ADA 2023 Final

Shortest Path

- Bellman-Ford, Lawler (DAG), Dijkstra
- Matrix Binary-lifting $O(n^3 \log n)$, Floyd-Warshall
- Johnson's: Assign height $h(u)$ s.t. $w(i,j) + h(i) - h(j) \geq 0$.
  Let $w(0,i) = 0$ and run Bellman-Ford, $h(i) = d(i)$. Because $d(u) + w(u,v) \geq d(v) \Rightarrow w(u,v) + d(u) - d(v) \geq 0$ and any shortest path P' from u to v is a shortest path.

Random

- Matrix multiplication check: pick random row matrix and check $ABu = Cu$.
- Polynomials check: pick random number r in $[1, 4n]$ and check $A(r)B(r) = C(r)$.

Problem Classes

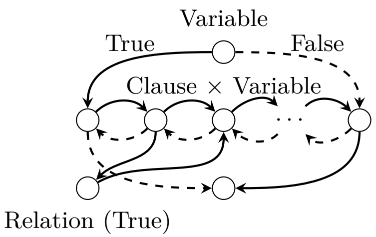
- NP-Complete list: CNF-SAT, 0-1 Integer Programming, Clique, Vertex Cover, Set Cover, Hamiltonian, 3-SAT, Graph Coloring, Clique Cover, Exact Cover, 3-Matching, Steiner Tree, Max Cut
- A can be reduced to B: all instance of A can be solved by B (probably as subroutine)

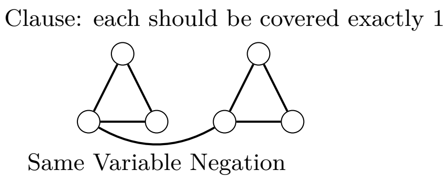
Approximation

- Vertex Cover 2-apx Pick any edge and add both endpoints.
- Metric TSP 2-apx Find MST or Euler Tour and get shortcut (keep first occurence)
- Metric TSP 1.5-apx (Christofide) Find MST and weighted edge matching of odd degree (Chinese post-man problem) and short cut them. Let C be the cycle of best solution, then odd degree has two choice (the smaller of them is only 0.5).
- Longest Simple Path **no** constant approximation.
- General TSP **no** approximation.

Reductions

- Hamiltonian Path $\iff$ Cycle / Independent Set $\iff$ Clique (take complement)
- SAT $\leq_p$ Hamiltonian Path (Directed) / 3-SAT $\leq_p$ Independent Set





- Hamiltonian Cycle Directed $\leq_p$ Undirected: Make every node $v \longrightarrow v_{in} - v - v_{out}$

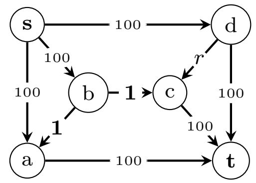
Flow

Definitions and lemmas

- A **flow** f is a function s.t. $f(u,v) \leq c(u,v)$ and $\sum f(u,\cdot) - \sum f(\cdot,u) = 0$ (where $u \neq s,t$)
- The residual graph $R(f)$ is defined as, for each edge (u,v) (directed),
  There are edge (u,v) of capacity $c(u,v) - f(u,v)$ and (v,u) of capacity $f(u,v)$.
- f is a flow of G and g is a flow of $R(f) \iff f + g$ is a flow of G.

Algorithms

- Ford-Fulkerson — Finding any avaliable flow in residual graph.
  - Pseudo-polynomial (exponential): $O((n+m)f)$ for rational capacity.
  - Wrong for irrational capacity.



$$r = \frac{\sqrt{5}-1}{2} \Longrightarrow r^2 = 1 - r$$

$$\left.\begin{array}{lcl} s \to b \to c \to t & \Rightarrow & 1 \\ s \to d \to c \to b \to a \to t & \Rightarrow & r \\ s \to b \to c \to d \to t & \Rightarrow & r \\ s \to d \to c \to b \to a \to t & \Rightarrow & r^2 \end{array}\right\} 1 + r + r + r^2 + r^2 + r^3 + \cdots$$

- Edmonds-Karp — Finding shortest avaliable flow in residual graph.
  - Observation. $d_f(s,v) \leq d_{f+g}(s,v)$ if g is a flow of $R(f)$.
    Proof. Let w be the some node such that $d_f(s,w) > d_{f+g}(s,w)$.
    There must be some edge (u,v) lies on the shortest path s-w on $R(f+g)$ such that

    $$d_f(s,u) \leq d_{f+g}(s,u), \quad d_f(s,v) > d_{f+g}(s,v)$$

    $$((u,v) \text{ is not in } R(f))\, d_f(s,v) \leq d_f(s,u) + 1 \leq d_{f+g}(s,u) + 1 = d_{f+g}(s,v)$$

    Therefore g auguments from v to u, however

    $$((v,u) \text{ is not in } R(f+g))\, d_f(s,v) + 2 = d_f(s,u) + 1 \leq d_{f+g}(s,u) + 1 = d_{f+g}(s,v)$$

    Ouch! Then the claim is wrong and $d_f(s,v)$ should be increasing.
  - Therefore for each bottleneck edge, next time the traversed direction between u,v will be different, and thus

    $$d_f(s,u) + 1 = d_f(s,v) \Longrightarrow d_{f+g}(s,v) + 1 = d_{f+g}(s,u)$$

    Each edge can only be bottleneck edge of $O(n)$ times $\Longrightarrow O(nm^2)$.