APPENDIX

A. Proof of Theorem 1

Proof: Based on (7), the perturbed received LLRs can be denoted as $\mathcal{L}'(y_0) = \mathcal{L}(y_0) + \mathcal{L}(n_0')$ and $\mathcal{L}'(y_1) = \mathcal{L}(y_1) + \mathcal{L}(n_1')$. Based on the f-function of (5), we have

$$\mathcal{L}'(u_0) = \mathrm{sign}\left(\mathcal{L}'(y_0) \cdot \mathcal{L}'(y_1)\right) \cdot \min\left(\left|\mathcal{L}'(y_0)\right|, \left|\mathcal{L}'(y_1)\right|\right).$$

According to [18], [23], the magnitude of artificial noise n_i' should be significantly smaller than that of the channel noise n_i, i.e., $|n_i'| \ll |n_i|$. Consequently, $|n_i'| \ll |y_i|$ and

$$\mathrm{sign}\left(\mathcal{L}'(y_0) \cdot \mathcal{L}'(y_1)\right) = \mathrm{sign}\left(\mathcal{L}(y_0) \cdot \mathcal{L}(y_1)\right). \tag{18}$$

Based on Taylor's formula, $|\mathcal{L}(y_i) + \mathcal{L}(n_i')|$ can be computed as

$$\left|\mathcal{L}(y_i) + \mathcal{L}(n_i')\right| \approx \left|\mathcal{L}(y_i)\right| + \mathrm{sign}\left(\mathcal{L}(y_i)\right) \cdot \mathcal{L}(n_i'). \tag{19}$$

If $|\mathcal{L}(y_0)| \leq |\mathcal{L}(y_1)|$,

$$\min\left(\left|\mathcal{L}'(y_0)\right|, \left|\mathcal{L}'(y_1)\right|\right) \approx \left|\mathcal{L}(y_0)\right| + \mathrm{sign}\left(\mathcal{L}(y_0)\right) \cdot \mathcal{L}(n_0').$$

Otherwise,

$$\min\left(\left|\mathcal{L}'(y_0)\right|, \left|\mathcal{L}'(y_1)\right|\right) \approx \left|\mathcal{L}(y_1)\right| + \mathrm{sign}\left(\mathcal{L}(y_1)\right) \cdot \mathcal{L}(n_1').$$

Hence, it can be obtained that

$$\begin{aligned}\min\left(\left|\mathcal{L}'(y_0)\right|, \left|\mathcal{L}'(y_1)\right|\right) \approx &\min\left(\left|\mathcal{L}(y_0)\right|, \left|\mathcal{L}(y_1)\right|\right) \\ &+ \mathrm{sign}\left(\mathcal{L}(y_i)\right) \cdot \mathcal{L}(n_i'),\end{aligned} \tag{20}$$

where $i = \arg\min\left(\left|\mathcal{L}(y_0)\right|, \left|\mathcal{L}(y_1)\right|\right)$. Based on eqs. (5), (18) and (20), we have

$$\mathcal{L}'(u_0) \approx \mathcal{L}(u_0) + \mathcal{L}(n_0^*),$$

where $n_0^* = \mathrm{sign}\left(\mathcal{L}(y_i)\right) \cdot n_i'$. Since $\{n_0', n_1'\} \sim \mathcal{N}(0, \sigma_p^2)$ and $\mathrm{sign}\left(\mathcal{L}(y_i)\right) \in \{-1, 1\}$, it follows that $n_0^* \sim \mathcal{N}(0, \sigma_p^2)$.

Based on the g-function of (6), we have

$$\mathcal{L}'(u_1) = (1 - 2\hat{u}_0) \cdot \mathcal{L}'(y_0) + \mathcal{L}'(y_1).$$

Since $\hat{u}_0 \in \{0, 1\}$,

$$\mathcal{L}'(u_1) = \begin{cases} \mathcal{L}(y_1) + \mathcal{L}(y_0) + \mathcal{L}(n_1' + n_0'), & \text{if } \hat{u}_0 = 0; \\ \mathcal{L}(y_1) - \mathcal{L}(y_0) + \mathcal{L}(n_1' - n_0'), & \text{if } \hat{u}_0 = 1. \end{cases} \tag{21}$$

For the conventional SC decoding, it follows that

$$\mathcal{L}(u_1) = \begin{cases} \mathcal{L}(y_1) + \mathcal{L}(y_0), & \text{if } \hat{u}_0 = 0; \\ \mathcal{L}(y_1) - \mathcal{L}(y_0), & \text{if } \hat{u}_0 = 1. \end{cases} \tag{22}$$

Based on (21) and (22), we have

$$\mathcal{L}'(u_1) = \mathcal{L}(u_1) + \mathcal{L}(n_1^*),$$

where $n_1^* \in \{n_1' + n_0', n_1' - n_0'\}$. Since $\{n_0', n_1'\} \sim \mathcal{N}(0, \sigma_p^2)$, it can be obtained that $n_1^* \sim \mathcal{N}(0, 2\sigma_p^2)$. □

B. Proof of Corollary 2

Proof: As mentioned in Sec. II-B, the SC decoding is performed by recursively invoking the f and the g functions. Based on Theorem 1, the perturbation power doubles only when performing the g-function. Otherwise, it stays the same. Therefore, it can be concluded that the power of n_i^*, i.e., σ_i^2, is 2^{δ_i} times that of the original perturbation power σ_p^2, whose artificial noise is applying on the received LLRs. Note that $\delta_i \in \{0, 1, 2, \ldots, n\}$ denotes the number of required g-functions for estimating u_i. That says

$$n_i^* \sim \mathcal{N}(0, \sigma_i^2),$$

where $\sigma_i^2 = 2^{\delta_i}\sigma_p^2$. Thus, based on (8), it can be obtained that

$$\mathcal{L}'(u_i) = \mathcal{L}(u_i) + \mathcal{L}(n_i^*),$$

where $n_i^* \sim \mathcal{N}(0, \sigma_i^2)$, and $\mathcal{L}(u_i)$ denotes the decoding *a posteriori* LLR obtained from the SC decoding. □

C. Proof of Theorem 3

Proof: Let P_c^{SC} denote the correct probability of SC decoding, i.e., the decoding accuracy, which satisfies

$$\begin{aligned}P_c^{\text{SC}} &= \Pr\left(\hat{u}_0^{N-1} = u_0^{N-1}\right) \\ &= \prod_{i=0}^{N-1} \Pr\left(\hat{u}_i = u_i \mid \hat{u}_0 = u_0, \hat{u}_1 = u_1, \ldots, \hat{u}_{i-1} = u_{i-1}\right) \\ &= \prod_{i \in \mathcal{A} \cup \mathcal{A}^c} \left(1 - P_e(u_i)\right),\end{aligned} \tag{23}$$

where $1 - P_e(u_i)$ denotes the probability that u_i has been correctly estimated, and $\mathcal{A} \cup \mathcal{A}^c = \{0, 1, \ldots, N-1\}$.

For polar codes, frozen bits are fixed as zero and known to the polar decoder [1]. Thus, the error probability of frozen bits is zero, i.e., $P_e(u_i) = 0$ for $\forall i \in \mathcal{A}^c$. That says

$$\prod_{i \in \mathcal{A}^c} \left(1 - P_e(u_i)\right) \triangleq 1. \tag{24}$$

Substituting (24) into (23), we obtain

$$P_c^{\text{SC}} = \prod_{i \in \mathcal{A}} \left(1 - P_e(u_i)\right). \tag{25}$$

Thus, based on (25), the error probability of SC decoding under GA can be computed by

$$\begin{aligned}P_e^{\text{SC}} &= 1 - P_c^{\text{SC}} \\ &= 1 - \prod_{i \in \mathcal{A}} \left(1 - P_e(u_i)\right).\end{aligned} \tag{26}$$

When computing P_e^{LRP}, it is assumed that the erroneous bit estimations occur only in the set of LRPs, i.e., $\mathcal{D}$. Therefore, it follows that $P_e(u_i) = 0$ for $\forall i \in \mathcal{A} \backslash \mathcal{D}$. That says

$$\prod_{i \in \mathcal{A} \backslash \mathcal{D}} \left(1 - P_e(u_i)\right) \triangleq 1 \tag{27}$$

Combining (23), (24), and (27), using the SC decoding, the decoding accuracy of LRPs can be defined as

$$P_c^{\text{LRP}} = \prod_{i \in \mathcal{D}} \left(1 - P_e(u_i)\right). \tag{28}$$

Based on (28), under GA, the error probability of LRPs, i.e., P_e^{LRP}, can be computed as

$$\begin{aligned} P_e^{\mathrm{LRP}} &= 1 - P_c^{\mathrm{LRP}} \\ &= 1 - \prod_{i \in \mathcal{D}} \left(1 - P_e\left(u_i\right)\right). \end{aligned} \tag{29}$$

$\square$

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