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Preface

The author of this slides is:
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This sides is based on the paper called

ATMOSPHERES OF PROTOPLANETARY CORES: CRITICAL MASS FOR NUCLEATED INSTABILITY

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from

The Astrophysical Journal, 648:666Y682, 2006 September 1

You can find this slides on 董桁宇(Hengyu Dong)'s github

Outline

Settings

Settings about the nebula:
MMSN (Minimum Mass Solar Nebula):
$\Sigma_p(a)$ is the particulate surface densities
$\Sigma_g(a)$ is the gas surface densities
$T_0(a)$ is the gas temperature
$a_n \equiv a/(n\,\mathrm{AU})$, while 'a' is the distance to the sun

$$\Sigma_g(a) \approx 100\Sigma_p(a) \approx 3000\,\mathrm{g\ cm}^{-2}\, a_1^{-3/2} \tag{1}$$

$$T_0(a) \approx 300\,\mathrm{K}\, a_1^{-1/2} \tag{2}$$

Settings

The nebula is isothermal in the vertical direction
$\rho(z, a) = \rho_0(a) \exp(-z^2/2h^2)$ is the density structure
$c_0 \equiv (kT_0/\mu)^{1/2}$ is the isothermal sound speed
k is the Boltzmann constant and μ is the mean molecular weight
$h \equiv c_0/\Omega$ is the vertical scale height
$\Omega \equiv (GM_\odot/a^3)^{1/2}$ is the orbital angular frequency
$\rho_0 \equiv (2\pi)^{-1/2}\Sigma_g/h$ is the midplane density
In terms of numbers:

$$c_0(a) \approx 10^5 \,\text{cm s}^{-1}\, a_1^{-1/4} \tag{3}$$

$$h(a)/a \approx 3.4 \times 10^{-2}\, a_1^{1/4} \tag{4}$$

$$\rho_0(a) \approx 2.4 \times 10^{-9} \,\text{g cm}^{-3}\, a_1^{-11/4} \tag{5}$$

Basic equations

Assume that:
the atmosphere is not rapidly rotating
the gas distribution as spherically symmetric
$M_p \gtrsim M_{atm}$
we can get:

$$\frac{\partial P}{\partial r} = -G\frac{M_p}{r^2}\rho \tag{6}$$

$M_p$means the mass of the core embedded in the nebular gas. M_{atm}means the mass of the atmosphere.

Basic equations

Assume that:
the planetesimal velocity far from the core is small compared to the core’ s escape speed
planetesimals penetrate to the core surface without much resistance from the envelope and release there all their kinetic energy
the dynamical and thermal timescales of the atmosphere are shorter than the core accretion timescale $\frac{Mp}{\dot{M}}$
we can get:

$$L = -G\frac{M_p \dot{M}}{R_p} \tag{7}$$

$R_p$means the radio of the core embedded in the nebular gas.
However, even in the most unfavorable case of small planetesimals that are quasi-statically lowered from the top of the atmosphere to its bottom, the luminosity is $1 - \frac{R_p}{r}L$
i.e., the luminosity varies only very near the core’ s surface. At $r \gtrsim R_p$, in the bulk of the atmosphere, we can safely assume L to be a constant given by the equation.

Basic equations

Assume that:
the envelope is chemically homogeneous
the envelope is nonrotating
we can get:

$$\frac{\partial lnT}{\partial lnP} \equiv \nabla < \nabla_{ad} \equiv \frac{\gamma - 1}{\gamma} \tag{8}$$

∇means the temperature gradient.
∇_{ad}means the temperature gradient under isentropic conditions.
γmeans the adiabatic index of the gas.
Imagining a small part of the gas was slightly disturbed.
This equation can be used to determine weather the accretion luminosity is transported by radiative diffusion or convection.

Basic equations

Assume that:
optically thick
the dynamical and thermal timescales of the atmosphere are shorter than the core accretion timescale $\frac{Mp}{\dot{M}}$
we can get:

$$\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial r} = -\frac{L}{4\pi r^2} \tag{9}$$

σ is the Stefan-Boltzmann constant.
κ is the opacity.
we assume that:

$$\kappa = \kappa_0 \frac{P}{P_0}^{\alpha} \frac{T}{T_0}^{\beta} \tag{10}$$

κ_0, P_0, T_0 are the opacity, pressure, and temperature in the nebular gas far from the core.
And we assume that κ_0 is independent of a.

Basic equations

Assume that:
the envelope is chemically homogeneous
the envelope is ideal gas
we can get:

$$P = C_{onst}\rho T \tag{0}$$

And if the equation(8)($\nabla < \nabla_{ad}$)is violated
$\nabla \approx \nabla_{ad}$ we can get:

$$P = C_{onst}\rho^{\gamma} \tag{11}$$

Length Scales

The mean free path of photons in the nebular gas:

$$\lambda = (\kappa_0 \rho_0)^{-1} \equiv 1.7 \times 10^9 \,\mathrm{cm}\, a_1^{11/4} \kappa_{0.1}^{-1} \tag{12}$$

The Hill radius:

$$R_H \equiv a \left(\frac{M_p}{M_\odot} \right)^{1/3} \approx 2 \times 10^{11} \,\mathrm{cm}\, a_1 \left(\frac{M_p}{M_\oplus} \right)^{1/3} \tag{13}$$

This equation can be used to determine whether the gas is controlled by the gravity of the core or the star. The Bondi radius:

$$R_B \equiv \frac{GM_p}{c_0^2} \approx 4 \times 10^{10} \,\mathrm{cm}\, a_1^{1/2} \frac{M_p}{M_\oplus} \tag{14}$$

is defined as the distance from the protoplanet at which the thermal energy of the nebular gas is of the order of its gravitational energy in the potential well of the core. ($P \approx P_0$ when $r \gtrsim R_B$)

Length Scales

The radius of the core:

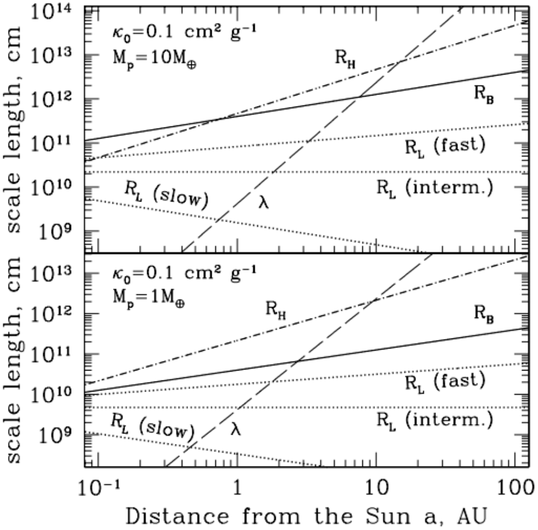
$$R_p \equiv \left(\frac{3}{4\pi}\frac{M_p}{\rho_p}\right)^{1/3} \approx 10^9\,\mathrm{cm}\left(\frac{M_p}{M_\oplus}\right)^{1/3}\rho_1^{-1/3} \tag{15}$$

$$p \equiv \frac{R_p}{R_H} = \left(\frac{3M_\odot}{4\pi\rho_p a^3}\right)^{1/3} \approx 5.2\times 10^{-3} a_1^{-1}\rho_1^{-1/3} \tag{16}$$

where $\rho_1 \equiv \rho_p(1\,\mathrm{g\,cm^{-3}})$ and $R_p \gg R_H$
The luminosity radius:

$$R_L \equiv \left(\frac{L}{16\pi\sigma T_0^4}\right)^{1/2} \tag{17}$$

The luminosity radius is 1/2 of the radius of the sphere that can radiate the accretion luminosity L at an effective temperature T



Dotted lines correspond to the luminosity radius R_L evaluated for three different planetesimal accretion regimes, fast, intermediate, and slow.

We define the "sphere of influence" as the region of space in which planetary gravity dominates over both the tidal field of the Sun and the unperturbed nebular pressure P_0.

It is $\min(R_H, R_B)$

The gas dynamics inside it is determined only by the gravity of the core and pressure gradients in the surrounding gas.

Mass Scales

Atmosphere exists $\iff R_B \gtrsim R_p \iff M_p \gtrsim M_a$:

$$M_a \equiv \frac{c_0^3}{G}\left(\frac{3}{4\pi G\rho_p}\right)^{1/2} \approx 4.5\times 10^{-3} M_\oplus a_1^{-3/4}\rho_1^{-1/2} \tag{18}$$

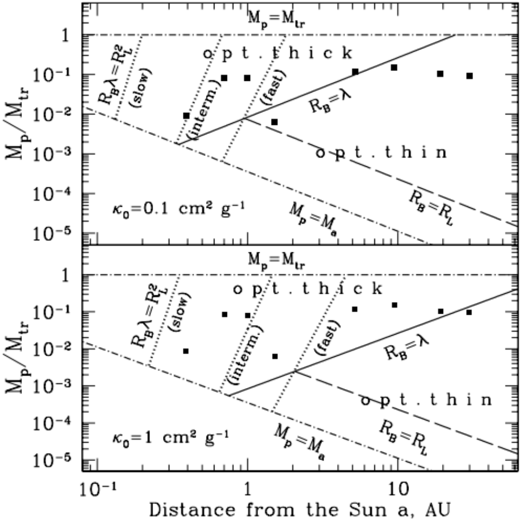
The nebula can be considered homogeneous on the scale of R_B $\iff R_B \gg h \iff M_p \gtrsim M_{tr}$

$$M_{tr} \equiv \frac{c_0^3}{G\Omega} \approx 12 M_\oplus a_1^{3/4} \tag{19}$$

We can see that: $R_B \lesssim R_H$ and $R_H \lesssim h$, if $R_B \gg h$
$R_B \gtrsim R_H$ and $R_H \gtrsim h$, if $R_B \ll h$
Now we only talk about the situation that:

$$M_a \lesssim M_p \lesssim M_{tr} \tag{20}$$

$$R_p \lesssim R_B \lesssim R_H \lesssim h \tag{21}$$



Since according to equations (18) and (19) $M_a = p^{3/2} M_{tr}$, there is a broad mass range in which equation (20) can be valid, spanning 4-5 orders of magnitude in mass depending on a (e.g., from 0.1 lunar mass to M_J at 10 AU).

Dotted lines correspond to the luminosity radius R_L evaluated for three different planetesimal accretion regimes, fast, intermediate, and slow.

Low-Luminosity Optically Thick

we define:

$$T(R_{out}) = T_0, P(R_{out}) = P_0, \rho(R_{out}) = \rho_0, \tag{22}$$

The core is "low-luminosity" if any of the following equations holds true. For the Optically Thick case:

$$\lambda \ll R_B, {R_L}^2 \ll \lambda R_B \tag{23}$$

For the Optically Thin case:

$$R_L \ll R_B \ll \lambda \tag{24}$$

using equations (6) (8) (9),we get:

$$\nabla(T) = \frac{3}{64\pi}\frac{L\kappa_0\rho_0{c_0}^2}{GM_p\sigma{T_0}^4}(\frac{T}{T_0})^{\beta-4}(\frac{P}{P_0})^{1+\alpha} = \nabla_\infty(\frac{T}{T_0})^{\beta-4}(\frac{P}{P_0})^{1+\alpha} \tag{25}$$

$$\nabla_\infty = \frac{3}{4}\frac{{R_L}^2}{R_B\lambda} \tag{26}$$

As we can see $\nabla(T) = \nabla_\infty$in the outer atmosphere.

Low-Luminosity Optically Thick

Using equation (6)(9)(10),we can get:

$$(\frac{P}{P_0})^{1+\alpha} - 1 = \frac{\nabla_0}{\nabla_\infty}[(\frac{T}{T_0})^{4-\beta} - 1] \tag{27}$$

$$\nabla_0 \equiv \frac{1+\alpha}{4-\beta} \tag{28}$$

Because of equation (23), $\frac{\nabla_0}{\nabla_\infty}$ is really big, which means that a large change of P results in only a small perturbation of T.

Later we will talk about the asymptotic behavior of the atmospheric properties in two regions:

in the "outer atmosphere", where $T - T_0 \lesssim T_0 (T \approx T_0)$,

and in the "deep atmosphere", where one expects $T \gg T_0$.

Low-Luminosity Optically Thick

In the "outer atmosphere", $T - T_0 \lesssim T_0$, substituting equation (27) into either equation (6) or equation (9)we can get:

$$\frac{P}{P_0} \approx \frac{\rho}{\rho_0} \approx e^{(\frac{R_B}{r} - \frac{R_B}{R_{out}})} \approx e^{\frac{R_B}{r}} \tag{29}$$

$$\frac{T - T_0}{T_0} \approx \frac{\nabla_\infty}{1+\alpha}(e^{(1+\alpha)(\frac{R_B}{r} - \frac{R_B}{R_{out}})} - 1) \approx \frac{\nabla_\infty}{1+\alpha}(e^{(1+\alpha)\frac{R_B}{r}} - 1) \tag{30}$$

Low-Luminosity Optically Thick

When $T - T_0 \sim T_0$ according to equation (30),"outer atmosphere" turn to "deep atmosphere",the radio is :

$$R_a \equiv R_B \frac{1+\alpha}{ln(R_B\lambda/{R_L}^2)} \approx R_B \frac{1+\alpha}{ln({\nabla_\infty}^{-1})} \tag{31}$$

so the pressure and density is

$$\frac{P_a}{P_0} \approx \frac{\rho_a}{\rho_0} \approx {\nabla_\infty}^{-1/(1+\alpha)} \gg 1 \tag{32}$$

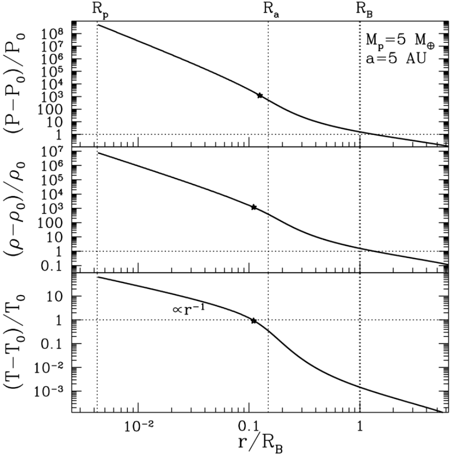
Low-Luminosity Optically Thick

In the "deep atmosphere", $T \gg T_0$, with boundary condition that $T \approx \xi T_0 at r = R_a$ substituting equation (27) into either equation (6) or equation (9)we can get:

$$\frac{T}{T_0} \approx \xi + \nabla_0(\frac{R_B}{r} - \frac{R_B}{R_a}) \tag{33}$$

$$\frac{P}{P_0} \approx (\frac{\nabla_0}{\nabla_\infty})^{1/(1+\alpha)}[\xi + \nabla_0(\frac{R_B}{r} - \frac{R_B}{R_a})]^{1/\nabla_0} \tag{34}$$

It can also be calculated numerically, the result is:



$M_p = 5M_\oplus, a = 5AU, P_0 = 1.3 \times 10^{-7}$bar, $\rho_0 = 3 \times 10^{-11}$g cm^{-3}, $T_0 = 130$K, $\alpha = 0, \beta = 1, \gamma = \frac{7}{5}, \kappa_0 = 0.1$cm^{2}g^{-1}, planetesimal accretion is in the intermediate regime.

$$\dot{M} = \Omega\Sigma_p R_P R_H \theta \qquad \text{(A1)}$$

$$\tau_{acc} \equiv \frac{M_p}{\dot{M}} = \frac{M_p}{\Omega\Sigma_p R_P R_H}\theta^{-1} \approx \left(\frac{M_p}{M_\oplus}\right)^{1/3} \begin{cases} 3\times10^{10} \text{ yr } a_{10}^3, \\ 1.4\times10^7 \text{ yr } a_{10}^2, \\ 3\times10^5 \text{ yr } a_{10}^{3/2}, \end{cases} \qquad \text{(A2)}$$

Low-Luminosity Optically Thick

We have written a c program and a shell to reproduce the image. The program can be found on 董桁宇(Hengyu Dong)'s github

```bash
#!/bin/bash

SOURCE_FILE="dhy23.c"
OUTPUT_FILE="output_program"

wget https://raw.githubusercontent.com/little-potato-loria/plot2/main/dhy23.c
icx -O3 -xHost -ipo -o $OUTPUT_FILE $SOURCE_FILE -lGL -lGLU -lglut -lm
./$OUTPUT_FILE
```

"1.sh" 9L, 230B 1,1 All

And the results fit well.

Low-Luminosity Optically Thin

When $\lambda \gtrsim R_B$ the boundary will be optically thin.Deep in the atmosphere (below the photosphere) the gas becomes optically thick because of the increasing density ,while the nebula is also optically thick when $r \gtrsim \lambda$ because it is "thick",so it is:

0 to R_{ph}	$R_{ph} to \lambda$	$\lambda to \infty$
thick	thin	think

And equation (9) can not be used in the optically thin area.Insteadly, it should be $T^4 \approx {T_0}^4 + L/(16\pi\sigma r^2)$ then we interpolating between the outer optically thick and intermediate optically thin regions,we get:

$$T^4(r) \approx {T_0}^4(r) + \frac{L}{16\pi\sigma r^2} + \frac{3L}{16\pi\sigma}\int_\infty^r \frac{\kappa\rho(r')dr'}{r'^2} \tag{35}$$

Because of $R_L \ll R_B \ll \lambda$,we can get that when $r \gtrsim R_B$

$$T(r) \approx T_0(1 + \frac{{R_L}^2}{r^2} + 3\frac{{R_L}^2}{\lambda r})^{1/4} \tag{36}$$

When $r = R_B$, from equation (36) ,we can get
$T(R_B) - T_0 \sim T_0(R_L/R_B)^2$
Because $T \approx T_0$.P and ρ follow equation(29).

Low-Luminosity Optically Thin

Define R_{ph} as the radius that $\lambda \sim \partial r/\partial lnP = r^2/R_B$, So R_{ph} actually means the radius that the atmosphere change from thin to thick.

$$R_{ph} \approx R_B \frac{1+\alpha}{ln(\lambda/R_B)} \tag{37}$$

So we can get: $T_{ph} \equiv T(R_{ph}), P_{ph} \equiv P(R_{ph})$

$$\frac{T_{ph} - T_0}{T_0} \approx \frac{{R_L}^2}{4{R_{ph}}^2} \approx [\frac{ln(\lambda/R_B)}{2(1+\alpha)} \frac{R_L}{R_B}]^2 \ll 1 \tag{38}$$

$$P_{ph}/P_0 \approx (\lambda/R_B)^{1/(1+\alpha)} \gg 1 \tag{39}$$

Low-Luminosity Optically Thin

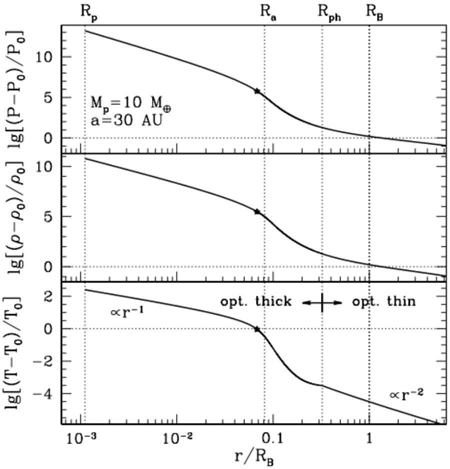
When$r < R_{ph}$,the atmosphere is optically thick,from equation (27) we can get:

$$(\frac{P}{P_0})^{1+\alpha} - (\frac{P_{ph}}{P_0})^{1+\alpha} = \frac{\nabla_0}{\nabla_\infty}[(\frac{T}{T_0})^{4-\beta} - (\frac{T_{ph}}{T_0})^{4-\beta}] \tag{40}$$

$$\frac{T - T_{ph}}{T_0} = \frac{\nabla_\infty}{(1+\alpha)}[e^{(1+\alpha)\frac{R_B}{r}} - (\frac{R_{ph}}{P_0})^{1+\alpha}] \tag{41}$$

Low-Luminosity Optically Thin

It can also be calculated numerically, the result is:



$M_p = 10M_\oplus, a = 30AU, P_0 = 4\times10^{-10}\text{bar}, \rho_0 = 2\times10^{-13}\text{g cm}^{-3}, T_0 = 55\text{K}, \alpha = 0, \beta = 1, \gamma = \frac{7}{5}, \kappa_0 = 0.1\text{cm}^2\text{g}^{-1}$, planetesimal accretion is in the intermediate regime.

For the "optically thick" case, from equation (25) and (27)we can get:

$$\nabla(T) = \nabla_0[1 - (\frac{T_0}{T})^{4-\beta}(1 - \frac{\nabla_\infty}{\nabla_0})] \tag{42}$$

We have said that $\frac{\nabla_\infty}{\nabla_0}$is really small,so:

$$\nabla(T) \approx \nabla_0[1 - (\frac{T_0}{T})^{4-\beta}] \tag{43}$$

And we can see that :
when $T \approx T_0, \nabla(T) \approx 0 \ll \nabla_{ad}$
when $T \gg T_0, \nabla(T) \approx \nabla_0$

For the "optically thick" and $r > R_{ph}$ case, using the definition of $\nabla = \frac{\partial lnT}{\partial lnP}$ the equation (29) and (36) with the approximate relationship $ln(1+d) \approx d$,we can get:

$$\nabla \approx \frac{1}{4}(2\frac{{R_L}^2}{R_B r} + 3\frac{{R_L}^2}{R_B \lambda}) \ll 1 \tag{44}$$

And for the "optically thick" and $r \lesssim R_{ph}$ case,same as equation (42),we can get:

$$\nabla(T) = \nabla_0[1 - (\frac{T_{ph}}{T})^{4-\beta}(1 - \frac{\nabla_\infty}{\nabla_0}(\frac{P_{ph}}{P_0})^{1+\alpha}(\frac{T_0}{T_{ph}})^{4-\beta})] \tag{45}$$

With the $T_{ph}R_{ph}$ we have already calculated, equation (45) equal to equation (43). Thus we prove that the upper atmosphere must be radiative .

As we have proved the temperature gradient in the deep atmosphere is ∇_0 ,the **whole** atmosphere is convectively stable if

$$\nabla_0 < \nabla_{ad} \tag{46}$$

With the equation (43) we can define T_{conv} as the temperature that $\nabla = \nabla_{ad}$, so we can get:

$$T_{conv} \approx T_0(1 - \frac{\nabla_{ad}}{\nabla_0})^{-1/(4-\beta)} \tag{47}$$

We have said that $\frac{\nabla_\infty}{\nabla_0}$is really small,so $T_{conv} \approx T_0$so r $\approx R_a$

Low-Luminosity Convection in the Deep Atmosphere

Taking the equation (47) into equation (40),we can get P_{conv}

$$
\begin{aligned}
P_{conv} &\equiv P_0[(\frac{P_{ph}}{P_0})^{1+\alpha} + \frac{\nabla_0}{\nabla_\infty}(\frac{1}{1-\nabla_{ad}/\nabla_0} - (\frac{T_{ph}}{T_0})^{4-\beta})]^{1/(1+\alpha)} \\
&\approx P_0[\frac{\nabla_{ad}\nabla_0}{\nabla_\infty(\nabla_0 - \nabla_{ad})}]^{1/(1+\alpha)}
\end{aligned}
\tag{48}
$$

Because equation (39) and $T_{ph} \approx T_0$ and ${\nabla_\infty}^{-1} \ll \lambda/R_B$
And from equation (32),we can get $P_{conv} \sim P_a \gg P_0$
And ρ_{conv} can be calculated by equation (6)and (11)with boundary condition $\rho_{conv} at R_a$:

$$
\begin{aligned}
\rho(r) &= \rho_{conv}[1 + \nabla_{ad}\frac{GM_p}{K{\rho_{conv}}^{\gamma-1}}(\frac{1}{r} - \frac{1}{R_a})]^{1/(\gamma-1)} \\
&= \rho_{conv}[1 + \nabla_{ad}(1 - \frac{\nabla_{ad}}{\nabla_0})^{1/(4-\beta)}(\frac{R_B}{r} - \frac{R_B}{R_a})]^{1/(\gamma-1)}
\end{aligned}
\tag{49}
$$

When $r \lesssim R_a(1 - R_a/R_B), \rho \gg \rho_{conv}$ we can get:

$$(\frac{\rho}{\rho_{conv}})^{\gamma-1} \approx \frac{T}{T_{conv}} \approx \nabla_{ad}\frac{T_0}{T_{conv}}(\frac{R_B}{r} - \frac{R_B}{R_a}) \tag{50}$$

Which is similar to equation (33) $\xi + \nabla_0(\frac{R_B}{r} - \frac{R_B}{R_a})$

High-Luminosity Optically Thin

The core is "high-luminosity" if any of the following equations holds true. For the Optically Thick case:

$$\lambda \gg R_B, {R_L}^2 \gtrsim \lambda R_B \tag{51}$$

For the Optically Thin case

$$R_L \gtrsim R_B, \lambda \gtrsim R_B \tag{54}$$

When it is Optically Thin,according to equation(36),T is strongly perturbed even $r > R_B$.

High-Luminosity Optically Thick

Now we will talk about the Optically Thick case, from equation (25),$\nabla_\infty \gtrsim \nabla_{ad}$,so the outer convective zone exists.
Same as what we did in equation (43) but $R_a$andρ_{conv}turn to R_{out}andρ_0 , besides, $T_0 = T_{conv}$

$$\rho(r) = \rho_0(\frac{R_B}{r} - \frac{R_B}{R_{out}})^{1/(\gamma-1)} \tag{52}$$

which shows that ρ, P, and T deviate by a factor of ~ 1 from ρ_0, P_0, T_0at $r \sim R_B$

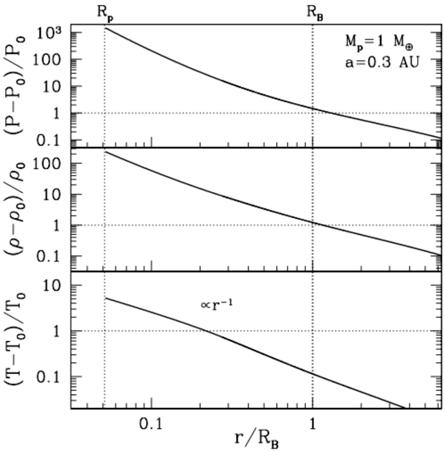
High-Luminosity Optically Thick

However, in the deep atmosphere it depends on $\nabla_0$and∇_∞.
When $\nabla_0 > \nabla_\infty$,the atmosphere is wholly convective;
when $\nabla_0 < \nabla_\infty$,the outer atmosphere is convective but the inner atmosphere is radiative.
We can calculate $\nabla(T)$ same as equation (45) with different boundary condition given by equation(52),then the radius when $\nabla(T) = \nabla_{ad}$ is R_{rad}:

$$R_{rad} \approx R_B \frac{\nabla_{ad}}{\Theta}, \Theta \equiv \left(\frac{\nabla_\infty}{\nabla_{ad}}\right)^{\nabla_{ad}/[(4-\beta)(\nabla_{ad}-\nabla_0)]} \tag{53}$$

We can see that$R_{rad} \ll R_B (\nabla_\infty \gtrsim \nabla_{ad})$.
$R_{rad} > R_B$ is also required for the establishment of the inner radiation zone.
When $\rho_{rad} \gg \rho_0$, $T_{rad}/T_0 = (\rho_{rad}/\rho_0)^{\gamma-1} \approx \Theta$

High-Luminosity Optically Thick



$M_p = 1M_\oplus, a = 0.3AU, P_0 = 10^{-3}$bar, $\rho_0 = 7 \times 10^{-8}$g cm^{-3}, $T_0 = 550$K, $\alpha = 0, \beta = 1, \gamma = \frac{7}{5}, \kappa_0 = 0.1$cm^{2}g^{-1}, planetesimal accretion is in the intermediate regime.

This case is High-Luminosity Optically Think Wholly Convective

Mass of Low-Luminosity Atmosphere

The mass of atmosphere M_{atm} is define as:

$$M_{atm} \equiv 4\pi \int_{R_p}^{R_B} \rho(r) r^2 dr \tag{55}$$

So for the optically thick case, ρ drops exponentially outside R_a ,so we can replace the R_B of equation (55) intoR_a,so that M_{atm} can be calculated by equation (33)(34)(radiative)or(47)(48)(50)(convective):

$$M_{atm} \approx 4\pi \Psi_1 \rho_0 {R_B}^3 (\frac{R_B \lambda}{R_L^2})^{1/(1+\alpha)} \tag{56}$$

$$\Psi_1 \approx C(\frac{R_a}{R_B})^{3-\varsigma} \int_0^1 z^2 (\frac{1}{z} - 1)^{\zeta} dz \tag{B1}$$

We have set R_p/R_B to 0.

For the radiative case :

$$C = \left(\frac{4\nabla_0}{3}\right)^{1/(1+\alpha)} \nabla_0^{(1-\nabla_0)/\nabla_0}, \quad \zeta = \frac{1}{\nabla_0} - 1, \tag{B2}$$

For the convective case :

$$C = \left(1 - \frac{\nabla_{ad}}{\nabla_0}\right)^{1/\nabla_{ad}(4-\beta)} \left(\frac{4\nabla_0\nabla_{ad}}{3\nabla_0 - \nabla_{ad}}\right)^{1/(1+\alpha)} \nabla_{ad}^{1/(\gamma-1)}, \quad \zeta = \frac{1}{\gamma - 1}. \tag{B3}$$

ζ is the power-law index of 1/r,and we will plus r^2 and then integrate.
When $\zeta < 3$,the main part of M_{atm} comes from $r \sim R_a$ and it is weakly depends on smaller r.
When $\zeta = 3, \alpha = 0$,each decade in radius have same mass, so we will change equation (B1) into $\Psi \sim In(R_a/R_p)$.
When $\zeta > 3$,the main part ofM_{atm} comes from the innermost part of the atmosphere near R_p and $M_{atm} \propto (R_a/R_p)^{\zeta-3}$.
We will only talk about $\zeta < 3 or \zeta = 3, \alpha = 0$,so we can safely assume $R_p/R_B = 0$.

Mass of Low-Luminosity Atmosphere

So now we can write down M_{atm} numerically at $\alpha = 0$:

$$M_{atm} = 64\pi^2 \Psi_1 \left(\frac{GM_p\mu}{k}\right)^4 \frac{\sigma}{\kappa_0 L} \approx \left[\ln\left(\nabla^{-1}\right)_\infty\right]^{-1/2} \left(\frac{M_p}{M_\oplus}\right)^{8/3} \kappa_{0.1}^{-1} \tag{57}$$

$$\times \left\{ \begin{array}{ll} 2.7\, M_\oplus\, a_{10}^3, & \text{slow,} \\ 1.4 \times 10^{-3}\, M_\oplus\, a_{10}^2, & \text{intermediate,} \\ 3.2 \times 10^{-5}\, M_\oplus\, a_{10}^{3/2}, & \text{fast.} \end{array} \right\} \tag{57}$$

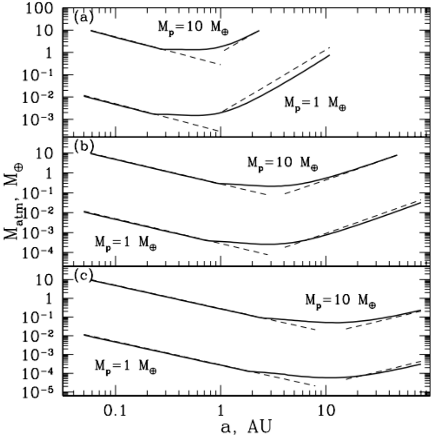
Mass of High-Luminosity Atmosphere

We can also calculate M_{atm} in High-Luminosity case by equation(52)

$$M_{atm} = 4\pi\Psi_2\rho_0 {R_B}^3 \approx 2.8\times10^{-4} M_\oplus (\frac{M_p}{M_\oplus})^3 {a_1}^{-5/4}\Psi_2 \equiv \int_0^1 z^2(\frac{\nabla_{ad}}{z} - 1)^{1/(\gamma} \tag{58}$$

The equation can also be used when the inner radiative region exists, because we have said that $R_{rad} \ll R_B$.
The numerical result of M_{atm} is based on $\gamma = 7/5$.

It can also be calculated numerically:



This picture shows the difference between out theory(Dashed lines left for equation (58) and right for equation (59)) and simulation(Solid curves).

$\alpha = 0, \beta = 1, \gamma = \frac{7}{5}, \kappa_0 = 0.1 \mathrm{cm}^2\mathrm{g}^{-1}$, different panels correspond to different planetesimal accretion regimes: (a) slow, (b) intermediate, and (c) fast.

The (a) part of the picture do not fits well, because of the finite size of the core: our extension of the integration in equation (B1) to zero (instead of R_p) leads to an overestimate of M_{atm}.

Critical Core Mass

The core will be instable when $M_p \sim M_{atm}$, assuming the exact proportion is $\eta \sim 1$

$$M_{atm}(M_{cr}) = \eta M_{cr} \tag{59}$$

In the Low-luminosity case using equations (7), (57), and (A1) we can find:

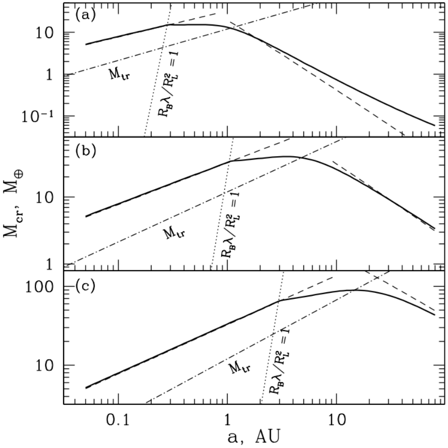
$$M_{\rm cr} \approx \left[\frac{\eta\theta\Omega\Sigma_p a\kappa_0}{64\pi^2\Psi_1\sigma G^3 M_\odot^{1/3}}\left(\frac{k}{\mu}\right)^{4/3}\right]^{3/5} \approx \eta_{0.3}^{3/5}\kappa_{0.1}^{3/5}\begin{cases} 0.26\, M_\oplus\, a_{10}^{-9/5}, & \text{slow,} \\ 25\, M_\oplus\, a_{10}^{-6/5}, & \text{interm} \\ 240\, M_\oplus\, a_{10}^{-9/10}, & \text{fast,} \end{cases} \tag{60}$$

And in the High-luminosity case using using equation (58), we can find:

$$M_{cr} = {c_0}^3\left(\frac{\eta}{4\pi\Psi_2\rho_0 G^3}\right)^{1/2} \approx 30 M_\oplus {\eta_{0.3}}^{1/2} {a_1}^{5/8} \tag{61}$$

Critical Core Mass

It can also be calculated numerically:



This picture shows the difference between out theory(Dashed lines left for equation (60) and right for equation (61)) and simulation(Solid curves),besides, the dot-dashed line shows the run of the transitional mass M_tr with a.

different panels correspond to different planetesimal accretion regimes: (a) slow, (b) intermediate, and (c) fast.

The (a) part of the picture do not fits well, because of the finite size of the core: our extension of the integration in equation (B1) to zero (instead of R_p) leads to an overestimate of M_{atm}.