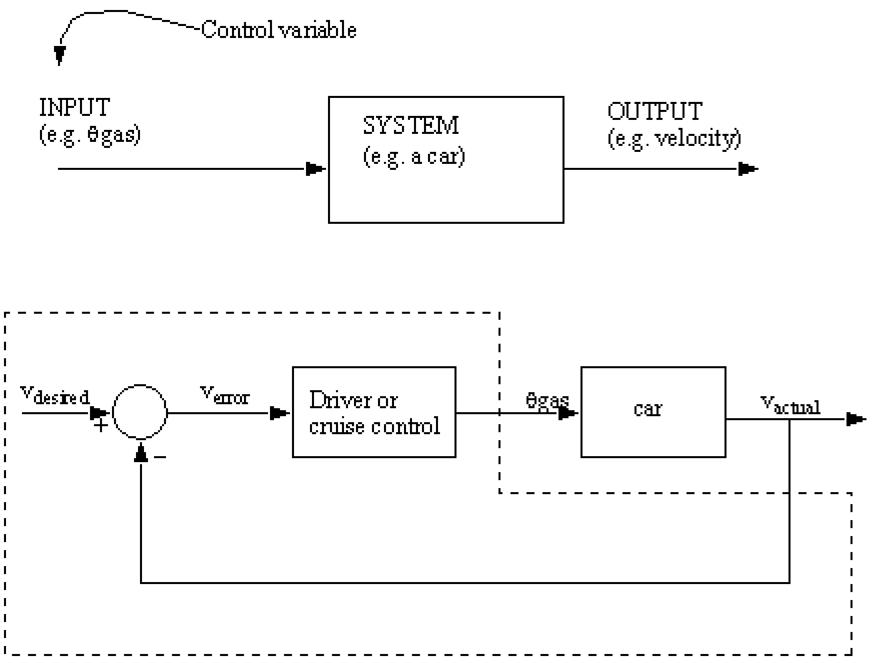
• Control systems use some output state of a system and a desired state to make control decisions.

• In general we use negative feedback systems because,

- they typically become more stable
- they become less sensitive to variation in component values
- it makes systems more immune to noise

• Consider the system below, and how it is enhanced by the addition of a control system.



The control system is in the box and could be a driver or a cruise control
(this type is known as a feedback control system)

Figure 19.1 An example of a feedback controller

Human rules to control car (also like expert system/fuzzy logic):

1. If v_{error} is not zero, and has been positive/negative for a while, increase/decrease θ_{gas}
2. If v_{error} is very big/small increase/decrease θ_{gas}
3. If v_{error} is near zero, keep θ_{gas} the same
4. If v_{error} suddenly becomes bigger/smaller, then increase/decrease θ_{gas}.
5. etc.

Figure 19.2 Rules for a feedback controller

• Some of the things we do naturally (like the rules above) can be done with mathematics

19.2.1 PID Control Systems

• The basic equation for a PID controller is shown below. This function will try to compensate for error in a controlled system (the difference between desired and actual output values).

$$u = K_c e + K_i \int e\,dt + K_d\left(\frac{de}{dt}\right)$$

Figure 19.3 The PID control equation

• The figure below shows a basic PID controller in block diagram form.

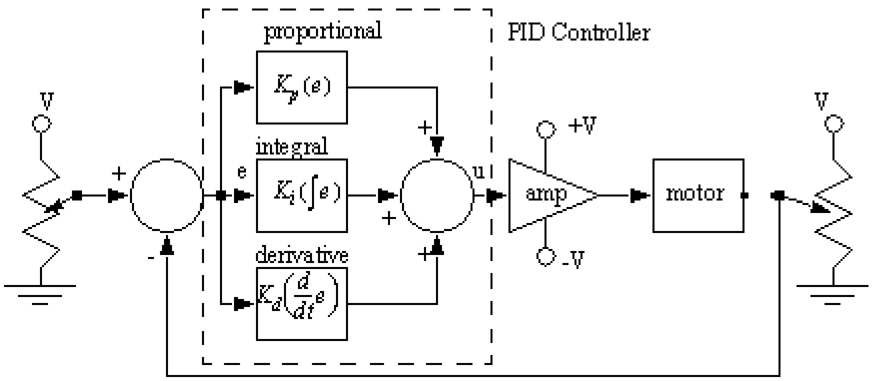


Figure 19.4 A block diagram of a feedback controller

e.g.

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt + K_d\left(\frac{dv_{error}}{dt}\right)$$

Rules 2 & 3
(general difference)

Rule 1
(Long term error)

Rule 4
(Immediate error)

Kc
Ki
Kd

Relative weights of components

This is a PID Controller

Proportional
Integral
Derivative

For a PI Controller

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt$$

For a P Controller

$$\theta_{gas} = K_c v_{error}$$

For a PD Controller

$$\theta_{gas} = K_c v_{error} + K_d \left(\frac{dv_{error}}{dt} \right)$$

• The PID controller is the most common controller on the market.

19.2.2 Analysis of PID Controlled Systems With Laplace Transforms

1. We can rewrite the control equation as a ratio of output to input.

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt + K_d \left(\frac{dv_{error}}{dt} \right)$$

$$\frac{\theta_{gas}}{v_{error}} = K_c + K_i \int dt + K_d \left(\frac{d}{dt} \right)$$

Then do a Laplace transform

$$\frac{d}{dt} \rightarrow s \qquad \frac{dx}{dt} \rightarrow sx$$

$$\int dt = \frac{1}{s} \qquad \int x dt = \frac{x}{s}$$

$$\boxed{L\left[\frac{\theta_{gas}}{v_{error}} \right] = K_c + \frac{K_i}{s} + K_d s}$$

The transfer function

2. We can also develop a transfer function for the car.

$$F = A\theta_{gas} = 10\theta_{gas}$$

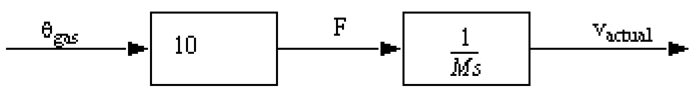
$$\boxed{\frac{F}{\theta_{gas}} = 10}$$ Transfer function for engine and transmission. (Laplace transform would be the same as initial value.)

$$F = Ma = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$

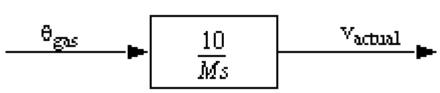
$$\frac{F}{v} = M\frac{d}{dt}$$

$$\boxed{L\left[\frac{v}{F}\right] = \frac{1}{Ms}}$$ Transfer function for acceleration of car mass

3. We want to draw the system model for the car.



- The 'system model' is shown above.
- If θ_{gas} is specified directly, this is called 'open loop control'. This is not desirable, but much simpler.
- The two blocks above can be replaced with a single one.

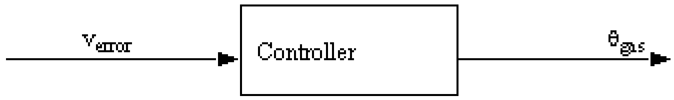


4. If we have an objective speed, and an actual speed, the difference is the 'system error'

$$v_{error} = v_{desired} - v_{actual}$$

'set-point' - desired system operating point

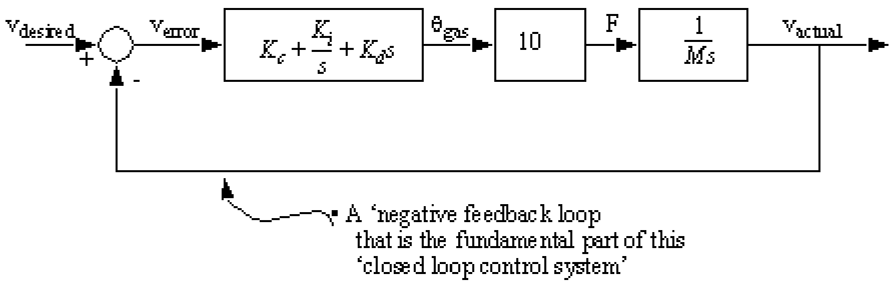
5. Finally, knowing the error is v_{error}, and we can control θ_{gas} (the control variable), we can select a control system.



$$L\left[\frac{\theta_{gas}}{v_{error}}\right] = K_c + \frac{K_i}{s} + K_d s$$

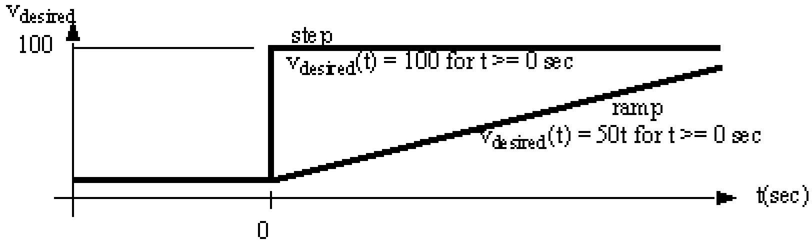
*The coefficients can be calculated using classical techniques, but they are more commonly approximated by trial and error.

6. For all the components we can now draw a 'block diagram'



19.2.3 Finding The System Response To An Input

- Even though the transfer function uses the Laplace 's', it is still a ratio of input to output.
- Find an input in terms of the Laplace 's'



Input type	Time function	Laplace function
STEP	$f(t) = Au(t)$	$f(s) = \frac{A}{s}$
RAMP	$f(t) = Atu(t)$	$f(s) = \frac{A}{s^2}$
SINUSOID	$f(t) = A\sin(\omega t)u(t)$	$f(s) = \frac{A\omega^2}{s^2+\omega^2}$
PULSE	$f(t) = A(u(t) - u(t-t_1))$	$f(s) =$ **?**
etc......		

Therefore to continue the car example, lets assume the input below,

$$v_{desired}(t) = 100 \qquad t \geq 0\sec \textbf{ ?}$$

$$v_{desired}(s) = L[v_{desired}(t)] = \frac{100}{s}$$

Next, lets use the input, and transfer function to find the output of the system.

$$v_{actual} = \left(\frac{v_{actual}}{v_{desired}}\right)v_{desired}$$

$$v_{actual} = \left(\frac{s^2(K_d) + s(K_c) + K_i}{s^2\left(\frac{M}{10} + K_d\right) + s(K_c) + K_i}\right)\left(\frac{100}{s}\right)$$

To go further, some numbers will be selected for the values.

$K_d = 10000$
$K_c = 10000$
$K_i = 1000$
$M = 1000$

$$v_{actual} = \left(\frac{s^2(10000) + s(10000) + 1000}{s^2(10100) + s(10000) + 1000}\right)\left(\frac{100}{s}\right)$$

At this point we have the output function, but not in terms of time yet. To do this we break up the function into partial fractions, and then find inverse Laplace transforms for each.

$$v_{actual} = 10^2\left(\frac{s^2+s+0.1}{s(s^2(1.01)+s+0.1)}\right)$$

Aside: We must find the roots of the equation, before we can continue with the partial fraction expansion.

recall the quadratic formula,

$$ax^2+bx+c = 0 \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2-4(1.01)(0.1)}}{2(1.01)} = -0.113, -0.877$$

$$v_{actual} = \frac{10^2}{1.01}\left(\frac{s^2+s+0.1}{s(s+0.113)s+0.877}\right)$$

$$v_{actual} = \frac{A}{s}+\frac{B}{s+0.114}+\frac{C}{s+0.795}$$

$$A = \lim_{s\to 0}\left[s\left(\frac{10^2}{1.01}\left(\frac{s^2+s+0.1}{s(s+0.113)(s+0.877)}\right)\right)\right] = \frac{10^2}{1.01}\left(\frac{0.1}{(0.113)(0.877)}\right)$$

$$\therefore A = 99.9$$

$$B = \lim_{s\to -0.113}\left[\left(\frac{10^2}{1.01}\left(\frac{s^2+s+0.1}{s(s+0.113)(s+0.877)}\right)\right)(s+0.113)\right]$$

$$\therefore B = \left(\frac{10^2}{1.01}\left(\frac{(-0.113)^2+(-0.113)+0.1}{(-0.113)(-0.113+0.877)}\right)\right) = 0.264$$

$$C = \lim_{s\to -0.877}\left[\left(\frac{10^2}{1.01}\left(\frac{s^2+s+0.1}{s(s+0.113)(s+0.877)}\right)\right)(s+0.877)\right]$$

$$\therefore C = \left(\frac{10^2}{1.01}\left(\frac{(-0.877)^2+(-0.877)+0.1}{(-0.877)(-0.877+0.113)}\right)\right) = -1.16$$

$$v_{actual} = \frac{99.9}{s}+\frac{0.264}{s+0.113}-\frac{1.16}{s+0.877}$$

Next we use a list of forward/inverse transforms to replace the terms in the partial fraction expansion.

$f(t)$	$f(s)$
A	$\frac{A}{s}$
At	$\frac{A}{s^2}$
$Ae^{-\alpha t}$	$\frac{A}{s+\alpha}$
$A\sin(\omega t)$	$\frac{A\omega}{s^2+\omega^2}$
$e^{-\xi\omega t}\sin(\omega_n t\sqrt{1-\xi^2})$	$\frac{\omega_n\sqrt{1-\xi^2}}{s^2+2\xi\omega_n s+\omega_n^2}$ $for(\xi < 1)$

etc.

To finish the problem, we simply convert each term of the partial fraction back to the time domain.

$$v_{actual} = \frac{99.9}{s} + \frac{0.264}{s+0.113} - \frac{1.16}{s+0.877}$$

$$\boxed{v_{actual} = 99.9 + 0.264e^{-0.113t} - 1.16e^{-0.877t}}$$

19.2.4 Controller Transfer Functions

• The table below is for typical control system types,

Type	Transfer Function	
Proportional (P)	$G_c = K$	
Proportional-Integral (PI)	$G_c = K\left(1 + \frac{1}{\tau s}\right)$	
Proportional-Derivative (PD)	$G_c = K(1 + \tau s)$	
Proportional-Integral-Derivative (PID)	$G_c = K\left(1 + \frac{1}{\tau s} + \tau s\right)$	
Lead	$G_c = K\left(\frac{1 + \alpha\tau s}{1 + \tau s}\right)$	$\alpha > 1$
Lag	$G_c = K\left(\frac{1 + \tau s}{1 + \alpha\tau s}\right)$	$\alpha > 1$
Lead-Lag	$G_c = K\left[\left(\frac{1 + \tau_1 s}{1 + \alpha\tau_1 s}\right)\left(\frac{1 + \alpha\tau_2 s}{1 + \tau_2 s}\right)\right]$	$\alpha > 1$ $\tau_1 > \tau_2$