MATH 2XX3 - Advanced Calculus II

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Contents

1 Introduction

In this course, we wish to study calculus using the concepts from linear algebra.

1.1 Vector norm

Definition 1.1. Euclidean norm *of* $\vec{x} = (x_1, x_2, \ldots, x_n)$ *is given as*

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

Theorem 1.1 (Properties of a norm)**.**

1. $\|\vec{x}\| \geq 0$ *and* $\|\vec{x}\| = 0$ *iff* $\vec{x} = \vec{0} = (0, 0, \ldots, 0)$.
2. *For all scalars* $a \in \mathbb{R}$, $\|a\vec{x}\| = |a| \cdot \|\vec{x}\|$.
3. *(Triangle inequality)* $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

We say that this is a property of a norm *because there are other norms, which measure* distance *in* $\mathbb{R}^n$ *in different ways!*

Example 1.1.1 (A non-pythagorian norm - *The Taxi Cab Norm*)**.** Consider the following vector $\vec{p} = (p_1, p_2) \in \mathbb{R}^2$. The euclidean norm gives the length of the diagonal line. On the other hand,

$$\|\vec{p}\|_1 = |p_1| + |p_2|$$

gives us the total distance in a rectangular grid system.

For $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, $\|\vec{p}\|_1 = \sum_{j=1}^{n} |p_j|$. Note that the Taxi Cab norm is a valid norm because it satisfies all properties of a norm above. So it also gives us a valid alternative way to measure *distance* in $\mathbb{R}^n$, $\mathrm{dist}(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\|$. This way of measuring distance gives $\mathbb{R}^n$ a *different geometry.*

Definition 1.2. Neighborhood *of a point* $\vec{p}$*, or disks centered at* $\vec{p}$ *is defined as*

$$D_r(\vec{p}) = \left\{\vec{x} \in \mathbb{R}^n \middle| \|\vec{x} - \vec{p}\| < r\right\}$$

Definition 1.3. *Sphere is defined as*

$$S_r(\vec{p}) = \left\{\vec{x} \in \mathbb{R}^n \middle| \|\vec{x} - \vec{p}\| = r\right\}$$

What neighboorhood and sphere look like depends on which norm you choose. First, let's start with the familiar euclidean norm. Then, the sphere is given by

$$\begin{aligned} &\|\vec{x}-\vec{p}\| = r \\ \iff &\sqrt{\sum_{j=1}^{n}(x_j-p_j)^2} = r \end{aligned}$$

Then, we have

$$\sum_{j=1}^{n}(x_j-p_j)^2 = r^2$$

If $n=3$, we have $(x_1-p_1)^2+(x_2-p_2)^2+(x_3-p_3)^2 = r^2$, usual sphere in $\mathbb{R}^3$ with center $\vec{p}=(p_1,p_2,p_3)$

If $n=2$, we have $(x_1-p_1)^2+(x_2-p_2)^2 = r^2$, usual circle in $\mathbb{R}^n$ with center $\vec{p}=(p_1,p_2)$.

If we replace Euclidean norm by the Taxi Cab norm (for simplicity, take $\vec{p}=\vec{0}$), we have

$$S_r^{\text{taxi}}(\vec{0}) = \left\{\vec{x}\in\mathbb{R}^n \middle| \|\vec{x}-\vec{0}\|_1 = r\right\}$$

In other words, we have

$$\vec{x}\in S_r^{\text{taxi}}(\vec{0}) \iff \sum_{j=1}^{n}|x_j| = r$$

Looking at it in $\mathbb{R}^2$, we have $\vec{x}=(x_1,x_2)$. Then, $r=|x_1|+|x_2|$. This, in fact, is a diamond.

Remark. Note that $|x_1|+|x_2|=r$ is a *circle* in $\mathbb{R}^2$ under the Taxi Cab norm. Then, we have

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{8r}{2r} = 4$$

1.2 Subset

Let's introduce some properties of *subsets* in $\mathbb{R}^n$. $A\subset\mathbb{R}^n$ means A is a *collection* of points $\vec{x}$, drawn from $\mathbb{R}^n$.

Definition 1.4. *Let $A\subset\mathbb{R}^n$, and $\vec{p}\in A$. We say $\vec{p}$ is an interior point of A if there exists a neighbourhood of $\vec{p}$, i.e. an open disk disk, which is entirely contained in A:*

$$D_r(\vec{p})\subset A.$$

Example 1.2.1.

$$A = \left\{\vec{x}\in\mathbb{R}^n | \vec{x}\neq\vec{0}\right\}$$

Take any $\vec{p} \in A$, so $\vec{p} \neq \vec{0}$. Then, let $r = \|\vec{p} - \vec{0}\| > 0$, and $D_r(\vec{p}) \subset A$, since $\vec{0} \notin D_r(\vec{p})$. (Notice: any smaller disk, $D_s(\vec{p}) \subset D_r(\vec{p}) \subset A$, where $0 < s < r$ works to show that $\vec{p}$ is an interior point).

So every $\vec{p} \in A$ is an interior point to A.

Definition 1.5. *If every $\vec{p} \in A$ is an interior point, we cal A an* open set*.*

Example 1.2.2. $A = \left\{\vec{x} \in \mathbb{R}^n | \vec{x} \neq \vec{0}\right\}$ is an open set.

Example 1.2.3. $A = D_R(\vec{0})$ is an open set.

Proof. If $\vec{p} = \vec{0}$, $D_r(\vec{0}) \subseteq A = D_R(\vec{0})$ provided $r \leq R$. So $\vec{p} = \vec{0}$ is interior to A. Consider any other $\vec{p} \in A$. It's evident that $D_r(\vec{p}) \subset A = D_R(\vec{0})$ provided that $0 \leq r \leq R - \|\vec{p}\|$. Therefore, $A = D_R(\vec{0})$ is an open set. ☐

Example 1.2.4. Suppose we use Taxi Cab disks instead of Euclidean disk. It does not change which points are interior to A since the diamond is inscribed in a circle. In other words,

$$D_r^{\text{taxi}}(\vec{p}) \subset D_r^{\text{Euclid}}(\vec{p})$$

Definition 1.6. *The complement of set A is*

$$A^c = \{\vec{x} | \vec{x} \notin A\}$$

Definition 1.7. *$\vec{b}$ is a* boundary point *of A if for every $r > 0$, $D_r(\vec{b})$ contains both points in A and points not in A:*

$$D_r(\vec{b}) \cap A \neq \emptyset \text{ and } D_r(\vec{b}) \cap A^c \neq \emptyset$$

In the example 1.2.3, the set of all boundary points of $A = D_R(\vec{0})$

$$\left\{\vec{b} \quad \big| \|\vec{b}\| = R\right\}$$

is a sphere of radius R.

Definition 1.8. *A set A is closed if A^c is open.*

Theorem 1.2. *A is cloased if and only if A contains all its boundary points.*

Example 1.2.5. Consider the following set:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \big| x_1 \geq 0, x_2 > 0\}$$

If $\vec{p_1} = (p_1, p_2)$, where $p_1 > 0, p_2 > 0$, then $\vec{p_1}$ is an interior point. Take $r = \min\{p_1, p_2\}$. Then, $D_r(\vec{p}) \subset A$. On the other hand, any $\vec{p}$ that lies on either axes (including $\vec{0}$) is a boundary point. Since there are boundary points in A, A can't be open.

1.3 Functions

In this section, we will be considering vector values functions such that

$$F : A \subseteq \mathbb{R}^n \to \mathbb{R}^k.$$

We can rewrite this using a matrix notation:

$$F(x_1, x_2, \dots, x_n) = \begin{bmatrix} F_1(x_1, x_2, \dots, x_n) \\ F_2(x_1, x_2, \dots, x_n) \\ \vdots \\ F_k(x_1, x_2, \dots, x_n) \end{bmatrix}$$

Example 1.3.1. For a $(k \times n)$ matrix M,

$$F(\vec{x}) = M\vec{x}$$

First, we wish to study limits. What does $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{L}$ mean? Note that it's not enough to treat the variables $x_1, x_2, \dots x_n$ separately.

Example 1.3.2. Consider the following function:

$$F(x, y) = \frac{xy}{x^2 + 4y^2},$$

where $(x, y) \neq (0, 0)$. First, we can attempt to find its limit by considering them separately.

$$\lim_{x \to 0} \left(\lim_{y \to 0} F(x, y) \right) = \lim_{x \to 0} \left(\frac{0}{x^2} \right) = \lim_{x \to 0} = 0$$

Similarly, we have

$$\lim_{y \to 0} \left(\lim_{x \to 0} F(x, y) \right) = 0$$

However, if $(x, y) \to (0, 0)$ along a straight line path with $y = mx$, where m is constant, we have

$$F(x, mx) = \frac{mx^2}{x^2 + 4m^2x^2} = \frac{m}{1 + 4m^2}$$

In this case, we have

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = mx}} F(x, y) = \frac{m}{1 + 4m^2}$$

Therefore, the values of $F(x, y)$ don't approach any particular value as $(x, y) \to (0, 0)$.

Example 1.3.3 (Worse)**.** Consider the following function:

$$F(x,y) = \frac{y^2}{x^4 + y^2}.$$

If we approach $(0,0)$ along $y = mx$, limit equals 1. However, if we approach along a parabola, $y = mx^2$, limit equals $m^2/(1+m^2)$. We get different limits alond different parabolas.

We showed that computing

$$\lim_{\vec{x}\to\vec{a}} = \vec{b}$$

is tricky because $\vec{x} \to \vec{a}$ has to be more precise. It can't depend on the path or direction on which $\vec{x}$ approaches $\vec{a}$, but only on *proximity*. In other words, we want $\|\vec{F}(\vec{x}) - \vec{b}\|$ to go to zero as $\|\vec{x} - \vec{a}\|$ goes to zero.

Definition 1.9. *We say* $\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x}) = \vec{b}$ *if for any given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that whenever* $0 < \|\vec{x} - \vec{a}\| < \delta$*, we have* $\|\vec{F}(x) - \vec{b}\| < \varepsilon$*. Therefore,*

$$\lim_{\vec{x}\to\vec{a}} \vec{F}(x) = \vec{b} \iff \lim_{\vec{x}\to\vec{a}} \|\vec{F}(\vec{x}) - \vec{b}\| = 0$$

Remark. Geometrically, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\vec{F}(\vec{x}) \in D_\varepsilon(\vec{b}),$$

where $\vec{x} \in D_\delta(\vec{a})$.

Before doing examples, here's a useful observations. Take $\vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. Then, we have

$$\|\vec{v}\| = \sqrt{\sum_{j=1}^{n} v_j^2} \geq \sqrt{v_i^2} = |v_i|$$

for each coordinate $i = 1, 2, \ldots, n$.

Example 1.3.4. Show

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$$

Proof. Note that $F : \mathbb{R} \setminus \{\vec{0}\} \to \mathbb{R}$, $b = 0$, $\vec{a} = (0,0)$. Call

$$R = \|\vec{x} - \vec{a}\| = \|\vec{x}\| = \sqrt{x^2+y^2}$$

Since $\vec{F}(\vec{x}) \in \mathbb{R}$, we have

$$
\begin{aligned}
\|\vec{F}(\vec{x}) - \vec{b}\| &= |F(\vec{x}) - b| \\
&= \left| \frac{2x^2 y}{x^2 + y^2} - 0 \right| \\
&= \frac{2|x|^2|y|}{x^2 + y^2} \\
&\leq \frac{2 \cdot R^2 \cdot R}{R^2} \\
&= 2R \\
&= 2\|\vec{x} - \vec{a}\|
\end{aligned}
$$

By letting $\|\vec{x} - \vec{a}\| = \|\vec{x}\| < \varepsilon/2$, we get $\|\vec{F}(\vec{x}) = \vec{b}\| < \varepsilon$. Therefore, definition is satisfied with $\delta \leq \varepsilon/2$ $\square$

Example 1.3.5. Consider the following function, $F : \mathbb{R}^3 \setminus \{\vec{0}\} \to \mathbb{R}$:

$$
\frac{3z^2 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2}.
$$

Determine whether $\lim\limits_{(x,y,z)\to(0,0,0)} \vec{F}(x, y, z) = 2$.

Example 1.3.6. *Proof.* We have

$$
\begin{aligned}
\|\vec{F}(x, y, z) - \vec{b}\| &= |F(x, y, z) - 2| \\
&= \left| \frac{3z^3 + 2x^2 + 4y^2 + 6z^2}{x^2 + 2y^2 + 3z^2} - 2 \right| \\
&= \frac{3|z|^3}{x^2 + 2y^2 + 3z^2} \\
&\leq \frac{3R^3}{x^2 + y^2 + z^3} \\
&= \frac{3R^3}{R^2} \\
&= 3R
\end{aligned}
$$

Then,

$$
\|\vec{F}(x, y, z) - \vec{b}\| < 3R < \varepsilon
$$

provided that

$$
R = \|\vec{x} - \vec{0}\| < \delta = \frac{\varepsilon}{3}
$$

$\square$

Definition 1.10. *We say* $\vec{F} : \mathbb{R}^n \to \mathbb{R}^k$ *is* continuous *at* $\vec{a}$ *in the domain of* $\vec{F}$ *iff*

$$
\vec{F}(a) = \lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{a})
$$

Example 1.3.7. Going back the example (TODO: ref), if we redefine F as follows,

$$\vec{F} = \begin{cases} \frac{3z^2+2x^2+4y^2+6z^2}{x^2+2y^2+3z^2} & (x,y,z) \neq (0,0,0) \\ 2 & (x,y,z) = (0,0,0) \end{cases}$$

then $\vec{F}$ is continuous at $(0,0,0)$ (and in fact at all $\vec{x} \in \mathbb{R}$).