{x})$. Find $DH(\vec{a})$ where $a = (1, -1, 0)$.

First, we have

$$DF(\vec{x}) = \begin{bmatrix} 2x_1 & x_3 & x_2 \\ 2x_1 & 0 & 2x_3 \end{bmatrix}, DF(1, -1, 0) = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

Similarly, we have

$$DG(\vec{y}) = \begin{bmatrix} 0 & -3y_2 \\ 1 & 1 \end{bmatrix}, DG(1, 1) = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

By Chain Rule, we get

$$DH(1, -1, 0) = DG(1, 1)DF(1, -1, 0)$$

$$= \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 & 0 \\ 4 & 0 & -1 \end{bmatrix}$$

3 Paths and Curves

3.1 Directional derivatve

Definition 3.1. *A* path *is* $\vec{C} : \mathbb{R} \to \mathbb{R}^n$ *is a vector-valued function of a scalar independent variable, usually,* t*:*

$$\vec{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

$\vec{c}(t)$ can be thought of as a moving vector. It takes out a curve in $\mathbb{R}^n$ as t increases. Basically, path is a way of describing a curve using functions. Note that this is not the only way to describe a curve.

Example 3.1.1. A unit circle in $\mathbb{R}^2$ described as a path is

$$\vec{c}(t) = (\cos t, \sin t),$$

where $t \in [0, 2\pi)$. But we could also describe the unit circle *non-parametrically* as

$$x^2 + y^2 = 1$$

Note that the same curve can be described by diffrent paths. Going back to unit circle, we can also write

$$\vec{b}(t) = \left(\sin(t^2), \cos(t^2) \right).$$

Using different paths can change (1) time dyanmics and (2) direction of the curve. This curve has a non-constant speed and reversed orientation.

If $\vec{c}$ is differentiable, $D\vec{c}(t)$ is an $(n \times 1)$ matrix. Since each compoenent $\vec{c}_j(t)$ is a real-valued function of only one variable, the *partial-derivative* is the usual derivative:

$$\frac{\partial c_j}{\partial t} = \frac{dc_j}{dt} = c_j'(t) = \lim_{h \to 0} \frac{c_j(t+h) - c_j(t)}{h}$$

So $D\vec{c}(t) = \vec{c}'(t)$ is written as a column vector:

$$D\vec{c}(t) = \begin{bmatrix} c_1'(t) \\ c_2'(t) \\ \vdots \\ c_3'(t) \end{bmatrix}$$

$$= \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}$$

which is a vector which is tangent to the curve traced out at $\vec{x} = \vec{x}(t)$. Physically, $\vec{c}'(t)$ is the velocity vector for motion along the path.

Example 3.1.2 (Lines)**.** Given two points, $\vec{p}_1, \vec{p}_2 \in \mathbb{R}^n$, there is a unique line connecting them. One path which represents this line is

$$\vec{c}(t) = \vec{p}_1 + t\vec{v},$$

where $\vec{v} = \vec{p}_2 - \vec{p}_1$. Velocity is then given by $\vec{c}'(t) = \vec{v}$, a constant.

Definition 3.2. $f : \mathbb{R}^n \to \mathbb{R}$ *is a* scalar-valued function*.*

If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, $Df(\vec{x})$ is a $(1 \times n)$ matrix:

$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

We use *paths* $\vec{c}(t)$ to explore $f(x)$ by looking at

$$h(t) = f \circ \vec{c}(t) = f\left(\vec{c}(t)\right).$$

where $h : \mathbb{R} \to \mathbb{R}$, By chain rule,

$$Dh(t) = h'(t) = \underbrace{Df(\vec{c}(t))}_{1 \times n} \underbrace{D\vec{c}(t)}_{n \times 1}$$

$$= Df\left(\vec{c}(t)\right) \vec{c}'(t)$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix}$$

We can think of this as a dot product of $\vec{c}'(t)$ with a vector $Df^T = \nabla f$, the gradient vector:

$$h'(t) = \nabla f(\vec{c}(t)) \cdot \vec{c}'(t)$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a} \in \mathbb{R}^n$, and we have a path $\vec{c} : \mathbb{R}^n \to \mathbb{R}$ with $\vec{c}(0) = \vec{a}$. Let $\vec{v} = \vec{c}'(0)$. Then, $h'(0)$ measures rate of change of f along the path as we cross through $\vec{a}$:

$$h'(0) = \nabla f(\vec{c}(0)) \cdot \vec{c}'(0)$$
$$= \nabla f(\vec{c}(0)) \cdot \vec{v}$$

Note that we get the same value for $h'(0)$ for *any* path $\vec{c}(t)$ going through $\vec{a}$ with velocity $\vec{c}'(t) = \vec{v}$. In other words, $h'(0)$ says something about f at $\vec{a}$, and not the path $\vec{c}(t)$.

Definition 3.3 (Directional derivative)**.** *The directional derivative of f at $\vec{a}$ in direction $\vec{v}$ is given by*

$$D_{\vec{v}}f(\vec{a}) = Df(\vec{a})\vec{v} = \nabla f(\vec{a}) \cdot \vec{v}.$$

Now, we can make some observations. Using the Chain Rule, directional derivatives can be rewritten as

$$D_{\vec{v}}f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t}.$$

Note the similarity to partial derivatives, where $\vec{v} = \vec{e}_j$.

Second, notice that $D_{2\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot (2\vec{v}) = 2D_{\vec{v}}f(\vec{a})$. To get the information on how fast f is changing at $\vec{a}$, we need to restrict to unit vectors $\|\vec{v}\| = 1$.

Directional derivatives also gives a geometrical interpretation of the gradient vector, $\nabla f(\vec{a})$. We use the Cauchy-Schwartz Inequality[2] to do so. By applying the Cauchy-Schwartz inequality, we get:

$$D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v} \le \|\nabla f(\vec{a})\| \|\vec{v}\| = \|f(\vec{a})\|.$$

Therefore, we can conclude that the length of $\|\nabla f(\vec{a})\|$ is the largest of $D_{\vec{v}}f(\vec{a})$ among all choices of unit directions $\vec{v}$. In other words, the direction $\vec{v}$ in which $f(\vec{x})$ increases most rapidly is the direction of $\nabla f(\vec{a})$, i.e.

$$\vec{v} = \frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|},$$

provided that $\nabla f(\vec{a}) \neq \vec{0}$.

Similarly, $-\nabla f(\vec{a})$ points in the direction of largest of $f(\vec{x})$, i.e.

$$\vec{v} = -\frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|},$$

gives the most negative directional derivative.

3.2   Parameterized curve

A path, $\vec{c}(t)$, is {continuous, differentiable, and C^1} provided that each component $c_j(t)$, $j = 1, 2, \ldots, n$ are. Note that $\{\vec{c}(t) : t \in [a, b]\}$ traces out a curve in $\mathbb{R}^n$, with initial endpoint, $\vec{a}$, and final endpoint, $\vec{b}$. Therefore, the path $\vec{c}(t)$ *parameterizes* the curve drawn out.

Recall that for any function $F : \mathbb{R}^k \to \mathbb{R}^n$, differentiability means that tangent (i.e. linearization) makes a good aproximation. For a differentiable path, $\vec{c}'(t)$ is a tangent vector to the curve drawn out when $\vec{c}'(t) \neq 0$. We call $\vec{v}(t) = \vec{c}'(t)$ the velocity vector ($v = \|\vec{v}\| = \|\vec{c}'(t)\|$ is the speed).

Finally, we can define the unit tangent vector:

Definition 3.4. *Unit tangent vector is defined as*

$$\vec{T}(t) = \frac{\vec{v}}{\|\vec{v}(t)\|} = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}$$

---

[2]For any vectors $\vec{u} \cdot \vec{v} \le \|\vec{y}\| \|\vec{v}\|$, and equality holds if and only if $\vec{u} = t\vec{v}$ for a scalar t.

Example 3.2.1. Consider a path, $\vec{c} : \mathbb{R} \to \mathbb{R}^2$:

$$\vec{c}(t) = \left(\cos^3 t, \sin^3 t \right), \ t \in [-\pi, \pi].$$

This is a C^1 path[3] whose velocity vector is given by

$$\vec{c}\,'(t) = \left(-3 \cos^2 t \sin t, 3 \sin^2 t \cos t \right).$$

To find the unit tangent, we have to find its speed first:

$$
\begin{aligned}
v &= \left\| \left(-3 \cos^2 t \sin t, 3 \sin^2 t \cos t \right) \right\| \\
&= 3 |\sin t \cos t| \, \| (-\cos t, \sin t) \| \\
&= 3 |\sin t \cos t|
\end{aligned}
$$

Then, its unit tangent is given by

$$\vec{T}(t) = \frac{\vec{v}(t)}{\| \vec{v}(t) \|} = \left(-|\cos t| \frac{\sin t}{|\sin t|}, |\sin t| \frac{\cos t}{|\cos t|} \right)$$

Note that its tangent is undefined when $\sin t = 0$ or $\cos t = 0$, i.e. at multiples of $\frac{\pi}{2}$. Worse, $\frac{\sin t}{|\sin t|}, \frac{\cos t}{|\cos t|}$ flip discontinuously as t crosses a multiple of $\pi/2$ from -1 to $+1$, or vice versa. Although the path is C^1, the curve is not smooth! When $\vec{v}(t) = \vec{c}\,'(t) = 0$, it allows the curve to have cusps.

Note that it is possible to have a nice tangent direction even when $\vec{c}\,'(t) = 0$:

Example 3.2.2. Consider a parameterized straight line:

$$\vec{c} = \vec{a} + \vec{w} t^3$$

Its velocity vector, $\vec{c}\,'(t) = 3\vec{w} t^2$, is equal to $\vec{0}$ when $t = 0$. However, it still has a tangent direction which is parallel to $\vec{w}$.

Definition 3.5. *We say a parameterized curve is* smooth[4] *(or* regular*) if its path is C^1, i.e. if it can be parameterized by a path $\vec{c}(t)$, and $\| \vec{c}\,'(t) \neq 0 \|$ for any t.*

If $\vec{c}(t)$ is twice-differentiable, $\vec{c}\,''(t) = \vec{a}(t)$ gives us the acceleration vector.

Theorem 3.1.

1. *If $f : \mathbb{R} \to \mathbb{R}$, $\vec{c} : \mathbb{R} \to \mathbb{R}^n$, both differentiable,*

$$
\begin{aligned}
\frac{d}{dt} \left(f(t) \vec{c}(t) \right) &= f(t) \vec{c}\,'(t) + f'(t) \vec{c}(t) \\
&= \sum_{j=1}^{n} \frac{d}{dt} \left(f(t) \vec{c}_j(t) \right) \vec{e}_j \\
&= \sum_{j=1}^{n} \frac{d}{dt} \left(f'(t) \vec{c}_j(t) + f(t) \vec{c}_j\,'(t) \right) \vec{e}_j
\end{aligned}
$$

---

[3]In fact, it is C^∞, differentiable to all orders!
[4]For a smooth curve, the unit tangent $\vec{T}(t)$ is continuous.

2. If $\vec{c}, \vec{d} : \mathbb{R} \to \mathbb{R}^n$ are differentiable,

$$\frac{d}{dt}\left(\vec{c}(t) \cdot \vec{d}(t)\right) = \vec{c}'(t) \cdot \vec{d}(t) + \vec{c}(t) \cdot \vec{d}'(t)$$

3. If $\vec{c}, \vec{d} : \mathbb{R} \to \mathbb{R}^3$ are differentiable,

$$\frac{d}{dt}\left(\vec{c}(t) \times \vec{d}(t)\right) = \vec{c}(t) \times \vec{d}'(t) + \vec{c}'(t)\vec{d}(t),$$

where $\vec{c} \times \vec{d} = \sum_{i,j,k=1}^{3} = c_i d_j \vec{e}_k \varepsilon_{ijk}$. ε_{ijk} is defined in the footnotes[5].

Example 3.2.3. Suppose $\vec{c}$ is a twice differentiable path and $\vec{a}(t) = k\vec{c}(t)$ for some constant $k \neq 0$. Show that $\vec{c}(t)$ describes a motion in a fixed plane.

Define a vector

$$\vec{n} = \vec{c}(t) \times \vec{v}(t) = \vec{c}(t) \times \vec{c}'(t)$$

Notice $\vec{n} \perp \vec{c}(t)$ and $\vec{v}(t)$, i.e. $\vec{n}$ is normal to the plane.

$$\frac{d\vec{n}}{dt} = \frac{d}{dt}\left(\vec{c}(t) \times \vec{c}'(t)\right) = \vec{c}(t) \times \underbrace{\vec{c}''(t)}_{\vec{a}(t)} + \underbrace{\vec{c}'(t) \times \vec{c}'(t)}_{\vec{0}}$$

$$= \vec{c}(t) \times k\vec{c}(t)$$

$$= \vec{0}$$

Therefore, $\vec{n}$ is constant in time!

So $\vec{c}(t)$ and $\vec{v}(t)$ are, for all t, perpendicular to the constant vector $\vec{n}$. Then,

$$P = \{\vec{w} \,|\, \vec{w} \cdot \vec{n} = 0\}$$

is the plane through $\vec{0}$. So $\vec{c}(t) \in P$ for all t.

Definition 3.6 (Arclength). *The arclength (or distance travelled along the parameterized curve) for $a \leq t \leq b$ is*

$$\int_a^b \underbrace{\|\vec{c}'(t)\|}_{speed} \, dt$$

For a variable time interval, the arclength function

$$s(t) = \int_a^t \|\vec{c}'(u)\| du$$

is a distance travelled from time a to time t.

---

[5] $\varepsilon_{ijk} = \begin{cases} 0 & \text{if } i = j \text{ or } j = k \text{ or } \mathrm{k} = \mathrm{l} \\ 1 & \text{if } (i,j,k) \text{ is positively ordered} \\ -1 & \text{if } (i,j,k) \text{ is negatively ordered} \end{cases}$

Example 3.2.4. Consider the following path:

$$\vec{c}(t) = (3\cos t, 3\sin t, 4t),\ t \in [0, 4\pi].$$

Its velocity vector is given by

$$\vec{v}(t) = (-3\sin t, 3\cos t, 4).$$

It follows that its speed is exactly equal to 5. Then, we can compute the arclength:

$$s(t) = \int_0^t v(t)dt = \int_0^t 5du = 5t$$

Therefore, total arclength is $s(4\pi) = 20\pi$.

Definition 3.7. *When the path $\vec{c}(t)$ traces out the curve with speed $\|\vec{v}(t)\| = 1$ for all t, we say that the curve is* arclength parameterized.

If a curve is arclength parameterized, arclength function becomes

$$s(t) = t$$

Then, we can use s instead of t as a parameterization in the path.

Example 3.2.5. In example 3.2.4, helix is not arclength parameterized but we can re-parameterize it so that it is. To do so, we need to solve for $t = \varphi(s)$ to invert the function, $s(t)$.

Going back the example, we had $s(t) = 5t$. It follows that $t = \frac{1}{5}s$. Then,

$$\vec{c}(s) = \vec{c}(\varphi(s)) = \vec{c}\left(\frac{s}{5}\right) = \left(3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\right)$$

is an arclength parameterization of the original helix, i.e. $\|\vec{c}'(s)\| = 1$, $\forall s$.

3.3 Geometry of curves in $\mathbb{R}^3$

Path,

$$\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = (x, y, z)(t),$$

traces out a curve, for $t \in [a, b]$, in space, and its velocity vector and speed are given by $\vec{c}'(t)$ and $\|\vec{c}'(t)\|$, respectively. This is a smooth parameterization if $\vec{c} \in C^1$ and $\|\vec{c}'(t)\| \neq 0$ for any $t \in [a, b]$.

We introduced the arclength function,

$$s(t) = \int_a^t \|\vec{c}'(u)\|du,$$

the total distance along the curve up to time t.

We also introduced the idea of arclength parameterization, where $s(t) = t$. Then, since

$$\frac{ds}{dt} = \|\vec{c}'(t)\|,$$

arclength parameterization is a path that travels along the curve with unit speed, $ds/dt = 1$, throughout. Therefore, any path with $\|\vec{c}'(t)\| \neq 0$ can be parameterized by arclength by inverting $s = s(t)$ such that $t = \varphi(s)$. Note that we can always do this for a smooth path ($ds/dt > 0$ so $s(t)$ is monotonically increasing). In practice, however, you may not be able to find an explicit formula for the arclength parameterization!

Example 3.3.1. Consider the following path:

$$\vec{c}(t) = (t, \frac{1}{2}t^2) :$$

Since $y = x^2/2$, it's a parabola. Then, we observe that

$$\vec{c}'(t) = (1, t), \|\vec{c}'(t)\| = \sqrt{1 + t^2} \geq 1 > 0.$$

So the path is smooth. Then, we have

$$s(t) = \int_0^t \|\vec{c}'(u)\| du = \int_0^t \sqrt{1 + u^2} du = \frac{1}{2}\left(\ln\left| \sqrt{1 + t^2} + t \right| + t\sqrt{1 + t^2} \right).$$

Clearly, there's no way we can solve for t as a function of s. The way out of this trouble is to treat all $\vec{c}$ as if they were parameterized by arclength and use Chain rule with $ds/dt = \|\vec{c}'(t)\|$ to compensate.

Recall that unit tangent vector to $\vec{c}(t)$ is

$$\vec{T}(t) = \frac{\vec{c}'(t)}{\|\vec{c}'(t)\|}.$$

We wish to understand how direction of the curve changes over time:

Definition 3.8. *The* curvature *of a curve is defined as rate of change of unit tangent:*

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|.$$

By chain rule,

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

So, in the original time parameter, t,

$$\kappa(t) = \left\| \frac{1}{\frac{ds}{dt}} \frac{d\vec{T}}{dt} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}$$

Example 3.3.2. Consider a circle of radius $R > 0$ in xy-plane:

$$\vec{c}(t) = (R \sin t, R \cos t).$$

Now, we can easily find its velocity vector and speed:

$$\vec{c}\,'(t) = (R \cos t, -R \sin t)$$
$$\|\vec{c}\,'(t)\| = R$$

Notice that this travels with constant speed but is not arclength parameterized.

We can also find its unit tangent:

$$\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \frac{\vec{c}\,'(t)}{R} = (\cos t, -\sin t)$$

Then,

$$\vec{N}(t) = \vec{T}\,'(t) = (-\sin t, -\cos t)$$

Again, notice that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$.

Finally, we have

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{c}\,'(t)\|} = \frac{1}{R}.$$

Therefore, circle with large radius has less curvature.

Example 3.3.3. Consider the following helix:

$$\vec{c}(t) = (3 \cos t, 3 \sin t, 4t).$$

Following the same approach as shown in the previous example, we get

$$\vec{c}\,'(t) = (-3 \sin t, 3 \cos t, 4)$$
$$\|\vec{c}\,'(t)\| = 5$$
$$\vec{T}(t) = \left(-\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right)$$
$$\vec{T}\,'(t) = \left(-\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right)$$

Then,

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{c}\,'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

This curve also has a constant curvature.

Definition 3.9 (Principal normal vector)**.**

$$\vec{N} = \frac{\vec{T}\,'(s)}{\|\vec{T}\,'(s)\|} = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

Since $\|\vec{T}(s)\| = 1$ for all s, $\vec{T}(s) \cdot \vec{T}(s) = \|\vec{T}(s)\|^2 = 1$. By implicit differentiation, we have

$$\frac{d}{ds}(1) = \frac{d}{ds}(\vec{T}(s) \cdot \vec{T}(s))$$
$$0 = \vec{T}'(s) \cdot \vec{T}(s) + \vec{T}(s) \cdot \vec{T}'(s)$$
$$= 2\vec{T}'(s) \cdot \vec{T}(s)$$

Therefore, $\vec{T}'(s) \perp \vec{T}(s)$ for all s. So as long as $\vec{T}'(s) \neq 0$, i.e. $\kappa \neq 0$, we have $\vec{N}(s) \perp \vec{T}(s)$. In fact, $\vec{T}'(s) = \|\vec{T}'(s)\|\vec{N} = \kappa\vec{N}$, so the tangent turns in the direction of $\vec{N}$. For motion in a line, where $\kappa(s) = 0$ for all s, $\vec{N}$ cannot be defined!

$\vec{T}, \vec{N}$ determines a plane in $\mathbb{R}^3$, the osculating plane. The normal vector to the osculating plane is given by

$$\vec{B} = \vec{T} \times \vec{N}.$$

Definition 3.10 (Binormal vector). $\vec{B} = \vec{T} \times \vec{N}$

We observe that $\vec{B} \perp \vec{T}$, $\vec{B} \perp \vec{N}$, and

$$\|\vec{B}\| = \|\vec{T}\|\|\vec{N}\||\sin\theta| = 1 \cdot 1 \cdot \sin(\pi/2) = 1$$

Therefore, $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ is a moving *orthonormal basis* for $\mathbb{R}^3$ at each point along the curve. This plane is also referred to as *moving frame* or *frenet frame*. Now, we introduce the following consequences:

(1). If curvature $\kappa(s) = 0$ for all s, then the curve is a straight line.

To see this, $\vec{T}'(s) = \kappa\vec{N}(s) = 0$ for all s. Therefore, $\vec{T}(s) = \vec{u}$ is a constant vector and

$$\vec{r}(s) = \vec{u}s + \vec{p},$$

a line thorugh $\vec{p} = \vec{r}(0)$ with direction vector $\vec{u}$.

(2). When $\kappa = 0$, $\vec{N}$ and $\vec{B}$ cannot be defined.

(3). If $\vec{B}(s)$ is a constant vector, then $\vec{c}(t)$ ($\vec{r}(t)$) move in a fixed plane, with normal vector $\vec{B}$.

Now, suppose $\vec{B}(s)$ isn't constant. First, $\|\vec{B}(s)\| = 1$ for all s. Then,

$$1 = \|\vec{B}(s)\|^2 = \vec{B}(s) \cdot \vec{B}(s)$$

holds for all s. So we can apply implicit differentiation:

$$0 = \frac{d}{dS}(1) = \frac{d}{dS}\left(\vec{B} \cdot \vec{B}\right) = 2\vec{B}' \cdot \vec{B}.$$

Then, it follows that $\vec{B}' \perp \vec{B}$, for every s.

Next, since $\vec{B}(s) \perp \vec{T}(s)$ for all s, we have $\vec{B} \cdot \vec{T} = 0$ for all s. Then,

$$\frac{d}{ds}\left(\vec{B} \cdot \vec{T}\right) = \vec{B}'(s) \cdot \vec{T}(s) + \vec{B}(s) \cdot \vec{T}'(s) = 0.$$

Since $\vec{T}' = \kappa \vec{N}$ and $\vec{B} \cdot \vec{N} = 0$, it follows that

$$\vec{B}'(s) \cdot \vec{T}(s) = 0 \iff \vec{B}'(s) \perp \vec{T}(s)$$

Since $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ form a orthonormal basis for $\mathbb{R}^3$, we must have $\vec{B}'(s)$ parallel to $\vec{N}$. Therefore,

$$\vec{B}'(s) = -\tau(s)\vec{N}(s)$$

for a function $\tau(s)$ called the *torsion*. Since $\tau = \|d\vec{B}/ds\|$, it measures how fast the normal $\vec{B}$ to the osculating plane is twisting.

Definition 3.11 (Torsion).

$$\tau = \left\|\frac{d\vec{B}}{ds}\right\| = \frac{\|\vec{B}'(t)\|}{\|\vec{c}'(t)\|}$$

Putting all the information together we get *Frenet formulas*:

Theorem 3.2 (Frenet formula).

$$\begin{cases} \frac{d\vec{T}}{ds} = \kappa \vec{N} \\ \frac{d\vec{B}}{ds} = -\tau \vec{N} \\ \frac{d\vec{N}}{ds} = -\kappa \vec{T} + \tau \vec{B} \end{cases}$$

Example 3.3.4. Consider the following helix:

$$\vec{c}(t) = (3\cos t, 3\sin t, 4t)$$

Then, we have

$$\|\vec{c}'(t)\| = 5,$$

$$\vec{T}(t) = \left(-\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right),$$

$$\vec{T}'(t) = \left(-\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right),$$

$$\kappa = \frac{3}{25},$$

$$\vec{N}(t) = (-\cos t, -\sin t, 0),$$

$$\vec{B}(t) = \vec{T} \times \vec{N} = \left(\frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5}\right),$$

$$\vec{B}' = \left(\frac{4}{5}\cos t, \frac{4}{5}\sin t, 0\right),$$

$$\tau = \frac{4}{25}.$$

3.4 Dynamics

How do these quantities relate to dynamical quantities? Given, $\vec{c}(t)$, a position vector along the curve, $\vec{c}\,'(t) = \vec{v}(t) = \vec{T}(t) \cdot ds/dt$ is its velocity vector and $\|\vec{c}\,'(t)\| = ds/dt$ is its speed.

Definition 3.12 (Acceleration). $\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$

First, observe that

$$\vec{v}(t) = \vec{c}\,'(t) = \frac{ds}{dt} \cdot \vec{T}(t)$$

Then,

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{ds}{dt} \cdot \vec{T}(t) \right) = \frac{d^2 s}{dt^2} \cdot \vec{T}(t) + \frac{ds}{dt} \cdot \vec{T}\,'(t)$$

$$= \frac{d^2 s}{dt^2} \cdot \vec{T} + \frac{ds}{dt} \cdot \left(\frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \right)$$

So we have

$$\vec{a}(t) = \underbrace{\frac{d^2 s}{dt^2} \cdot \vec{T}}_{\text{Linear acceleration}} + \underbrace{\kappa \left(\frac{ds}{dt} \right)^2 \vec{N}}_{\text{Steering-term}}$$

By looking at the steering term, we see that acceleration to turn on a curve is proportional to the curvature and $(\text{speed})^2$.

Example 3.4.1. Consider the following path

$$\vec{c}(t) = (e^t \cos t, e^t \sin t, e^t)$$

that drwas a spiral in xy direction and monotonically increases along z coordinate.

First, observe that

$$\vec{v}(t) = \vec{c}\,'(t) = (-e^t \sin + e^t \cos t, e^t \cos t + e^t \sin t, e^t)$$

$$\frac{ds}{dt} = \|\vec{c}\,'(t)\| = \sqrt{3} e^t$$

Then, we have

$$\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \frac{1}{\sqrt{3}}(-\sin t + \cos t, \cos t + \sin t, 1),$$

$$\vec{T}\,'(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \frac{1}{\sqrt{3}}(-\cos t - \sin t, -\sin t + \cos t, 0).$$

Since $\|\vec{T}\,'(t)\| = \sqrt{2/3}$, we can easily find the principal normal vector:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{1}{\sqrt{2}}(-\cos t - \sin t, -\sin t + \cos t, 0)$$

Then,

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|} = \frac{\sqrt{2}}{3}e^{-t}.$$

Furthermore,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \cdots = \frac{1}{\sqrt{6}}(\cos t - \sin t, -\sin t - \cos t, 2)$$

$$\vec{B}'(t) = \frac{1}{\sqrt{6}}(-\sin t - \cos t, -\cos t + \sin t, 0)$$

Therefore, torsion of the curve is given by

$$\tau(t) = \frac{\|\vec{B}'(t)\|}{\|\vec{c}'(t)\|} = \frac{1}{3}e^{-t}$$

We can then veriy formula for $\vec{a}$ in terms of $\vec{T}, \vec{N}, \kappa$, (and verify that it agrees with $\vec{a} = \vec{v}'(t)$ calculated directly).

Now, we present an alternative equation for curvature using dynamical quantities:

Theorem 3.3.
$$\kappa(t) = \frac{\|\vec{c}'(t) \times \vec{c}''(t)\|}{\|\vec{c}'(t)\|^3} = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

Proof. To verify it, we use the decomposition of $\vec{a}$:

$$\vec{v} \times \vec{a} = \vec{v} \times \left(\frac{d^2s}{dt^2} \cdot \vec{T} + \kappa \left(\frac{ds}{dt} \right)^2 \vec{N} \right)$$

$$= \frac{d^2s}{dt^2} \left(\vec{v} \times \vec{T} \right) + \kappa \left(\frac{ds}{dt} \right)^2 \left(\vec{v} \times \vec{N} \right)$$

$$= \kappa \left(\frac{ds}{dt} \right)^3 \left(\vec{T} \times \vec{N} \right)$$

$$= \kappa \left(\frac{ds}{dt} \right)^3 \vec{B}$$

Then, $\kappa(ds/dt)^3\|\vec{B}\| = \|\vec{v} \times \vec{a}\|$. Since $\vec{B}$ is a unit vector, the desired result has been achieved. $\square$

4 Implicit functions

4.1 The Implicit Function Theorem I

Often, we have an *implicit* relationship between variables,

$$F(x_1, x_2, \ldots, x_n) = 0,$$

rather than an *explicit* function relation, such as

$$x_n = f(x_1, x_2, \ldots, x_{n-1}).$$

Example 4.1.1. Look at a familiar example in $\mathbb{R}^2$,

$$x^2 + y^2 = 1.$$

This fails vertical line test $(y \neq f(x))$ as well as horizontal line test $(x \neq g(y))$; globally, this relation does not define a function. Locally, we can write this as a function, i.e. by restricting attention to small pieces of the curve.

First, define

$$F(x, y) = x^2 + y^2 - 1$$

If $y_0 > 0$, $x_0^2 + y_0^2 = 1$, i.e. $F(x_0, y_0) = 0$, and we look at a window (or *neighborhood*) around (x_0, y_0), which lies entirely in the upper half plane, we can solve for $y = f(x)$,

$$y = \underbrace{\sqrt{1 - x^2}}_{f(x)}$$

We could calculate $y' = f'(x)$ from the explicit formula but we can also get it via *implicit differentiation*:

$$\frac{d}{dx}(F(x, f(x))) = \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot f'(x)$$

$$= 2x + 2y f'(x),$$

so $f'(x) = -x/y$.

For a general $F(x, y) = 0$, we can solve for $f'(x)$ where its coefficient

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0,$$

where y is the variable we want to solve for. This gives the limitation on which we can solve for $y = f(x)$ locally! For the circle example,

$$\frac{\partial F}{\partial y} = 2y.$$

When $y = 0$, the vertical line test fails in every neighborhood of $(x_0, y_0) = (\pm 1, 0)$.

In general, suppose we have a C^1 function,

$$F : \mathbb{R}^{n+1} \to \mathbb{R},$$

and consider all functions of $F(\vec{x}, y) = 0$. In order that $y = g(\vec{x})$, i.e. we can solve for y as a differentiable function of $\vec{x}$, we do the same implicit differentiation, with the chain rule,

$$\frac{\partial}{\partial x_i}\left(F\left(x_1, x_2, \ldots, x_n, f(\vec{x})\right)\right) = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial y}\frac{\partial f}{\partial x_i}$$

for each $i = 1, 2, \ldots, n$. We can then solve for each

$$\frac{\partial f}{\partial x_i} = -\frac{-\dfrac{\partial F}{\partial x_i}}{\dfrac{\partial F}{\partial y}},$$

provided $\partial F/\partial y \neq 0$. This is a sufficient condition to solve for $y = f(\vec{x})$.

Theorem 4.1 (Implicit Function Theorem I). *Assume $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1 in a neighborhood of $(\vec{x}_0, y_0)$ with $F(\vec{x}_0, y_0) = 0$. If $\frac{\partial F}{\partial y}(\vec{x}_0, y_0) \neq 0$, then there exists neighborhood U of $\vec{x}_0$ and V of y_0 and a C^1 function*

$$f : U \subset \mathbb{R}^n \to V \subset \mathbb{R},$$

for which $F(\vec{x}, f(\vec{x})) = 0$ for all $\vec{x} \in U$. In addition,

$$Df(\vec{x}) = \frac{-1}{\dfrac{\partial F}{\partial y}(\vec{x}, y)} D_{\vec{x}} F(\vec{x}, y),$$

where

$$D_{\vec{x}} F(\vec{x}, y) = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}.$$

Example 4.1.2. Consider the following function:

$$xy + y^2 z + z^3 = 1.$$

For which parts on this surface ca we write $z = f(x, y)$, i.e.

$$F : \mathbb{R}^3 \to \mathbb{R}, F(x, y, z) = xy + y^2 z + z^3 - 1$$

is C^1?

We want to solve for z, so we look at

$$\frac{\partial F}{\partial z} = y^2 + 3z^2$$

We observe that $\partial F/\partial z = 0$ iff $y = 0$ and $z = 0$. However, $y = 0$ and $z = 0$ is not defined on this surface. At all points on this surface, $\partial F/\partial z \neq 0$. So at

every (x_0, y_0, z_0) with $F(x_0, y_0, z_0) = 0$, we can solve for $z = f(x, y)$ locally near (x_0, y_0)!

We can then use the implicit differentiation formula in the theorem to calculate $Df(x, y)$:

$$D_{(x,y)}F = \begin{bmatrix} y & (x + 2yz) \end{bmatrix},$$

so we get

$$Df(x, y) = \frac{-D_{(x,y)}F}{\partial F/\partial z} = \begin{bmatrix} -\dfrac{y}{y^2 + 3z^2} & -\dfrac{x + 2yz}{y^2 3z^2} \end{bmatrix}$$

or

$$\nabla f(x, y) = \left(-\frac{y}{y^2 + 3z^2}, -\frac{x + 2yz}{y^2 3z^2} \right).$$

Example 4.1.3. Consider the following equation:

$$x^4 + xz^2 + z^4 = 1.$$

Show that we can solve for $z = g(x)$ near $(x_1, z_1) = (-1, 1)$ but not near $(x_2, z_2) = (1, 0)$.

Proof. First, let

$$F(x, z) = x^4 + xz^2 + z^4 - 1.$$

Clearly, $F : \mathbb{R}^2 \to \mathbb{R}$ is C^1 for all $(x, z) \in \mathbb{R}^2$. Observe that

$$\frac{\partial F}{\partial z} = 2xz + 4z^3,$$

and so $\partial F(-1, 1)/\partial z \neq 0$.

By the Implicit Function Theorem, we can solve for $z = g(x)$ locally near $(x_1, z_1) = (-1, 1)$. In addition, we can get an explicit formula for its derivative:

$$Dg(x) = g'(x) = -\frac{\partial F(x, z)/\partial x}{\partial F/\partial z} = -\frac{4x^3 + z^2}{2xz + 4z^3}$$

Finally, since $\partial F(1, 0)/\partial z = 0$, the Implicit Function Theorem does not apply near $(1, 0)$. $\square$

Example 4.1.4. Consider the following equation:

$$x - z^3 = 0$$

Clearly, $F(x, z) = x - z^3$ is C^1 for all $(x, z) \in \mathbb{R}^2$. Note that

$$\frac{\partial F}{\partial z} = -3z^2.$$

Clearly, $\partial F/\partial z = 0$ at $(x, z) = (0, 0)$. However, we can write $z = x^{1/3}$ globally. So $z = g(x) = x^{1/3}$ exists but isn't differentiable at $(x_0, z_0) = (0, 0)$.

Example 4.1.5. Suppose we have a system of equations with more unknowns:

$$\begin{cases} u^2 - v^2 - x^3 = 0 \\ 2uv - y^5 = 0 \end{cases}$$

Can we solve for (u, v) as functions of (x, y)?

First, consider a C^1 function, $F : \mathbb{R}^4 \to \mathbb{R}^2$, that is defined as follows:

$$\begin{cases} F_1(x, y, u, v) = u^2 - v^2 - x^2 = 0 \\ F_2(x, y, u, v) = 2uv - y^5 = 0 \end{cases}$$

Following what we did before, we can assume $(u, v) = g(x, y)$ and see when we can calculate Dg. Note that

$$0 = \frac{\partial}{\partial x} F_1(x, y, u(x, y), v(x, y)) = \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v}\frac{\partial v}{\partial x}$$

$$0 = \frac{\partial}{\partial x} F_2(x, y, u(x, y), v(x, y)) = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F_2}{\partial v}\frac{\partial v}{\partial x}$$

Then, we can solve for $\partial u/\partial x$ and $\partial v/\partial x$. Rearranging,

$$\begin{bmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial F_1}{\partial x} \\ -\dfrac{\partial F_2}{\partial x} \end{bmatrix}$$

This can be solved if $D_{(u,v)}F$ is invertible, i.e. $\det\left[D_{(u,v)}F\right] \neq 0$.

Simiarly, we can also solve for $\partial u/\partial y$ and $\partial v/\partial y$. As a result, we get a different linear system to solve but with the same matrix $\left[D_{(u,v)}F\right]$. The second version of the Implicit Function Theorem says that this is the correct condition to solve for $g(x)$ in this setting.

4.2   The Implicit Function Theorem II

Implicit differentiation allows us to look at an underdetermined system of (non-linear) equations. Given a following function,

$$F_1(x_1, \ldots, x_n, u_1, \ldots u_m) = 0$$
$$F_2(x_1, \ldots, x_n, u_1, \ldots u_m) = 0$$
$$\vdots$$
$$F_m(x_1, \ldots, x_n, u_1, \ldots u_m) = 0$$

we want to solve for $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$ as a function, $\vec{u} = g(\vec{x})$, of $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Via implicit differentiation, for the case of $n = m = 2$, we arrived at an appropriate condition where this is possible.

Theorem 4.2 (Implicit Function Theorem II - General Form). *Let*

$$F : \mathbb{R}^{n+m} \to \mathbb{R}^m$$

be a C^1 function in a neighborhood of $(\vec{x}_0, \vec{u}_0) \in \mathbb{R}^{n+m}$, with $F(\vec{x}_0, \vec{u}_0) = \vec{0}$. If, in addition, $D_{\vec{u}}F(\vec{x}_0, \vec{u}_0)$ is invertible, then there exists neighborhoods $\mathcal{V}$ of $\vec{x}_0$ and $\mathcal{U}$ of $\vec{u}_0$, for which solutions of $F(\vec{x}, \vec{u}) = \vec{0}$ lie on a C^1 graph, $\vec{u} = g(\vec{x})$,

$$g : \mathcal{V} \subset \mathbb{R}^n \to \mathcal{U} \in \mathbb{R}^m$$

Example 4.2.1. Consider the following set of equations:

$$\begin{cases} 2xu^2 + yv^4 = 2 \\ xy(u^2 - v^2) = 0 \end{cases}$$

Can we solve for $(u, v) = g(x, y)$ near $(x_0, y_0, y_0, v_0) = (1, 1, -1, -1)$?

Let

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \vec{x} = (x, y), \vec{u} = (u, v),$$

where F is defined as

$$F_1(\vec{x}, \vec{u}) = 2xu^2 + yv^4 - 2 = 0$$
$$F_2(\vec{x}, \vec{u}) = xy(u^2 - v^2 = 0)$$

Then, we get the following Jacobian

$$D_{\vec{u}}F = \frac{\partial(F_1, F_2)}{\partial(u, v)}$$

$$= \begin{bmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} 4xu & 4yv^3 \\ 2uxy & -2vxy \end{bmatrix}$$

Substituting the given values, we have

$$D_{\vec{u}}F(1, 1, -1, -1) = \begin{bmatrix} -4 & -4 \\ -2 & 2 \end{bmatrix}$$

Since $\det D_{\vec{u}}F = -16 \neq 0$, the Implicit Function Theorem does apply, and we can solve for $\vec{u} = (u, v) = g(\vec{x}) = g(x, y)$ near $(x_0, y_0, u_0, v_0) = (1, 1, -1, -1)$.

Remark. In general, we can't get an explicit formula for g, but we can get a formula for $Dg(x, y)$, /ie its partial derivatives, using implicit differentiation.

Example 4.2.2. Consider the following set of equations:

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

Note that this example fits the form of the Implicit Function Theorem, but it's a special case. We want to invert this relationtion, i.e. given, $\vec{x} = f(\vec{u})$, we want to solve for $\vec{u} = g(\vec{x})$.

To get a nice theorem for this special case, we can use the framework of the Implicit Function Theorem:

$$F_1(\vec{x}, \vec{u}) = f_1(\vec{u}) - x = 0 \quad F_2(\vec{x}, \vec{u}) = f_2(\vec{u}) - x = 0$$

Since

$$D_{\vec{u}} F(\vec{x}, \vec{u}) = Df(\vec{u}),$$

we can do this locally near a point $(\vec{x}_0, \vec{u}_0)$ provided that

$$\det\left(Df(\vec{u})\right) \neq 0$$

Note that if we had a linear system, $\vec{x} = M\vec{u}$, we can solve $\vec{u} = M^{-1}\vec{x}$ provided $\vec{M} \neq 0$. This is why we call this derivative matrix, $Df(\vec{x})$ the *linearization* of $f(\vec{u})$.

4.3 Inverse Function Theorem

In general, suppose we have a C^1 function, $f : \mathbb{R}^n \to \mathbb{R}^n$, where $\vec{x} = f(\vec{u})$. How do we solve for $\vec{u} = g(\vec{x})$?

First, let's In single-variable calculus, a function $f : \mathbb{R} \to \mathbb{R}$ is *one-to-one* on an interval $[a, b]$ if and only if f is strictly monotone on $[a, b]$. For these functions, f has an inverse $g = f^{-1}$,

$$g(f(x)) = x, \quad \forall x \in [a, b]$$

If f is differentiable on $[a, b]$, and $f'(x) > 0$ on $[a, b]$ (or $f'(x) < 0$ on $[a, b]$), then the inverse $g(x)$ is also differentiable, and

$$g'(f(x)) = \frac{1}{f'(x)}, \quad \forall x \in [a, b]$$

If, for example, $f'(x) > 0$ for all $x \in \mathbb{R}$, then it's globally invertible, i.e. $g(f(x)) = x$ for all $x \in \mathbb{R}$. How do we apply this for $f : \mathbb{R}^n \to \mathbb{R}^n$ with $n \geq 2$?

Theorem 4.3 (Inverse Function Theorem)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ which is C^1 in a neighborhood of $\vec{u}_0$, with $f(\vec{u}_0) = \vec{x}_0$. If $\det\left(Df(\vec{u}_0)\right) \neq 0$, then there exist neighborhoods $\mathcal{U}$ of $\vec{u}_0$ and $\mathcal{V}$ of $\vec{x}_0$ and a C^1 function $g : \mathcal{V} \to \mathcal{U}$, with*

$$\underset{\text{with } \vec{u} \in \mathcal{U}}{\vec{x} = f(\vec{u})} \iff \underset{\text{with } \vec{x} \in \mathcal{V}}{\vec{u} = g(\vec{x})},$$

i.e. near $\vec{x}_0$ and $\vec{u}_0$, g is the inverse of f.

Example 4.3.1. Apply the Inverse Function Theorem to the function that was defined in the previous example:

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

Observe that

$$\det\left(Df(u,v)\right) = \det \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4u^2 + 4v^2 \neq 0$$

as long as $(u_0, v_0) \neq (0,0)$. So we can invert the variables and solve for $(u,v) = g(x,y)$, locally near any $(u_0, v_0) \neq (0,0)$.

Notice that

$$f_1(-u,-v) = x = f_1(u,v)$$
$$f_2(-u,-v) = y = f_2(u,v)$$

So in any neighborhood of $(0,0)$ there are 2 values of (u,v) corresponding to each (x,y). So f is not invertible near $(u,v) = (0,0)$.

Example 4.3.2. Consider the following equations:

$$\begin{cases} x = e^y \cos v \\ y = e^u \sin v \end{cases}$$

For which (u,v,x,y) can we solve for u,v as functions of x,y?

Call

$$f(u,v) = \begin{bmatrix} e^u \cos v \\ e^u \sin v \end{bmatrix}.$$

Then, we have

$$Df(u,v) \begin{bmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{bmatrix}$$

Then, we can compute $\det(Df(u,v))$, (or $\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$):

$$\det(Df(u,v)) = e^{2u}.$$

Clearly, $\det(Df(u,v)) > 0$ for all u,v. By the Inverse Function Theorem, we can invert and solve for $(u,v) = g(x,y)$, near any (u_0, v_0).

We can invert locally near any point; can we find a global inverse, i.e. a g for which $(u,v) = g(x,y)$ for every $(u,v) \in \mathbb{R}^2$? If so, then f would have to be a one-to-one function. However,

$$f(u, v + 2\pi k) = f(u,v)$$

for all $k \in \mathbb{Z}$. Therefore, f can't be globally inverted.

Example 4.3.3. Consider the following equations:

$$\begin{cases} x = f_1(u, v) = u^3 - 3uv^2 \\ y = f_2(u, v) = -v^3 + 3u^2v \end{cases}$$

Since they're polynomials, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1. Then, we have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} 3u^2 - 3v^2 & -6uv \\ 6uv & -3v^2 + 3u^2 \end{bmatrix}$$

$$\det\left(\frac{\partial(x, y)}{\partial(u, v)}\right) = (3u^2 - 3v^2) + (6uv)^2$$

Clearly, $\det(\partial(x, y)/\partial(u, v)) = 0$ iff $u = v = 0$. So, Inverse Function Theorem holds for all $(u_0, v_0) \neq (0, 0)$, and we can solve for $(x, y) = g(u, v)$ around any $(u_0, v_0) \neq (0, 0)$.

5 Taylor's Theorem

5.1 Taylor's Theorem in one dimension

Consider a one-dimensional function:

$$g : \mathbb{R} \to \mathbb{R},$$

which is C^{k+1}, i.e. it is $(k+1)$ times continuously differentiable; i.e., each derivative,

$$\frac{d^j g}{dx^j}(x), \; j = 1, 2, \ldots, k+1, \; \text{(of order up to and including the } (k+1)^{\text{st}}),$$

exists and is a continuous function (in some interval). Then, we can approximate $g(x)$ locally near $x = a$ by a polynomial of degree k, *Tayloer's polynomial*, $P_k(x)$:

$$P_k(x) = g(a) + g'(a)(x-a) + \frac{1}{2!}g''(a)(x-a)^2 + \cdots + \frac{1}{k!}\frac{d^k g}{dx^k}(a)(x-a)^k$$

There are chosen to match $g(x)$ up to the kth derivative at $x = 0$,

$$\frac{d^j P_k}{dx^j}(a) = \frac{d^j g}{dx^j}(a), \; j = 0, 1, 2, \ldots, k.$$

For example, $P_1(x) = g(a) + g'(a)(x-a)$ is the tangent line. Since we know that g is differentiable,

$$\lim_{x \to a} \frac{|g(x) - P_1(x)|}{|x-a|} = 0 \text{ or } g(x) = P_1(x) + o(|x-a|)$$

Theorem 5.1 (Taylor's Theorem). *Assume $g : \mathbb{R} \to \mathbb{R}$ is C^{k+1} in a neighborhood $\mathcal{N}$ around $x = a$. Then, for each $x \in N$, there is a c between a and x for which*

$$g(x) = P_k(x) + \underbrace{\frac{1}{(k+1)!}\frac{d^{k+1}g}{dx^{k+1}}(c)(x-a)^{k+1}}_{\text{Remainder term } R_k(a,x)}$$

Since we assume g is continuous, we have

$$\lim_{x \to a} \frac{R_k(a,x)}{|x-a|^k} = 0,$$

i.e. $R_k(a,x) = o\left(|x-a|^k\right)$. So $R_k(a,x)$ is small compared with each of the terms in $P_k(x)$.

Remark. Locally, $g(x)$ is well approximated by its Tayloer polynomials, but only near $x = a$.

Example 5.1.1. Notice that

$$g(x) = \cos x = \underbrace{1 - \frac{1}{2}x^2 + 0x^3}_{P_3(x)} + o(x^4)$$

This tells us that $\cos x$ is quadratic near $a = 0$. However, this clearly doesn't work for x that is not near $a = 0$.

5.2 Taylor's Theorem in higher dimensions

Can we apply Taylor's Theorem for $f : \mathbb{R}^n \to \mathbb{R}$, i.e., approximate a smooth function locally near $\vec{x} = \vec{a}$ via polynomial? We can do so by restricting our attention to each line, $\vec{x} = \vec{a} + t\vec{u}$, through $\vec{a}$ in direction $\vec{u}$.

Assume $f \in C^3$ near $\vec{x}_0 \in \mathbb{R}^n$. Let's sample $f(\vec{x})$ along a line running through x_0. Take a unit vector $\vec{u}$, $\|\vec{u}\| = 1$, and the line,

$$\vec{l}(t) = \vec{x}_0 + t\vec{u},$$

that goes through $\vec{x}_0$ at $t = 0$ in the direction of $\vec{u}$. Then, we get

$$g(t) = f(\vec{l}(t)) = f(\vec{x}_0 + t\vec{u}),$$

so $g : \mathbb{R} \to \mathbb{R}$.

By chain rule, if f is C^3 near $\vec{x}_0$, then g is C^3 near $t = 0$. So we use Taylors Theorem in g:

$$g(0) = f(\vec{x}_0),$$
$$g'(t) = Df(\vec{x}_0 + t\vec{u}) \cdot \vec{l}'(t) = Df(\vec{x}_0 + t\vec{u})\vec{u}.$$

So $g'(0) = Df(\vec{x}_0)\vec{u} = \nabla f(\vec{x}_0) \cdot \vec{u}$. Using coordinates,

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})u_i$$

so

$$g''(t) = \sum_{i=1}^{n} \underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(\vec{x}_0 + t\vec{u})\right)}_{D\frac{\partial f}{\partial x_i} \cdot \vec{u}} u_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0 + t\vec{u})u_j u_i$$

Therefore,

$$g''(0) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)u_j u_i$$

Now, call

$$H(\vec{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\right]_{i,j=1,\ldots,n},$$

the *Hessian matrix* of f at $\vec{x}_0$. For f, a C^2 function, $f_{x_i x_j} = f_{x_j x_i}$, so $H(\vec{x}_0)$ is a symmetric matrix.

So $g''(0) = \vec{u} \cdot H(\vec{x}_0)\vec{u}$. Using Taylor's Theorem, for g, we get:

Theorem 5.2 (Second order Taylor's approximation). *Assume $f : \mathbb{R}^n \to \mathbb{R}$ and is C^3 in a neighborhood of $\vec{x}_0$. Then,*

$$\underbrace{f(\vec{x})}_{g(\vec{x}_0 + t\vec{u})} = \underbrace{f(\vec{x}_0)}_{g(0)} + \underbrace{Df(\vec{x}_0)(\vec{x} - \vec{x}_0)}_{g'(0)(t-0)} + \frac{1}{2!}\underbrace{(\vec{x} - \vec{x}_0) \cdot H(\vec{x}_0)(\vec{x} - \vec{x}_0)}_{g''(0)(t-0)^2} + R_2(\vec{x}_0, \vec{x}),$$

where

$$H(\vec{x}_0) = D^2 f(\vec{x}_0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=1,2,\ldots,n}$$

is the hessian and

$$\lim_{\vec{x} \to \vec{x}_0} \frac{R_2(\vec{x}_0, \vec{x})}{\|\vec{x} - \vec{x}_0\|^2} = 0,$$

i.e. $R_2(\vec{x}_0 \vec{x}) = o(\|\vec{x} - \vec{x}_0\|^2)$. Alternatively, the second order Taylor's approximation can be written as

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h} + R_2(\vec{a}, \vec{h}),$$

with

$$\lim_{\vec{h} \to \vec{0}} = \frac{R_2(\vec{a}, \vec{h})}{\|\vec{h}\|^2} = 0.$$

Example 5.2.1. Find the second order Taylor polynomial of the following functions:

$$f(x, y) = \cos\left(xy^2\right)$$

near $\vec{a} = (\pi, 1)$.

First, we compute the derivatives:

$$f(\vec{a}) = f(\pi, 1) = \cos(\pi) = 1,$$
$$\frac{\partial f}{\partial x} = -y^2 \sin\left(xy^2\right),$$
$$\frac{\partial f}{\partial y} = -2xy \sin\left(xy^2\right),$$
$$\frac{\partial^2 f}{\partial x^2} = -y^2 \cos\left(xy^2\right) y^2,$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2y \sin\left(xy^2\right) - 2xy^3 \cos\left(xy^2\right),$$
$$\frac{\partial^2 f}{\partial y^2} = -2x \sin\left(xy^2\right) - 2xy \cos\left(xy^2\right).$$

Then, at $\vec{a} = (\pi, 1)$, we find that

$$Df(\vec{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$D^2 f(\vec{a}) = \begin{bmatrix} 1 & 2\pi \\ 2\pi & 4\pi^2 \end{bmatrix}$$

So, we have

$$f(\vec{a} + \vec{h}) = -1 + \frac{1}{2}\vec{h} \cdot \begin{bmatrix} 1 & 2\pi \\ 2\pi & 4\pi^2 \end{bmatrix} \vec{h} + R_2$$

$$f(\pi + h_1, 1 + h_2) = -1 + \frac{1}{2}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \cdot \begin{bmatrix} h_1 + 2\pi h_2 \\ 2\pi h_1 + 4\pi^2 h_2 \end{bmatrix}$$

$$= -1 + \frac{1}{2}\left(h_1^2 + 4\pi h_1 h_2 + 4\pi^2 h_2^2\right) + o(\|\vec{h}^2\|)$$

In terms of a point $\vec{x}$ (near $\vec{a}$), we can write $\vec{x} = \vec{a} + \vec{h}$, so $\vec{h} = \vec{x} - \vec{a}$, and then

$$\cos\left(xy^2\right) = -1 + \frac{1}{2}\left((x - a_1)^2 + 4\pi(x - a_1)(y - a_2) + 4\pi^2(y - a_2)^2\right) + R_2.$$

Advantage to the $f(\vec{a} + \vec{h})$ form is that it makes it easier to guess the behaviour of $f(\vec{x})$ near $\vec{x} = \vec{a}$.

5.3   Local minima/maxima

Definition 5.1. *We say $\vec{a}$ is a* local minimum *for f if there exists an open disk $D_r(\vec{a})$ for which*

$$f(\vec{a}) \leq f(\vec{x})$$

for all $\vec{x} \in D_r(\vec{a})$. $\vec{a}$ is a strict local minimum *if*

$$f(\vec{a}) < f(\vec{x})$$

for all $\vec{x} \neq \vec{a}, \vec{x} \in D_r(\vec{a})$.

Definition 5.2. *We say $\vec{a}$ is a local maximum for f if $\exists r > 0$ with $f(\vec{a}) \geq f(\vec{x})$, $\forall \vec{x} \in D_r(\vec{a})$. $\vec{a}$ is a strict local max if $f(\vec{a}) > f(\vec{x})$, $\forall \vec{x} \in D_r(\vec{a}) \setminus \{\vec{a}\}$.*

Note that if f is differentiable, we have a necessary condition for local maxima and minima.

Theorem 5.3. *If f has a local maxima or minima at $\vec{a}$ and is differentiable at $\vec{a}$, then $Df(\vec{a}) = \vec{0}$.*

Proof. Again, we start by restricting to line through $\vec{a}$:

$$g(t) = f(\vec{a} + t\vec{u}),$$

where $\vec{u}$ is a unit vector. If f has a local minima at $\vec{a}$, then

$$g(0) = f(\vec{a}) \leq f(\vec{a} + t\vec{u}) = g(t),$$

for all t with $|t| < r$. So $g(t)$ has a local minima at $t = 0$. By a calculus theorem, $g'(0) = 0$. But,

$$0 = g'(0) = Df(\vec{a})\vec{u},$$

for all $\vec{u}$. Then, by taking $\vec{u} = \vec{e}_1, \vec{e}_2 \ldots, \vec{e}_n$, we get

$$\frac{\partial f}{\partial x_j}(\vec{a}) = 0,$$

for each $j = 1, 2, 3, \ldots, n$. Therfore, $Df(\vec{a}) = 0$. $\qquad\square$

Definition 5.3. *An $\vec{a}$ for which $Df(\vec{a}) = 0$ is called a* critical point.

Example 5.3.1. In the previous example, $\vec{a} = (\pi, 1)$ was a critical point.

Now, we want to combine Taylor's Theorem and linear algebra to classify critical points as local minima, maxima, or others[6]. Taylor's theorem states that for $\vec{x} = \vec{a} + \vec{h}$, if $\|\vec{h}\|$ is small,

$$f(\vec{x}) = f(\vec{a} + \vec{h}) = f(\vec{a}) + \underbrace{Df(\vec{h})}_{0} + \underbrace{\frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h}}_{\text{quadratic form}} + \underbrace{R_2(\vec{a}, \vec{h})}_{o(\|\vec{h}\|^2)}$$

So we expect the behaviour of $f(\vec{x})$ near $\vec{a}$ to be determined by the quadratic term.

Notice that the Hessian matrix, $M = D^2 f(\vec{a})$, is a symmetric matrix. This allows us to apply the following theorem:

Theorem 5.4 (Spectral Theorem). *Assume M is a symmetric $(n \times n)$ matrix. Then,*

- *All eigenvalues of M are real, $\lambda_i \in \mathbb{R} \, \forall i = 1, 2, \ldots, n$.*

- *There is an orthonormal basis composed ot eigenvalues of M,*

$$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}, M\vec{u}_i = \lambda_i \vec{u}_i, \|\vec{u}_i\| = 1, \vec{u}_i \cdot \vec{u}_j = 0 \text{ for } i \neq j$$

- *In the basis of eigenvalues, M is a diagonal matrix. In other words, if we let U be the matrix whose columns are the $\vec{u}_i$; then*

$$MU = U\Lambda,$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

---

[6] **Cramer's Rule.** Given a system of linear equations that is represented by 2×2 matrices,

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases},$$

solution of the system is given by

$$x = \frac{\det \begin{pmatrix} s & b \\ t & d \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}, y = \frac{\det \begin{pmatrix} a & s \\ c & t \end{pmatrix}}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

Remark. Note that since the eigenvalues are real, they can be ordered, smallest to largest:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

However, they may not be necessarily distinct.

Written in the orthonormal basis of eigenvalues, the quadratic form, $\vec{h} \cdot M\vec{h}$, has an easy expression. Firt, we write

$$\vec{h} = \sum_{i=1}^{n} c_i \vec{u}_i, \; c_i \in \mathbb{R}, \; \forall i = 1, 2, \ldots, n$$

Notice that

$$\|\vec{h}\| = \sqrt{\sum_{i=1}^{n} c_i^2}$$

Then, we have:

$$\vec{h} \cdot M\vec{h} = \vec{h} \cdot \sum_{i=1}^{n} c_i M \vec{u}_i$$

$$= \vec{h} \cdot \sum_{i=1}^{n} \lambda_i c_i \vec{u}_i$$

$$= \sum_{i=1}^{n} \lambda_i c_i (\underbrace{\vec{h} \cdot \vec{u}_i}_{c_i})$$

$$= \sum_{i=1}^{n} \lambda_i c_i^2$$

Theorem 5.5. *Suppose M is a symmetric matrix with eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Then,

$$\lambda_1 \|\vec{h}\|^2 \leq \vec{h} \cdot M\vec{h} \leq \lambda_n \|\vec{h}\|^2$$

Proof. First, we have

$$\vec{h} \cdot M\vec{h} = \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\leq \sum_{i=1}^{n} \lambda_n c_i^2$$

$$= \lambda_n \sum_{i=1}^{n} c_i^2 = \lambda_n \|\vec{h}\|^2,$$

which proves the right hand inequality. For the let hand one,

$$\vec{h} \cdot M\vec{h} \geq \sum_{i=1}^{n} \lambda_1 c_i^2 = \lambda_1 \|\vec{h}\|^2.$$

This proves both sides of the inequality. $\qquad\square$

Now we apply this idea to the Hessian via Taylor's Theorem to get the following theorem:

Theorem 5.6 (Second derivative test). *Suppose f is C^3 in a neighborhood of a critical point $\vec{a}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $D^2 f(\vec{a})$. Then,*

1. *If all eigenvalues are positive, then $\vec{a}$ is a strict local minima of f.*

2. *If all eigenvalues are negative, then $\vec{a}$ is a strict local maxima of f.*

3. *If $D^2 f(\vec{a})$ has at least one positive and at least one negative eigenvalue, then $\vec{a}$ is a saddle point. In other words, $\exists r_0 > 0$ for which in $D_r(\vec{a}), 0 < r < r_0$, there are points with $f(\vec{x}) > f(\vec{a})$ and points with $f(\vec{x}) < f(\vec{a})$.*

Proof. Let's verify (1). By Taylor's Theorem, with $\vec{x} = \vec{a} + \vec{h}$, we have

$$f(\vec{x}) = f(\vec{a} + \vec{h}) = f(\vec{a}) + \frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h} + R_2(\vec{a}, \vec{h})$$

Notice that

$$\frac{1}{2}\vec{h} \cdot D^2 f(\vec{a})\vec{h} \geq \frac{1}{2}\lambda_1 \|\vec{h}\|^2,$$

where λ_1 is the smallest eigenvalue.

Now, we look at the rectangular term: $R_2(\vec{a}, \vec{h}) = o(\|\vec{h}\|^2)$. By taking $\epsilon = \lambda_1/4 > 0$, there exists $\delta > 0$ for which

$$\frac{|R_1|}{\|\vec{h}\|^2} < \epsilon = \frac{1}{4}\lambda_1,$$

if $0 < \|\vec{h}\| < \delta$, i.e. if $\vec{h} \in D_\delta(\vec{0})$, then $|R_2| < \frac{1}{4}\lambda_1 \|\vec{h}\|^2$. This implies that $R_2 > -\frac{1}{4}\lambda_1 \|\vec{h}\|^2$.

Combining these two results with Taylor expansion, if $\vec{x} \in D_\delta(\vec{a}), \vec{h} \in D_\delta(\vec{0})$, we get

$$\begin{aligned} f(\vec{x}) &\geq f(\vec{a}) + \frac{1}{2}\lambda_1 \|\vec{h}\|^2 - \frac{1}{4}\lambda_1 \|\vec{h}\|^2 \\ &\geq f(\vec{a}) + \frac{1}{4}\lambda_1 \|\vec{h}\|^2 \\ &> f(\vec{a}) \end{aligned}$$

if $\vec{h} \neq 0, \vec{h} \in D_\delta(\vec{0})$, i.e. $\vec{x} \in D_\delta(\vec{a})$. $\qquad\square$

Remark. When $D^2 f(\vec{a})$ has zero as an eigenvalue, things can get complicated. For example, if $\lambda_i \geq 0$ for all i, you *could* still have a local minima. In this case, the behaviour would be determined by higher order terms in Taylor Series. We call this *Degenerate critical point*.

Example 5.3.2. Consider

$$f(x, y, z) = x^3 - 3xy + y^3 + \cos z$$

Find all critical points and classify them using the Hessian.

First, observe that

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3y \\ \frac{\partial f}{\partial y} = -3x + 3y^2 \\ \frac{\partial f}{\partial z} = -\sin z \end{cases}$$

Critical points are defined as $\nabla f(\vec{a}) = \vec{0}$ so we get the following critical points

$$(0, 0, n\pi), (1, 1, n\pi),$$

where $n \in \mathbb{Z}$.

Then, we want to compute the Hessian at each point.

$$D^2 f(\vec{a}) = \begin{bmatrix} 6x & -3 & 0 \\ -3 & 6y & 0 \\ 0 & 0 & -\cos z \end{bmatrix}$$

Notice that at $(0, 0, n\pi)$, we get

$$D^2 f(0, 0, 2k\pi) = \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$D^2 f(0, 0, (2k+1)\pi) = \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When n is even, we find that its eigenvalues are

$$\lambda = -3, -1, 3$$

so we get a saddle at $(0, 0, 2k\pi), k \in \mathbb{Z}$. Similarly, when n is odd, we find that its eigenvalues are

$$\lambda = -3, 1, 3$$

which is also a saddle. Thus, we get a saddle at $(0, 0, n\pi)$ for all $n \in \mathbb{Z}$.

At $(1, 1, n\pi)$, we get

$$D^2 f(1, 1, n\pi) = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & (-1)^{n+1} \end{bmatrix}$$

By observation, we find that $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector with $\lambda = (-1)^{n+1}$.

Then, the two eigenvalues are eigenvalues of $\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$. Since its trace is 12

and determinant is 27, its characteristic equation is given by

$$\lambda^2 - 12\lambda + 27 = 0.$$

So we find that two other eigenvalues are $\lambda = 3, 9$. Therefore, $(1, 1, (2k + 1)\pi)$ is a local minima, and $(1, 1, 2k\pi)$ are saddles.

Example 5.3.3. Consider

$$f(x, y) = x^2 + y^4$$

We find that

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 4y^3 \end{bmatrix}$$

so we get only one critical point, $(x, y) = (0, 0)$. Ntice that

$$D^2 f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} \text{ so } D^2 f(0, 0) \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

So we get $\lambda = 2, 0$. Since the quadractic doesn't dominate the remainder, we call this a *degenerate* case.

Still, $f(0, 0) < f(x, y)$ for all $(x, y) \neq (0, 0)$ so its a minima even if the Hessian test doesn't tell us so.

Example 5.3.4. Consider

$$g(x, y) = x^2 - y^4$$

This has the same second order Taylor expansion as the previous example but has a different ramainder, $R_2 = -y^4$. This is a degenerate saddle.

Notice that for the converse, eigenvalues don't have to be strictly larger or smaller than 0. In other words, if $\vec{a}$ is a local minima, then $\vec{a}$ is a critical point and all the eigenvalues of $D^2 f(\vec{a})$ must be greater than equal to 0 (not necessarily strictly greater than 0).

6 Calculus of Variations

6.1 Single variable

In this section, we look at extremal problems, i.e. maxima, minima, and saddle points, where the unknown is a function (of one or several variables) which should optimize some real-valued expression.

Example 6.1.1. Set $[a, b] \subset \mathbb{R}$ and choose values $c, d \in \mathbb{R}$. Consider all C^2 functions, $u(x)$, joining $P_1 = (a, c)$ and $P_2 = (b, d)$, i.e. $u(a) = c$ and $u(b) = d$. Among all C^2 curves, $u(x)$, connecting P_1 to P_2, find the one with shortest arclength.

Let $\vec{c}(t) = (t, u(t))$. Then, we have

$$\|\vec{c}'(t)\| = \sqrt{1 + u'(t)^2}.$$

This allows us to compute the arclength:

$$I(u) = \in_a^b \sqrt{1 + u'(x)^2} dx.$$

Now, call $\mathcal{A} = \{u : [a, b] \to \mathbb{R} | u \in C^2, u(a) = c, u(b) = d\}$. Then, $I : \mathcal{A} \to \mathbb{R}$ is a function of functions, or *functional*. We want to minimize $I(u)$ over all $u \in \mathcal{A}$.

Example 6.1.2. For the same class $\mathcal{A}$ of functions, take $u \in \mathcal{A}$ and rotate around the x axis, creating a surface of revolution. Which $y = u(x)$ generate the surface of least area?

This time, we let

$$I(u) = 2\pi \int_a^b u(x)\sqrt{1 + u'(x)^2} dx.$$

A function which is a local minima of this $I(u)$ is called a *minimal surface*

Example 6.1.3. Find $y = u(x)$ for which one object sliding along the curvature from point P_1 to P_2 in the shortest time (with gravity).

For this problem, it is convenient to chose $P_1 = (0, 0)$. Then, we want to minimize

$$I(u) = \int_0^b \frac{\sqrt{1 + u'(x)^2}}{\sqrt{2g(u(x))}} dx$$

So the idea is to think of u as vectors and use analogy to $f(\vec{a})$. Recall that a local maxima or minima of a function $g(t) = f(\vec{a} + t\vec{v})$ should have a critical point at $t = 0$:

$$0 = g'(0) = Df(\vec{x})\vec{v},$$

for all directions $\vec{v}$. We can apply the same idea for $I(u)$. We assume u is a local minima or maxima for $I(u)$, and let $g(t) = I(u + tv)$, where v is a function which creates *variation* of u.

Problem is that we minimize (or maximize) under the assumption that u connects P_1 to P_2, i.e. $(i + tv) \in a$ is required and end points cannot change. So we insist that $v \in C^1$ and $v(a) = v(b) = 0$.

Now, call $\mathcal{A}_0$ the set of all $v(x)$ (notice that $\mathcal{A}_0$ is a vector space). Then, if $g(t) = I(u + tv)$ and $I(u)$ has an extreme value, we get

$$0 = g'(0) = \frac{d}{dt}(I(u + tv))\bigg|_{t=0}.$$

This equation is called the *first variation* and must hold for all variations, $v \in \mathcal{A}_0 = \{v \in C^1 | v(a) = v(b) = 0\}$.

Example 6.1.4. Solve example 6.1.1.

We had $I(u) = \int_a^b \sqrt{1 + u'(x)^2}dx$. Then, we have

$$
\begin{aligned}
g'(0) &= \frac{d}{dt}(I(u + tv))\bigg|_{t=0} \\
&= \frac{d}{dt}\left(\int_a^b \sqrt{1 + (u' + tv')^2}dx\right)\bigg|_{t=0} \\
&= \int_a^b \frac{d}{dt}\sqrt{1 + (u' + tv')^2}dx\bigg|_{t=0} \\
&= \int_a^b \frac{1}{2}\left(1 + (u' + tv')^2\right)^{-1/2} \cdot 2(u' + tv') \cdot v'dx\bigg|_{t=0} \\
&= \int_a^b \frac{u'(x)v'(x)}{\sqrt{1 + u'(x)^2}}dx = 0,
\end{aligned}
$$

for all $v \in \mathcal{A}_0$. So we can call $DI(u)v = 0$ its *critical point equation*.

Now we want to express the critical point equation in terms of $v(x)$ instead of $v'(x)$. To do so, we are going to integrate by parts:

$$
\begin{aligned}
\int_a^b \frac{u'(x)v'(x)}{\sqrt{1 + u'(x)^2}}dx &= \frac{u'}{\sqrt{1 + (u')^2}}v(x)\bigg]_a^b - \int_a^b \frac{d}{dx}\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)v(x)dx \\
&= \int_a^b -\frac{d}{dx}\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)v(x)dx \\
&= 0
\end{aligned}
$$

Notice that the equation holds for all $v \in \mathcal{A}_0$. This implies that the derivative of $u'/\sqrt{1 + (u')^2}$ must be identically equal to 0 on $[a, b]$ by the following lemma:

Lemma 6.1 (The Fundamental Lemma of the Calculus of Variations)**.** *Assume* $h(x)$ *is continuous on* $[a, b]$ *and* $\int_a^b h(x)v(x)dx = 0$ *for all* $v \in \mathcal{A}_0$. *Then,* $h(x) \equiv 0$ *on* $[a, b]$.

Proof. Assume $\int_a^b h(x)v(x)dx = 0$ for all $v \in \mathcal{A}_0$. Let $h(x_0) \neq x_0$ for some x_0 and assume that $h(x) > 0$ on an interval (α, β), which contains x_0.

Now, choose $v(x)$ with $v(x) > 0$ in (α, β) and zero outside. Then, we have

$$h(x)v(x) = \begin{cases} > 0 & \text{in } (\alpha, \beta) \\ = 0 & \text{outside } (\alpha, \beta) \end{cases}$$

So

$$0 = \int_a^b h(x)v(x)dx$$
$$= \int_\alpha^\beta h(x)v(x)dx > 0,$$

yielding a contradiction. Therefore, $h(x) = 0$ for all $x \in [a, b]$. $\qquad\square$

Going back to the critical point equation, we get a second order ODE, which is also referred to as *the Euler-Lagrange Equation*:

$$-\frac{d}{dx}\left(\frac{u'}{\sqrt{1+(u')^2}}\right) = 0$$

Then, we get

$$\frac{u'}{\sqrt{1+(u')^2}} = C,$$

where C is a constant. Solving, we find that $u'(x) = C_1 = \pm\sqrt{C^2/(1-C^2)}$, which yields

$$u(x) = C_1 x + C_2.$$

Therefore, we can conclude that a straight line is the path of least arc length.

Example 6.1.5. Solve example 6.1.2.

We were given

$$I(u) = 2\pi \int_a^b u(x)\sqrt{1+(u'(x))^2}dx$$

Notice that this functional has the form of

$$I(u) = \int_a^b F(u', u, x)dx,$$

where $F : \mathbb{R}^3 \to \mathbb{R}$. For surface area, we have $F(p, u, x) = 2\pi u(1 + p^2)$

In general, the first variation is given by

$$0 = \frac{d}{dt}(I(u+tv))\Big|_{t=0}$$
$$= \frac{d}{dt}\left(\int_a^b F(u'(x) + tv'(x), u(x) + tv(x), x)dx\right)\Big|_{t=0}$$
$$= \int_a^b \left(\frac{\partial F}{\partial p}v'(x) + \frac{\partial F}{\partial u}v(x)\right)dx$$

where $p = u'(x) + tv'(x)$. Now, we apply integration by parts to obtain

$$\int_a^b \left(\frac{\partial F}{\partial p}v'(x) + \frac{\partial F}{\partial u}v(x)\right)dx = \frac{\partial F}{\partial p}v(x)\Big]_a^b - \int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial p}\right)v - \frac{\partial F}{\partial u}v\right]dx$$

$$= \int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial u}\right]vdx = 0$$

Then, by the Fundamental Lemma of the Calculus of Variation, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial p}(u'(x), u(x), x)\right) - \frac{\partial F}{\partial u}(u'(x), u(x), x) = 0,$$

which is the general solution to the Euler-Lagrange Equation.

Here, we also present an alternate derivation of a general solution to the Euler-Lagrange equation. Assume $F = F(p, u)$ but not explicitly a function of x. Then, we can use the following trick. Notice that

$$\frac{d}{dx}\left[\frac{\partial F}{\partial p}(u', u)u'(x) - F(u', u)\right] = \frac{d}{dx}\left(\frac{\partial F}{\partial p}\right)u' + \frac{\partial F}{\partial p}u'' - \left(\frac{\partial F}{\partial p}u + \frac{\partial F}{\partial u}u'\right)$$

$$= \left[\frac{d}{dx}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial F}{\partial u}\right]u' = 0$$

Thus, by integrating the last equation, we get

$$\frac{\partial F}{\partial p}(u', u)u' - F(u', u) = C$$

Now, we apply this to the surface area equation. Since $F(p, u) = 2\pi u\sqrt{1 + p^2}$, we have

$$C = \frac{\partial F}{\partial p}(u', u)u' - F(u', u)$$

$$= 2\pi u \frac{u'}{\sqrt{1 + (u')^2}} - 2\pi u\sqrt{1 + (u')^2}$$

$$= 2\pi\left(\frac{u}{\sqrt{1 + (u')^2}}\right)$$

By squaring both sides, we get

$$\frac{C^2}{4\pi^2} = \frac{u^2}{1 + (u')^2}$$

$$u' = \sqrt{\frac{u^2}{c_1^2} - 1}$$

Since this is a separable equation, we can solve it by substituting $u/c_1 = \cosh\theta$:

$$\int \frac{du}{\sqrt{u^2/c_1^2 - 1}} = \int 1\,dx$$

$$\int \frac{c_1 \sinh\theta}{\sinh\theta}\,d\theta = x + c_2$$

$$c_1 \cosh^{-1}\left(\frac{u}{c_1}\right) = c_1\theta = x + c_2$$

Therefore, we get

$$u(x) = c_1 \cosh\left(\frac{x + c_2}{c_1}\right)$$

We call this surface of revolution a *Catenoid* – a minimal surfaces!

Example 6.1.6 (The brachistochrone). A brachistochrone is a curve on which motion downward from P_1 to P_2 travels in least time.

To find the brachistochrone curve, we first write the function we want to minimize:

$$I(u) = \int_0^b \frac{\sqrt{1 + (u'(x))^2}}{\sqrt{2gu(x)}}\,dx,$$

for $u \in \mathcal{A}$. We can rewrite this as

$$F(p, u) = \frac{\sqrt{1 + p^2}}{\sqrt{2gu}}.$$

Since this is of the same special form, we can use the *integrated* form of the Euler-Lagrange equation!

$$C = \frac{\partial F}{\partial p}(u', u, x)u' - F(u', u)$$

$$= \frac{u'}{\sqrt{1 + (u')^2}\sqrt{2gu}}u' - \frac{\sqrt{1 + (u')^2}s}{\sqrt{2gu}}$$

$$= \frac{-1}{\sqrt{1 + (u')^2}\sqrt{2gu}}$$

Now, we take the reciprocal and take the square on both sides:

$$u(1 + (u')^2) = \frac{1}{2gc^2} = k$$

We get a separable first order ODE. Unfortunately, the integral needed cannot expressed in *closed form*. However, we can still write it as a parametric curve:

$$\begin{cases} x = \frac{k}{2}(t - \sin t) \\ y = \frac{k}{2}(1 - \cos t) \end{cases} , t \in \mathbb{R}$$

Notice that $(\sin t, \cos t)$ describes a circle whereas $(t, 1)$ describes a horizontal motion to the right with constant speed of 1. So we get a *cycloid*.

Notice that

$$u'(x) = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

Then,

$$u(1 + (u')^2) = \frac{k}{2}(1 - \cos t)\left(1 + \frac{(1 - \cos^2 t)}{(1 - \cos t)^2}\right)$$

$$= k$$

6.2 Multi-variable

We wish to expand the idea to a function of several variables.

Definition 6.1 (Closure)**.** *Let D be a bounded, open set in $\mathbb{R}^n$ with* smooth *boundary ∂D (∂D is a closed curve for $D \subset \mathbb{R}^2$ and is a surface for $D \subset \mathbb{R}^3$). Then, $\bar{D} = D \cup \partial D$ is the* closure *of D (a closed set).*

For C^1 functions $u : \bar{D} \subset \mathbb{R}^n \to \mathbb{R}$, we define a functional,

$$I(u) = \int_D F(\nabla u(\vec{x}), u(\vec{x}), \vec{x})dx_1 \cdots dx_n$$

Then, the task is to minimize (or maximize) $I(u)$ over some class of C^1 function $u(\vec{x})$.

For example, all having the same given boundary values,

$$u(\vec{x}) = g(\vec{x}), \forall \vec{x} \in \partial D$$

is the boudary condition where g is a fixed, given function. We'll do as we did before: define a class of *variations*, $v(\vec{x})$, and look at the directional derivative of I,

$$\frac{d}{dt}I(u + tv)\bigg|_{t=0} = 0.$$

Let's start with an example:

Example 6.2.1. In $\mathbb{R}^2$, let the bounded, open set D represent an elastic membrane, at rest. Asssume that the membrane is attached to its ∂D in the $z = 0$ plane, like a drum skin to its rim. We then apply a loading force (i.e., a weight) to the membrane, represented by $f(x, y)$.

Call $u(x, y)$ the graph of the membrane shape, under loading. According to linear elasticity theory, the equilibrium shape of the membrane minimizes the elastic energy functional,

$$I(u) = \iint_D \left[\frac{1}{2}\|\nabla u\|^2 - f(x, y)u(x, y)\right] dxdy,$$

over all $u \in C^1(\bar{D})$ with $u(x, y) = 0, \forall (x, y) \in \partial D$.

Let $\mathcal{A}_0 = \{v \in C^1(\bar{D}) | v(\vec{x}) = 0, \forall \vec{x} \in \partial D\}$, the class which we're minimizing but also the usual class of admissible variations. So, if $u, v \in \mathcal{A}_0$ and $t \in \mathbb{R}$, then $u + tv \in \mathcal{A}_0$. As before, if u is extremal for $I(u)$, then

$$\frac{d}{dt}I(u + tv)\bigg|_{t=0} = 0, \forall v \in \mathcal{A}_0$$

Now, let's calculate it directly (rather than develop of general formula for the $F(\nabla u, u, \vec{x})$ case):

$$
\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0} \iint_D \left[\frac{1}{2}\|\nabla u + t\nabla v\|^2 - f(u + tv)\right] dxdy \\
&= \iint_D \frac{d}{dt}\bigg|_{t=0} \left[\frac{1}{2}\|\nabla u\|^2 + t\nabla u \cdot \nabla v + \frac{t^2}{2}\|\nabla v\|^2 - fu - tfv\right] dxdy \\
&= \iint_D \left[\nabla u \cdot \nabla v - fv + t\|\nabla v\|^2\right]\bigg|_{t=0} dxdy
\end{aligned}
$$

So we get

$$0 = \iint_D [\nabla u \cdot \nabla v - fv]dxdy, \forall v \in \mathcal{A}_0$$

Before we proceed, we introduce integration by parts for multiple integrals:

Lemma 6.2. *Suppose $D \subset \mathbb{R}^n$ ($n = 2$ or 3) with smooth ∂D, $u \in C^1(\bar{D})$, and $\vec{A} : \bar{D} \to \mathbb{R}^n$ is a C^1 vector field. Let $\vec{n}$ be the unit exterio normal vector to ∂D. Then,*

$$\iint_D \vec{A}(\vec{x}) \cdot \nabla u \, dxdy = \underbrace{\int_{\partial D} u\vec{A}(\vec{x}) \cdot \vec{n}ds}_{line\ integral} - \iint_D u \operatorname{div}(\vec{A})dxdy.$$

In 3D,

$$\iiint_D \vec{A} \cdot \nabla u \, dxdydz = \underbrace{\iint_{\partial D} u\vec{A}(\vec{x}) \cdot \vec{n}dS}_{surface\ integral} - \iiint_D u \operatorname{div}(\vec{A})dxdydz$$

Proof. This comes from combining a product rule,

$$\operatorname{div}\left(\vec{A}u\right) = \vec{A} \cdot \nabla u + \operatorname{div}(\vec{A})u$$

with the Divergence Theorem,

$$\iint_D \operatorname{div}(\vec{A}u)dxdy = \int_{\partial D}(\vec{A}u) \cdot \vec{n}ds.$$

We can then double integrate both sides of the identity and rearrange to get the integration by parts formula. $\square$

Continuing with our example, integrate $\iint\limits_{D} \nabla u \cdot \nabla v dx dy$ by parts:

$$0 = \int_{\partial D} \underbrace{v}_{=0} \nabla u \cdot \vec{n} ds - \iint\limits_{D} v \text{div}(\nabla u) dx dy - \iint\limits_{D} f v dx dy$$

So for all $v \in \mathcal{A}_0$, $\Delta u = \text{div}(\nabla u)$ is the Laplacian:

$$0 = - \iint\limits_{D} \left[\Delta u + f(x,y) \right] v dx dy$$

By FLCoV, we find that

$$\Delta u(x,y) + f(x,y) = 0$$

at each $(x,y) \in D$.

Remark. The expansion we did is a Taylor polynomial,

$$I(u + tv) = I(u) + tDI(u)v + \frac{1}{2} t^2 \iint\limits_{D} \|\nabla v\|^2 dx dy,$$

so the solution u is a minimum!

7 Fourier Series

Fourier series was developed by using solutions to the heat equation of a special form,

$$e^{kt}\sin(wx),$$

and linear combinations of functions of this form. Fourier claimed that any function can be expressed as a series of functions of this form:

$$\sum_{n=1}^{\infty}\sin(w_n x),$$

where w_n is a constant. Fourier was right; almost any functions can be expressed as a series of the following function:

$$\sum_{n=1}^{\infty} e_n \sin(nw_0 x)$$

Notice that $f(x) = A\sin(wx + \varphi)$ is a periodic function with period $T = 2\pi/w$, i.e. $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Similarly, if $k \in \mathbb{N}$, then $f_k(x) = A\sin(kwx + \varphi)$ is also T-periodic. So any linaer combination of these $f_k(x)$ are T-periodic and we can't represent non-periodic function by Fourier series, at least not on the whole real line $\mathbb{R}$. So the solution is to restrict to an interval of length T.

Remark. If $f(x)$ is T-periodic, it's enough to study it on any interval of length T, i.e. $[a, a + T]$, for any a. By this identity, we get

$$A\sin(wx + \varphi) = \alpha\cos(wx) + \beta\sin(wx),$$

where $A = \sqrt{\alpha^2 + \beta^2}$ and $\alpha = A\sin\varphi, \beta = A\cos\varphi$.

It will be convenient to use both sin and cos and use a symmetricinterval of length $T = 2L$ where $x \in [-L, L]$. We follow Fourier and look at integer multiple of the frequency, $w = \pi/L$, i.e.

$$\alpha_k \cos\left(\frac{k\pi x}{L}\right) + \beta_k \sin\left(\frac{k\pi x}{L}\right),$$

where $k = 1, 2, 3, \ldots$ as our basic building blocks. Using these building blocks, we want to answer 2 questions: (1) given a function $f(x)$ on $[-L, L]$, can we choose the coefficients α_k, β_k such that f is represented by the following form?

$$\alpha_0 + \sum_{k=1}^{\infty}\left(\alpha_k \cos\left(\frac{k\pi x}{L}\right) + \beta_k \sin\left(\frac{k\pi x}{L}\right)\right)$$

(2) does the series converge to $f(x)$ and in what sense does it converge?

7.1 Orthogonality

Consider the vector space of all continuous, T-periodic functions:

$$f : \mathbb{R} \to \mathbb{R},$$

which we call C_T. We introduce an inner product (or scalar product) on C_T:

Definition 7.1 (Inner product). $\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx$

We can easily verity the symmetry and linearity of the inner product. Now, we also define a norm:

Definition 7.2 (Norm). $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-L}^{L} (f(x))^2 \, dx}$

Using this definition of norm, we can define distance between f and g as $\|f(x) - g(x)\|$, which is equal to square root of area under the graph of $(f(x) - g(x))^2$.

This definition of norm should have the property that $\|f\| = 0$ iff $f(x) = 0$. This is true by an arrangement which is similar to the FLCoU!

With this definition of inner product, we find that

$$\left\{ 1, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \mid k = 1, 2, 3, \dots \right\}$$

is an orthogonal family of functions. In other words,

$$\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = 0$$

for $m \neq k$.

7.2 Fourier Series

So the idea is to take a function $f(x)$ defined on an interval of length $T = 2L$, typically $(-L, L)$ and associate to it a series of trigonometric function of period $T = 2L$:

$$f(x) \sim S(x)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right))$$

Further, we want to derive the coefficients a_k and b_k from $f(x)$ and test whether the series converges or not.

To find the coefficients, we use orthogonality. Define an inner product on C_T, a set of continuous functions with period T:

$$\langle f, g \rangle \int_{-L}^{L} f(x)g(x).$$

Then, $f \perp g$ when $\langle f, g \rangle = 0$ and we find that

$$\left\{ \frac{1}{2}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \,\middle|\, k = 1, 2, 3, \dots \right\}$$

is an orthogonal family on C_T. Further, we find that

$$\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \left\langle \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \begin{cases} \sqrt{L} & m = k \\ 0 & m \neq k \end{cases}$$

Recall for vectors $u, v \in \mathbb{R}^n$, the projection of v onto u is

$$\frac{v \cdot u}{\|u\|^2} u.$$

By analogy, we get

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

So the Fourier series of $f(x)$ is

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right).$$

If we look at a partial sum of this series,

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right),$$

we find that $S_n(x)$ is a trigonometric polynomial. Then, for large values of n, this should be a good approximation to $f(x)$ on $(-L, L)$.

Example 7.2.1. Consider $f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ with $L = \pi$ on $[-\pi, \pi]$. Then,

$$\begin{cases} a_0 = 1, a_1 = 0, a_2 = 1/2, a_{k \geq 3} = 0 \\ b_k = 0, \forall k \end{cases}$$

Any function of the form $\cos^m x, \sin^m x$ or $\cos^m x \sin^m x$ is a trig polynomial and it can be written as a linear combination using trigonometric identities.

Example 7.2.2. Consider $f(x) = x^2$ for $x \in (-\pi, \pi)$. Then, we get

$$b_k = \frac{1}{\pi} \int_{-\pi}^{p} ix^2 \sin(kx) dx = 0, \forall k \in \mathbb{N}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx = \cdots = \frac{4}{k^2}(-1)^k$$

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So

$$f(x) \sim S(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx)$$

Example 7.2.3. Consider $f(x) = x^2$ where $x \in (0, 2\pi)$. This periodic extension is different from the one above. Especially, this extension is discontinuous at all $2\pi m$. So even in the best case, where the series, $S(x)$, converges everywhere, it's discontinuous function at $x = 2\pi m$, for all $m \in \mathbb{Z}$. Then does it converge there, and if so, to which values?

We calculuate the Fourier coefficients, remembering that a 2π-periodic function can be integrated over any interval over length 2π, and give the same answer. So

$$a_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(kx) dx$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(kx) dx$$

In this case, we get $a_0 = 8\pi^2/3, a_k = 4/k^2, b_k = -4\pi/k$. Since $b_k \neq 0$, we know that this function is not symmetric.

Example 7.2.4. Consider $f(x) = x$ where $x \in (-\pi, \pi)$ and $L = \pi$. Then, we get

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(kx) dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin(kx) dx$$

$$= \frac{2}{k} (-1)^{k+1}$$

So as $f(x) = x$ on $(-\pi, \pi)$ is odd, we have a sine series:

$$f(x) \sim S(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

$S(x)$ has discontinuities, $f(-\pi) = -\pi \neq \pi = f(\pi)$. So what happens at the discontinuities, $x = (2m-1)\pi$?

$$S(n\pi) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kn\pi) = \sum_{k=1}^{\infty} 0 = 0$$

It will be a general fact that at a *jump discontinuity*, the Fourier series, $S(x)$, takes the midpoint value. It does not choose the value of $f(x)$ from either the right or the left at a jump discontinuity.

7.3 Convergence

Let's deal with convergence! Start with numerical seris, $\sum_{k=1}^{\infty} c_k$. The series converges to a number s if

$$\lim_{n \to \infty} \sum_{k=1}^{n} c_k = s.$$

We call $s_n = \sum_{k=1}^{n} c_k$, the *n-th partial sum* of the series. If the limit, $\lim_{n \to \infty} S_n$, does not exist, we say the series diverges.

Example 7.3.1 (Geometric Series). Consider

$$\sum_{k=0}^{\infty} r^k.$$

This is exceptional, because we can explicitly calculate s_n for these,

$$s_n = \frac{1 - r^{n+1}}{1 - r}, r \neq 1.$$

So series converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $r = 1$, $s_n = n$ and it diverges as $n \to \infty$. When $r = -1$, $s_n = 1 - 1 + 1 - 1 + 1 - \cdots \pm 1$. So

$$s_n = \begin{cases} 1, & \text{if even} \\ 0, & \text{if odd} \end{cases}$$

and s_n diverges. In conclusion, geometric series converges iff $|r| < 1$ and $s = 1/(1-r)$.

Example 7.3.2 (P-series). $\sum_{k=1}^{\infty} 1/k^p$ converges iff $p > 1$.

Using the comparison teset, we can determine convergence and divergence of many series using these exmaples.

Definition 7.3 (Absolute and conditional convergence). *A series $\sum_{k=1}^{\infty} c_k$ converges* absolutely *if $\sum_{k=1}^{\infty} |c_k|$ converges. The series converges* conditionally *if $\sum_{k=1}^{\infty} c_k$ converges but $\sum_{k=1}^{\infty} |c_k|$ diverges.*

Example 7.3.3 (Alternating series). Consider

$$\sum_{k=1}^{\infty} (-1)^k b)k,$$

with $b_k \to 0$ and $b_k \geq b_{k+1}$ for all k, the series converges conditionally. For example, if we let $b_k = 1/k$, the alternating series converges but the absolute series diverges.

Conditional convergence is delicate and the series converges slowly.

Let's apply it to Fourier Series. Fourier series are series of functions not just numbers:

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)) = \sum_{k=0}^{\infty} g_k(x).$$

There are several different (and important) notions of convergence for series of functions.

Definition 7.4 (Pointwise convergence). *FOr each fixed value of x, $\sum_{k=1}^{\infty} g_k(x)$ is a numerical series, which converges or doesn't. If is converges for each indi-vidual $x \in [a,b]$, we say the series converges* pointwise.

Example 7.3.4. Consider

$$\sum_{k=1}^{\infty}(1-x)x^k, x \in [0,1].$$

Then, we get the following partial sums:

$$S_n = \sum_{k=1}^{n}(1-x)x^k$$

$$= (1-x)\sum_{k=1}^{n} x^k$$

If $x \in [0,1)$, the series converges:

$$\lim_{n\to\infty} S_n(x) = (1-x)\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n} x^k}_{1/(1-x)} = 1$$

However, when $x = 1$, we have $S_n(1) = 0$. So at $x = 1$, the series converges to 0, i.e.

$$\sum_{k=1}^{\infty}(1-x)x^k = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1 \end{cases}$$

A series of continuous functions can converge pointwise to a discontinuous function![7] Fortunately, there is a nice theorem about pointwise convergence of trigonometric Fourier series, which is easy to apply.

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[7] Also, there are examples for which

$$\int_a^b \left(\sum_{k=1}^{\infty} g_n(x)\right) dx \neq \sum_{k=1}^{\infty} \int_a^b g_n(x)dx.$$

We say that $f(x)$ is piecewise C^1, or *piecewise smooth*, on $[a, b]$ if (i) f is differentiable and $f'(x)$ is continuous except maybe at finitely many points; (ii) At each exceptional point, $f(x)$ and $f'(x)$ have jump discontinuities.[8]

Theorem 7.1 (Pointwise Convergence Theorem for Trigonometric Fourier Series). *Assume f is a piecewise C^1, $(2L)$-periodic function. Then, its trigonometric Fourier Series,*

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right],$$

converges pointwise at every $x \in \mathbb{R}$ to

$$S(x) = \frac{1}{2} \left(f\left(x^+\right) + f\left(x^-\right) \right),$$

where

$$f\left(x^+\right) = \lim_{t \to x^+} f(t),$$
$$f\left(x^-\right) = \lim_{t \to x^-} f(t).$$

In other words, if f is continuous, $S(x) = f(x)$. If $f(x)$ jumps, $S(x)$ averages the jump values.

Remark. Just assuming $f(x)$ is continuous is not enough to calculate that

$$\lim_{n \to \infty} S_n(x) = f(x)$$

Example 7.3.5. Consider $f(x) = x$, $x \in (-\pi, \pi)$. Then,

$$S(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$
$$= x,$$

if $x \in (-\pi, \pi)$.

Example 7.3.6. Consider $f(x) = x$ for $x \in (-\pi, \pi)$, which is extended 2π-periodically. Then, $f(x)$ is discontinuoous. It takes the value of $S((2m-1)\pi) = 0$ and we get the following Fourier series:

$$S(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx).$$

Observe that
$$|b_k| = \left| \frac{2(-1)^{k+1}}{k} \right| = \frac{2}{k}.$$

Since $\sum_{k=1}^{2} 2/k$ diverges, the series converges conditionally, and b_k is an alternating series. So we see how slowly it converges by looking at graphs of S_n, for large n.

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[8] Even if f is $C^1([-\pi, \pi])$, when we do Fourier Seires, we get 2π-periodic extension of f to $\mathbb{R}$, which might create discontinuities $x = \pm\pi$.

Example 7.3.7. Consider $f(x) = x^2$ with $x \in (-\pi, \pi)$ extended 2π-periodically. In this case, the extension is continuous, so $S(x) = f(x) \, \forall x \in \mathbb{R}$. Further, we find that

$$S(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx).$$

Then, by looking at the individual terms,

$$|a_k \cos(kx)| = \left| \frac{4(-1)^k}{k^2} \cos(kx) \right| \leq \frac{4}{k^2},$$

we find that $S(x)$ converges absolutely for all $x \in \mathbb{R}$ by the Comparison Test.

Since it converges for all $x \in \mathbb{R}$, let's try some values. First, when $x = 0$, we get

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} = 0$$

So we find that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$$

Likewise, when $x = \pi$, we find that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

This example illustrates a second notion of convergence, uniform convergence.

Theorem 7.2 (Uniform Convergence). *Consider $S(x) = \sum_{k=1}^{\infty} g_k(x)$ with $x \in [a, b]$. We say that the series converges uniformly to $S(x)$ if*

$$\lim_{n \to \infty} \left(\max_{x \in [a,b]} \left| S(x) - \sum_{k=1}^{n} g_k(x) \right| \right) = 0.$$

In other words, for any $\epsilon > 0$, there is an N for which

$$\left| S(x) - \sum_{k=1}^{n} g_k(x) \right| < \epsilon,$$

whenever $n \geq N$.

Theorem 7.3 (Weierstress). *Assume $g_k(x)$ are continuous on $[a, b]$, where $k = 1, 2, 3, \ldots$.*

- ***M-Test.** If $|g_k(x)| < M_k$, for all $x \in [a, b]$, and $\sum_k M_k$ converges, then $\sum_k g_k(x)$ converges uniformly on $[a, b]$.*

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- If $S(x) = \sum_{k=1} g_k(x)$ converges uniformly on $[a, b]$, then $S(x)$ is continuous, and

$$s \int_a^b S(x)dx = \sum_{k=1}^{\infty} \sum_a^b g_k(x)dx$$

Going back to the examples, the series for $f(x) = x$ does not converge uniformly because $f(x)$ extended periodically is not continuous, whereas the series for $f(x) = x^2$ extended periodically does converge uniformly because

$$|g_k(x)| = \left| \frac{4(-1)^k}{k^2} \cos(kx) \right| \leq \frac{4}{k^2} \; \forall x$$

Remark. For Fourier Series, we get uniform convergnece if

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

converges. In other words, both $\sum_k |a_k|$ and $\sum_k |b_k|$ must converge.

Example 7.3.8. Consider $f(x) = |\sin x|$ with $x \in (-\pi, \pi)$. Then, we get

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \sin() ksdx$$

$$= 0$$

$$a_1 = 0$$

$$a_k = \frac{2}{\pi} \left((-1)^{k+1} - 1 \right) \frac{1}{k^2 - 1}$$

So

$$S(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} \cos(2jx),$$

which converges to $f(x)$ by the THeorem.

Since

$$|a_{2j}| = \left| -\frac{4}{\pi} \frac{1}{4j^2 - 1} \right|$$

$$= \frac{4}{\pi(4j^2 - 1)}$$

$$< \frac{4}{\pi} \frac{1}{4j^2 - j^2}$$

and $\sum 1/j^2$ converges, by the M-Test, the series converges uniformly.

We can now use it to calculate some numerical series. When $x = 0$, by the convergence theorem,

$$0 = f(0) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1}$$

$$\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} = \frac{1}{2}$$

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7.4 Orthogonal functions

So where do orthogonal functions come from?

7.4.1 Gram-schmidt process

Take a linearly independent collection $\{v_1, v_2, v_3, \dots\}$ in an independent space. By an iterative proecedure, we can create an orthogoanl collection $\{u_1, u_2, u_3 \dots\}$ by projection.

Take $u_1 = v_1$. Then,

$$u_2 = v_2 - \frac{\langle v_2, U_1 \rangle}{\|u_1\|^2} u_1$$

is orthogonal to u_1.

Proof.

$$\langle u_1, u_2 \rangle = \langle u_1, v_2 \rangle - \left\langle u_1 \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 \right\rangle$$

$$= \langle u_1, v_2 \rangle - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \underbrace{\langle u_1, u_1 \rangle}_{\|u_1\|^2}$$

$$= 0.$$

$\square$

Then, if we already have the following orthogoanl set:

$$\{u_1, u_2, \dots, u_k\},$$

we find that the vector obtained by the following procedure

$$u_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{j+1}, u_j \rangle}{\|u_j\|^2} u_j$$

is orthogoanl to the given set and we still get the same span.

A common application is to polynomials, $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. A basis of polynomials is $\{1, x, x^2, \dots\}$. For any choice of inner product, we get different orthogonal families of polynomials (Legendre, Chebysev, ...). If we take a familiar one, $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, then $P_0(x) = 1$. In homework, you'll also see that $P_1(x) = x \perp P_0(x) = 1$. Further,

$$P_2(x) = x^2 - \frac{\langle x^2, P_0 \rangle}{\|P_0\|^2} P_0 - -\frac{\langle x^2, P_1 \rangle}{\|P_1\|^2} P_1$$

$$= x^2 - \frac{\int_{-1}^{1} x^2 \cdot 1 dx}{\int_{-1}^{1} 1^2 dx} - \underbrace{\frac{\int_{-1}^{1} x^3 \cdot 1 dx}{\int_{-1}^{1} x^2 dx}}_{=0}$$

$$= x^2 - \frac{1}{3}$$

To get other families, we have to use different inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)w(x)dx.$$

7.4.2 Eigenvalue problems

In $\mathbb{R}^n$, a symmetric matrix, M, has n eigenvectors which forman orthogonal basis. In other words, if $Mu = \lambda u$ and $Mv = \mu v$ with $\lambda \neq \mu$, we get $\langle u, v \rangle = 0$.

Proof. Notice that due to symmetry, we get

$$\langle \lambda u, v \rangle = \langle Mu, v \rangle = \langle u, Mv \rangle = \langle u, \mu v \rangle$$

So we get $\lambda \langle u, v \rangle = \mu \langle u, v \rangle$. So since $\lambda \neq \mu$, we conclude that $\langle u, v \rangle = 0$. $\qquad \square$