CS201: Discrete Math for Computer Science
2020 Fall Semester   Written Assignment # 5
Due: Dec. 22nd, 2020, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b) | a \text{ and } b \text{ have no letters in common}\}$

(2) $R_2 = \{(a, b) | a \text{ and } b \text{ are not the same length}\}$

(3) $R_3 = \{(a, b) | a \text{ is longer than } b\}$

Solution:

(1) Irreflexive, symmetric

(2) Irreflexive, symmetric

(3) Irreflexive, antisymmetric, transitive

□

Q.2 Show that a subset of an *antisymmetric* relation is also *antisymmetric*.
Solution: Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a, b) \in R_1$ and $(b, a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

□

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

□

Q.4 Prove or give a counterexample to the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Solution: Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

Then R is symmetric and reflexive, but not transitive.

□

Q.5 Give an examples of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

Solution: We fix the ground set $S = \{a, b, c, d\}$, and we consider the relation $R = \{(a, b), (b, c), (c, d)\}$. Then the transitive closure of R equals $R^* = \{(a, b), (b, c), (c, d), (a, c), (b, d), (a, d)\}$. On the other hand, $R^2 = \{(a, c), (b, d)\}$, and $R^3 = \{(a, d)\}$. Hence, R^3 is necessary to get R^*.

$\square$

Q.6 Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.
Solution: Suppose that $(a, b) \in R$. Because $(b, b) \in R$, it then follows that $(a, b) \in R^2$. Thus, R is a subset of R^2.

$\square$

Q.7 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Solution:

(1) Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.

(2) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cap R_2$

(3) Yes. Since $(a, a) \in R_1$ and $(a, a) \in R_2$ for all $a \in A$, it follows that $(a, a) \notin R_1 \cup R_2$

$\square$

Q.8 Suppose that R is a *symmetric* relation on a set A. Is $\overline{R}$ also symmetric? Explain your answer.
Solution: Under this hypothesis, $\overline{R}$ must also be symmetric. If $(a, b) \in \overline{R}$, then $(a, b) \notin R$, whence (b, a) cannot be in R since R is symmetric. In other words, (b, a) is also contained in $\overline{R}$. Thus, $\overline{R}$ is symmetric.

$\square$

Q.9

(1) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

(2) Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution:

(1) Let $R = \{(a, b), (a, c)\}$. The transitive closure of the symmetric closure of R is $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ and is different from the symmetric closure of the transitive closure of R, which is $\{(a, b), (a, c), (b, a), (c, a)\}$.

(2) Suppose that (a, b) is in the symmetric closure of the transitive closure of R. We must show that (a, b) is in the transitive closure of the symmetric closure of R. We know that at least one of (a, b) and (b, a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward.

Hence, (a, b) is in the transitive closure of the symmetric closure of R.

$\square$

Q.10 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)\mathbb{R}(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Solution: $((a, b), (a, b)) \in R$ because $a + b = a + b$. Hence R is reflexive.

If $((a, b), (c, d)) \in R$ then $a + d = b + c$, so that $c + b = d + a$. It then follows that $((c, d), (a, b)) \in R$. Hence R is symmetric.

Suppose that $((a, b), (c, d))$ and $((c, d), (e, f))$ belong to R. Then $a + d = b + c$ and $c + f = d + e$. Adding these two equations and subtracting $c + d$ from both sides gives $a + f = b + e$. Hence $((a, b), (e, f))$ belongs to R. Hence, R is transitive.

$\square$

Q.11 Which of the following are equivalence relations on the set of all people?

(1) $\{(x, y)|x$ and y have the same sign of the zodiac$\}$

(2) $\{(x, y)|x$ and y were born in the smae year$\}$

(3) $\{(x, y)|x$ and y have been in the same city$\}$

Solution:

(1) This is an equivalence relation.

(2) This is an equivalence relation.

(3) This is not an equivalence relation, since it is not transitive.

$\square$

Q.12 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?
Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2}/2 = 25$.

$\square$

Q.13 Show that $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?
Solution: This relation is reflexive, since $x - x = 0 \in \mathbb{Q}$. To see that it is symmetric, suppose that $x-y \in \mathbb{Q}$. Then $y-x = -(x-y)$ is again a rational number. For transitivity, if $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$, then their sum, namely $x-z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of $1/2$ are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, in other words, $\{\pi + r|r \in \mathbb{Q}\}$.

$\square$

Q.14 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Solution:

(a) Yes. (It is the smallest partial order: reflexivity ensures that very partial order contains at least all pairs (a, b).)

(b) No. It is not reflexive.

(c) Yes.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.15 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

$\square$

Q.16 Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Solution: The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$.

$\square$

Q.17 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Solution:

(a) Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

(b) No, because $33 \preceq 26$ and $26 \preceq 33$, but $26 \neq 33$.

(c) No, because $33 \preceq 35$ and $35 \preceq 13$, but we do not have $33 \preceq 13$.

$\square$

Q.18 Answer these questions for the partial order represented by this Hasse diagram.

(a) Find the maximal elements.

(b) Find the minimal elements.

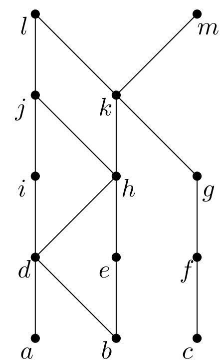

Figure 1: Q.18

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) The maximal elements are the ones with no other elements above them, namely l and m.

(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

(c) There is no greatest element, since neither l nor m is greater than the other.

(d) There is no least elements, since neither a nor b is less than the other.

(e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l and m are the elements fitting this description.

(f) Since k is less than both l and m, it is the least upper bound of a, b and c.

(g) No element is less than both f and h, so there are no lower bounds.

(h) Since there is no lower bound, there cannot be greatest lower bound.

□

Q.19 Let G be a simple graph. Show that the relation R on the set of vertices of G such that rRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Solution:

If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. Thus, by definition, R is a symmetric relation. A simple graph does not allow loops; therefore uRu never holds, and so by definition R is irreflexive.

□

Q.20 A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Solution: If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

□

Q.21 Let G be a *simple* graph with n vertices.

(a) What is the *maximum* number of edges G can have?

(b) If G is connected, what is the *minimum* number of edges G can have?

(c) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Solution:

(a) $\binom{n}{2}$

(b) $n-1$

(c) We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

$\square$

Q.22 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Solution:

(a) omitted.

(b) The degree of each vertex is $\binom{n-2}{2}$.

$\square$

Q.23 Let $G = (V, E)$ be a graph on n vertices. Construct a new graph, $G' = (V', E')$ as follows: the vertices of G' are the edges of G (i.e., $V' = E$), and two distinct edges in G are adjacent in G' if they share an endpoint.

(a) Draw G' for $G = K_4$.

(b) Find a formula for the number of edges of G' in terms of the vertex degrees of G.

Solution:

(a) omitted.

(b) The degree $\deg(v)$ of a vertex v in the graph G means that the number of $\deg(v)$ edges share the same vertex v. This will generate $\binom{\deg(v)}{2}$ edges in G'. In all the number of edges in G' is

$$\sum_{v \in V} \binom{\deg(v)}{2}.$$

$\square$

Q.24 Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Solution:

(1) If $x \in N(A \cup B)$, then x is adjacent to some vertex $v \in A \cup B$. W.l.o.g., suppose that $v \in A$. Then $x \in N(A)$ and therefore also in $N(A) \cup N(B)$. Conversely, if $x \in N(A) \cup N(B)$, then w.l.o.g. suppose that $x \in N(A)$. Thus, x is adjacent to some vertex $x \in A \subseteq A \cup B$, so $x \in N(A \cup B)$.

(2) If $x \in N(A \cap B)$, then x is adjacent to some vertex $v \in A \cap B$. Since both $v \in A$ and $v \in B$, it follows that $x \in N(A)$ and $x \in N(B)$, whence $x \in N(A) \cap N(B)$. For the counterexample, let $G = (\{u, v, w\}, \{\{u, v\}, \{v, w\}\})$, $A = \{u\}$, and $B = \{w\}$.

$\square$

Q.25 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Solution:

(i) By definition, the $dist(u, v)$ is the minimal length of a path from u to v, and the length if the number of edges in the path. Thus, $dist(u, v)$ cannot be negative. Furthermore, $dist(u, v) = 0$ if and only if there is a path of length 0 from u to v, which means that $u = v$.

(ii) Suppose that P is path from u to v of the minimal length. We reverse all the edges in the path P, and will get a path P' from v to u. Note that P' must be the minimal path from v to u. Otherwise, we reverse P' and will get a shorter path from u to v, which is a contradiction. Thus, $dist(u, v) = dist(v, u)$.

(iii) By definition, $dist(u, v) = \#$ of edges in the path P, where P is the path from u to v with the minimum length. Suppose that P_1 and P_2 are the paths of minimal length from u to w, and from w to v, respectively. Then $u\tilde{w}\tilde{v}$ is a new path P' from u to v. By the minimality of P, we must have $dist(u, v) \leq dist(u, w) + dist(w, v)$.

$\square$

Q.26 Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.
Solution: Suppose that the parts are of sizes k and $v-k$, respectively. Then the maximum number of edges of the graph may have is $k(v-k)$. By algebra, we know that the function $f(k) = k(v-k)$ achieves its maximum value when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

$\square$

Q.27 Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.
Solution: The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

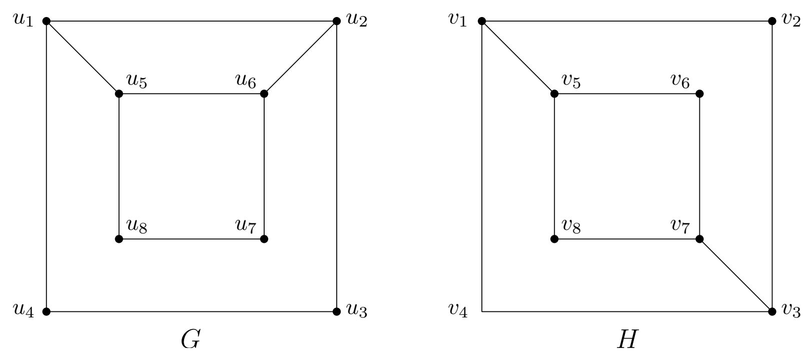

Figure 2: Q.27

□

Q.28 Show that isomorphism of simple graphs is an equivalence relation.
Solution:

G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive.

□

Q.29 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.
Solution: The isomorphism need not hold. For the simplest counterexample, let G_1, G_2 and H_1 each be the graph consisting of the single vertex v, and let H_2 be the graph consisting of the single vertex w. Then of course G_1 and H_1 are isomorphic, as are G_2 and H_2. But $G_1 \cup G_2$ is a graph with one vertex, and $H_1 \cup H_2$ is a graph with two vertices.

□

Q.30 How can the adjacency matrix of $\overline{G}$ be found from the adjacency matrix of G, where G is a simple graph?

Solution: Since a 1 in the adjacency matrix indicates the presence of an edge and a 0 the absence of an edge, to obtain the adjacency matrix for $\overline{G}$ we change each 1 in the adjacency matrix for G to a 0, and we change each 0 not on the main diagonal to a 1 (make sure that there is no loop).

□

Q.31 Show that if G is simple graph with at least 11 vertices, then either G or $\overline{G}$, the complement of G, is nonplanar.

Solution: If G is planar, then because $e \leq 3v - 6$, G has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\overline{G}$ has at most 27 edges. However, the union of G and $\overline{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$.

□

Q.32 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Solution:

As in the argument in the proof of Corollary 1, we have $2e \geq 5r$ and $r = e - v + 2$. Thus $e - v + 2 \leq 2e/5$, which implies that $e \leq (5/3)v - (10/3)$.

□

Q.33 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Solution:

(1) K_6: The diameter is clearly 1, since the maximum distance between two vertices is 1; the radius is also 1, with any vertex serving as the center.

(2) $K_{4,5}$: The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2 with any vertex serving as the center.

(3) Q_3: Vertices at diagonally opposite corners of the cube are a distance of 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

(4) C_6: Vertices at opposite corners of the hexagon are a distance 3 from each other, and this it the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Note that despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

□

Q.34 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Solution:

We draw a bipartite graph in the following way:

The vertex set contains two sets of vertices, V_1 and V_2, with each containing n vertices. For the ith vertex in V_1, it is connected to the first i vertices in the set V_2. In this way, the ith vertex in V_1 has degree i, and the ith vertex in V_2 has degree $n-i+1$, since the previous $i-1$ vertices in $V-1$ are not connected to the ith vertex in V_2 by the adjacency rule. The constructed graph is connected in an obvious way: $(1, 1, 2, 2, 3, 3, 4, 4, \ldots, n, n)$, where the first i denotes the ith vertex in V_1 and the second i denotes the ith vertex in V_2 (see the following figure).

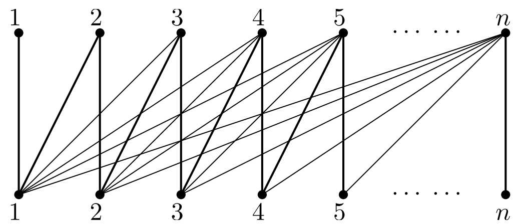

Figure 3: Q.34

□

Q.35

An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Show that every n-cube has a Hamilton circuit.

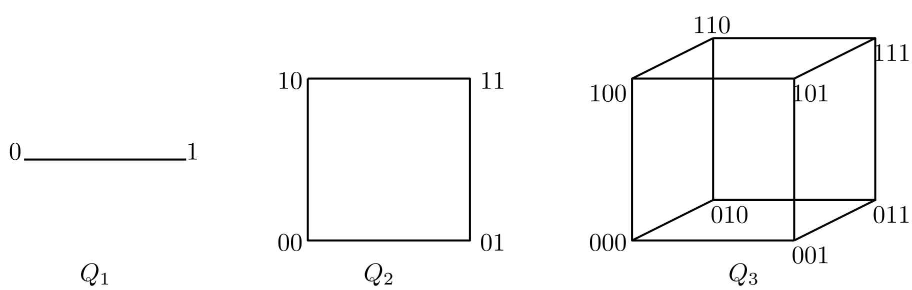

Figure 4: Q.35

Solution: We proof by induction.

If $n = 1$, we simply need to visit each vertex of a two-vertex graph with an edge connecting them.

Assume that it is true for $n = k$. To build a $(k+1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take that Hamilton circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the $(k+1)$-cube.

□

Q.36 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

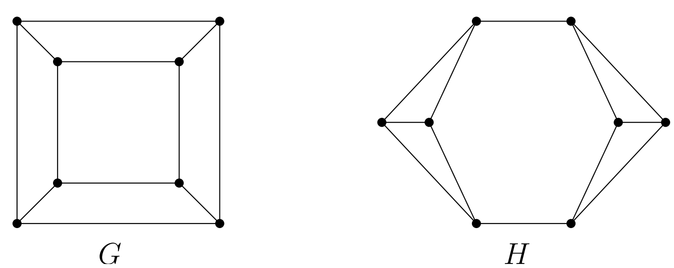

Figure 5: Q.36

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Solution:

(1) The graph G is bipartite, and the graph H is not. It is checked below that the graph G can be 2-colored, which H cannot.

(2) The two graphs are not isomorphic, since there exists a circuit of length 3 in H, which does not exist in G.

(3) Since both of the two graphs have only degree-3 vertices, neither of these two graphs has an Euler circuit.

□

Q.37 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Solution: We use vertices to denote the 17 students and edges to denote the communication among these students. In addition, we use 3 different colors to color the edges to denote the 3 problems they are discussing. For one fixed student A, A communicates with the other 16 students. By the Pigeonhole Principle, at least 6 edges are of the same color, w.l.o.g., we assume that the edges AB, AC, AD, AE, AF, AG are all of color red.

If among the six students B, C, D, E, F, G there is one edge, e.g., BC whose color is also red, then all the three edges of the triangle ABC are red.

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these five edges, so at least there are three of these five edges of the same color. W.l.o.g., assume that the three edges BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If there is one yellow edge of the triangle CDE, say, CD is yellow, then the triangle BCD is a triangle with three edges all yellow. If the triangle CDE does not have yellow edge, which means all edges of CDE are blue, we again have a triangle with three edges of the same color.

□

Q.38

The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

Solution:

The number of vertices in the tree T_n satisfies the recurrence relation $v_n = v_{n-1} + v_{n-2} + 1$ (the "+1" is for the root), with $v_1 = v_2 = 1$. Thus the sequence begins $1, 1, 3, 5, 9, 15, 25, \ldots$. It is easy to prove by induction that $v_n = 2f_n - 1$, where f_n is the n-th Fibonacci number. The number of leaves satisfies the recurrence relation $l_n = l_{n-1} + l_{n-2}$, with $l_1 = l_2 = 1$, so $l_n = f_n$. Since $i_n = v_n - l_n$, we have $i_n = f_n - 1$. Finally, it is clear that the height of the tree T_n is one more than the height of the tree T_{n-1} for $n \geq 3$, with the height of T_2 being 0. Therefore the height of T_n is $n - 2$ for all $n \geq 2$ (and of course the height of T_1 is 0).

$\square$

Q.39

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

(c) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Solution:

We exhibit the answers by showing with parentheses the operation that is applied next, working from left to right (it always involves the first occurrence of an operator symbol).

(a) $5\ (2\ 1\ -)\ -3\ 1\ 4\ +\ +\ * = (5\ 1\ -)3\ 1\ 4\ +\ +\ * = 4\ 3\ (1\ 4\ +)\ +\ * = 4\ (3\ 5\ +) * = (4\ 8\ *) = 32$

(b) $(9\ 3\ /)\ 5 + 7\ 2\ -\ * = (3\ 5\ +)\ 7\ 2\ -\ * = 8(\ 7\ 2\ -) * = (8\ 5\ *) = 40$

(c) $(3\ 2\ *)\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ - = (6\ 2\ \uparrow)\ 5\ 3\ -\ 8\ 4\ /\ *\ - = 36\ (5\ 3\ -)\ 8\ 4\ /\ *\ - = 36\ 2(8\ 4\ /)\ *\ - = 36(2\ 2\ *) - = (36\ 4\ -) = 32$

$\square$

Q.40

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

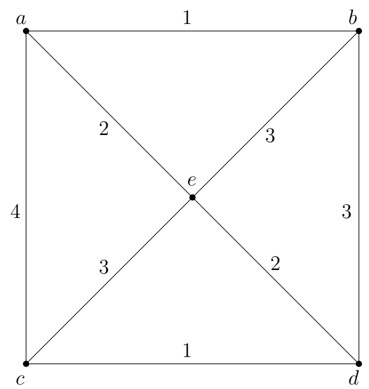

Figure 6: Q.40

Solution:

We start with the minimum weight edge $\{a, b\}$. The least weight edge incident to the tree constructed so far is edge $\{a, e\}$, with weight 2, so we add it to the tree. Next we add edge $\{d, e\}$, and then edge $\{c, d\}$. This completes the tree, whose total weight is 6.

□

Q.41

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Q.40.

Solution:

With Kruskal's algorithm, we add at each step the shortest edge and will not complete a simple circuit. Thus we pick edge $\{a, b\}$ first, and then edge $\{c, d\}$ (alphabetical order breaks ties), followed by $\{a, e\}$ and $\{d, e\}$. The total weight is 6.

□