Lecture 10: Characteristic Functions

1. <u>Definition of characteristic functions</u>

2. Applications

1. <u>Definition of characteristic function</u>

1.1 Complex random variables

(1) Complex numbers has a form $u = a + ib$ where a, b are real numbers and i is so-called imaginary unit which has a property $i^2 = -1$.

(2) The summation of complex numbers $u = a+ib$ and $v = c+id$ is performed according the rule $u + v = (a + c) + i(b + d)$ and the multiplication as $uv = (ac - bd) + i(ad + bc)$.

(3) For a complex number $u = a + ib$, the conjugate complex number is defined as $\bar{u} = a - ib$ and the norm is defined by the formula $|u|^2 = u\bar{u} = a^2 + b^2$; $|uv| = |u||v|$; $|u + v| \leq |u| + |v|$.

(4) The basic role in what follows plays the formula

$$e^{it} = \cos t + i \sin t, \ \ t \in R_1.$$

Note that $|e^{it}| = \sqrt{\cos^2 t + \sin^2 t} = 1$.

A complex random variable is defined as $Z = X + iY$ where $X = X(\omega)$ and $Y = Y(\omega)$ are usual real-valued random variables defined on some probability space $< \Omega, \mathcal{F}, P >$.

The distribution of a complex random variable Z is determined by the distribution of the random vector (X, Y).

The expectation of the complex random variable $Z = X + iY$ is defined as the complex number $EZ = EX + iEY$.

Two complex random variables $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ are independent if the random vectors (X_1, Y_1) and (X_2, Y_2) are independent.

(5) The product of two complex random variables $Z_1Z_2 = (X_1X_2 - Y_1Y_2) + i(X_1Y_2 + Y_1X_2)$ then $EZ_1Z_2 = E(X_1X_2 - Y_1Y_2) + iE(X_1Y_2 + Y_1X_2)$. If the random variables Z_1 and Z_2 are independent, then

$EZ_1Z_2 = EZ_1EZ_2$.

$EZ_1Z_2 = E(X_1X_2 - Y_1Y_2) + iE(X_1Y_2 + Y_1X_2) = (EX_1EX_2 - EY_1EY_2) + i(EX_1EY_2 + EY_1EX_2) = EZ_1EZ_2$.

---

(6) If $|X| \leq M$ then $|EX| \leq M$.

Let $X = Y + iZ$, then $|EX|^2 = |EY + iEZ|^2 = (EY)^2 + (EZ)^2 \leq EY^2 + EZ^2 = E|X|^2 \leq M^2$.

---

(7) $|EX| \leq E|X|$.

1.2 Definition and basic properties of characteristic functions

Definition 10.1. Let X be a random variable with the distribution $F_X(A)$ and the distribution function $F_X(x)$. The characteristic function of the random variable X (distribution $F_X(A)$ and the distribution function $F_X(x)$) is called the complex-valued function $\phi_X(t)$ defined on a real line as function of t by the formula,

$$\phi_X(t) = E\exp\{itX\} = \int_{-\infty}^{\infty} e^{itx} F_X(dx)$$

$$= E\cos tX + iE\sin tX = \int_{-\infty}^{\infty} \cos tx F_X(dx) + i\int_{-\infty}^{\infty} \sin tx F_X(dx).$$

(1) If X is a discrete random variable with the discrete distribution $p_X(x_n) = P(X = x_n), n = 0, 1, \ldots$ ($\sum_n p_X(x_n) = 1$), then

$$\phi_X(t) = E\exp\{itX\} = \sum_n e^{itx_n} p_X(x_n).$$

(2) If X is a continuous random variable with the probability density function $f_X(x)$, then

$$\phi_X(t) = E\exp\{itX\} = \int_{\infty}^{\infty} e^{itx} f_X(x)dx,$$

where the integration the Riemann integration is performed if the density $f_X(x)$ is Riemann integrable (otherwise the Lebesgue integration should be used).

(3) In the latter case the characteristic function is also known as the Fourier transform for function $f_X(x)$.

Theorem 10.1. The characteristic function $\phi_X(t)$ of a random variable X possesses the following basic properties:

(1) $\phi_X(t)$ is a continuous function in $t \in R_1$;
(2) $\phi_X(0) = 1$
(3) $\phi_X(t)$ is positively defined function, i.e., a quadratic form $\sum_{1\leq j,l\leq n} c_j \bar{c}_l \phi_X(t_j - t_l) \geq 0$ for any complex numbers $c_1, \ldots, c_n$ and real $t_1, \ldots, t_n$ and $n \geq 1$.

(a) $|\phi_X(t+h) - \phi_X(t)| \leq |Ee^{i(t+h)X} - Ee^{itX}| = |Ee^{itX}(e^{ihX} - 1)| \leq E|e^{itX}(e^{ihX} - 1)| = E|e^{ihX} - 1|$.

(b) $|e^{ihx} - 1| \leq 2$ and also $|e^{ihx} - 1| \leq |hx|$. Choose $A > 0$ and get the following estimates

$$E|e^{ihX} - 1| \leq \int_{|x|>A} |e^{ihx} - 1| F_X(dx) + \int_{-A}^{A} |e^{ihx} - 1| F_X(dx)$$

$$2\int_{|x|>A} F_X(dx) + \int_{-A}^{A} |hx| F_X(dx)$$

$$\leq 2P(|X| \geq A) + hAP(|X| \leq A) \leq 2P(|X| \geq A) + hA.$$

(c) Choose an arbitrary $\varepsilon > 0$. Then choose A so large that $2P(|X| \geq A) \leq \frac{\varepsilon}{2}$ and then h so small that $hA \leq \frac{\varepsilon}{2}$. Then the expression on the right hand side in the estimate given in (b) will be less or equal ε.

(d) $\phi_X(0) = Ee^{i0X} = E1 = 1$.

(e) To prove (3) we get

$$\sum_{1\leq j,l\leq n} c_j\bar{c}_l\phi_X(t_j - t_l) = \sum_{1\leq j,l\leq n} c_j\bar{c}_l Ee^{i(t_j-t_l)X}$$

$$= E\sum_{1\leq j\leq n} c_j e^{it_jX}\overline{\sum_{1\leq j\leq n} c_j e^{it_jX}} = E|\sum_{1\leq j\leq n} c_j e^{it_jX}|^2 \geq 0.$$

Theorem 10.2 (Bochner)**. A complex-valued function $\phi(t)$ defined on a real line is a characteristic function of some random variable X, i.e., can be represented in the form $\phi(t) = Ee^{itX} = \int_{-\infty}^{\infty} e^{itx}F_X(dx), t \in R_1$ if and only if it possesses the properties (1) - (3) formulated in Theorem 1.

An additional properties of characteristic functions are:

(4) $|\phi_X(t)| \leq 1$ for any $t \in R_1$;
(5) $\phi_{a+bX}(t) = e^{iat}\phi_X(bt)$;

(6) $|e^{itX}| \equiv 1$ and, thus, $|\phi_X(t)| = |Ee^{itX}| \leq 1$;

(7) $\phi_{a+bX}(t) = Ee^{it(a+bX)} = e^{iat}Ee^{itbX} = e^{iat}\phi_X(bt)$;

1.3 Examples

(a) If $X \equiv a$ where $a =$ const, then $\phi_X(t) = e^{iat}$.

(1) Binomial random variable.

$X = B(m, p), m = 1, 2, \ldots, 0 \leq p \leq 1;$
$p_X(n) = C_m^n p^n (1-p)^{m-n}, n = 0, 1, \ldots, m;$
$\phi_X(t) = (pe^{it} + (1-p))^m.$

(2) Poisson random variable.

$X = \mathcal{P}(\lambda), \lambda > 0;$
$p_X(n) = \frac{\lambda^n}{n!} e^{-\lambda}, n = 0, 1, \ldots;$
$\phi_X(t) = e^{\lambda(e^{it}-1)}.$

(3) Normal random variable.

$X = N(m, \sigma^2), m \in R_1, \sigma^2 > 0;$
$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in R_1;$
$\phi_X(t) = e^{-\frac{t^2}{2}}$

$Y = m + \sigma X;$
$f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, x \in R_1$
$\phi_Y(t) = e^{iat - \frac{\sigma^2 t^2}{2}}$

(4) Gamma random variable.

$X = Gamma(\alpha, \beta), \alpha, \beta > 0;$
$f_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, x \geq 0$ where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x};$
$\phi_X(t) = (\frac{1}{1-i\beta t})^\alpha.$

1.4 Inversion formulas

$F(A)$ and $G(A)$ are probability measures on a real line; $\phi(t) = \int_{-\infty}^{\infty} e^{itx} F(dx)$ and $\psi(t) = \int_{-\infty}^{\infty} e^{itx} G(dx)$ are their characteristic functions.

Lemma 10.1 (Parseval identity). The following so-called Parseval identity takes place,

$$\int_{-\infty}^{\infty} e^{-ity} \phi(t) G(dt) = \int_{-\infty}^{\infty} \psi(x-y) F(dx).$$

(a) The following equality follows from the definition of a characteristic function,

$$e^{-ity} \phi(t) = \int_{-\infty}^{\infty} e^{it(x-y)} F(dx)$$

(b) Integrating it with respect to the measure $G(A)$ and using Fubinni theorem we get,

$$\int_{-\infty}^{\infty} e^{-ity} \phi(t) G(dt) = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} e^{it(x-y)} F(dx)] G(dt)$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} e^{it(x-y)} G(dt)] F(dx) = \int_{-\infty}^{\infty} \psi(x-y) F(dx).$$

Theorem 10.3 (inversion). Different characteristic function correspond to different distributions and vise versa.

This theorem is a corollary of the inversion formula given below.

Theorem 10.4. Let $F(x)$ be a distribution function and C_F is the set of all point of continuity of this distribution function. Let also $\phi(t) = \int_{-\infty}^{\infty} e^{itx} F(dx)$ be the characteristic function that corresponds to this distribution function. The following inversion formula takes place

$$F(z) = \lim_{\sigma^2 \to 0} \frac{1}{2\pi} \int_{-\infty}^{z} [\int_{-\infty}^{\infty} e^{-ity} \phi(t) e^{-\frac{t^2\sigma^2}{2}} dt] dy, \ z \in C_F.$$

(a) Let $G(A)$ is a normal distribution with parameters $m = 0$ and $\frac{1}{\sigma^2}$. Then Parseval identity takes the following form

$$\int_{-\infty}^{\infty} e^{-ity} \phi(t) \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{t^2\sigma^2}{2}} dt = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2}} F(dx),$$

or equivalently

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi(t) e^{-\frac{t^2\sigma^2}{2}} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} F(dx).$$

(b) Integrating with respect to the Lebesgue measure (Riemann integration can be used!) over the interval $(-\infty, z]$ we get

$$\frac{1}{2\pi} \int_{-\infty}^{z} [\int_{-\infty}^{\infty} e^{-ity} \phi(t) e^{-\frac{t^2\sigma^2}{2}} dt] dy = \int_{-\infty}^{z} [\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} F(dx)] dy$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-y)^2}{2\sigma^2}} dy] F(dx) = P(N(0, \sigma^2) + X \leq z),$$

where $N(0, \sigma^2)$ is a normal random variable with parameters 0 and σ^2 independent of the random variable X.

(c) If $\sigma^2 \to 0$ then the random variables $N(0, \sigma^2)$ converge in probability to 0 ($P(|N(0, \sigma^2)| \geq \varepsilon) \to 0$ for any $\varepsilon > 0$) and, in sequel, the random variables $N(0,\sigma^2)+X$ weakly converge to the random variable X that is $P(N(0,\sigma^2)+X \leq z) \to P(X \leq z)$ as $\sigma^2 \to 0$ for any point z that is a point of continuity for the distribution function of the random variable X.

Theorem 10.5. Let $F(x)$ be a distribution function and $\phi(t) = \int_{-\infty}^{\infty} e^{itx} F(dx)$ be the characteristic function that corresponds to this distribution function. Let $\phi(t)$ is absolutely integrable at real line, i.e., $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$. Then the distribution function $F(x)$ has a bounded continuous probability density function $f(y)$ and the following inversion formula takes place

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi(t) dt, \; y \in R_1.$$

(a) The proof can be accomplished by passing $\sigma^2 \to 0$ in the Parseval identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi(t) e^{-\frac{t^2\sigma^2}{2}} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} F(dx).$$

(b) Both expressions on the left and right hand sides of the above identity represent the probability density function $f_\sigma(y)$ for the distribution function $F_\sigma(y)$ of random variable $N(0,\sigma^2)+X$.

(c) $F_\sigma(\cdot) \Rightarrow F(\cdot)$ as $\sigma \to 0$.

(d) From the left hand side expression in the above identity, we see that.

$$f_\sigma(y) \to f(y) \text{ as } \sigma \to 0.$$

(e) Note that $f_\sigma(y)$ and $f(y)$ are bounded continuous functions;

(f) It follows from (c), (d) and (e) that for any $x' < x''$

$$F_\sigma(x'') - F_\sigma(x') = \int_{x'}^{x''} f_\sigma(y)dy \to \int_{x'}^{x''} f(y)dy$$

(g) But, if x', x'' are continuity point of $F(x)$ that

$$F_\sigma(x'') - F_\sigma(x') \to F(x'') - F(x') \text{ as } \sigma \to 0.$$

(h) Thus,

$$F(x'') - F(x') = \int_{x'}^{x''} f(y)dy,$$

i.e., indeed, $f(y)$ is a probability density of the distribution function $F(x)$.

———————————————-

2. Applications

2.1 Characteristic function for sum of independent random variables

Lemma 10.2. If a random variable $X = X_1 + \cdots + X_n$ is a sum of independent random variables $X_1, \ldots, X_n$ then

$$\phi_X(t) = \phi_{X_1}(t) \times \cdots \times \phi_{X_n}(t).$$

———————————————-

$\phi_X(t) = Ee^{it(X_1+\cdots X_n)} = E\prod_{k=1}^n e^{itX_k} = \prod_{k=1}^n Ee^{itX_k}$.

———————————————-

Examples

(1) Let random variables $X_k = N(m_k, \sigma_k^2), k = 1, \ldots, n$ be independent normal random variables. In this case $X = X_1 + \cdots + X_n$ is also a noramal random variable with parameters $m = m_1 + \cdots + m_n$ and $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

$\phi_X(t) = \prod_{k=1}^n e^{itm_k - \frac{\sigma_k^2 t^2}{2}} = e^{itm - \frac{\sigma^2 t^2}{2}}$.

---

(2) Let random variables $X_k = \mathcal{P}(\lambda_k), k = 1, \ldots, n$ be independent Poissonian random variables. In this case $X = X_1 + \cdots + X_n$ is also a Poissonian random variable with parameters $\lambda = \lambda_1 + \cdots + \lambda_n$.

$\phi_X(t) = \prod_{k=1}^n e^{\lambda_k(e^{it}-1)} = e^{\lambda(e^{it}-1)}$.

2.2 Characteristic functions and moments of random variables

Theorem 10.6. Let X be a random variable such that $E|X|^n = \int_{-\infty}^{\infty} |x|^n F_X(dx) < \infty$ for some $n \geq 1$. Then, there exist n-th continuous derivatives for the characteristic function $\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} F_X(dx)$ for all $t \in R_1$ given by the formula,

$$\phi_X^{(k)}(t) = i^k \int_{-\infty}^{\infty} x^k e^{itx} F_X(dx),\ k = 1, \ldots, n,$$

and, therefore, the following formula takes place

$$EX^k = i^{-k}\phi_X^{(k)}(0), k = 1, \ldots, n.$$

----

(1) $\frac{\phi_X(t+h)-\phi_X(t)}{h} = \int_{-\infty}^{\infty} e^{itx}\frac{e^{ihx}-1}{h}F_X(dx)$;
(2) $|\frac{e^{ihx}-1}{h}| \leq |x|$ that let to perform the limit transition in the above relation as $h \to 0$ using also that $\frac{e^{ihx}-1}{h} \to ix$ as $h \to 0$;
(3) In general case, the formula for $\phi_X^{(k)}(t)$ can be proved by induction.

Examples

(1) Let $X = N(m, \sigma^2)$ be a normal random variable. Then $\phi_X(t) = e^{itm - \frac{\sigma^2 t^2}{2}}$ and, therefore,
(1) $\phi'_X(t) = (im - \sigma^2 t)e^{itm - \frac{\sigma^2 t^2}{2}}$ and $EX = i^{-1}\phi'_X(0) = m$.
(2) $\phi''_X(t) = -\sigma^2 e^{itm - \frac{\sigma^2 t^2}{2}} + (im - \sigma^2 t)^2 e^{itm - \frac{\sigma^2 t^2}{2}}$ and $EX^2 = i^{-2}\phi''_X(0) = \sigma^2 + m^2$.

(2) Let random variables $X = \mathcal{P}(\lambda)$ be a Poissonian random variable. Then $\phi_X(t) = e^{\lambda(e^{it}-1)}$ and, therefore,
(1) $\phi'_X(t) = i\lambda e^{it}e^{\lambda(e^{it}-1)}$ and, thus, $EX = i^{-1}\phi'_X(0) = \lambda$.
(2) $\phi''_X(t) = i^2\lambda e^{it}e^{\lambda(e^{it}-1)} + i^2\lambda^2 e^{2it}e^{\lambda(e^{it}-1)}$ and $EX^2 = i^{-2}\phi''_X(0) = \lambda + \lambda^2$.

2.3 Continuity theorem

$X_n, n = 0, 1, \ldots$ is a sequence of random variables with distribution functions, respectively, $F_n(x), n = 0, 1, \ldots$;

$\phi_n(t) = Ee^{itX_n} = \int_{-\infty}^{\infty} e^{itx} F_n(dx), n = 0, 1, \ldots$ are the corresponding characteristic functions;
C_F denotes the set of continuity points for a distribution function $F(x)$.

Theorem 10.7 (continuity)**. Distribution functions $F_n \Rightarrow F_0$ as $n \to \infty$ if and only if the corresponding characteristic functions $\phi_n(t) \to \phi_0(t)$ as $n \to \infty$ for every $t \in R_1$.

(a) Let $F_n \Rightarrow F_0$ as $n \to \infty$.

(b) Then by Helly theorem, for every $t \in R_1$,

$$\phi_n(t) = \int_{-\infty}^{\infty} e^{itx} F_n(dx) \to \int_{-\infty}^{\infty} e^{itx} F_0(dx) = \phi_0(t) \text{ as } n \to \infty.$$

(c) Let $\phi_n(t) \to \phi_0(t)$ as $n \to \infty, t \in R_1$.

(d) As was proved in Lecture 9 using Cantor diagonal method, for any subsequence $n_k \to \infty$ as $k \to \infty$, one can select a subsequence $n'_r = n_{k_r} \to \infty$ as $r \to \infty$ such that

$$F_{n'_r}(x) \to F(x) \text{ as } n \to \infty, x \in C_F,$$

where $F(x)$ is proper or improper distribution function (non-decreasing, continuous from the right and such that $0 \leq F(-\infty) \leq F(\infty) \leq 1$).

(e) Let us write down the Parseval identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi_{n'_r}(t) e^{-\frac{t^2\sigma^2}{2}} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} F_{n'_r}(dx).$$

(f) By passing to the limit as $r \to \infty$ in the above identity and takin into account (c), and Lebesque theorem, and the fact that $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-x)^2}{2\sigma^2}} \to 0$ as $x \to \pm\infty$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi_0(t)e^{-\frac{t^2\sigma^2}{2}}dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-x)^2}{2\sigma^2}}F(dx).$$

(g) By multiplying both part in (f) by $\sqrt{2\pi}\sigma$ and taking $y = 0$, we get

$$\int_{-\infty}^{\infty} \phi_0(t)\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{t^2\sigma^2}{2}}dt = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}F(dx) \leq F(+\infty) - F(-\infty).$$

(g) By passing to the limit as $\sigma \to 0$ in the above equality and taking into account that (a) $\int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{t^2\sigma^2}{2}}dt = 1$, (b) $\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{t^2\sigma^2}{2}} \to 0$ as $\sigma \to \infty$, for $t \neq 0$, and (c) $\phi(t)$ is continuous in zero and $\phi(0) = 1$, we get,

$$1 = \lim_{\sigma\to\infty}\int_{-\infty}^{\infty} \phi_0(t)\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{t^2\sigma^2}{2}}dt \leq F(+\infty) - F(-\infty).$$

(h) This, $F(+\infty) - F(-\infty) = 1$, i.e., the distribution $F(x)$ is a proper distribution.

(i) In this case, the weak convergence $F_{n'_r} \Rightarrow F$ as $n \to \infty$ implies by the first part of the theorem that

$$\phi_{n'_r}(t) \to \phi(t) = \int_{-\infty}^{\infty} e^{itx}F(dx) \text{ as } n \to \infty,\ t \in R_1.$$

(j) This relation together with (c) implies that

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}F(dx) = \phi_0(t), t \in R_1.$$

(k) Therefore due to one-to-one correspondence between characteristic functions and distribution functions $F(x) \equiv F_0(x)$.

(l) Therefore, by the subsequence definition of the weak convergence $F_n \Rightarrow F_0$ as $n \to \infty$.

Theorem 10.8* (continuity). Distribution functions $F_n(x)$ weakly converge to some distribution function $F(x)$ if and only if the corresponding characteristic functions $\phi_n(t)$ converge pointwise (for every $t \in R_1$) to some continuous function $\phi(t)$. In this case, $\phi(t)$ is the characteristic function for the distribution function $F(x)$.

(a) Let $\phi_n(t) \to \phi(t)$ as $n \to \infty$ for every $t \in R_1$. Then $\phi(0) = 1$;

(b) $\phi(t)$ is positively defined since functions $\phi_n(t)$ possess these properties.

(c) Since $\phi(t)$ is also continuous, the properties (a)-(c) pointed in Theorem 10.1 hold and, therefore, $\phi(t)$ is a characteristic function for some distribution function $F(x)$. Then $F_n(x)$ weakly converge to $F(x)$ by Theorem 10.7.

2.2 Weak law of large numbers (LLN)

$< \Omega, \mathcal{F}, P >$ is a probability space;
$X_n, n = 0, 1, 2, \ldots$ are independent identically distributed (i.i.d.)

random variables defined on a probability space $< \Omega, \mathcal{F}, P >$.

Theorem 10.9 (weak LLN). If $E|X_1| < \infty, EX_1 = a$, then random variables

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n} \xrightarrow{d} a \text{ as } n \to \infty.$$

(a) Let $\phi(t) = Ee^{itX_1}, \phi_n(t) = Ee^{it\overline{X}_n}$ be the corresponding characteristic functions. Then $\phi_n(t) = (\phi(\frac{t}{n}))^n$.

(b) Since $E|X_1| < \infty$, there exists the continuous first derivative $\phi'(t)$ and, therefore, Taylor expansion for $\phi(t)$ in point $t = 0$ can be written down that is $\phi(t) = 1 + \phi'(0)t + o(t)$

(c) Since $\phi'(0) = iEX_1 = ia$, the above Taylor expansion can be re-written. in the following form $\phi(t) = 1 + iat + o(t)$.

(d) Therefore, $\phi_n(t) = (\phi(\frac{t}{n}))^n = (1 + ia\frac{t}{n} + o(\frac{t}{n}))^n$.

(e) If $u_n \to 0$ and $v_n \to \infty$ then $\lim(1 + u_n)^{v_n}$ exists if and only if there exists $\lim u_n v_n = w$ and, in this case, $\lim(1+u_n)^{v_n} = e^w$.

(f) Using this fact, we get $\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty}(1 + ia\frac{t}{n} + o(\frac{t}{n}))^n = e^{iat}$, for every $t \in R_1$.

(g) Thus by the continuity theorem for characteristic functions $\overline{X}_n \xrightarrow{d} a$ as $n \to \infty$.

10.10 Central Limit Theorem (CLT)

$< \Omega, \mathcal{F}, P >$ is a probability space;

$X_n, n = 0, 1, 2, \ldots$ are independent identically distributed (i.i.d.)

random variables defined on a probability space $< \Omega, \mathcal{F}, P >$; $Z = N(0,1)$ is standard normal random variable with the distribution function $F(x) = \frac{1}{\sqrt{n}\sigma} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, -\infty < x < \infty$.

Theorem 10.10 (CLT). If $E|X_1|^2 < \infty, EX_1 = a, VarX_1 = \sigma^2 > 0$, then random variables

$$Z_n = \frac{X_1 + \cdots + X_n - an}{\sqrt{n}\sigma} \xrightarrow{d} Z \text{ as } n \to \infty.$$

(1) Since, the limit distribution $F(x)$ is continuous, the above relation means that $P(Z_n \leq x) \to F(x)$ as $n \to \infty$ for every $-\infty < x < \infty$.

(a) Let introduce random variables $Y_n = \frac{X_n - a}{\sigma}, n = 1, 2, \ldots$, then $Z_n = \frac{X_1 + \cdots + X_n - an}{\sqrt{n}\sigma} = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}}$.

(b) Let $\psi(t) = Ee^{itY_1}, \psi_n(t) = Ee^{it\overline{Z}_n}$. Then $\psi_n(t) = (\psi(t/\sqrt{n}))^n$.

(c) If $E|X_1|^2 < \infty$ then $E|Y_1|^2 < \infty$ and, moreover, $EY_1 = \sigma^{-1}(EX_1 - a) = 0, E(Y_1)^2 = \sigma^{-2}E(X_1 - a)^2 = 1$.

(d) Since $E|Y_1|^2 < \infty$, there exist the continuous second derivative $\psi''(t)$ and, therefore, Taylor expansion for $\psi(t)$ in point $t = 0$ can be written down that is $\psi(t) = 1 + \psi'(0)t + \frac{1}{2}\psi''(0)t^2 + o(t^2)$.

(e) Since $\psi'(0) = iEY_1 = 0$ and $\psi''(0) = i^2E(Y_1)^2 = -1$, the above Taylor expansion can be re-written in the following form $\psi(t) = 1 - \frac{1}{2}t^2 + o(t^2)$.

(f) Therefore, $\psi_n(t) = (\psi(\frac{t}{\sqrt{n}}))^n = (1 - \frac{t^2}{2n} + o(\frac{t^2}{n}))^n$.

(g) Using the above formula, we get $\lim_{n\to\infty} \psi_n(t) = \lim_{n\to\infty}(1 - \frac{t^2}{2n} + o(\frac{t^2}{n}))^n = e^{-\frac{t^2}{2}}$, for every $t \in R_1$.

(h) Thus by the continuity theorem for characteristic functions $Z_n \xrightarrow{d} Z$ as $n \to \infty$.

2.6 Law of small numbers

Let $X_{n,k}, k = 1, 2, \ldots$ are i.i.d. Bernoulli random variables defined on a probability space $< \Omega_n, \mathcal{F}_n, P_n >$ and taking values 1 and 0, respectively, with probabilities p_n and $1 - p_n$, where $0 \leq p_n \leq 1$.
$N(\lambda) = \mathcal{P}(\lambda)$ is a Poissonian random variable with parameter λ, i.e., $P(N(\lambda) = r) = \frac{\lambda^r}{r!}e^{-\lambda}, r = 0, 1, \ldots$.

Theorem 10.11 (LSN). If $0 < p_n \cdot n \to \lambda < \infty$, then random variables

$$N_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{d} N(\lambda) \text{ as } n \to \infty.$$

(1) The random variable N_n is the number of successes in n Bernoulli trials.

(2) The random variables $N_n = X_{n,1} + \cdots + X_{n,n}, n = 1, 2, \ldots$ are determined by series of random variables
$X_{1,1}$
$X_{2,1}, X_{2,2}$
$X_{3,1}, X_{3,2}, X_{3,3}$
$\ldots\ldots$

This representation gave the name to this type of models. as so-called *triangle array mode*.

(a) Let $\phi_n(t) = Ee^{itN_n}$ The characteristic function $Ee^{itX_{n,1}} = p_ne^{it} + 1 - p_n$ and, thus, the characteristic function $\phi_n(t) = Ee^{itN_n} = (p_ne^{it} + 1 - p_n)^n$.

(b) Using the above formula, we get $\phi_n(t) = (p_ne^{it} + 1 - p_n)^n \sim e^{p_n(e^{it}-1)n} \to e^{\lambda(e^{it}-1)}$ as $n \to \infty$, for every $t \in R_1$.

(c) Thus by the continuity theorem for characteristic functions $N_n \xrightarrow{d} N(\lambda)$ as $n \to \infty$.

LN Problems

1. What is the characteristic function of random variable X which takes value -1 and 1 with probability $\frac{1}{2}$.

2. Prove that the function $\cos^n t$ is a characteristic function for every $n = 1, 2, \ldots$.

3. Let X is a uniformly distributed random variable on the interval $[a, b]$. Please find the characteristic function of the random variable X.

4. Let $f(x)$ is a probability density for a random variable X. Prove that the characteristic function is a real-valued function if $f(x)$ ia a symmetric function, i.e. $f(x) = f(-x), x \geq 0$.

5. Let random variables X_1 and X_2 are independent and have gamma distributions s parameters α_1, β and α_2, β, respectively. Prove that the random variable $Y = X_1 + X_2$ also has gamma distribution with parameters $\alpha_1, +\alpha_2, \beta$.

6. Let X has a geometrical distribution with parameter p. Find EX and $VarX$ using characteristic functions.

7.Let X_n are geometrical random variables with discrete distribution $p_{X_n}(k) = p_n(1-p_n)^{k-1}, k = 1, 2, \ldots$, where $0 < p_n \to 0$ as $n \to \infty$. Prove that $p_n X_n \xrightarrow{d} X$ as $n \to \infty$, where $X = Exp(1)$ is an exponential random variable with parameter 1.