CS201: Discrete Math for Computer Science
2022 Spring Semester Written Assignment # 6
Due: June 2th, 2022, please submit one pdf file through Sakai
Please answer questions in English. Using any other language will lead to a zero point.

Plagiarism in an Assignment or a Quiz:

- **For the first time: the score of the assignment or quiz will be zero**
- **For the second time: the score of the course will be zero**
- **When two assignments are nearly identical, the policy will apply to BOTH students, unless one confesses having copied without the knowledge of the other.**

Any late submission will lead to a zero point with no exception.

Q. 1. (5 points) Let G be a *simple* graph with n vertices.

(a) What is the *maximum* number of edges G can have?

(b) If G is connected, what is the *minimum* number of edges G can have?

(c) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Solution:

(a) $\binom{n}{2}$

(b) $n-1$

(c) We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

$\square$

Q. 2. (5 points) Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Solution:

(a) omitted.

(b) The degree of each vertex is $\binom{n-2}{2}$.

$\square$

Q. 3. (5 points) The complementary graph $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in G if and only if they are not adjacent in G. A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Solution: If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of edges in G must be $n(n-1)/4$. Since this number must be an integer, n may be congruent to either 0 or 1 modulo 4.

$\square$

Q. 4. (10 points) Suppose that G is a graph on a finite set of n vertices. Prove the following:

(a) If every vertex of G has degree 2, then G contains a cycle.

(b) If G is disconnected, then its complement $\bar{G}$ is connected.

Solution:

(a) Assume for contradiction that G has no cycle, and consider the longest path P in G (one must exist, since the graph is finite). Let v be the final vertex in P. Since v has degree 2, it must have two edges e_1 and e_2 incident on it, of which one, say e_1, is the last edge of the path P. Then e_2, cannot be incident on any other vertex of P since that would create a cycle $(v, e_2, [\text{section of } P \text{ ending in } e_1], v)$. So the other endpoint of e_2 is not part of P, and hence e_2 can be appended to P to give a strictly longer path, which contradicts our choice of P. Hence, G must contain a cycle.

(b) Let $\overline{G}$ denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G, then no edge of G connects them, so there will be an edge $\{u, v\}$ in $\overline{G}$. If u and v are in the same connected component in G, then consider any vertex w in a different connected component (since G is disconnected, there must be at least one other connected component). By our first argument, the edges $\{u, w\}$ and $\{v, w\}$ exist in $\overline{G}$, so u and v are connected by the path (u, w, v). Hence, any two vertices are connected in $\overline{G}$, so the whole graph is connected.

$\square$

Q. 5. (5 points) Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Solution:

(1) If $x \in N(A \cup B)$, then x is adjacent to some vertex $v \in A \cup B$. W.l.o.g., suppose that $v \in A$. Then $x \in N(A)$ and therefore also in $N(A) \cup N(B)$. Conversely, if $x \in N(A) \cup N(B)$, then w.l.o.g. suppose that $x \in N(A)$. Thus, x is adjacent to some vertex $x \in A \subseteq A \cup B$, so $x \in N(A \cup B)$.

(2) If $x \in N(A \cap B)$, then x is adjacent to some vertex $v \in A \cap B$. Since both $v \in A$ and $v \in B$, it follows that $x \in N(A)$ and $x \in N(B)$, whence $x \in N(A) \cap N(B)$. For the counterexample, let $G = (\{u, v, w\}, \{\{u, v\}, \{v, w\}\})$, $A = \{u\}$, and $B = \{w\}$.

□

Q. 6. (5 points) Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution: Suppose that the parts are of sizes k and $v-k$, respectively. Then the maximum number of edges of the graph may have is $k(v-k)$. By algebra, we know that the function $f(k) = k(v-k)$ achieves its maximum value when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

□

Q. 7. (10 points) Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\mathrm{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. That is, $\max_{u,v\in V} d_G(u, v)$. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v of G is defined as $\max_{u\in V} d_G(u, v)$. Finally, the *radius* $\mathrm{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v\in V} \mathrm{ecc}(v)$. Prove the following.

(a) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

(b) For every positive integer n, there are connected graphs G_1 and G_2 with $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$ and $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$.

Solution:

(a) As $\mathrm{rad}(G) = \min_{v\in V}[\max_{u\in V} d_G(u, v)]$, obviously $\mathrm{rad}(G) \leq \mathrm{diam}(G)$.

Now suppose that $\mathrm{diam}(G)$ goes from vertices v_1 to v_2, where $v_1, v_2 \in V$. Recall that $\mathrm{rad}(G) = \min_{v\in V} \mathrm{ecc}(v)$. Let the chosen v for minimal eccentricity be v^*. Note that $\mathrm{diam}(G) \leq d_G(v^*, v_1) + d_G(v^*, v_2)$. Since $\mathrm{diam}(G)$ is the shortest path from v_1 to v_2, any other path from v_1 to v_2 is either as long or longer. Also note that $d_G(v^*, v_1) + d_G(v^*, v_2) \leq \mathrm{rad}(G) + \mathrm{rad}(G) = 2\mathrm{rad}(G)$ since $\mathrm{rad}(G)$ is the maximum distance of any other vertex from v^* in G. Hence, we have

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G).$$

(b) Consider a cycle with $2n$ or $2n + 1$ vertices. This will always have $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$. For $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$, consider a line graph with $2n + 1$ vertices.

□

Q. 8. (5 points) Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

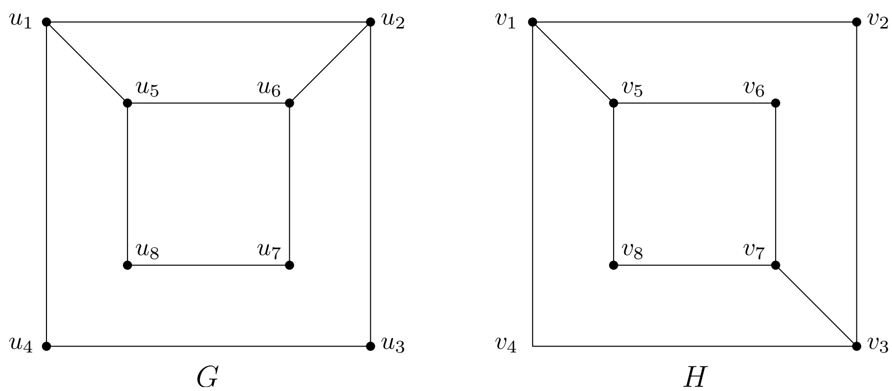


Figure 1: Q.19

Solution: The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

□

Q. 9. (10 points) Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

Solution: First, suppose that the directed multigraph has an Euler circuit. Since this circuit provides a path from every vertex to every other vertex, the graph must be strongly connected (and hence also weakly connected). Also, we can count the in-degree and out-degree of the vertices by following this circuit; as the circuit passes through a vertex, it adds to the count of both the in-degree (as it comes in) and the out-degree (as it leaves). Therefore the two degrees are equal for each vertex.

Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem we talked about in class and construct an Euler circuit.

□

Q. 10. (5 points) Which of the these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph? a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

Solution: a) and c).

□

Q. 11. (5 points) Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Solution:

If both m and n are at least 2, then clearly there is a simple circuit of length 4 in $K_{m,n}$. On the other hand, $K_{m,1}$ is cleraly a tree (as is $K_{1,n}$). Thus we conclude that $K_{m,n}$ is a tree if and only if $m = 1$ or $n = 1$.

□

Q. 12. (10 points) An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Show that every n-cube ($n \geq 2$) has a Hamilton circuit.

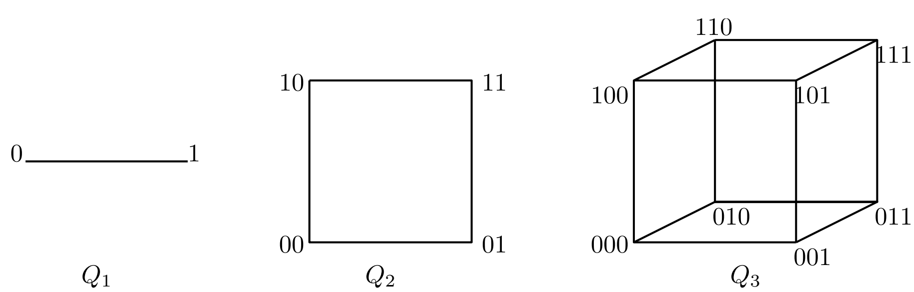
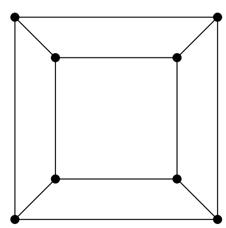


Figure 2: Q.36

Solution: We proof by induction. If $n = 2$, then there is a Hamilton circuit, i.e., 10, 00, 01, 11, 10. Assume that it is true for $n = k$. To build a $(k+1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take that Hamilton circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the $(k+1)$-cube.

$\square$

Q. 13. (5 points) Consider the two graphs G and H. Answer the following three questions, and explain your answers.



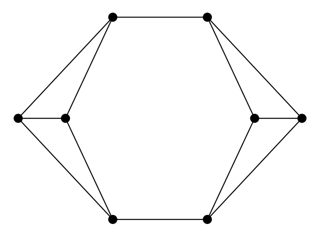


Figure 3: Q.37

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Solution:

(1) The graph G is bipartite, and the graph H is not. It is checked below that the graph G can be 2-colored, which H cannot.

(2) The two graphs are not isomorphic, since there exists a circuit of length 3 in H, which does not exist in G.

(3) Since both of the two graphs have only degree-3 vertices, neither of these two graphs has an Euler circuit.

□

Q. 14. (5 points) There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Solution: We use vertices to denote the 17 students and edges to denote the communication among these students. In addition, we use 3 different colors to color the edges to denote the 3 problems they are discussing. For one fixed student A, A communicates with the other 16 students. By the Pigeonhole Principle, at least 6 edges are of the same color, w.l.o.g., we assume that the edges AB, AC, AD, AE, AF, AG are all of color red.

If among the six students B, C, D, E, F, G there is one edge, e.g., BC whose color is also red, then all the three edges of the triangle ABC are red.

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these five edges, so at least there are three of these five edges of the same color. W.l.o.g., assume that the three edges BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If there is one yellow edge of the triangle CDE, say, CD is yellow, then the triangle BCD is a triangle with three edges all yellow. If the triangle CDE does not have yellow edge, which means all edges of CDE are blue, we again have a triangle with three edges of the same color.

□

Q. 15. (5 points) How many different spanning trees does each of these simple graphs have? a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Solution:

a) 3 b) 16 c) 4 d) 5

□

Q. 16. (5 points) How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

Solution:

a) 1 b) 2 c) 3

□