RGPV SOLUTION OF BT-1002-MATHEMATICS-1-CBGS-DEC-2017

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1. (a) Expand $\sin^{-1} x$ is power of x by Maciaurin's theorem.

Solution : Suppose $y = f(x) = \sin^{-1} x \quad \Rightarrow \quad y_0 = 0$

Differentiate successively w.r.l., x we get

$$y_1 = \frac{1}{\sqrt{1-x^2}} = \left[1-x^2\right]^{-1/2}$$

$$\Rightarrow \quad y_1 = 1 + \frac{1}{2\underline{|1}}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{\underline{|1}}x^4 + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{\underline{|3}}x^6 + ...$$

$$\left[\Theta\ (1-x)^{-n} = 1 + \frac{n}{\underline{|1}}x + \frac{n(n+1)}{\underline{|2}}x^2 + \frac{n(n+1)(n+2)}{\underline{|3}}x^3 + ...\right]$$

$$\Rightarrow \quad y_1 = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{5}{16}x^6 + ... \quad \Rightarrow \quad (y_1)_0 = 1$$

and $\quad y_2 = x + \frac{3}{2}x^3 + \frac{15}{8}x^5 + ... \quad \Rightarrow \quad (y_2)_0 = 0$

$$y_3 = 1 + \frac{9}{2}x^2 + \frac{75}{8}x^4 + ... \quad \Rightarrow \quad (y_3)_0 = 1$$

$$y_4 = 9x + \frac{75}{2}x^3 + ... \quad \Rightarrow \quad (y_4)_0 = 0$$

$$y_5 = 9 + \frac{225}{2}x^2 + ... \quad \Rightarrow \quad (y_5)_0 = 9$$

By Maclurin's theorem, we have

$$y = y_0 + \frac{x}{\underline{|1}}(y_1)_0 + \frac{x^2}{\underline{|2}}(y_2)_0 + \frac{x^3}{\underline{|3}}(y_3)_0 + \frac{x^4}{\underline{|4}}(y_4)_0 + \frac{x^5}{\underline{|5}}(y_5)_0 + ...$$

Putting the values we get

$$\sin^{-1} x = 0 + \frac{x}{\underline{|1}}(1) + \frac{x^2}{\underline{|2}}(0) + \frac{x^3}{\underline{|3}}(1) + \frac{x^4}{\underline{|4}}(0) + \frac{x^5}{\underline{|5}}(9) + ...$$

$$\Rightarrow \quad \boxed{\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + ...} \qquad \text{Answer}$$

b) Show that the curvature of a circle is constant and is equal to the reciprocal of its radius.

Solution : Suppose the equation of circle with centre origin and radius a is,

$$x^2 + y^2 = a^2 \qquad \ldots\ldots(1)$$

Differentiate w.r.t., x on both sides, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{x}{y} \qquad \ldots\ldots\ldots(2)$$

and $$\frac{d^2y}{dx^2} = -\left[\frac{y.1 - x\frac{dy}{dx}}{y^2}\right] = -\left[\frac{y.1 - x\left(-\frac{x}{y}\right)}{y^2}\right]$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = -\frac{1}{y^3}\left[y^2 + x^2\right] = -\frac{x^2}{y^3} \qquad \text{[From (1)]}$$

Since the radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(-\frac{x}{y}\right)^2\right]^{3/2}}{-\frac{a^2}{y^3}}$$

$$\Rightarrow \qquad = -\frac{\left[y^2 + x^2\right]^{3/2}}{a^2y^3}xy^3 = -\frac{\left[a^2\right]^{3/2}}{a^2} \qquad \text{[From (1)]}$$

$$\Rightarrow \qquad \rho = -a = a \qquad \text{[Since a>0]}$$

$$\therefore \qquad Curvature = \frac{1}{\rho} = \frac{1}{a}$$ **Hence Proved**

2. (a) Write statement of Rolle's and Lagrange's theorem and explain their geometrical meaning.

Solution : 1. Rolle's theorem

If $f(x)$ be a real valued function of x such that

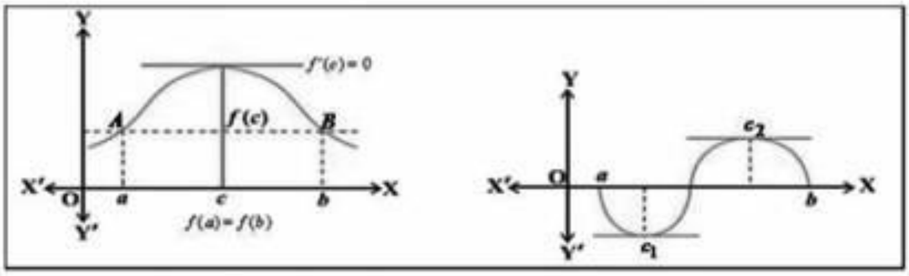
1. $f(x) = f(b)$

2. $f(x)$ is a continuous function in the closed interval $[a,b] i.e. a \le x \le b$

3. $f(x)$ is differentiable in the open interval $(a,b) i.e., a < x < b$

Then there exit at least one real value of c ((a, b) such that

$f'(c) = 0$



Rolle's Theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to X-axis, abscissa of the point lying in (a,b).

2. Lagrange's Mean value theorem (L.M.V.T.)

If $f(x)$ be real valued function of x such that

1. $f(a) \neq f(b)$

2. $f(x)$ is continuous function in the closed interval *[a, b] i.e.* $a \leq x \leq b$

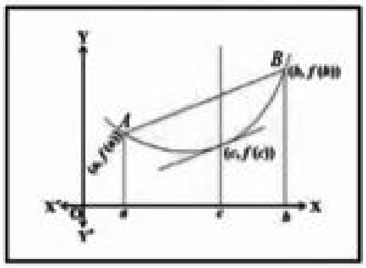
3. $f(x)$ is differentiable in the open interval *(a, b) i.e.,* $a \leq x \leq b$.

Then there exit at least one real value c $\in (a,b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

GEOMETRICAL INTERPRETATION OF LAGRANGE'S MEAN VALUE THEOREM

The MVT states that there is a point c in (a, b) such that the slope of the tangent at $(c, f(c))$ is same as the slope of the secant between $(a, f(a))$ and $(b, f(b))$. In other words, there is a point c in (a, b) such that the tangent at $(c, f(c))$ is parallel to the secant between $(a, f(a))$ and $(b, f(b))$.



(b) Discuss the maximum and minimum of $x^3 + y^3 - 3xy$

Solution : Suppose $u = x^3 + y^3 - 3xy$

Partially differentiate w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3y \quad \text{........ (1)}$$

and $$\frac{\partial u}{\partial y} = 3y^2 - 3x \quad \text{..........(2)}$$

Taking, $$\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 - 3y = 0 \; i.e. x^2 = y \quad \text{..........(3)}$$

and $$\frac{\partial u}{\partial y} = 0 \Rightarrow 3y^2 - 3x = 0 \; i.e. y^2 = x \quad \text{..........(4)}$$

Squaring both sides of equation (4), we get

$$y^4 = x^2$$

$$\Rightarrow \quad y^4 = y \qquad \text{[From (3)]}$$

$$\Rightarrow \quad y(y^3 - 1) = 0 \; i.e. \; y(y-1)(y^2 + y + 1) = 0$$

$$\Rightarrow \quad y = 0, 1$$

Putting the values in equation (3), we get

$$x^2 = 1 \Rightarrow x = 1, -1 \text{ and } x = 0$$

The require stationary points are (1, 1), (1-1) and (0, 0)

Again equation (1) partially differentiate w.r.t x and y respectively, we get

$$r = \frac{\partial^2 u}{\partial x^2} = 6x \text{ and } s = \frac{\partial^2 u}{\partial y \partial x} = -3$$

Equation (2) partially differentiate w.r.t. y, we get

$$t = \frac{\partial^2 u}{\partial y^2 = 6y}$$

Case 1 : at (1, 1)

$r = 6 > 0, s = -3$ and $t = 6$

$$\therefore \quad rt = s^2 = (6)\,(6) - (-3)^2 = 36 - 9 = 27 > 0$$

Therefore, u is minimum at (1, 1) and minimum value is $M_{\min}(1,1) = 1 + 1 - 3 = -1$

Case 2: at (– 1, 1)

$r = -6 > 0, s = -3$ and $t = 6$

$$\therefore \quad rt = s^2 = (-6)\,(6) - (-3)^2 = 36 - 9 = -45$$

Therefore function neither maximum nor minimum at (-1, 1) and point is called saddle point.

Case 3 : at (0, 0)

$r = 0, s = -3$ and $t = 0$

$$\therefore \quad rt = s^2 = (0)\,(0) - (-3)^2 = 0 - 9 = -9$$

Therefore function neither maximum nor minimum at (0, 0) and point is called saddle point.

3. (a) If $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$**, then show that** $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$

Solution : Given $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$

$\Rightarrow$ $$\tan u = \frac{x^2+y^2}{x-y}$$

Suppose $z = \tan u$

$\therefore$ **t-test:** $z(xt, yt = \dfrac{(t\,x)^2+(t\,y)^2}{(t\,x)-(t\,y)} = t\left(\dfrac{x^2+y^2}{x-y}\right) = t^1 z(x,y)$

Therefore, z be homogeneous function in x and y with degree 1, then by Euler theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z$$

$\Rightarrow$ $$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$$

$\Rightarrow$ $$\sec^2 u\left[d\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \tan u$$

$\Rightarrow$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}\times\cos^2 u = \frac{1}{2}\sin 2u$$

Thus, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$$ **Answer**

(b) Find the percentage error in the area of an ellipse if 1% error is made in measuring the major and minor axis.

Solution : Suppose a and b length of major and minor axes of an ellipse respectively, then

Given, $E_p(a) = 1$ and $E_p(b) = 1$

Since Area of Ellipse $A = \pi\, ab$

Taking log on both sides, we get

$$\log A = \log 2\pi + \log a + \log b$$

Differentiating on both sides, we get

$$\frac{\delta A}{A} = 0 + \frac{\delta a}{a} + \frac{\delta b}{b}$$

$\Rightarrow$ $$\frac{\delta A}{A}\times 100\% = \frac{\delta a}{a}\times 100\% + \frac{\delta b}{b}\times 100\%$$

$\Rightarrow$ $$E_p(A) = E_p(a) + E_p(b)$$

$\therefore \qquad E_p(A) = 1 + 1 = 2$

Thus, Percentage error in area of an ellipse is 2% **Answer**

4. (a) Evaluate $\int_a^b x\,dx$ from the definition of integral as limit of sum.

Solution : Suppose $f(d) = x$ and $nh = b - a$

We know that by definition of definite integral as limit of sum,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(x) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h]$$

$$\Rightarrow \qquad \int_a^b x\,dx = \lim_{h\to 0} h\,[a + (a+h) + (a+2h) + ...(a+(n-1)h]$$

$$\Rightarrow \qquad = \lim_{h\to 0} h\left[na + h\left(1 + 2 + 3 + ...\overline{n-1}\right)\right]$$

$$\Rightarrow \qquad = \lim_{h\to 0} h\left[na + h\frac{(n-1)n}{2}\right] = \lim_{h\to 0}\left[(nh)a + \frac{(nh-h)nh}{2}\right]$$

$$\Rightarrow \qquad = \lim_{h\to 0}\left[(b-a)a + \frac{(b-a-b)(b-a)}{2}\right] = (b-a)\left[a + \frac{b-a-0}{2}\right]$$

$$\Rightarrow \qquad = (b-a)\left(\frac{b+a}{2}\right) = \frac{b^2-a^2}{2}$$ **Answer**

Thus $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ **Answer**

(b) Evaluate the Limit $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$

Solution : Suppose

$$I = \lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$$

$$\Rightarrow \qquad = \lim_{n\to\infty}\left[\frac{1,2,3.......n}{n.n.n.....n}\right]^{1/n}$$

$$\Rightarrow \qquad = \lim_{n\to\infty}\left[\left(\frac{1}{n}\right).\left(\frac{2}{n}\right).\left(\frac{3}{n}\right).....\left(\frac{n}{n}\right)\right]^{1/n}$$

Taking log, on both sides, we get

$$\Rightarrow \qquad \log I = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \log\left(\frac{3}{n}\right) + ... + \log\left(\frac{n}{n}\right)\right]$$

$$\Rightarrow \qquad \log I = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(\frac{r}{n}\right) \qquad [r^{th} term]$$

Upper Limit : $\lim_{n\to\infty}\left[\frac{r}{n}\right]_{r=n} = \lim_{n\to\infty}\frac{n}{n} = 1$

Lower Limit : $\lim_{n\to\infty}\left[\frac{r}{n}\right]_{r=1} = 0\,(Fixed)$

By Summation of series, we get

$$\log I = \int_0^1 \log x dx$$

$$\Rightarrow \qquad = [\log x(d)]_0^1 - \int_0^1 \frac{1}{x}\times x dx$$

$$\Rightarrow \qquad = [0-1] - \int_0^1 1\,dx = -[x]_0^1 = -1$$

$$\Rightarrow \qquad \log I = -1$$

$$\Rightarrow \qquad I = e^{-1} = \frac{1}{e}$$

Hence $\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = \frac{1}{e}$ **Answer**

5. (a) Evaluate : $\iint_R e^{2x+3y} dx\,dy$, **where R is a triangle bounded by x=0, y = 0, and x+y=1**

Solution : Given: $I = \iint_R e^{2x+3y} dx\,dy$

Since Region R is bounded by x = 0, y = 0, and x + y =1, then the limits are

$$0 \le x \le 1 \text{ and } 0 \le y \le 1-x$$

$$\therefore \qquad I = \int_0^1\int_0^{1-x} e^{2x+3y} dx\,dy$$

$$\Rightarrow \qquad = \int_0^1 e^{2x}\left[\int_0^{1-x} e^{3y} dy\right]dx$$

$$\Rightarrow \qquad = \int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x} dx = \frac{1}{3}\int_0^1 e^{2x}\left[e^{3(1-x)} - 1\right]dx$$

$$\Rightarrow \qquad = \frac{1}{3}\int_0^1 \left[e^{3-x} - e^{2x}\right]dx = \frac{1}{3}\left[\frac{e^{3-x}}{-1} - \frac{e^{2x}}{2}\right]_0^1$$

$$\Rightarrow \qquad = \frac{1}{3}\left[\left\{-e^2 - \frac{e^2}{2}\right\}\right] - \left[\left\{-e^3 - \frac{1}{2}\right\}\right] = \frac{1}{3}\left[\left\{-\frac{3e^2}{2} + e^3 + \frac{1}{2}\right\}\right]$$

Thus, $\iint_R e^{2x+3y}dx\,dy = \frac{1}{3}\left[-\frac{3e^2}{2}+e^3+\frac{1}{2}\right]$ **Answer**

(b) Evaluate : $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dx\,dy\,dz$

Solution: Given : $I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}dx\,dy\,dz$

$$= \int_0^1\int_0^1 e^{x+y}\left[\int_0^1 e^z dz\right]dx\,dy$$

$\Rightarrow$ $= \int_0^1\int_0^1 e^{x+y}\left[e^z\right]_0^1 dx\,dy = \int_0^1\int_0^1 e^{x+y}\left[e-1\right]dx\,dy$

$\Rightarrow$ $= (e-1)\int_0^1 e^x\left[\int_0^1 e^y dy\right]dx = (e-1)\int_0^1 e^x\left[e^y\right]_0^1 dx$

$\Rightarrow$ $= (e-1)\int_0^1 e^x\left[e-1\right]dx$

$\Rightarrow$ $= (e-1)^2\int_0^1 e^x dx = (e-1)^2\left[e^x\right]_0^1 = (e-1)^2(e-1)$

$\Rightarrow$ $= (e-1)^3$

Thus, $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dx\,dy\,dz = (e-1)^3$ **Answer**

6. (a) By triple integration determine the volume of a hemisphere of radius 'a'.

Solution : Suppose the equation sphere of centre at origin and radius a is $x^2+y^2+z^2=a^2$.

Therefore the volume of sphere is

$$V = \iiint_v 1.dv$$

Using the Spherical coordinate system,

$$x = r\cos\theta\sin\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \text{ and } dv = r^2\sin\theta\,dr\,d\phi\,d\theta$$

The limits are $0 \le r \le a, 0 \le \phi 2\pi \; and \; 0 \le \theta \le \pi$

$\therefore$ $V = \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2\sin\theta\,dr\,d\phi\,d\theta$

$\Rightarrow$ $= \int_{r=0}^{a}\int_{\phi=0}^{2\pi} r^2\left[\int_{\theta=0}^{\pi}\sin\theta\,dr\,\theta\right]dr\,d\phi$

$\Rightarrow$ $= \int_{r=0}^{a}\int_{\phi=0}^{2\pi} r^2\left[-\cos\theta\right]_d^{\pi} dr\,d\phi$

$$\Rightarrow \qquad = \int_{r=0}^{a}\int_{\phi=0}^{2x} r^2[\cos\pi - \cos 0]\,dr\,d\phi = -\int_{r=0}^{a}\int_{\phi=0}^{2x} r^2[-1-1]dr\,d\phi$$

$$\Rightarrow \qquad = 2\left[\frac{r^3}{3}\right]_0^a \times [\phi]_0^{2\pi} = 2\left[\frac{a^3}{3}\right]\times[2\pi - 0] = \frac{4\pi a^3}{3}$$

Now, Volume of Hemisphere = $\frac{V}{2} = \frac{2\pi a^3}{3}$ **Answer**

(b) Evaluate : $\int_0^2\int_0^3 (x^2+y^2)dx\,dy$

Solution : $\int_0^2\int_0^3 (x^2+y^2)dx\,dy = \int_0^2\left[x^2y + x\left(\frac{y^3}{3}\right)\right]_0^3 dx$

$$= \int_0^2 [(3x^2+9x)-(0+0)]dx = 3\int_0^2 (x^2+3x)dx$$

$$= 3\left[\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^2 = 3\left[\frac{8}{3}+6\right] = 26$$

Thus $\int_0^2\int_0^3 (x^2+y^2)\,dx\,dy = 26$ **Answer**

7. (a) Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Beta functions and hence evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$

Solution : Given : $I = \int_0^1 x^m(1-x^n)^p\,dx$ ……….(1)

Putting, $x^n = t \Rightarrow x = t^{1/n}$

$$dx = \frac{t^{\left(\frac{1}{n}-1\right)}}{n}dt$$

$\therefore$ From equation (1), we get

$$I = \int_0^1 t^{\frac{m}{n}}(1-t)^p\,\frac{1}{n}t^{\left(\frac{1}{n}-1\right)}dt$$

$$\Rightarrow \qquad = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}}(1-t)^p\,dt$$

$$\Rightarrow \qquad I = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{p+1-1}dt$$

$$\Rightarrow \qquad I = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right) \qquad \text{……… (2)}$$

Putting, $m = 5, n = 3$ and $p = 10,$ we get

$$\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{3}\beta\left(\frac{5+1}{3},10+1\right)$$

$$= \frac{1}{3}\beta(2,11) = \frac{1}{3}\frac{|\underline{2}\,|\overline{11}}{|\overline{(2+11)}}$$

$$\Rightarrow \qquad = \frac{1}{3}\frac{|\underline{1}\ \ |\underline{10}}{|\underline{12}} = \frac{1\ |\underline{10}}{3\cdot 12\cdot 11\ |\underline{10}} = \frac{1}{396}$$

Thus, $\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{396}$ **Answer**

(b) Prove that : $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}, m>0$ and $n>0$ **7**

Solution: We know that if z is independent of x, then

$$\Gamma n = z^n \int_0^\infty e^{-1zx} x^{n-1} dx \qquad \ldots\ldots\ldots\ldots(1)$$

Where x is independent of z and vice versa.

Equation (1), both sides multiplying by $z^{m-1}e^{-z}$ and integrating w.r.t. z from 0 to ∞, then we get

$$\int_0^\infty \Gamma n\left(z^{m-1}e^{-z}\right)dz = \int_0^\infty \left[\left(z^{m-1}e^{-z}\right)z^n \int_0^\infty e^{-zx}x^{n-1}dx\right]dx$$

$$\Rightarrow \qquad \Gamma n \int_0^\infty z^{m-1}e^{-z}dz = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-z-zx}z^{m+n-1}dz\right]dx \qquad \left[\Theta \quad \Gamma n = \int_0^\infty e^{-x}x^{n-1}dx\right]$$

$$\Rightarrow \qquad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-z(1+x)}x^{m+1-1}dx\right]dx$$

Putting $\quad y = z(1+x) \quad \Rightarrow \quad z = \frac{y}{1+x}$ $i.e. dx = \frac{dy}{1+x}$

$$\therefore \qquad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left[\int_0^\infty e^{-y}\left(\frac{y}{1+x}\right)^{m+n-1}\frac{dy}{1+x}\right]dx$$

$$\Rightarrow \qquad \Gamma n\,\Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\int_0^\infty e^{-y}y^{m+n-1}dy\right]dx$$

$$\Rightarrow \qquad \Gamma n\,\Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\Gamma(m+n)\right]dx \qquad \left[\Theta \quad \Gamma n = \int_0^\infty e^{-x}x^{n-1}dx\right]$$

$$\Rightarrow \qquad = \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \Gamma(m+n)\,\beta(m,n)$$

Hence, $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$ **Proved**

8. (a) Find the equation of the tangent and normal at the point *(at²,* *2at)* on the parabola *y² = 4ax*.

Solution: Given the equation of parabola is

$$y^2 = 4ax \qquad \text{……………(1)}$$

Differentiate w.r.t., x we get

$$2y\frac{dy}{dx} = 4d$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t} \qquad \textit{at } (ar^2, 2at)$$

Equation of Tangent:

The equation of tangent at $(at^2, 2at)$ is

$$y - 2at = \frac{dy}{dx}\left(x - at^2\right)$$

$$\Rightarrow \qquad y - 2at = \frac{1}{t}\left(x - at^2\right)$$

$$\Rightarrow \qquad yt - 2at^2 = x - at^2$$

$$\Rightarrow \qquad yt - x = at^2 \qquad \textbf{Answer}$$

Equation of normal:

The equation of normal at $(ar^2, 2at)$ is

$$y - 2at = \frac{1}{dx/dx}\left(x - at^2\right)$$

$$\Rightarrow \qquad y - 2at = -t\left(x - at^2\right)$$

$$\Rightarrow \qquad y - 2at = -t\,x - at^3$$

$$\Rightarrow \qquad y + xt = at^3 + 2at \qquad \textbf{Answer}$$

(b) Define Gamma Function and prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Solution : Define gamma function:

If n > 0, then gamma function of n is denoted by Γn and it is defined as

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$$

Such that $\Gamma 0=\infty$

Derivation of $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$:

We know that

$$\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m,n)}$$

$$\Rightarrow \qquad 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta=\frac{\Gamma m\Gamma n}{\Gamma(m+n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Taking, $2m-1=0 \Rightarrow m=\frac{1}{2}$

and $2n-1=0 \Rightarrow n=\frac{1}{2}$

Putting in equation (1), we get

$$2\int_0^{\pi/2}\sin^0\theta\cos^0\theta\, d\,\theta=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\frac{\left[\Gamma\frac{1}{2}\right]^2}{\Gamma 1}$$

$$\Rightarrow \qquad 2\int_0^{\pi/2} 1.d\theta=\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{1}$$

$$\Rightarrow \qquad 2[\theta]_0^{\pi/2}=\left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$$\Rightarrow \qquad 2\left[\frac{\pi}{2}-0\right]=\left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$$\Rightarrow \qquad \left[\Gamma\left(\frac{1}{2}\right)\right]=\sqrt{\pi}$$ **Proved**