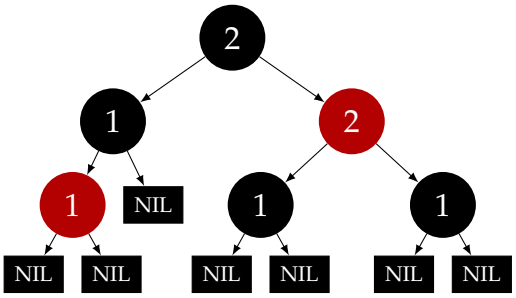
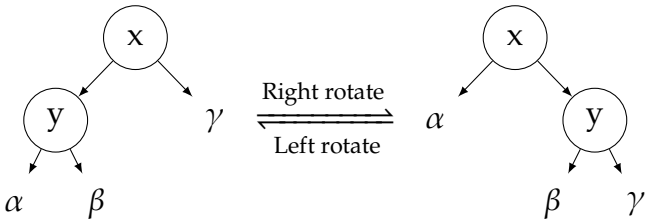
1 RB Tree

1.1 Black Height



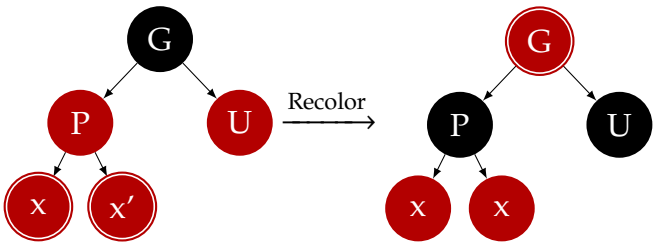
1.2 Rotation



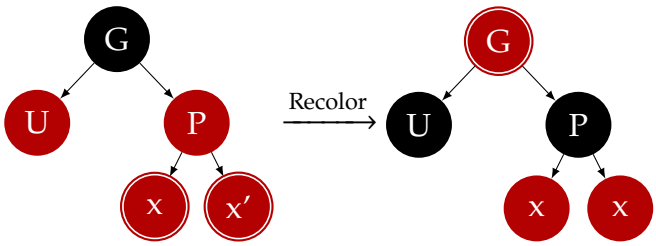
1.3 Insert Fixup

Loop invariant: Always red violation.

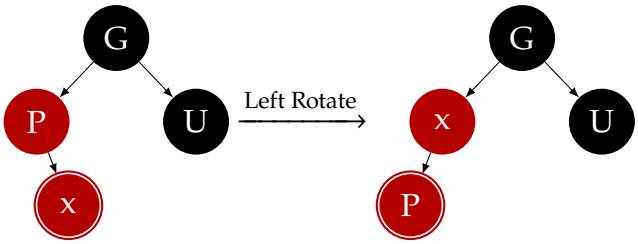
Case 1: **U**ncle is red (My **P**arent is left child)



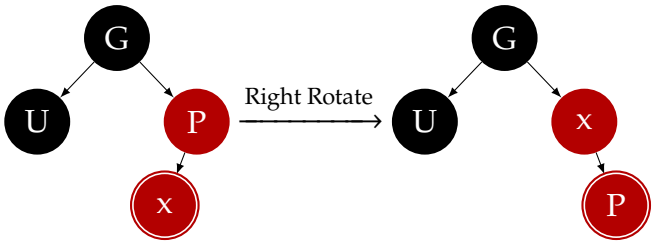
Case 4: **U**ncle is red (My **P**arent is right child)



Case 2: **U**ncle is black, I am his near nephew

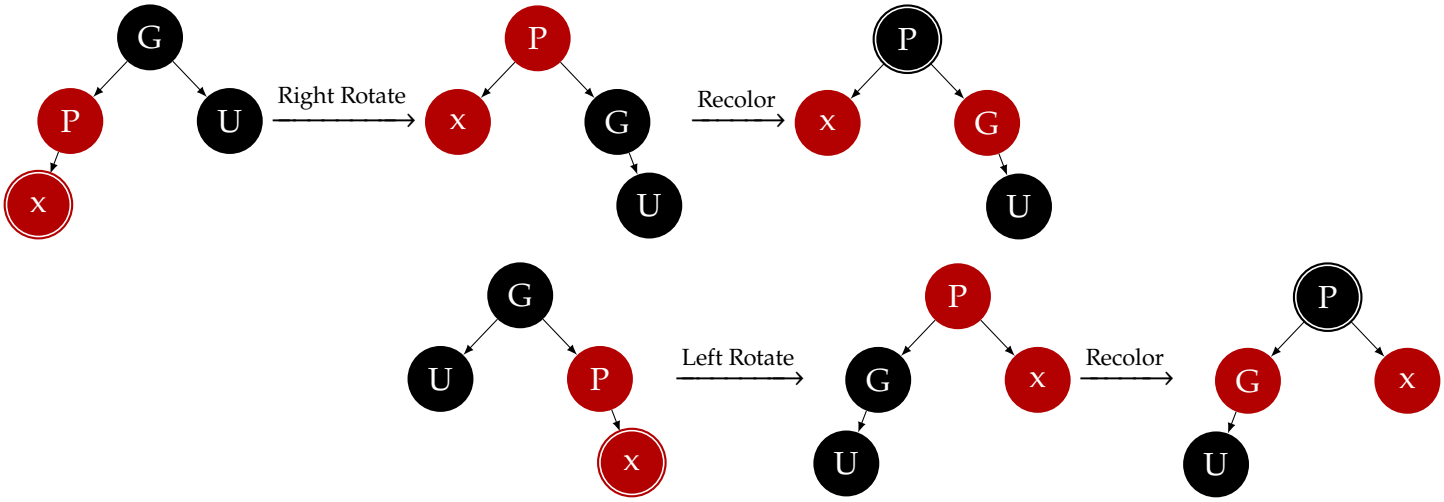


Case 5: **U**ncle is black, I am his near nephew



Case 3: **U**ncle is black, I am his distant nephew (my **P**arent is left child)

Case 6: **U**ncle is black, I am right child (my **P**arent is right child)



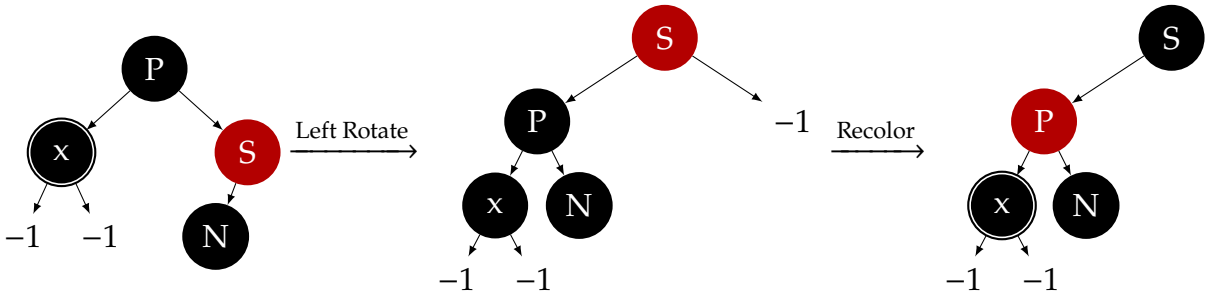
1.4 Delete

If target node has only one child, then move it up and call fixup. Otherwise, let current be z, next node be y, y has only a child x. We move y's key to z, and remove y (child moves up). Call fixup on y.

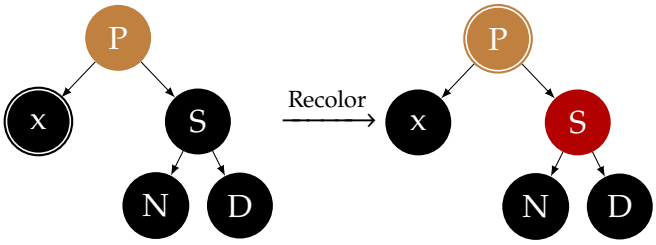
1.5 Delete Fixup

Loop invariant: Subtree of current node always missing one black height.
Case 0: I am red or root — color myself to black, and terminate (fixup only need for black)

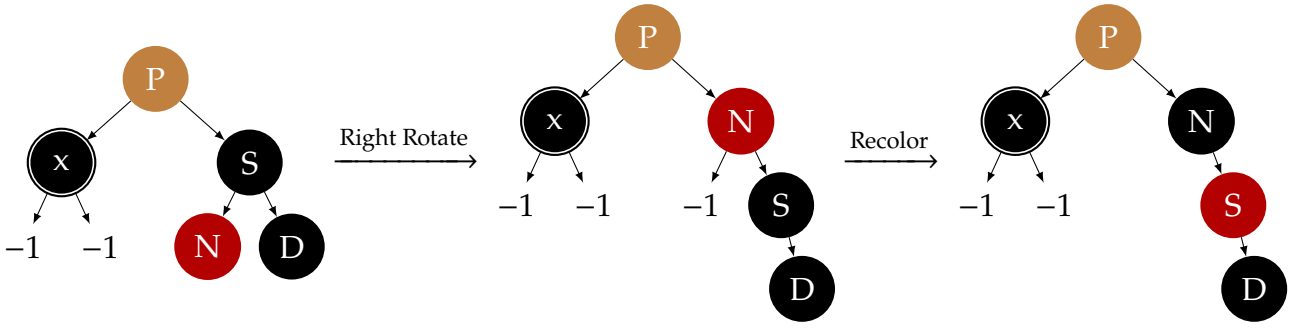
Case 1: My **S**ibling is red



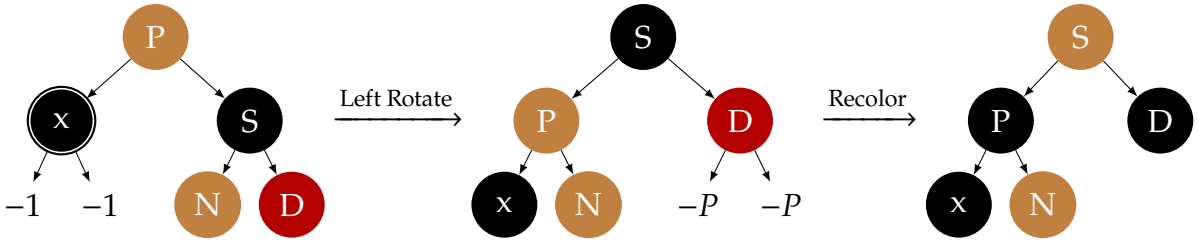
Case 2: My **S**ibling is black, and both its children are black



Case 3: My **S**ibling is black, and the **D**istant child is black



Case 4: My **S**ibling is black, and the **D**istant child is red



1.6 Pseudo Codes

```
LEFT-ROTATE(T, x)
 1  y = x.right
 2  x.right = y.left          // turn y's left subtree into x's right subtree
 3  if y.left ≠ T.nil         // if y's left subtree is not empty ...
 4      y.left.p = x          // ... then x becomes the parent of the subtree's root
 5  y.p = x.p                 // x's parent becomes y's parent
 6  if x.p == T.nil           // if x was the root ...
 7      T.root = y            // ... then y becomes the root
 8  elseif x == x.p.left      // otherwise, if x was a left child ...
 9      x.p.left = y          // ... then y becomes a left child
10  else x.p.right = y        // otherwise, x was a right child, and now y is
11  y.left = x                // make x become y's left child
12  x.p = y
```

```
RB-INSERT(T, z)
 1  x = T.root                // node being compared with z
 2  y = T.nil                 // y will be parent of z
 3  while x ≠ T.nil           // descend until reaching the sentinel
 4      y = x
 5      if z.key < x.key
 6          x = x.left
 7      else x = x.right
 8  z.p = y                   // found the location—insert z with parent y
 9  if y == T.nil
10      T.root = z            // tree T was empty
11  elseif z.key < y.key
12      y.left = z
13  else y.right = z
14  z.left = T.nil            // both of z's children are the sentinel
15  z.right = T.nil
16  z.color = RED             // the new node starts out red
17  RB-INSERT-FIXUP(T, z)     // correct any violations of red-black properties
```

```
RB-INSERT-FIXUP(T, z)
 1  while z.p.color == RED
 2      if z.p == z.p.p.left              // is z's parent a left child?
 3          y = z.p.p.right               // y is z's uncle
 4          if y.color == RED             // are z's parent and uncle both red?
 5              z.p.color = BLACK
 6              y.color = BLACK                          } case 1
 7              z.p.p.color = RED
 8              z = z.p.p
 9          else
10              if z == z.p.right
11                  z = z.p
12                  LEFT-ROTATE(T, z)                    } case 2
13              z.p.color = BLACK
14              z.p.p.color = RED                        } case 3
15              RIGHT-ROTATE(T, z.p.p)
16      else // same as lines 3–15, but with "right" and "left" exchanged
17          y = z.p.p.left
18          if y.color == RED
19              z.p.color = BLACK
20              y.color = BLACK
21              z.p.p.color = RED
22              z = z.p.p
23          else
24              if z == z.p.left
25                  z = z.p
26                  RIGHT-ROTATE(T, z)
27              z.p.color = BLACK
28              z.p.p.color = RED
29              LEFT-ROTATE(T, z.p.p)
30  T.root.color = BLACK
```

```
RB-TRANSPLANT(T, u, v)
1  if u.p == T.nil
2      T.root = v
3  elseif u == u.p.left
4      u.p.left = v
5  else u.p.right = v
6  v.p = u.p
```

```
RB-DELETE(T, z)
 1  y = z
 2  y-original-color = y.color
 3  if z.left == T.nil
 4      x = z.right
 5      RB-TRANSPLANT(T, z, z.right)        // replace z by its right child
 6  elseif z.right == T.nil
 7      x = z.left
 8      RB-TRANSPLANT(T, z, z.left)         // replace z by its left child
 9  else y = TREE-MINIMUM(z.right)          // y is z's successor
10      y-original-color = y.color
11      x = y.right
12      if y ≠ z.right                      // is y farther down the tree?
13          RB-TRANSPLANT(T, y, y.right)    // replace y by its right child
14          y.right = z.right               // z's right child becomes
15          y.right.p = y                   //     y's right child
16      else x.p = y                        // in case x is T.nil
17      RB-TRANSPLANT(T, z, y)              // replace z by its successor y
18      y.left = z.left                     // and give z's left child to y,
19      y.left.p = y                        //     which had no left child
20      y.color = z.color
21  if y-original-color == BLACK            // if any red-black violations occurred,
22      RB-DELETE-FIXUP(T, x)               //     correct them
```

```
RB-DELETE-FIXUP(T, x)
 1  while x ≠ T.root and x.color == BLACK
 2      if x == x.p.left                  // is x a left child?
 3          w = x.p.right                 // w is x's sibling
 4          if w.color == RED
 5              w.color = BLACK
 6              x.p.color = RED                          } case 1
 7              LEFT-ROTATE(T, x.p)
 8              w = x.p.right
 9          if w.left.color == BLACK and w.right.color == BLACK
10              w.color = RED                            } case 2
11              x = x.p
12          else
13              if w.right.color == BLACK
14                  w.left.color = BLACK
15                  w.color = RED                        } case 3
16                  RIGHT-ROTATE(T, w)
17                  w = x.p.right
18              w.color = x.p.color
19              x.p.color = BLACK
20              w.right.color = BLACK                    } case 4
21              LEFT-ROTATE(T, x.p)
22              x = T.root
23      else // same as lines 3–22, but with "right" and "left" exchanged
24          w = x.p.left
25          if w.color == RED
26              w.color = BLACK
27              x.p.color = RED
28              RIGHT-ROTATE(T, x.p)
29              w = x.p.left
30          if w.right.color == BLACK and w.left.color == BLACK
31              w.color = RED
32              x = x.p
33          else
34              if w.left.color == BLACK
35                  w.right.color = BLACK
36                  w.color = RED
37                  LEFT-ROTATE(T, w)
38                  w = x.p.left
39              w.color = x.p.color
40              x.p.color = BLACK
41              w.left.color = BLACK
42              RIGHT-ROTATE(T, x.p)
43              x = T.root
44  x.color = BLACK
```

2 B Tree

2.1 Pseudo Codes

```
B-TREE-SPLIT-CHILD(x, i)
1  y = x.c_i                          // full node to split
2  z = ALLOCATE-NODE()                // z will take half of y
3  z.leaf = y.leaf
4  z.n = t − 1
5  for j = 1 to t − 1                 // z gets y's greatest keys ...
6      z.key_j = y.key_{j+t}
7  if not y.leaf
8      for j = 1 to t                 // ... and its corresponding children
9          z.c_j = y.c_{j+t}
10 y.n = t − 1                        // y keeps t − 1 keys
11 for j = x.n + 1 downto i + 1       // shift x's children to the right ...
12     x.c_{j+1} = x.c_j
13 x.c_{i+1} = z                      // ... to make room for z as a child
14 for j = x.n downto i               // shift the corresponding keys in x
15     x.key_{j+1} = x.key_j
16 x.key_i = y.key_t                  // insert y's median key
17 x.n = x.n + 1                      // x has gained a child
18 DISK-WRITE(y)
19 DISK-WRITE(z)
20 DISK-WRITE(x)
```

```
B-TREE-SPLIT-ROOT(T)
1  s = ALLOCATE-NODE()
2  s.leaf = FALSE
3  s.n = 0
4  s.c_1 = T.root
5  T.root = s
6  B-TREE-SPLIT-CHILD(s, 1)
7  return s
```

```
B-TREE-INSERT-NONFULL(x, k)
1  i = x.n
2  if x.leaf                          // inserting into a leaf?
3      while i ≥ 1 and k < x.key_i    // shift keys in x to make room for k
4          x.key_{i+1} = x.key_i
5          i = i − 1
6      x.key_{i+1} = k                // insert key k in x
7      x.n = x.n + 1                  // now x has 1 more key
8      DISK-WRITE(x)
9  else while i ≥ 1 and k < x.key_i   // find the child where k belongs
10         i = i − 1
11     i = i + 1
12     DISK-READ(x.c_i)
13     if x.c_i.n == 2t − 1           // split the child if it's full
14         B-TREE-SPLIT-CHILD(x, i)
15         if k > x.key_i             // does k go into x.c_i or x.c_{i+1}?
16             i = i + 1
17     B-TREE-INSERT-NONFULL(x.c_i, k)
```

```
B-TREE-INSERT(T, k)
1  r = T.root
2  if r.n == 2t − 1
3      s = B-TREE-SPLIT-ROOT(T)
4      B-TREE-INSERT-NONFULL(s, k)
5  else B-TREE-INSERT-NONFULL(r, k)
```

2.2 Rules

Node	Min	Max Deg	Min	Max Keys
Root	0	$2t$	1	$2t-1$
Internal	t	$2t$	$t-1$	$2t-1$
Leaf		0	$t-1$	$2t-1$

2.3 Insertion

Start from root, split any full nodes. Then we can directly insert.

2.4 Deletion

Case 1 At leaf — delete

Case 2 Found in internal.

Case 2a Preceding child has t keys: steal.

Case 2b Succeeding child has t keys: steal.

Case 2c Adjacent children have $t-1$ keys: merge into a $(2t-1)$-key node, and recurse.

Case 3 Not found yet — ensure next node has t node for safe Deletion

Case 3a Preceding sibling has at least t keys: steal.

Case 3b Succeeding child has at least t keys: steal.

Case 3c Adjacent children have $t-1$ keys: merge into a $(2t-1)$-key node, and recurse.

In case 2c and 3c, root can be empty after operation. We remove it.

3 Disjoint Set

Method	Description	MAKE-SET	FIND	UNION
Array	Keep set ID	?	$O(1)$	$O(n)$
Tree	Keep tree	?	$O(n)$	$O(n)$
Linked-list	Keep head	$O(1)$	$O(1)$	$O(n)$
Array Small-to-Large		?	$O(1)$	$O(n)$ worst case, $O(\log n)$ amortized
Linked-list Small-to-Large		?	$O(1)$	$O(n)$ worst case, $O(\log n)$ amortized
Tree Small-to-Large		?	$O(\log n)$	$O(\log n)$
Full		$O(1)$	$O(\log n)$ worst case, $O(\alpha(n))$ amortized	$O(\log n)$ worst case, $O(\alpha(n))$ amortized

4 Hashing

- **Direct-address tables**: Array ($h(x) = x$).
- ***Uniform* hash function**: Probability of any probing sequence is the same (on slide).
- ***Simple uniform* hash function**: $P(h(x) = a) = |U|^{-1}$
- **Load factor** α: keys / slots
- **Division method**: $h(x) \equiv k \mod m$
- **Multiplication method**: $h(x) = \lfloor m(kA \mod 1) \rfloor = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$
- **Quadratic probing**: $h(x, i) = (h'(x) + ai + bi^2) \mod m$
- **Double hashing**: $h(x, i) = (h_1(x) + ih_2(x)) \mod m$
- **Primary clustering**: Consecutive filled slots produced by open addressing probing
- **Secondary clustering**: IDK
- **Dynamic hashing using directories**: (left) When overflow occurs, duplicate table with unchanged pointers. Lazy resolve correct new hash until touched.
- **Directoryless Dynamic hashing**: (right) Lazy resolve collision, branch cell from 0 to full, and starts from 0 again (space doubled every scan), collision is solved only when the cell is duplicated.