RGPV SOLUTION BE-3001 (ME) MATHEMATICS-3 JUN 2018

1. a) Express $f(x) = x$ as a half range sine series in $0 < x < 2$

Solution : Given $f(x) = x, \quad 0 < x < 2$ ....(1)

Here, $L = 2$

Suppose the Half range sine series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \qquad [Since\ L = \pi] \qquad ....(2)$$

Now, $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$

$$\Rightarrow \quad = \left[x\left(\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right) - 1\left(-\frac{4}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\Rightarrow \quad = \left\{-\frac{4}{n\pi}(-1)^n - 0\right\} - \{0 - 0\}$$

$$\Rightarrow \quad b_n = -\frac{4}{n\pi}(-1)^n$$

Putting in equation (1), we get

$$\boxed{f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)} \qquad \textbf{Answer}$$

b) Obtain the Fourier series for the function $f(x) = x^2$ in $-\pi < x < \pi$

Solution : Given : $f(x) = x^2, -\pi < x < \pi$

Here, $2L = \pi - (-\pi) i.e. 2L = 2\pi \Rightarrow L = \pi$

Suppose the Fourier series of $f(x)$ with period $2L$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad [\text{ Since } L = \pi\ ] \qquad ....(2)$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$

$$\Rightarrow \quad = 2\int_0^{\pi} x^2 dx \qquad [\text{Since } x^2 = \text{Even}]$$

$$\Rightarrow \quad a_0 = 2\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\left[\pi^3 - 0\right] = \frac{2\pi^2}{3}$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \qquad \{ x\cos nx = \text{odd}]$

$$\Rightarrow \quad = 2\int_0^{\pi} x^2 \cos nx dx$$

$$\Rightarrow \quad = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$\Rightarrow \quad a_n = \frac{2}{\pi}\left[\left\{0 + \frac{2\pi(-1)^n}{n^2} - 0\right\} - \{0 - 0 - 0\}\right] = \frac{4(-1)^n}{n^2}$$

and $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx$

$\Rightarrow \qquad [x^2 \sin nx = \text{odd}]$

Putting in equation (1), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$\Rightarrow \quad \boxed{f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ...\right]} \qquad \dots (3)$$

Proved

2. a) Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Solution : Suppose $f(x) = \frac{e^{-ax}}{x}$

By Fourier sine Transform,

$$F\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx dx$$

$$\Rightarrow \quad f\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left(\frac{e^{-ax}}{x}\right)\sin sx dx = I \qquad \dots(1)$$

Differentiate w.r.t. s, on both sides, we get

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\frac{d}{dx}\left[\int_0^{\infty}\left(\frac{e^{-ax}}{x}\right)\sin sx dx\right]$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left(\frac{e^{-ax}}{x}\right)\frac{\partial}{\partial s}(\sin sx)dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left(\frac{d^{-ax}}{x}\right)(x\cos sx)dx$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{(-a)^2 + s^2}\{-a\cos sx + s\sin sx\}\right]_0^{\infty}$$

$$\Rightarrow \quad \frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+a^2}\right)[\{0\} - \{-a+0\}] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating both sides, w.r.t.s, we get

$$I = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right)\right] + c \qquad \text{….(2)}$$

For the initial condition, putting s = 0, then c = 0

$\therefore$ from (2), we have

$$I = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] \Rightarrow F\{f(x)\} = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad F(s) = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] \Rightarrow F\{f(x)\} = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad \boxed{F(s) = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]}$$ From (1) **Answer**

$$f_s\left\{\frac{e^{-ax}}{x}\right\} = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$ [By definition of Sine Transform]

Putting $a = 0,$ we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}(\infty)\right]$$

$$\Rightarrow \quad f_s\left\{\frac{1}{x}\right\} = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \quad \boxed{f_s\left\{\frac{1}{x}\right\} = \sqrt{\frac{\pi}{2}}}$$ **Answer**

b) Find Fourier cosine transform of $f(x) = e^{-x}$

Solution : Given the function : $F(x) = e^{-ax}$

The Fourier cosine transform of $F(x)$ is given by,

$$f_c(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos px\,dx$$

$$\therefore \quad f_c(P) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos px\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{(-1)^2+p^2}\{-\cos px + p\sin px\}\right]_0^\infty$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}\right]$$

$$f_c(p) = \frac{-1}{p^2+1}\sqrt{\frac{2}{\pi}}\left[e^{-x}\{\cos px - p\sin px\}\right]_0^\infty = \frac{-1}{p^2+1}\sqrt{\frac{2}{\pi}}\left[\{0-1(1+p.0)\}\right]$$

Thus $\quad f_c(p) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{p^2+1}\right]$ **Answer**

3. a) Find the cosine transform of $\frac{1}{a^2+a^2}$

Solution : Suppose $F(x) = \frac{1}{x^2+1}$

The Fourier cosine transform of $F(x)$ is,

$$f_c(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos px\,dx$$

$$\therefore \quad f_c(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x^2+1}\cos px\,dx = I\,[\text{Say}] \qquad \ldots(1)$$

Differentiating w.r.t., p, we get

$$\frac{d}{dp}I = \frac{d}{dp}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x^2+1}\cos px dx$$

$$\Rightarrow \quad = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x^2+1}\frac{\partial}{\partial p}(\cos px)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{-x}{x^2+1}\sin px\,dx$$

$$\Rightarrow \quad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x^2}{x(x^2+1)}\sin px\,dx$$

$$\Rightarrow \quad = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+x^2-1)}{x(x^2+1)}\sin px\,dx \qquad \text{[Adding and subtract 1]}$$

$$\Rightarrow \quad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x}dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(x^2+1)}dx$$

$$\Rightarrow \quad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) + 1\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(x^2+1)}dx$$

$$\Rightarrow \quad \frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(x^2+1)}dx \qquad \ldots(2) \quad \left[\Theta \int_0^\infty \frac{\sin ax}{x}dx = \frac{\pi}{2}\right]$$

Again differentiating w.r.t., p, we get

$$\frac{d^2I}{dp^2} = 0 + 1\frac{d}{dp}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(x^2+1)}dx$$

$$\Rightarrow \quad = a^2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\cos px}{x(x^2+1)}dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos px}{x^2+1}dx = I \qquad \text{From (1)}$$

$$\Rightarrow \quad \frac{d^2I}{dp^2} - I = 0 \qquad \ldots(3)$$

This is Linear differential equation of higher order.

$\therefore$ The solution of (3) is,

$$I = c_1 e^{p} + c_2 e^{-p} \qquad \ldots(4)$$

Differentiating w.r.t., p, we get

$$\frac{dI}{dp} = c_1 e^{p} - c_2 e^{-p} \qquad \ldots(5)$$

Putting p=0, in equation (1) and (4) we get

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x^2+1}\,dx = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}(x)\right]_0^\infty = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}$$

and $\quad c_1 + c_2 = I \Rightarrow c_1 + c_2 = \sqrt{\frac{\pi}{2}} \qquad \ldots(6)$

Again Putting p=0, in equation (2) and (5) we get

$$\frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} + 0 \Rightarrow \frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} \text{ and } c_1 - c_2 = -\sqrt{\frac{\pi}{2}} \qquad \ldots.(7)$$

Solve (6) and (7), we get

$$c_1 = 0 \text{ and } c_2 = \sqrt{\frac{\pi}{2}}$$

$\therefore$ From (4), we get

$$I = \sqrt{\frac{\pi}{2}}e^{-p}$$

$\Rightarrow \quad \boxed{i.e.,\ F_c\left\{\frac{1}{x^2+1}\right\} = \sqrt{\frac{\pi}{2}}e^{-p}}$ **Answer**

b) Develop $\sin\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the range $0 < x < l$.

Solution : Given : $f(x) = \sin\left(\frac{\pi x}{l}\right)$ and $0 < x < l$

Here, $\quad L = l$

Suppose the Half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \ldots(1)$$

Now, $\quad a_0 = \frac{2}{1}\int_0^l f(x)dx = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)dx$

$$a_0 = \frac{2}{l} \times \frac{l}{\pi}\left[-\cos\left(\frac{\pi x}{l}\right)\right]_0^l = -\frac{2}{\pi}[-1-1] = \frac{4}{\pi}$$

and $$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow \quad = \frac{1}{l}\int_0^l \left[\sin\left\{\frac{(n+1)\pi x}{l}\right\} - \sin\left\{\frac{(n-1)\pi x}{l}\right\}\right]dx = \frac{1}{l}\left[-\frac{\cos\left\{\frac{(n+1)\pi x}{l}\right\}}{\frac{(n+1)\pi}{l}} + \frac{\cos\left\{\frac{(n-1)\pi x}{l}\right\}}{\frac{(n-1)\pi}{l}}\right]_0^l \quad \ldots(2)$$

$$\Rightarrow \quad = \frac{1}{\pi}\left[\left\{\frac{-\cos(n\pi+\pi)}{n+1} + \frac{\cos(n\pi-\pi)}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right] = \frac{1}{\pi}\left[\left\{\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}\right\} + \left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}\right]$$

$$\Rightarrow \quad \boxed{a_n = \frac{-2}{(n^2-1)\pi}\left[(-1)^n + 1\right]; \quad n \neq 1}$$

Putting $n = 1$, in equation (2), we get

$$a_1 = \frac{1}{l}\int_0^l \sin\left\{\frac{2\pi x}{l}\right\}dx = \frac{1}{l}\times\frac{l}{2\pi}\left[-\cos\left(\frac{2\pi x}{l}\right)\right]_0^l = -\frac{1}{2\pi}[1-1] = 0$$

Putting in equation (1), we get

$$\boxed{f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\left[(-1)^n+1\right]}{n^2-1}\cos\left(\frac{n\pi x}{l}\right)}$$ **Answer**

4. a) Find the Laplace Transform of $te^{-4t}\sin 3t$

Solution : Since $L\{\sin 3t\} = \dfrac{3}{p^2+9} = f(p)$

By Multiplication property, we have

$$L\{t\sin 3t\} = (-1)\frac{d}{dp}f(p)$$

$$\Rightarrow \quad = (-1)\frac{d}{dp}\left[\frac{3}{p^2+9}\right]$$

$$\Rightarrow \quad = (-3)\left[-\frac{2p}{(p^2+9)^2}\right] = \frac{6p}{(p^2+9)^2} = f_1(p)$$

$$\Rightarrow \quad = \frac{6(p+4)}{\left[(p+4)^2+9\right]^2} = \frac{6p+24}{(p^2+8p+25)^2}$$

Thus, $$\boxed{L\{e^{-4t}(t\sin 3t)\} = \frac{6p+24}{(p^2+8p+25)^2}}$$ **Answer**

b) Find inverse Laplace transform of $\frac{5s+3}{(s-1)(s^2+2s+5)}$

Solution : Now $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=L^{-1}\left\{\frac{1}{s-1}-\frac{s-2}{s^2+2s+5}\right\}$ [By Partial fraction]

$$=L^{-1}\left\{\frac{1}{s-1}\right\}-L^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+4}\right\}$$

$$=e^t-e^{-t}L^{-1}\left\{\frac{s-3}{s^2+4}\right\}$$

$$=e^t-e^{-t}\left[\cos 2t-\frac{3}{2}\sin 2t\right]$$

Thus, $\boxed{L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}=e^t-e^{-t}\left[\cos 2t-\frac{3}{2}\sin 2t\right]}$ **Answer**

5. a) Find Laplace transform

(i). $\frac{e^{-at}-e^{-bt}}{t}$ **(ii).** $\sin at-at\cos at$

Solution : (i). Suppose $F(t)=e^{-at}-e^{-bt}$

Taking laplace transform on both sides, we get

$$L\{F(t)\}=L\{e^{-at}\}-L\{e^{-bt}\}$$

$$\Rightarrow \quad =\frac{1}{p+a}-\frac{1}{p+b}=f(p)$$

By Division property of Laplace transform, we have

$$L\left\{\frac{F(t)}{t}\right\}=\int_p^\infty f(p)dp$$

$$\therefore \quad L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_p^\infty\left[\frac{1}{p+a}-\frac{1}{p+b}\right]dp$$

$$\Rightarrow \quad =[\log(p+a)-\log(p+b)]_p^\infty=\left[\log\left(\frac{p+a}{p+b}\right)\right]_p^\infty$$

$$\Rightarrow \quad =0-\log\left(\frac{p+a}{p+b}\right)=-\log\left(\frac{p+a}{p+b}\right)$$

Thus, $\boxed{L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\log\left(\frac{p+b}{p+a}\right)}$ **Answer** (ii). Now $L\{\sin at - at\cos at\} = L\{\sin at\} - aL\{t\cos at\}$

$$\Rightarrow \quad = \frac{a}{p^2+a^2} - a(-1)\frac{d}{dp}L\{\cos at\}$$ [By Multiplication of *t*]

$$\Rightarrow \quad = \frac{a}{p^2+a^2} + a\frac{d}{dp}\left[\frac{p}{p^2+a^2}\right]$$

$$\Rightarrow \quad = \frac{a}{p^2+a^2} + a\left[\frac{(p^2+a^2).1 - p(2p)}{(p^2+a^2)}\right]$$

$$\Rightarrow \quad = \frac{a}{p^2+a^2} + a\left[\frac{a^2-p^2}{(p^2+a^2)^2}\right]$$

$$\Rightarrow \quad = \frac{ap^2+a^3+a^3-ap^2}{(p^2+a^2)^2} = \frac{2a^3}{(p^2+a^2)^2}$$ **Answer**

b) Using Laplace transform, solve the differential equation

$y''-3y'+2y = 4t+e^{3t}$ **when** $y(0)=1$ **and** $y'(0)=-1$

Solution : Given the differential equation is,

$$y''(t) - 3y'(t) + 2y(t) = 4t + e^{3t} \quad \ldots(1)$$

With initial condition are: $y(0)=1$ and $y'(0)=-1$

Taking Laplace transform of (1) on both sides, we get

$$L\{y''(t)\} - 3L\{y'(t)\} + 2L\{y(t)\} = 4L\{t\} + L\{e^{3t}\}$$

$$\Rightarrow \quad [p^2 y(p) - p\,y(0) - y'(0)] - 3[p\,y(p) - y(0)] + 2\,y(p) = \frac{4}{p^2} + \frac{1}{p-3} \qquad [\Theta \quad L\{y(t)\} = y(p)]$$

Putting the initial values,

$y(0)=1$ and $y'(0)=-1,$ we get

$$\therefore \quad [p^2 y(p) - p + 1] - 3[p\,y(p) - 1] + 2y(p) = \frac{4}{p^2} + \frac{1}{p-3}$$

$$\Rightarrow \quad (p^2 - 3p + 2)y(p) - p + 4 = \frac{4}{p^2} + \frac{1}{p-3}$$

$$\Rightarrow \quad (p-1)(p-2)y(p) = \frac{4}{p^2} + \frac{1}{p-3} + p - 4$$

$$\Rightarrow \quad (p-1)(p-2)y(p) = \frac{4}{p^2} + \frac{p^2-7p+13}{p-3}$$

$$\Rightarrow \quad L\{p(t)\} = \frac{4}{p^2(p-1)(p-2)} + \frac{p^2-7p+13}{(p-1)(p-2)(p-3)}$$

$$\Rightarrow \qquad y(t) = 4L^{-1}\left\{\frac{1}{p^2}\left[\frac{1}{p-2} - \frac{1}{p-1}\right]\right\}$$

$$+L^{-1}\left\{\frac{1-7+13}{\underline{(p-1)}(1-2)(1-3)} + \frac{4-7(2)+13}{(2-1)\underline{(p-2)}(2-3)} + \frac{3^2-7(3)+13}{(3-1)(3-2)\underline{(p-3)}}\right\}$$

[Using **"Cover up"** Method]

$$\Rightarrow \qquad y(t) = 4\left(\frac{e^{2t}}{4} - e^t + \frac{t}{2} + \frac{3}{4}\right) + \frac{7}{2}L^{-1}\left\{\frac{1}{p-1}\right\} - 3L^{-1}\left\{\frac{1}{p-2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p-3}\right\}$$

Θ Suppose $f(p) = \dfrac{1}{p-2} - \dfrac{1}{p-1} \Rightarrow L^{-1}\{f(p)\} = L^{-1}\left\{\dfrac{1}{p-2}\right\} - L^{-1}\left\{\dfrac{1}{p-1}\right\}$

$$= e^{2t} - e^t$$

By division of p, we get

$$L^{-1}\left\{\frac{f(p)}{p^{-2}}\right\} = \int_0^t\left[\int_0^t\left(e^{2t} - e^t\right)dt\right]dt = \int_0^t\left[\frac{e^{2t} - e^t}{2}\right]_0^t dt = \int_0^t\left[\frac{e^{2t}}{2} - e^t - \frac{1}{2} + 1\right]dt$$

$$= \int_0^t\left(\frac{e^{2t}}{2} - e^t + \frac{1}{2}\right)dt = \left[\frac{e^{2t}}{4} - e^t + \frac{t}{2}\right]_0^t = \frac{e^{2t}}{4} - e^t + \frac{t}{2} - \frac{1}{4} + 1 = \frac{e^{2t}}{4} - e^t + \frac{t}{2} + \frac{3}{4}$$

$$\Rightarrow \qquad y(t) = e^{2t} - 4e^t + 2t + 3 + \frac{7}{2}e^t - 3e^{2t} + \frac{1}{2}e^{3t} = \frac{1}{2}e^{3t} - 2e^{2t} - \frac{1}{2}e^t + 2t + 3$$

Thus $\boxed{y(t) = \frac{1}{2}e^{3t} - 2e^{2t} - \frac{1}{2}e^t + 2t + 3}$

6. a) Discuss the analyticity of the function $f(z) = \dfrac{1}{z}$

Solution : The given function is, $f(z) = \dfrac{1}{z}$

$$\Rightarrow \qquad u + iv = \frac{1}{x + iy}$$

$$\Rightarrow \qquad u + iv = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Equating the real and imaginary part, we get

$$u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2} \qquad \ldots (1)$$

Differentiating w.r.t. and y, we get

$$\frac{\partial u}{\partial x} = \frac{\left(x^2 + y^2\right)1 - x(2x)}{\left(x^2 + y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \;\&\; \frac{\partial u}{\partial y} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

and $\frac{\partial v}{\partial x}=\frac{2xy}{\left(x^2+y^2\right)^2}$ & $\frac{\partial v}{\partial y}=-\frac{\left(x^2+y^2\right).1-y(2y)}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$

Clearly, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Hence, C-R equations are satisfied.

Therefore f (z) is analytic.

b) Determine the poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ and evaluate residue of each pole.

Solution : Given, $f(z)=\frac{z^2}{(z-1)^2(z+2)}$

Taking, $(z-1)^2(z+2)=0$

$\Rightarrow \quad z=1(order\,2),-2$

(i). $\left[\text{Res } f(z)\right]_{z=1}=\lim_{z\to 1}\frac{1}{\underline{|2-1}}\left[\frac{d}{dz}(z-1)^2 f(x)\right]=\lim_{z\to 1}\left[\frac{d}{dz}(z-1)^2\times\frac{z^2}{(z-1)^2(z+2)}\right]$

$\Rightarrow \quad =\lim_{z\to 1}\left[\frac{d}{dz}\frac{z^2}{(z+2)}\right]=\lim_{z\to 1}\left[\frac{(z+2)(2z)-z^2(1)}{(z+2)^2}\right]$

$\Rightarrow \quad =\lim_{z\to 1}\left[\frac{z^2+4z}{(z+2)^2}\right]=\frac{(1)^2+4(1)}{(1+2)^2}=\frac{5}{9}$

Thus, $\boxed{\left[\text{Res } f(z)\right]_{z=1}=\frac{5}{9}}$ **Answer**

(ii). $\left[\text{Re}\,s\ f(z)\right]_{z=-2}=\lim_{z\to -2}(z+2)f(z)=\lim_{z\to -2}\left[(x+2)\times\frac{z^2}{(z-1)^2(z+2}\right]$

$=\lim_{z\to -2}\left[\frac{z^2}{(z-1)^2}\right]=\frac{(-2)^2}{(-2-1)^2}=\frac{4}{9}$ **Answer**

7. a) Evaluate $\int_C\frac{e^z}{(z+1)^2}dz$ where C is circle $|z-1|=3$

Solution : Given, $I=\int_C\frac{e^z}{(z+1)}dz$

The pole of integrand is given by,

$(z+1)^2=0\Rightarrow z=-1,$ of order 2.

Now, $z=-1\Rightarrow|z-1|=|-1-1|=2<3$

Clearly $z=-1,$ is a pole which inside of C, then by Cauchy integral derivative formula,

$$\int_C \frac{e^z}{(z+1)^2}\,dz = \frac{2\pi i}{\underline{|2-1}} \lim_{z\to -1}\left[\frac{d}{dz}(z+1)^2 \frac{e^z}{(z+1)^2}\right]$$

$$= 2\pi u \lim_{z\to -1}\left[\frac{d}{dz} e^z\right] = 2\pi i \lim_{z\to -1}\left[e^z\right]$$

$$= 2\pi i\left(e^{-1}\right) = \frac{2\pi i}{e}$$

Thus, $\boxed{\int_C \frac{e^z}{(z+1)^2}\,dz = \frac{2\pi i}{e}}$ **Answer**

b) Find the imaginary part of the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution : Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Partially differentiate w.r.t. x and y respectively

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \text{ and } \frac{\partial u}{\partial y} = -6xy - 6y \qquad \ldots(1)$$

To Find Conjugate function v

Now, $dv = \left(\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy$

$\Rightarrow \quad dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$ [by Cauchy-Riemann Equation]

$\Rightarrow \quad = (6xy + 6y)dx + (3x^2 - 3y^2 + 6x)dy$

Integrating on both sides, we get

$$v = \int_{y\,\text{constant}} (6xy + 6y)dx + \int_{\text{Independent of } x} (-3y^2)dy + c$$

$\boxed{v = 3x^2 y + 6xy - y^3 + c}$ **Answer**

8. a) Using Picard's method to approximate y when x = 0.2, given that y=1 when x = 0 and $\frac{dy}{dx} = x - y$

Solution : The given initial value problem is,

$$\frac{dy}{dx} = x - y = f(x, y);\ \text{with } y_0 = 1 \text{ at } x_0 = 0 \qquad \ldots(1)$$

1st Approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$\Rightarrow \quad y_1 = 1 + \int_0^x [x - y_0]dx = 1 + \int_0^x (x-1)dx = 1 + \frac{x^2}{2} - x$$

at $x = 0.2$, we get $y_1 = 1 + \frac{(0.2)^2}{2} - 0.2 = 0.82$

2nd Approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$\Rightarrow \quad y_2 = 1 + \int_{x_0}^{x} [x - y_1]dx = 1 + \int_0^x \left[x - \left(1 + \frac{x^2}{2} - x\right)\right]dx$$

$$\Rightarrow \quad y_2 = 1 + \int_0^x \left[2x - 1 - \frac{x^2}{2}\right]dx = 1 + x^2 - x - \frac{x^3}{6}$$

at $x = 0.2$, we get $y_2 = 1 + (0.2)^2 - 0.2 - \frac{(0.2)^3}{6} = 0.8386$ **Answer**

8. a) Apply Runge-Kutta method to find approximate value of y when x = 0.2 given that

$\frac{dy}{dx} = x + y$ **and** $y = 1$ **when** $x = 0.$

Solution : Given differential equation is,

$$\frac{dy}{dx} = x + y = f(x, y)$$

With initial condition, $y_0 = 1, x_0 = 0$ and $x = 0.2$ (Given)

Taking, $h = \frac{x = x_0}{n} = \frac{0.2 - 0}{2} = 0.1$, such that $x_1 = x_0 + h = 0.1$ and $x_2 = x_0 + 2h = 0.2$

1. First Approximation :

Here, $x_0 = 0$ and $y_0 = 1$

$\therefore \quad k_1 = h\,f(x_0,\ y_0)$

$\Rightarrow \quad = h[x_0 + y_0] = 0.1[0 + 1] = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$\Rightarrow \quad = 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1f(0.05,\ 1.05)$$

$\Rightarrow \quad = 0.1[0.05 + 1.05] = 0.11$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$\Rightarrow \quad = 0.1f\left(0 + \frac{0.1}{2},\ 1 + \frac{0.11}{2}\right) = 0.1f(0.05,\ 1.055)$$

$\Rightarrow \quad = 0.1[0.05 + 1.055] = 0.1105$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

$\Rightarrow \quad 0.1f(0 + 0.1,\ 1 + 0.1105) = 0.1f(0.1,\ 1.1105)$

$\Rightarrow$ =0.1 [0.1 + 1.1105] = 0.12105

$\therefore \qquad y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$\Rightarrow \qquad y_1 = 1 + \frac{1}{6}[0.1 + 2(0.11) + 2(0.1105) + 0.12105] = 1.11034$, at $x_1 = 0.1$

2. Second Approximation :

Here, $x_1 = 0.1$ and $y_1 = 1.11034$

$\therefore \qquad k_1 = hf(x_1, y_1)$

$\Rightarrow \qquad = h[x_1 + y_1] = 0.1[0.1 + 1.11034] = 0.121034$

$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$

$\Rightarrow \qquad = 0.1f\left(0.1 + \frac{0.1}{2}, 1.11034 + \frac{0.121034}{2}\right) = 0.1f(0.15,\ 1.17085)$

$\Rightarrow \qquad = 0.1[0.15 + 1.17085] = 0.13208$

$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$

$\Rightarrow \qquad 0.1f\left(0.1 + \frac{0.1}{2}, 1.11034 + \frac{0.13208}{2}\right) = 0.1f(0.15, 1.17638)$

$\Rightarrow \qquad = 0.1[0.15 + 1.7638] = 0.13263$

$k_4 = hf(x_1 + h,\ y_1 + k_3)$

$\Rightarrow \qquad = 0.1f(0.1 + 0.1,\ 1.11034 + 0.13263) = 0.1f(0.2,\ 1.24297)$

$\Rightarrow \qquad = 0.1[0.2 + 1.24297] = 0.14429$

$\therefore \qquad y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$\Rightarrow \qquad y_2 = 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208) + 2(0.13263) + 0.14429] = 1.24279$, at $x_2 = 0.2$

Thus, y(0.2) = 1.24279 **Answer**