RGPV SOLUTION BE-3001 (CE-TX) MATHEMATICS-3 JUN 2018

1. a) Obtain the Fourier series for the function: $f(x)=x$ in the interval $(-\pi,\pi)$.

Solution : Given: $f(x)=x,\ \ \pi<x<\pi$ …(1)

Here, $2L=\pi-(-\pi) i.e. 2L=2\pi\ \Rightarrow L=\pi$

Suppose the Fourier series of $f(x)$ with period $2L$ is,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad [\text{Since } L=\pi\,] \qquad \ldots(2)$$

Now, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}xdx$

$\Rightarrow \quad =0$ [Since x = Odd]

$\Rightarrow \quad \boxed{a_0=0}$

and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nxdx$

$\Rightarrow \quad =0$ [$x\cos nx=\text{odd}$]

$\Rightarrow \quad \boxed{a_n=0}$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nxdx$

$\Rightarrow \quad =\frac{2}{\pi}\int_0^{\pi}x\sin nxdx$ [$x\sin nx=\text{Even}$]

$$\Rightarrow \quad =\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-2x\left(=\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$\Rightarrow \quad b_n=\frac{2}{\pi}\left[\left\{-\frac{\pi(-1)^n}{n}-0\right\}-\{0-0-0\}\right]=-\frac{2(-1)^n}{n}$$

Putting in equation (1), we get

$$f(x)=0+0-\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx$$

$$\Rightarrow \quad \boxed{f(x)=2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}...\right]}$$

b) Obtain half range sine series for e^x in the interval $0<x<l$

Solution : Given : $f(x)=e^x;\quad 0<x<l$

Here $L=l$

Suppose the Half range cosine series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x)=\sum^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad \ldots(1) \qquad [\text{Since } L=I\,]$$

Now $\quad b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx=\frac{1}{l}\int_0^l e^x \sin\left(\frac{n\pi x}{l}\right)dx$

$$\Rightarrow \quad =\frac{2}{l}\left[\frac{e^x}{1^2+\frac{n^2\pi^2}{l^2}}\left\{1.\sin\left(\frac{n\pi x}{l}\right)-\frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right\}\right]_0^l$$

$$\Rightarrow \quad =\frac{2l}{n^2\pi^2+l^2}\left[\left\{e^l\left(0-\frac{n\pi(-1)^n}{l}\right)\right\}-\left\{1\left(0-\frac{n\pi}{l}\right)\right\}\right]$$

$$\Rightarrow \quad =\frac{2l}{n^2\pi^2+l^2}\times\frac{n\pi}{l}\left[1-(-1)^n\times e^l\right]$$

$$\therefore \quad \boxed{b_n=\frac{2n\pi}{n^2\pi^2+l^2}\left[1-(-1)^n\times e^l\right]}$$

Putting the values in equation (1), we get

$$\boxed{f(x)=\sum_{n=2}^{\infty}\frac{2n\pi}{n^2\pi^2+l^2}\left[1-(-1)^2\times e^l\right]\sin\left(\frac{n\pi x}{l}\right)} \qquad \textbf{Answer}$$

2. a) Find the Fourier transform of $F(x)=\begin{cases}1 & ; \quad |x|<a\\ 0 & ; \quad |x|<a\end{cases}$

Solution : Given the function $F(x)=\begin{cases}1 & ; \quad -a<x<a\\ 0 & ; \quad |x|>a\end{cases}$ $\qquad \ldots(1)$

The Fourier transform of a function $F(x)$ is given by

$$f(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{ipx}dx$$

$$\Rightarrow \quad f(p)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}1.e^{ipx}dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$\Rightarrow \quad =\frac{1}{\sqrt{2\pi}}\left(\frac{2}{p}\right)\left[\frac{e^{ipa}-e^{-ipa}}{2i}\right]=\sqrt{\frac{2}{\pi}}\frac{\sin ap}{p}$$

Thus, $\boxed{f(p)=\sqrt{\frac{2}{\pi}}\frac{\sin ap}{p}}$

b) Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$.

Solution : Given, $f(x) = \frac{e^{-ax}}{x}$

By Fourier sine Transform,

$$F\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$\Rightarrow \quad F\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x} \right) \sin sx \, dx = I \qquad \ldots (1)$$

Differentiate w.r.t. s, on both sides, we get

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \left[\int_0^\infty \left(\frac{d^{-ax}}{x} \right) \sin sx \, dx \right]$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x} \right) \frac{\partial}{\partial s} (\sin sx) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x} \right) (x \cos sx) dx$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(-a)^2 + s^2} \{-a \cos sx + s \sin sx\} \right]_0^\infty$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \left(\frac{1}{s^2 + a^2} \right) [\{0\} - \{-a + 0\}] = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2 + a^2} \right)$$

Integrating both sides, w.r.t.s, we get

$$I = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \left(\frac{s}{a} \right) \right] + c \qquad \ldots (2)$$

For the initial condition, putting $s = 0$, then $c = 0$

$\therefore$ from (2), we have

$$I = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \left(\frac{x}{a} \right) \right] \Rightarrow F\{f(x)\} = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \left(\frac{x}{a} \right) \right]$$

$$\Rightarrow \quad \boxed{F(s) = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \left(\frac{x}{a} \right) \right]} \qquad \text{From (1)}$$

3. a) Find the Laplace Transform of the following functions:

(i). $6 \sin 2t - 5 \cos 2t$ **(ii).** $\frac{e^{at} - 1}{a}$

Solution : (i). $L\{6 \sin 2t - 5 \cos 2t\} = 6L\{\sin 2t\} - 5L\{\cos 2t\}$

$$= 6 \left[\frac{2}{p^2 + 4} \right] - 5 \left[\frac{p}{p^2 + 4} \right] = \frac{12 - 5p}{p^2 + 4}$$ **Answer**

(ii). $L\left\{\dfrac{e^{at}-1}{a}\right\}=\dfrac{1}{a}L\{e^{at}-1\}$

$$=\frac{1}{a}\left[\frac{1}{p-a}-\frac{1}{p}\right]=\frac{p-p+1}{ap(p-a)}$$

$$=\frac{1}{ap(p-a)}$$ **Answer**

b) Find inverse Laplace transform of the following functions:

(i). $\dfrac{1}{s^2-6s+10}$ **(ii).** $\dfrac{3s-2}{s^2-4s+20}$

Solution : (i). $L^{-1}\left\{\dfrac{1}{s^2-6s+10}\right\}=L^{-1}\left\{\dfrac{1}{(s-3)^2+1}\right\}$ [By First Shifting theorem]

$$=e^{3t}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=e^{3t}\sin t$$ **Answer**

(ii). $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\}=L^{-1}\left\{\dfrac{3(s-2)+4}{(s-2)^2+16}\right\}$ [By First Shifting theorem]

$$=e^{2t}L^{-1}\left\{\frac{3s+4}{s^2+16}\right\}$$

$$=e^{2t}\left[3\cos 4t+16\sin 4t\right]$$

4. a) Use Convolution theorem to find $L^{-1}\left\{\dfrac{1}{(p-2)(p+1)}\right\}$

Solution : Suppose $f(s)=\dfrac{1}{p-2}$ and $g(s)=\dfrac{1}{p+1}$

$\therefore$ $L^{-1}\{f(s)\}=L^{-1}\left\{\dfrac{1}{p-2}\right\}=e^{2t}=F(t)$

and $L^{-1}\{g(s)\}=L^{-1}\left\{\dfrac{1}{p+1}\right\}=e^{-t}=G(t)$

By Convolution theorem of Inverse Laplace transform, we have

$$L^{-1}\{f(s)g(s)\}=\int_0^t F(x)G(t-x)dx$$

$\therefore$ $L^{-1}\left\{\dfrac{1}{(p-2)(p+1)}\right\}=\int_0^t\left[e^{2x}\right]\left[e^{-(t-x)}\right]dx$

$$=e^{-t}\int_0^t e^{3x}dx$$

$$= e^{-t}\left[\frac{e^{3x}}{3}\right]_0^t = \frac{e^{-t}}{3}\left[e^{3t}-1\right]$$

$$= \frac{1}{3}\left[e^{2t}-e^{-t}\right]$$

Thus $\boxed{L^{-1}\left\{\frac{1}{(p-2)(p+1)}\right\} = \frac{1}{3}\left[e^{2t}-e^{-t}\right]}$ **Answer**

b) Find Laplace transform of the followings :

(i). $L\{e^t \sin^2 t\}$ **(ii).** $L\{t^2 \sin at\}$

Solution : Suppose $F(t) = \sin^2 t = \frac{1}{2}(1-\cos 2t)$

Taking Laplace transform on both sides, we get

$$L\{F(t)\} = \frac{1}{2}L\{1-\cos 2t\}$$

$$= \frac{1}{2}\left[\frac{1}{p}-\frac{p}{p^2+4}\right] = \frac{1}{2}\left[\frac{p^2+4-p^2}{p(p^{2+}4)}\right]$$

$$= \frac{2}{p(p^{2+}4)} = f(p) \ [Say]$$

Using first shifting theorem, we get

$$L\{e^t \sin^2 t\} = f(p-1)$$

$$\Rightarrow \quad = \frac{2}{(p-1)[(p-1)^{2+}4]} = \frac{2}{(p-1)(p^2-2p+5)}$$ **Answer**

(ii). Suppose $F(t) = \sin at$

$$\therefore \quad L\{F(t)\} = L\{\sin at\} = \frac{a}{p^2+a^2} = f(p)$$

Differentiating w.r.t. p, on both sides, we get

$$f'(p) = a\left[-\frac{2p}{(p^2+a^2)^2}\right] = -\frac{2ap}{(p^2+a^2)^2}$$

Again Differentiating w.r.t. *p,* we get

$$f''(p) = -2a\left[\frac{(p^2+a^2)^2.1-p(p^2+a^2)(2p)}{(p^2+a^2)^4}\right]$$

$$\Rightarrow \quad f''(p) = 2a\left[\frac{p^2-a^2}{(p^2+a^2)^3}\right] = \frac{2ap^2-2a^3}{(p^2+a^2)^3}$$

By multiplication of t^2 in Laplace transform, we have

$$L\{t^2F(t)\}=(-1)^2 f''(p)$$

$\Rightarrow$ $\boxed{L\{t^2 \sin at\}=\dfrac{2ap^2-2a^3}{(p^2+a^2)^3}}$ **Answer**

5. a) Show that the function $e^x(\cos y + i\sin y)$ is an analytic function. Find its derivative.

Solution: Suppose $f(z)=e^x(\cos y+i\sin y)$

$=\quad u+iv=e^x\cos y+ie^x\sin y$

Equation on both sides, we get

$$u=e^x\cos y \text{ and } v=e^x\sin y$$

Partially differentiating with respect to, x and y, we get

$$\frac{\partial u}{\partial x}=e^x\cos y \qquad \frac{\partial v}{\partial x}=e^x\sin y$$

$$\frac{\partial u}{\partial y}=-e^x\sin y \qquad \frac{\partial v}{\partial y}=e^x\cos y$$

Clearly, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Therefore, C-R equation is satisfied, then given function is analytic everywhere.

Since $f(z)=u+iv$

Partially differentiating w.r.t. x we get

$$f'(x)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$\Rightarrow \quad =e^x\cos y+i(e^x\sin y)=e^x(\cos y+i\sin y)$

$\Rightarrow \quad =e^xe^{iy}=e^{x+iy}=e^z$ **Answer**

b) Show that the function $u(x,y)=x^2-y^2+2y$ is harmonic and find its conjugate.

Solution: Given : $u(x,y)=x^2-y^2+2y$

Partially differentiate successively *w.r.t.* x and y respectively, we get

$$\frac{\partial u}{\partial x}=2x \quad\Rightarrow\quad \frac{\partial^2 u}{\partial x^2}=2 \qquad \text{....(1)}$$

and $$\frac{\partial u}{\partial y}=-2y+2 \quad\Rightarrow\quad \frac{\partial^2 u}{\partial y^2}=-2 \qquad \text{...(2)}$$

Adding (1) and (2) we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2-2=0$$

Hence u is harmonic function.

Now, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$\Rightarrow \quad dv = \left(-\frac{\partial u}{\partial y} \right) dx + \frac{\partial u}{\partial x} dy$

$\Rightarrow \quad dv = -(-2y + 2)dx + (2x)dy$

$\Rightarrow \quad dv = (2y - 2)dx + (2x)dy$

Integrating on both sides, we get

$$v = \int (2y-2)dx + \int (2x)dy + c$$

$= \quad = 2xy - 2x + 2xy + c$

Thus, $\boxed{v = 4xy - 2x + c}$ **Answer**

6. a) Evaluate $\int_C \frac{e^z}{(z-1)(z-4)} dz$, where C is the circle $|z| = 2$ by using Cauchy's integral formula.

Solution: Given, $I = \int_C \frac{e^z}{(z-1)(z-4)} dz$

The pole of integrand is given by,

$$(z-1)(z-4) = 0 \Rightarrow z = 1, 4$$

Now, $z = 1 \Rightarrow |x| = 1 < 2$ [Lies within C]

and $z = 4 \Rightarrow |x| = 4 > 2$ [Outside of C]

By Cauchy integral formula,

$$\int_C \frac{e^z}{(z-1)(z-4)} dz = \int_{C_1} \frac{\frac{e^z}{z-4}}{z-1} dz$$

$\Rightarrow \quad = 2\pi i \left[\frac{e^z}{z-4} \right]_{z=1}$

$\Rightarrow \quad = 2\pi i \left[\frac{e^1}{1-4} \right] = \frac{2\pi i e}{3}$

Thus, $\boxed{\int_C \frac{e^z}{(z-1)(z-4)} dz = -\frac{2\pi i e}{3}}$ **Answer**

b) Find poles and residues of the function $\frac{z^2}{(z-1)(z-2)(z-3)}$

Solution : Given $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$

Taking, $(z-1)(z-2)(z-3)=0$

$\Rightarrow$ $z=1,2,3$ are simple pole of order 1

(i). $[\text{Res } f(z)]_{z=1}=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\left[\frac{z^2}{(z-1)(z-2)(z-3)}\right]$

$\Rightarrow$ $=\lim_{z\to 1}\frac{z^2}{(z-2)(z-3)}=\frac{1}{(1-2)(1-3)}=\frac{1}{2}$ **Answer**

(ii). $[\text{Res } f(z)]_{z=2}=\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}(z-2)\left[\frac{z^2}{(z-1)(z-2)(z-3)}\right]$

$\Rightarrow$ $=\lim_{z\to 2}\frac{z^2}{(z-1)(z-3)}=\frac{1}{(2-1)(2-3)}=-4$ **Answer**

(iii). $[\text{Res } f(z)]_{z=3}=\lim_{z\to 3}(z-2)f(z)=\lim_{z\to 3}(z-3)\left[\frac{z^2}{(z-1)(z-2)(z-3)}\right]$

$\Rightarrow$ $=\lim_{z\to 3}\frac{z^2}{(z-1)(z-3)}=\frac{1}{(3-1)(3-2)}=-4$ Ans**wer**

7. a) Find the real root of the equation $x^3-5x-7=0$ which lies between 2 and 3 by the method of false position. (Upto 3 iteration).

Solution : Given : $f(x)=x^3-5x-7$

Since root lies between 2 and 3, then

Taking $x=2$ $f(2)=2^3-5(2)-7=-9(-ve)$

and $x=3$ $f(3)=3^3-5(3)-7=5(-ve)$

Therefore, the root lies between 2 and 3.

1st Approximation :

Say, $a=2, b=3$ and $f(2)=-9, f(3)=5,$ by False position formula,

$$x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}=\frac{2f(3)-3f(2)}{f(3)-f(2)}$$

$\Rightarrow$ $x_1=\frac{2(5)-3(-9)}{(5)-(-9)}=2.6428$

$\therefore$ $f(2.6428)=(2.6428)^3-5(2.6428)-7=-1.7556(-ve)$

So, the root lies between 2.6428 and 3.

2nd Approximation :

Say, $x_1=2.6428,$ $b=3$ and $f(2.6428)=-1.7556, f(3)=5,$ by False position formula,

$$x_2=\frac{x_1f(b)-bf(x_1)}{f(b)-f(x_1)}=\frac{2.6428f(3)-3f(2.6428)}{f(3)-f(2.6428)}$$

$$\Rightarrow \qquad x_2 = \frac{2.6428(5) - 3(-1.7556)}{5 - (1.7556)} = 2.7356$$

$$\therefore \qquad f(2.7356) = (2.7356)^3 - 5(2.7356) - 7 = -0.2061(-ve)$$

So, the root lies between 2.7356 and 3.

3rd Approximation :

Say, $x_2 = 2.736,\ b = 3$ and $f(2.7356) = -0.2061, f(3) = 5$, by False position formula,

$$x_3 = \frac{x_2 f(b) - bf(x_2)}{f(b) - f(x_2)} = \frac{2.08126 f(3) - 3f(2.08126)}{f(3) - f(2.08126)}$$

$$\Rightarrow \qquad x_3 = \frac{2.7356(5) - 3(-0.2061)}{5 - (-0.2061)} = 2.7460$$

$\therefore$ Required root after three approximations is 2.7460.

(b) Apply Newton-Raphson method to solve 3x-cosx-1 = 0 (upto 3 iteration only).

Solution : Given : $f(x) = \cos x - 3x + 1$

Taking $x = 0,\ f(0) = \cos(0) - 3(0) + 1 = 2(+ve)$

and $x = 1\ \ f(1) = \cos(1) - 3(1) + 1 = -1.4596(-ve)$

Therefore a root lies between 0 and 1 and it is nearer to 1.

Now, $f'(x) = -\sin x - 3$

Taking $x_0 = \frac{0+1}{2} = 0.5$, such that $f'(0.5) \neq 0$

The n^{th} iteration formula of Newton-Raphson method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \qquad x_{n+1} = x_n + \frac{\cos(x_n) - 3x_n + 1}{\sin(x_n) + 3} \qquad \ldots(1)$$

Iteration table :

No. Iteration	Value of n	The value of **x** for next iteration x_n Where n=0, 1, 2…..	Iterative formula $x_{n+1} = x_n + \frac{\cos(x_n) - 3x_n + 1}{\sin(x_n) + 3}$
1	0	$x_0 = 0.5$	$x_1 = x_0 + \frac{\cos(x_0) - 3x_0 + 1}{\sin(x_0) + 3}$ $\Rightarrow x_1 = 0.5 + \frac{\cos(0.5) - 3(0.5) + 1}{\sin(0.5) + 3} = 0.608518$

2	1	$x_1 = 0.608518$	$x_2 = x_1 + \frac{\cos(x_1) - 3x_1 + 1}{\sin(x_1) + 3}$ $\Rightarrow x_2 = 0.608518 + \frac{\cos(0.608518) - 3(0.608518) + 1}{\sin(0.608518) + 3} = 0.607101$
3	2	$x_2 = 0.607101$	$x_3 = x_2 + \frac{\cos(x_2) - 3x_2 + 1}{\sin(x_2) + 3}$ $\Rightarrow x_3 = 0.607101 + \frac{\cos(0.607101) - 3(0.607101) + 1}{\sin(0.607101) + 3} = 0.607101$

Hence, a real root of equation is 0.60710 correct to five decimal places.

8. a) Using Bisection method, find the root of the equation $x^3 + x - 1 = 0$ near $x = 0$. (upto three iteration only).

Solution : Suppose $f(x) = x^3 + x - 1$ ....(1)

Taking, $x = 0$ $f(0) = 0^3 + 0 - 1 = -1(-ve)$

and $x = 1$ $f(1) = 1^3 + 1 - 1 = 1(+ve)$

Clearly $f(0).f(1) < 0$

$\therefore$ Root lies between 0 and 1. Say a = 0 and b=1

1. First Approximation :

$$x_0 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

Putting in equation (1), we get

$$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375(-ve)$$

Clearly $f(0.5).f(1) < 0$

$\therefore$ root lies between 2 and 2.25.

3. Third Approximation:

$$x_2 = \frac{a + x_1}{2} = \frac{2 + 2.25}{2} = 2.125$$

Putting in equation (1), we get

$$f(2.125) = (2.125)^3 - 2(2.125) - 5 = 0.3457(+ve)$$

Clearly $f(2).f(2.124) < 0$

$\therefore$ root lies between 2 and 2.125.

b) Find a Fourier series to represent $f(x) = x - x^2$ **from** $x = -\pi$ **to** $x = \pi$

Solution : Given : $f(x) = x - x^2, \; -\pi \le x \le \pi$ ...(1)

Here, $2L = \pi - (-\pi) i.e. 2L = 2\pi \;\Rightarrow\; L = \pi$

Suppose the Fourier series of $f(x)$ with period $2L$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{[Since } L = \pi \text{]} \qquad ...(2)$$

Now, $a_0 \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x - x^2\right)dx$

$$\Rightarrow \quad = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = 0 - 2\int_0^{\pi} x^2 dx \qquad \text{[Since x = Odd and } x^2 = \text{Even]}$$

$$\Rightarrow \quad a_0 = -2\left[\frac{x^3}{3}\right]_0^{\pi} = -\frac{2}{3}\left[\pi^3 - 0\right] = -\frac{2\pi^2}{3}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x - x^2\right)\cos nx\, dx \qquad [x \cos nx = \text{odd}]$

$$\Rightarrow \quad = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = 0 - 2\int_0^{\pi} x^2 \cos nx dx$$

$$\Rightarrow \quad = -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$\Rightarrow \quad a_n = -\frac{2}{\pi}\left[\left\{0 + \frac{2\pi(-1)^n}{n^2} - 0\right\} - \{0 - 0 - 0\}\right] = -\frac{4(-1)^n}{n^2}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(x - x^2\right)\sin nx dx$

$$\Rightarrow \quad = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx - 0 \qquad [x^2 \sin nx = \text{odd}]$$

$$\Rightarrow \quad = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$\Rightarrow \quad b_n = \frac{2}{\pi}\left[\left\{-\frac{\pi(-1)^n}{n} - 0\right\} - \{0 - 0 - 0\}\right] = -\frac{2(-1)^n}{n}$$

Putting in equation (1), we get

$$f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin nx$$

$$\Rightarrow \quad \boxed{f(x) = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \ldots\right]} \qquad \textbf{Answer}$$

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