

# Numerical Method for Ordinary differential equations and Its Application in Physics

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## Abstract

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## 1 Introduction

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## 2 Examples of ordinary differential problems in physics

### 2.1 The Earth-Sun system

The Earth-Sun system is a two-body system governed by the gravitational force

$$\vec{F}_G = \frac{GM_S M_E \hat{r}}{r^2}, \quad (1)$$

where  $G$  is the gravitational constant;  $M_S$  and  $M_E$  are mass of the Sun and the Earth;  $\vec{r}$  is the displacement between them.

Given the fact the Sun is about  $10^6$  heavier than the Earth, we can safely keep the Sun as the center of mass (C.M.) in this problem. With proper coordinate setup, the orbit of the Earth is co-planar in  $xy$ -plane. Using Newton's second law we get the following first-order ordinary differential equations (ODEs) for the Earth

$$\begin{cases} \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dv_x}{dt} = -\frac{GM_S x}{r^3} \\ \frac{dv_y}{dt} = -\frac{GM_S y}{r^3} \end{cases} \quad (2)$$

By solving above equations, we can obtain the informations about the Earth's orbit we need.

### 2.2 Many body problem

In the above Sec. 2.1, we simplify the calculation of the orbit of the Earth by taking into account only its interaction with the Sun. This simplification is reasonable as the Sun is much heavier than other planets in the solar system.

However, if we want to go a step further to get a more precise description of the Earth's orbit. We have to include distortions from other seven planets as well as the Pluto. We should also abandon our

previous static Sun setup but using the real center of mass of the solar system. Until now, we have a new set of ODEs for the Earth

$$\begin{cases} \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dv_x}{dt} = \sum_{i=1}^n -\frac{GM_i x_i}{r_i^3} \\ \frac{dv_y}{dt} = \sum_{i=1}^n -\frac{GM_i y_i}{r_i^3}, \end{cases} \quad (3)$$

where  $i$  runs over all other celestial bodies except the Earth itself. Besides the Earth, we have similar sets of ODEs for every celestial body.

By solving these coupled ODEs, we can obtain a full description for the solar system.

### 3 Numerical methods

#### 3.1 Euler forward method

Suppose a first-order ordinary differential equation

$$\frac{dx}{dt} = f(x, t); \quad t \in [t_0, t_1] \quad (4)$$

with a initial value  $x_0$  at time  $t_0$ .

In order to solve this problem, we first discretize the region  $[t_0, t_1]$  into  $N$  subintervals with a step  $h$ , so we get the relation

$$h = \frac{t - t_0}{N}. \quad (5)$$

Then, We have discretized  $x_i = x(t_i = t_0 + ih)$  where  $i$  is a integer between 0 and  $N$ .

Using Taylor expansion, we get

$$x_{i+1} = x_i + \frac{dx_i}{dt}h + \frac{d^2x_i}{dt^2}h^2 + O(h^3), \quad (6)$$

where  $i$  goes from 0 to  $N - 1$ . The Euler forward method truncates at the second term of the above equation. Thus, Eq. 6 becomes

$$\begin{aligned} x_{i+1} &= x_i + \frac{dx_i}{dt}h + O(h^2) \\ &= x_i + f(x_i, t_i)h + O(h^2). \end{aligned} \quad (7)$$

We can see it's a one-step method with a local error  $O(h^2)$ .

Getting back to the Earth-Sun system, we can formulate Eq. 2 to

$$\begin{cases} x_{i+1} = x_i + v_x^i h \\ y_{i+1} = y_i + v_y^i h \\ v_x^{i+1} = v_x^i - \frac{4\pi^2 x_i}{r_i^3} h \\ v_y^{i+1} = v_y^i - \frac{4\pi^2 y_i}{r_i^3} h. \end{cases} \quad (8)$$

in unit of AU for length, year (yr) for time and  $M_S$  for mass. We will keep using these units in our report calculations. Starting from some initial conditions, we can simply solve out time evolution of the

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Input:  $x_0 = 1, y_0 = 0, v_x^0 = 0, v_y^0 = 4\pi^2, M_E, M_S, G$ 
Output:  $\vec{x}=(x_0, x_1, \dots, x_N), \vec{y}, \vec{v}_x, \vec{v}_y, \vec{E}_k, \vec{E}_p, \vec{E}, \vec{L}_z$ 
1  $r = \text{sqrt}(x_0^2 + y_0^2);$ 
2 //  $i$  is deferent time points  $t_i = t_0 + ih;$ 
3 for  $i = 1; i \leq N; i++$  do
4    $x_i = x_{i-1} + v_x^{i-1}h; y_i = y_{i-1} + v_y^{i-1}h;$ 
5    $v_x^i = v_x^{i-1} - \frac{4\pi^2 x_{i-1}}{r_{i-1}^3}h; v_y^i = v_y^{i-1} - \frac{4\pi^2 y_{i-1}}{r_{i-1}^3}h;$ 
6    $r = \text{sqrt}(x_i^2 + y_i^2);$ 
7   // Kinetic energy  $E_k$ , Potential energy  $E_p$ , Total energy  $E$ , Angular momentum in  $\hat{z}$   $L_z$ ;
8    $E_k^i = 0.5M_E((v_x^i)^2 + (v_y^i)^2); E_p^i = -\frac{GM_E M_S}{r};$ 
9    $E^i = E_k^i + E_p^i; L_z^i = M_E(x_i v_y^i - y_i v_x^i);$ 
10 end
11 return  $\vec{x}, \vec{y}, \vec{v}_x, \vec{v}_y, \vec{E}_k, \vec{E}_p, \vec{E}, \vec{L}_z;$ 

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**Algorithm 1:** The Euler forward method for the Earth-Sun system. It initials from a circular orbit.

Earth iteratively. Our implementation of this method with circular orbit initial conditions is shown in Algorithm ??.

We can see that the Euler forward method is easy to be realized. However, it has a vital defects that it violates the energy conservation and time reversibility. The total energy increases with time in the Euler forward method. That's why we need the Velocity-Verlet method to describe physical systems.

### 3.2 Velocity-Verlet method

The velocity-Verlet method is widely used in molecular dynamics calculation as it overcomes these defects. It conserves energy with small round-off errors[2].

Staring from the Taylor expansions under same discretization

$$\begin{aligned} x_{i+1} &= x_i + x_i^{(1)}h + x_i^{(2)}h^2 + O(h^3), \\ v_x^{i+1} &= v_x^i + v_x^{i(1)}h + v_x^{i(2)}h^2 + O(h^3), \end{aligned} \quad (9)$$

with a initial value  $x_0$  and  $v_x^0$  at time  $t_0$ . We truncate at the third term and evaluate  $v_x^{i(2)}h \approx v_x^{i+1(1)} - v_x^{i(1)}$ . We can see that velocity-Verlet method is a two-step method with a local error  $O(h^3)$

In the Earth-Sun system, with this method, we can formulate Eq. 2 to

$$\begin{cases} x_{i+1} = x_i + v_x^i h - \frac{4\pi^2 x_i}{r_i^3} \frac{h^2}{2} \\ y_{i+1} = y_i + v_y^i h - \frac{4\pi^2 y_i}{r_i^3} \frac{h^2}{2} \\ v_x^{i+1} = v_x^i - \left( \frac{4\pi^2 x_i}{r_i^3} + \frac{4\pi^2 x_{i+1}}{r_{i+1}^3} \right) \frac{h}{2} \\ v_y^{i+1} = v_y^i - \left( \frac{4\pi^2 y_i}{r_i^3} + \frac{4\pi^2 y_{i+1}}{r_{i+1}^3} \right) \frac{h}{2}. \end{cases} \quad (10)$$

We show our realization of the velocity-Verlet method in Algorithm ??. Compared with the Euler forward method, we can see the calculations in this method is more complicated.

### 3.3 Object oriented code development

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Input:  $x_0 = 1, y_0 = 0, v_x^0 = 0, v_y^0 = 4\pi^2, M_E, M_S, G$ 
Output:  $\vec{x}=(x_0, x_1, \dots, x_N), \vec{y}, \vec{v}_x, \vec{v}_y, \vec{E}_k, \vec{E}_p, \vec{E}, \vec{L}_z$ 
1  $r = \text{sqrt}(x_0^2 + y_0^2);$ 
2  $a_x^0 = \frac{4\pi^2 x_{i-1}}{r_{i-1}^3}; a_y^0 = \frac{4\pi^2 y_{i-1}}{r_{i-1}^3};$ 
3 //  $i$  is deferent time points  $t_i = t_0 + ih;$ 
4 for  $i = 1; i \leq N; i++$  do
5    $x_i = x_{i-1} + v_x^{i-1}h - \frac{a_x^0 h^2}{2}; y_i = y_{i-1} + v_y^{i-1}h - \frac{a_y^0 h^2}{2};$ 
6    $r = \text{sqrt}(x_i^2 + y_i^2);$ 
7    $a_x^1 = \frac{4\pi^2 x_i}{r_i^3}; a_y^1 = \frac{4\pi^2 y_i}{r_i^3};$ 
8    $v_x^i = v_x^{i-1} - \frac{(a_x^0 + a_x^1)h}{2}; v_y^i = v_y^{i-1} - \frac{(a_y^0 + a_y^1)h}{2};$ 
9    $a_x^0 = a_x^1; a_y^0 = a_y^1;$ 
10  // Kinetic energy  $E_k$ , Potential energy  $E_p$ , Total energy  $E$ , Angular momentum in  $\hat{z}$   $L_z;$ 
11   $E_k^i = 0.5M_E((v_x^i)^2 + (v_y^i)^2);$ 
12   $E_p^i = -\frac{GM_E}{r}; E^i = E_p^i + E_k^i;$ 
13   $L_z^i = M_E(x_i v_y^i - y_i v_x^i);$ 
14 end
15 return  $\vec{x}, \vec{y}, \vec{v}_x, \vec{v}_y, \vec{E}_k, \vec{E}_p, \vec{E}, \vec{L}_z;$ 

```

**Algorithm 2:** The Velocity-Verlet method for the Earth-Sun system. It initials from a circular orbit.

## 4 Results and discussion

### 4.1 Comparison between two methods

To test the stability of Euler forward (Euler) and the velocity-Verlet (VV) method, we initialize the Earth-Sun system with a circular orbit as stated in Algorithms ??.

We varies the step size  $h$  starting from 0.02 yr to 0.001 yr. The Earth's orbits calculated by these two methods for 10 years are show in Fig. 1. Globally speaking, we see orbits given by the Euler method expand in time. On the other hand, VV methods' orbits keep circular with some tiny fluctuation hardly seen in Fig. 1a. It justifies the our statements in Sec. 3 that the VV method conserves energy but the Euler method increase energy.

For a large step size in Fig. 1a, the Euler method is very unstable. Its orbit deviates from circle both in distance and shape apparently. As the step size becomes smaller, we can see that the Euler method becomes better; as the orbits expand slower and slower from 1a to Fig. 1d. The trends we observed agree with the statement that the error in the Euler method goes down with decreasing  $h$ . We can hardly see differences between orbits yielded from the VV method, which indicates the it's stability.

For detailed check and performance comparison, we list distances, energies, angular momentums together with execution times for these two methods in Table 1&2 with precision up to  $10^{-6}$ . As conservation laws predicted by classical mechanics, physical quantities including kinetic, potential, total energies and angular momentum should be conserved in the Earth-Sun system. From these two tables, we can see conservations in the VV method but not in the Euler method. Thus, we can conclude that the VV method is stable and the Euler method is not.

Compared the execution time of two methods, we find the Euler method is faster. Counting the FLOPS from these two algorithms, we find there is about  $30N$  for the Euler method and  $45N$  for the VV method. It explains speed advantage of the Euler method.

For physically meaningful, we will keep using the VV method in our further calculations.

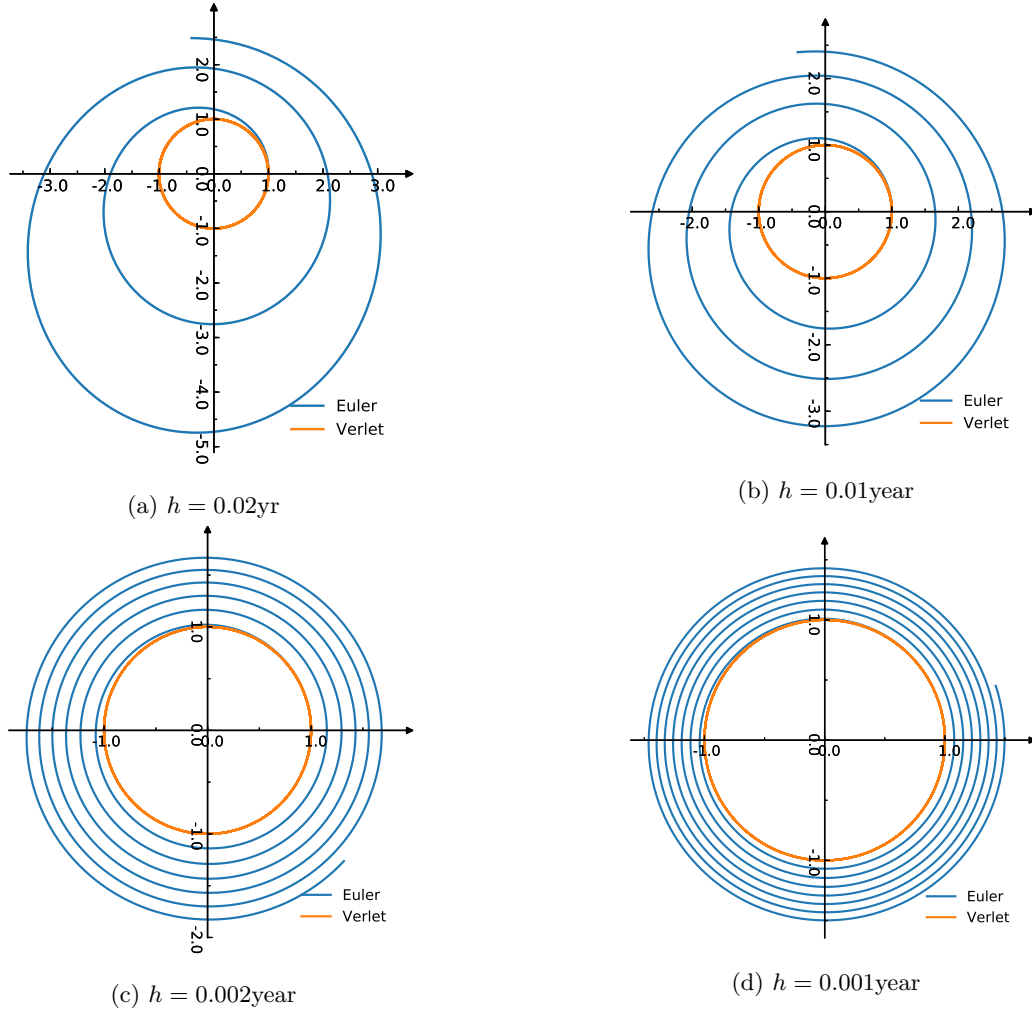


Figure 1: Comparison of different step size  $h$  of two methods for 10 years.

## 4.2 Escape velocity against the Sun

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## 4.3 Extension to whole solar system

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## 4.4 The perihelion precession of Mercury

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## 5 Conclusions

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Table 1: The Euler forward method: step size; kinetic energy; potential energy; total energy; angular momentum in z direction, distance from the Sun and execution time from left to the right.

$h$	$E_{kin}$	$E_{pot}$	$E_{tot}$	$L_z$	$r$	execution time (ms)
2.000E-02	3.300E-05	-4.700E-05	-1.400E-05	3.500E-05	2.520E+00	2.500E-02
1.000E-02	2.900E-05	-4.900E-05	-2.000E-05	3.200E-05	2.432E+00	5.000E-02
2.000E-03	3.200E-05	-6.500E-05	-3.300E-05	2.500E-05	1.824E+00	9.400E-02
1.000E-03	4.000E-05	-7.900E-05	-4.000E-05	2.300E-05	1.493E+00	1.680E-01

Table 2: The Velocity-Verlet method: step size; kinetic energy; potential energy; total energy; angular momentum in z direction, distance from the Sun and execution time from left to the right.

$h$	$E_{kin}$	$E_{pot}$	$E_{tot}$	$L_z$	$r$	execution time (ms)
2.000E-02	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000014	2.600E-02
1.000E-02	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	5.600E-02
2.000E-03	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	1.840E-01
1.000E-03	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	3.350E-01

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## References

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