Assignment 5

4.1 Exercises: 1 (a)

Q:

Solve the normal equations to find the least squares solution and 2-norm error for the following inconsistent systems:

(a)

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

A:

The components of the normal equations are

$$A^T A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

The normal equations

$$\begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$(\overline{x_1}, \overline{x_2}) = (-\frac{1}{7}, \frac{10}{7})$$

Substituting the least squares solution into the original problem yields

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{7} \\ \frac{10}{7} \end{bmatrix} = \begin{bmatrix} \frac{18}{7} \\ \frac{10}{7} \\ \frac{8}{7} \end{bmatrix}$$

To measure our success at fitting the data, we calculate the residual of the least squares solution $\overline{x}$ as

$$r = b - A\overline{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{19}{7} \\ \frac{10}{7} \\ \frac{8}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -\frac{3}{7} \\ -\frac{1}{7} \end{bmatrix}$$

$$||r||_2 = \sqrt{(\frac{2}{7})^2 + (-\frac{3}{7})^2 + (-\frac{1}{7})^2} = \frac{\sqrt{14}}{7}$$

4.1 Computer Problems: 1

Q:

Form the normal equations, and compute the least squares solution and 2-norm error for the following inconsistent systems:

$$(a) \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -5 \\ 15 \\ 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$

A:

(a)

code:

```
A = [3,-1,2;4,1,0;-3,2,1;1,1,5;-2,0,3];
b = [10;10;-5;15;0];
x = (A'*A)\(A'*b)
r = A*x-b;
norm = norm(r,2)
```

results:

```
x =

    2.5246
    0.6616
    2.0934


norm =

    2.4135
```

(b)

code:

```
A = [4,2,3,0;-2,3,-1,1;1,3,-4,2;1,0,1,-1;3,1,3,-2];
b = [10;0;2;0;5];
x = (A'*A)\(A'*b)
r = A*x-b;
norm = norm(r,2)
```

results:

```
x =

    1.2739
    0.6885
    1.2124
    1.7497


norm =

    0.8256
```

4.3 Exercises: 2(a)

Q:

Apply classical Gram–Schmidt orthogonalization to find the full QR factorization of the following matrices:

$$(a)\begin{bmatrix}4 & 0\\3 & 1\end{bmatrix}$$

A:

$$y_1 = A_1 = \begin{bmatrix}4\\3\end{bmatrix}$$

$$r_{11} = ||y_1||_2 = \sqrt{4^2 + 3^2} = 5$$

$$q_1 = \frac{y_1}{||y_1||_2} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

$$y_2 = A_2 - q_1 q_1^T A_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \frac{3}{5} = \begin{bmatrix} -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}$$

$$q_2 = \frac{y_2}{||y_2||_2} = \frac{5}{4} \begin{bmatrix} -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$$

$$r_{12} = q_1^T A_2 = 0.6, r_{22} = ||y_2||_2 = 0.8$$

$$A = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 & 0.6 \\ 0 & 0.8 \end{bmatrix} = QR$$

4.3 Exercises: 6(a)

Q:

Apply Householder reflectors to find the full QR factorization of the matrices in Exercise 2.

$$(a) \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$$

A:

We need to find a Householder reflector that moves the first column x = [2,-2,1] to the vector $w = [||x||_2, 0, 0]$. Set v = w − x = [3,0,0]−[2,-2,1]=[1,2,−1]. Referring to Theorem 4.4, we have

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$H_1 A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & -6 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$H_2 H_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} A = \begin{bmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$A = H_1 H_2 \begin{bmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} = QR$$

4.3 Computer Problems: 1

Q:

Write a Matlab program that implements classical Gram–Schmidt to find the reduced QR factorization. Check your work by comparing factorizations of the matrices in Exercise 1 with the Matlab qr(A,0) command or equivalent. The factorization is unique up to signs of the entries of Q and R.

A:

code:

QR.m:

```
function [Q,R]=QR(A,m,n)
Q=zeros(m,n);
R=zeros(n,n);
for j=1:n
    y=A(:,j);
    for i=1:j-1
        R(i,j)=Q(:,i)'*A(:,j);
        y=y-R(i,j)*Q(:,i);
    end
    R(j,j)=norm(y,2);
    Q(:,j)=y./R(j,j);
end
```

main.m:

```
A=[1,-4;2,3;2,2];
[Q,R]=QR(A,3,2)
[x,y]=qr(A)
```

results;

```
Q =

    0.3333   -0.9333
    0.6667    0.3333
    0.6667    0.1333


R =

    3.0000    2.0000
        0    5.0000


x =

-0.3333    0.9333   -0.1333
-0.6667   -0.3333   -0.6667
-0.6667   -0.1333    0.7333


y =

    -3    -2
    0    -5
    0     0
```

The results calculated by Classical Gram–Schmidt orthogonalization is different from full QR factorization.

4.3 Computer Problems: 3

Q:

Repeat Computer Problem 1, but implement Householder reflections.

A:

code:

HH.m:

```
function R=HH(A)
[m,n]=size(A);
a=zeros(1,n);
b=zeros(1,n);
c=zeros(1,n);
v=zeros(m,n);
e=eye(m,n);
for i=1:n
    a(i)=-1*sign(A(i,i))*sqrt(sum(A(i:m,i).*A(i:m,i)));
    v(:,i)=[zeros(i-1,1);A(i:m,i)]-a(i).*e(:,i);
    b(i)=v(:,i)'*v(:,i);
    if b(i)==0
        continue;
    end
    for j=i:n
        c(j)=v(:,i)'*A(:,j);
        A(:,j)=A(:,j)-(2*c(j)/b(i)).*v(:,i);
    end
end
R=A;
```

main.m:

```
A = [1,-4;2,3;2,2];
R = HH(A)
[X,Y] = qr(A)
```

results:

```
R =

    -3    -2
    0    -5
    0     0


X =

-0.3333    0.9333   -0.1333
-0.6667   -0.3333   -0.6667
-0.6667   -0.1333    0.7333


Y =

    -3    -2
    0    -5
    0     0
```