Assignment 3

2.5 Computer Problems: 2

Q:

Use the Jacobi Method to solve the sparse system within three correct decimal places (forward error in the infinity norm) for n = 100. The correct solution is [1,−1,1,−1,...,1,−1]. Report the number of steps needed and the backward error. The system is

$$\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}.$$

A:

code:

sparsesetup.m:

```
function [a,b] = sparsesetup(n)
e = ones(n,1);
a = spdiags([e 2*e e],-1:1,n,n);           % Entries of a
b=zeros(n,1);                              % Entries of r.h.s. b
b(1)=1;b(n)=-1;
```

jacobi.m:

```
function [x,j,be] = jacobi(a,b,tol)
n=length(b);    % find n
d=diag(a);      % extract diagonal of a
r=a-diag(d);    % r is the remainder
x=zeros(n,1);   % initialize vector x
x_true = ones(n,1);
for i=2:2:100
    x_true(i)=-1;
end
for j=1:50000   % loop for Jacobi iteration
    x = (b-r*x)./d;
    error = norm(x-x_true,inf); % FE
    be = norm(b-a*x,inf);       % BE
    if (error < tol)
        break
    end
end             % End of Jacobi iteration loop
```

main.m:

```
format long
[a,b]=sparsesetup(100);
[x, steps,be] = jacobi(a,b,0.001);
steps
be
```

results:

```
steps =

    14776


be =

    9.674122976033317e-07
```

Therefore, the number of steps needed is 14776, and the backward error is 9.674122976033317e-07.

2.5 Computer Problems: 6

Q:

Carry out the steps of Computer Problem 2 for (a) Gauss–Seidel Method and (b) SOR with ω = 1.5.

A:

(a)

code:

sparsesetup.m:

```
function [a,b] = sparsesetup(n)
e = ones(n,1);
a = spdiags([e 2*e e],-1:1,n,n);            % Entries of a
b=zeros(n,1);                               % Entries of r.h.s. b
b(1)=1;b(n)=-1;
```

GS.m:

```
function [x,j,be] = GS(a,b,tol)
n=length(b);    % find n
d=diag(a);      % extract diagonal of a
d=diag(d);
u=triu(a)-d;
l=tril(a)-d;
x=zeros(n,1);   % initialize vector x
x_true = ones(n,1);
for i=2:2:100
    x_true(i)=-1;
end

for j=1:50000   % loop for GS iteration
    x = (d+l)\(b-u*x);
    error = norm(x-x_true,inf); % FE
    if (error < tol)
        break
    end
    be = norm(b-a*x,inf);       % BE
end             % End of GS iteration loop
```

main.m:

```
format long
[a,b]=sparsesetup(100);
[x, steps,be] = GS(a,b,0.001);
steps
be
```

results:

```
steps =

        7389


be =

    9.688609748925714e-07
```

Therefore, the number of steps needed is 7389, and the backward error is 9.688609748925714e-07.

(b)

sparsesetup.m:

```
function [a,b] = sparsesetup(n)
e = ones(n,1);
a = spdiags([e 2*e e],-1:1,n,n);            % Entries of a
b=zeros(n,1);                               % Entries of r.h.s. b
b(1)=1;b(n)=-1;
```

SOR.m:

```
function [x,j,be] = SOR(a,b,omega,tol)
n=length(b);    % find n
d=diag(a);      % extract diagonal of a
d=diag(d);
u=triu(a)-d;
l=tril(a)-d;
x=zeros(n,1);   % initialize vector x
x_true = ones(n,1);
for i=2:2:100
    x_true(i)=-1;
end

for j=1:50000   % loop for GS iteration
    x = (omega*l+d)\((1-omega)*d*x-omega*u*x)+omega*((d+omega*l)\b);
    error = norm(x-x_true,inf); % FE
    if (error < tol)
        break
    end
    be = norm(b-a*x,inf);       % BE
end             % End of GS iteration loop
```

main.m:

```
format long
[a,b]=sparsesetup(100);
[x,steps,be] = SOR(a,b,1.25,0.001);
steps
be
```

results:

```
steps =

        4433


be =

    9.695012020971561e-07
```

Therefore, the number of steps needed is 4433, and the backward error is 9.695012020971561e-07.

2.6 Exercises: 13(a)

Q:

Solve the problems by carrying out the Conjugate Gradient Method by hand.

$$(a) \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A:

$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$r_0 = d_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\alpha_0 = \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{1}{5}$

$x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$

$r_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \end{bmatrix}$

$\beta_0 = 0.16$

$d_1 = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \end{bmatrix} + 0.16 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.56 \\ -0.24 \end{bmatrix}$

$$\alpha_1 = \frac{\begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \end{bmatrix}^T \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \end{bmatrix}}{\begin{bmatrix} 0.56 \\ -0.24 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0.56 \\ -0.24 \end{bmatrix}} = \frac{0.32}{0.064} = 5$$
$$x_2 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} + 5 \begin{bmatrix} 0.56 \\ -0.24 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
$$r_2 = \begin{bmatrix} 0.4 \\ -0.4 \end{bmatrix} - 5 \begin{bmatrix} 0.08 \\ -0.08 \end{bmatrix} = 0$$
Since $r_2 = 0$, the solution is $x_2 = [3, -1]$.

2.6 Computer Problems: 6

Q:

Let A be the n × n matrix with n = 1000 and entries A(i, i) = i, A(i, i + 1) = A(i + 1,i) = 1/2,A(i, i + 2) = A(i + 2,i) = 1/2 for all i that fit within the matrix.

(a) Print the nonzero structure spy(A).

(b) Let x_e be the vector of n ones. Set $b = Ax_e$, and apply the Conjugate Gradient Method, without preconditioner, with the Jacobi preconditioner, and with the Gauss–Seidel preconditioner. Compare errors of the three runs in a plot versus step number.

A:

(a)

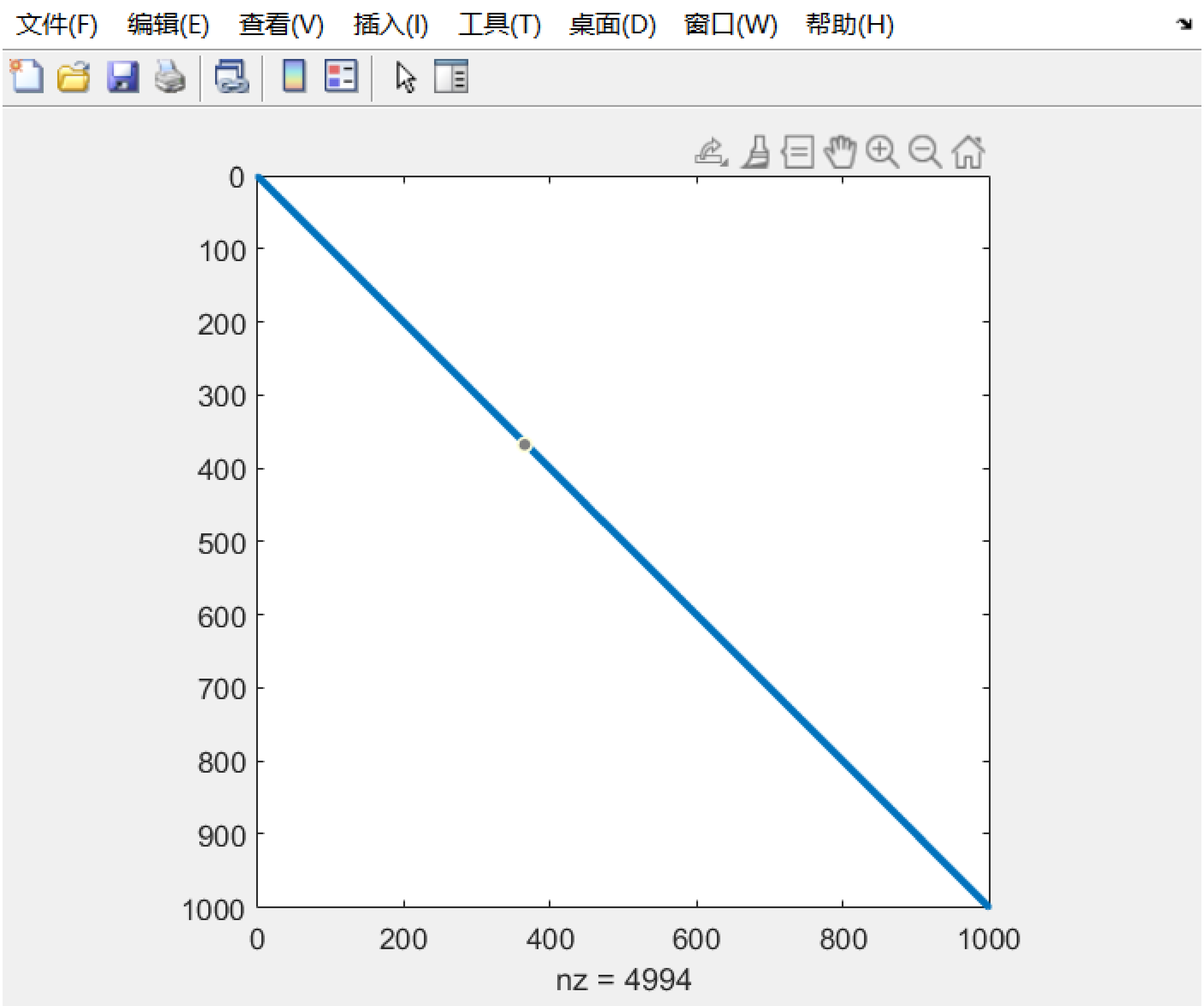
sparsesetup.m:

```
function a = sparsesetup(n)
e = ones(n,1);
a = spdiags([0.5*e 0.5*e e 0.5*e 0.5*e],-2:2,n,n);
for i=1:n
    a(i,i)=i;
end
```

main.m:

```
format long
a=sparsesetup(1000);
spy(a);
```

results:


文件(F) 编辑(E) 查看(V) 插入(I) 工具(T) 桌面(D) 窗口(W) 帮助(H)
0
100
200
300
400
500
600
700
800
900
1000
0
200
400
600
800
1000
nz = 4994