Intuitionistic Fuzzy Analytic Hierarchy Process

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***Abstract*—The intuitionistic fuzzy set has shown definite advantages in handling vagueness and uncertainty over a fuzzy set. Taking the powerfulness of the analytic hierarchy process (AHP) and the fuzzy AHP (FAHP) into account when tackling comprehensive multi-criteria decision-making problems, in this paper, we extend the classic AHP and the FAHP into the intuitionistic fuzzy AHP (IFAHP) in which the preferences are represented by intuitionistic fuzzy values. The IFAHP can be used to handle more complex problems, where the decision maker has some uncertainty in assigning preference values to the objects considered. The paper proposes a new way to check the consistency of an intuitionistic preference relation and then introduces an automatic procedure to repair the inconsistent one. It is worth pointing out that our proposed method can improve the inconsistent intuitionistic preference relation without the participation of the decision maker, and thus, it can save much time and show some advantages over the AHP and the FAHP. This paper also develops a novel normalizing rank summation method to derive the priority vector of an intuitionistic preference relation, on which the priorities of the hierarchy in the IFAHP are derived. The procedure of the IFAHP is given in detail, and an example concerning global supplier development is used to demonstrate our results.**

***Index Terms*—Decision making, intuitionistic hierarchy analysis process, intuitionistic fuzzy set, intuitionistic preference relation.**

I. Introduction

THE analytic hierarchy process (AHP), which was originally developed by Saaty [1], is a decision-making procedure widely used in management science and operations research for establishing priorities within the context of multicriteria decision making. It assists the decision maker to solve the problem by decomposing a complex problem into a multilevel hierarchic structure of objectives, criteria, subcriteria and alternatives and provides a fundamental scale of relative magnitudes expressed in dominance units to represent judgments in the form of pairwise comparisons. After deriving a ratio scale of relative magnitudes that are expressed in priority units from each set of comparisons, an overall ratio scale of priorities is synthesized to rank the alternatives [2]. Three principles can be used to summarize the procedure of an AHP, which are decomposition, comparative judgments, and synthesis of priorities [3].

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In spite of its popularity and simplicity in handling multicriteria decision-making problems, the AHP is often criticized for its inability to adequately tackle the inherent uncertainty and vagueness. In the classic AHP model, the comparison values over different criteria that are provided by the decision maker are represented by crisp numbers within the 1–9 scale. However, in some realistic situations, the decision makers might be reluctant or unable to assign the crisp evaluation values to the comparison judgments due to his/her limited knowledge or the subjectivity of the qualitative evaluation criteria or the variations of individual judgments in group decision making [4]. Hence, the conventional AHP seems to be inadequate to explicitly capture the important assessments for deriving the priorities in these situations. A person's decision-making process is generally affected by the information that is available to them about the state of the world [5]. To overcome this issue, the fuzzy set theory [6] was introduced to the AHP, and then, the traditional AHP was extended to the fuzzy AHP (FAHP), where each pairwise comparison judgment is represented as a fuzzy number that is described by a membership function. The membership function denotes the degree to which elements considered belong to the preference set. The earliest study of the FAHP was done by Van Laarhoven and Pedrycz [7], who directly extended Saaty's AHP with triangular fuzzy numbers and used a logarithmic least-squares method to derive fuzzy weights and fuzzy performance scores for ranking alternatives. Later on, Buckley [8] extended the classical AHP with trapezoidal fuzzy numbers and obtained the fuzzy weights and fuzzy performance scores via a geometric mean method. Boender *et al.* [9] modified Van Laarhoven and Pedrycz's method and proposed a more robust approach to the normalization of the local priorities. After that, Chang [10] used a row mean method to derive priorities for comparison ratios in the context of triangular fuzzy numbers. Recently, Wang *et al.* [11] reexamined Chang's method through three numerical examples. Since Chang's method is relatively easier than the other FAHP approaches and similar to the conventional AHP, it has been applied, as a representation of the FAHP, to many different areas, such as quality function deployment [12], transportation management [13], technology management [14], global supplier development [15], human capital management [16]. and safety management [17]. Nguyen and Gordon-Brown [18] employed the constrained fuzzy arithmetic during the FAHP application concerning portfolio selection. Although the computation of the FAHP is tedious, it is suitable to capture and represent a human's appraisal of ambiguity when the complex multi-criteria decision-making problems are considered.

Note that the FAHP that is mentioned in this paper means the analytical hierarchy process in which the pairwise comparison judgments are represented by the two kinds of ordinary fuzzy numbers, i.e., triangular fuzzy numbers (see [7] and [9]–[11]) and trapezoidal fuzzy numbers (see [8]). An ordinary fuzzy set A

in a set X is characterized by a membership function μ_A which takes the values in the interval $[0,1]$, i.e., $\mu_A : X \to [0,1]$. The value of μ_A at $x, \mu_A(x)$, named fuzzy number, represents the grade of membership (grade, for short) of x in A and is a point in $[0,1]$. For example, we can use the fuzzy set as

$$\begin{aligned} A &= \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + +\mu_A(x_3)/x_3 + \mu_A(x_4)/x_4 \\ &= 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3 \end{aligned} \tag{1}$$

to represent the linguistic term "low," where the operation "$+$" stands for logical sum (or).

The triangular fuzzy number and the trapezoidal fuzzy number are both fuzzy numbers, where the membership function of a triangular fuzzy number is [10]

$$\mu_A(x) = \begin{cases} \dfrac{x-l}{m-l}, & x \in [l, m] \\ \dfrac{u-x}{u-m}, & x \in [m, u] \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and the membership function of a trapezoidal fuzzy number is [8]

$$\mu_A(x) = \begin{cases} 0, & x \in [-\infty, l] \\ \dfrac{x-l}{m_1-l}, & x \in [l, m_1] \\ 1, & x \in [m_1, m_2] \\ \dfrac{u-x}{u-m_2}, & x \in [m_2, u] \\ 0, & x \in [u, +\infty] \end{cases} \tag{3}$$

where $l \le m \le u, l$ and u stand for the lower and upper values of the support of A, respectively, and m, m_1, and m_2 stand for the modal values. When $l = m = u$, the triangular fuzzy number reduces to a crisp number by convention, and when $m_1 = m_2$, the trapezoidal fuzzy number reduces to a triangular fuzzy number.

Central to the FAHP is using the fuzzy numbers (including triangular fuzzy numbers and trapezoidal fuzzy numbers) as the pairwise comparison scale, and it also shows some shortcomings due to the limitation of fuzzy set itself. Since the membership function of a fuzzy set is only single-valued function, it cannot be used to express the support and objection evidences simultaneously in many practical situations. If not possessing a precise or sufficient level of knowledge of the problem domain in cognition of things due to the complexity of the socio-economic environment, people usually have some uncertainty in assigning the preference evaluation values to the objects considered, which makes the judgments of cognitive performance exhibit the characteristics of affirmation, negation, and hesitation. For example, when getting together to evaluate an alternative, the decision makers may not only present "agreement" or "disagreement", but "abstention" as well, which indicates the hesitation and indeterminacy over the alternative. The same situation can be found in a voting process as well. In order to depict such sorts of realistic situations and to model human's perception and cognition more comprehensively, Antanassov [19]–[21] extended Zadeh's fuzzy set to intuitionistic fuzzy set (IFS), which is characterized by a membership function, a nonmembership function, and a hesitancy function. In practical applications, when evaluating some candidate alternatives, the decision makers may not be able to express their preferences accurately due to the fact that they may not grasp sufficient knowledge of the alternatives [22], or they are unable or unwilling to discriminate explicitly the degree to which alternative is better than others especially at the beginning of evaluation [23]. In other words, there is a certain degree of hesitation [24]. In such cases, the decision makers may provide their preferences for the alternative to a certain degree, and it is possible that they are not sure about it [25]. Thus, it is suitable to express the decision makers' preferences information in IFSs [26]. Up to now, the IFS has attracted increasingly scholars' attention and has been applied to many different fields, such as decision making [23], [26]–[29], fuzzy logics [30], fuzzy cognitive maps [31], fuzzy hardware [32], topological space [33], medical diagnosis [34], and pattern recognition [35].

To make our motivation more convincing and easier to understand, let us compare IFS with fuzzy set that is used in the FAHP deeply.

1) In fact, the IFS can be seen as a particular case of Type-2 fuzzy set. The type 2 fuzzy set, as an extension of the ordinary fuzzy set, is introduced to tackle the situations we usually encounter in reality that the grade itself is frequently ill-defined, as in the statement that the grade is "high," "low," "middle," "not high," or "very low." In this case, the ordinary fuzzy set cannot be used to represent these preferences. A type 2 fuzzy set $\tilde{A}$ in a set X is the fuzzy set which is characterized by a fuzzy membership function $\mu_{\tilde{A}}$ as $\mu_{\tilde{A}} : X \to [0,1]^J$, with the value $\mu_{\tilde{A}}(x)$ being called a fuzzy grade and being a fuzzy set in $[0,1]$ or in the subset J of $[0,1]$. The ordinary fuzzy set has a grade of membership that is crisp, whereas a type 2 fuzzy set has the grades of membership that are fuzzy. For example, suppose that $X = \{Lily, Lucy, Susie\}$ is a set of women and that $\tilde{A}$ is a type 2 fuzzy set of "beautiful" women in X, then we may have

$$\tilde{A} = \text{beautiful} = \text{middle/Lily} + \text{not low/Lucy} + \text{high/Susie} \tag{4}$$

where the fuzzy grades labeled "middle," "not low," and "high" are formed into a fuzzy set shown in analogy to (1). Since IFS can represent membership degree, nonmembership degree, and hesitancy degree by the three grades of membership function respectively, it can be seen as a particular case of type 2 fuzzy set. However, the triangular fuzzy numbers and the trapezoidal fuzzy numbers do not have this property, and each of them can only represent one grade of membership that is crisp in the unit interval $[0,1]$.

2) Despite the IFS has been proven to be equivalent to the interval-valued fuzzy set to some extent, a number of references we can find in the literature on IFSs (around 1000 papers actually; see [36]) may suggest that many researchers

find the advantages of IFSs over the equivalent interval-valued fuzzy sets.

3) Note that all of triangular fuzzy numbers, trapezoidal fuzzy numbers, and interval-valued fuzzy numbers can only be used to depict the fuzziness of "agreement" with a triple or quadruple or value range, respectively, but cannot reflect the "disagreement" of the decision maker.
4) Furthermore, they do not leave any room for indeterminacy between each fuzzy set and its negation as well. However, in the realist recognition of human beings, "disagreement" and indeterminacy are very common and useful in describing their opinions in decision making. For example, when intending to evaluate a person's credit, using the above three mentioned fuzzy numbers (i.e., triangular fuzzy numbers, trapezoidal fuzzy numbers, and interval-valued fuzzy numbers), respectively, to represent the membership degree with respect to positive aspect such as "trustworthy" may not be adequate and sometimes the decision makers also do not want to give their assessments from the positive point of view but from the negative aspect due to their personal considerations. The ignorance (or indeterminacy) is associated to the lack of information. Not including an extra ignorance stage in a model subject to preference learning may sometimes be unrealistic. The introduction of this ignorance statement, which is represented as the hesitancy function in an IFS, is the most characteristic of the IFS. Describing the opinions from three sides (i.e., membership degree, nonmembership degree, and indeterminacy degree (or hesitancy degree)) can ensure the preference information more comprehensive.

Considering that the preferences are essentially judgments of human beings that are based on perception, the FAHP, where the judgments are denoted by fuzzy set, may not satisfy the requirements of the comprehensive modern society [37]. As the IFS theory has been expanded in both depth and scope, it is urgent and natural to investigate the AHP within the context of intuitionistic fuzzy information. In this paper, we extend the classical AHP and the FAHP to the intuitionistic fuzzy circumstances and develop the IFAHP procedure for handling comprehensive multi-criteria decision-making problems. To do so, the remainder of this paper is constructed as follows: The next section describes how to decompose the comprehensive multicriteria decision-making problem into a hierarchy and then gives the interpretation of pairwise comparison with intuitionistic fuzzy values. Some basic knowledge of intuitionistic preference relation is also set out briefly at the end of this section. Section III presents the consistency checking method for the intuitionistic preference relation. The priority vector deriving method can be found in Section IV. In Section V, the procedure of the IFAHP is proposed. An illustrative example about global supplier development is shown in Section VI. Section VII gives some concluding remarks.

II. Decomposition and Comparative Judgment With Intuitionistic Fuzzy Values

As presented in the introduction, the AHP and the FAHP involve three phases: 1) decomposition, 2) pairwise comparison, and 3) synthesis of priorities. To extend the conventional AHP and the FAHP to intuitionistic fuzzy circumstances, we investigate these phases in the context of IFS one by one next.

A. Decomposition

The decomposition principle calls for structuring the hierarchy to capture the basic elements of the problem. This step in the context of the IFAHP is the same as in the AHP and the FAHP. Hence, we do not pay much attention to this phase in this paper.

The AHP, FAHP, and IFAHP are used to handle a variety of complex and comprehensive multi-criteria decision-making problems which have several alternatives to implement and quite a lot of criteria are used to check and evaluate the implementation. In order to apply the IFAHP, initially, the comprehensive problem needs to be structured into different hierarchical levels with regards to the properties or attributes of the problem considered. Philosophers often make a distinction between properties and attributes, but in this paper, we take these terms as interchangeable and refer to them as criteria. An affinity diagram, a tree diagram, or cluster analysis can be applied to construct the hierarchy [12]. For simplifying the presentation, we suppose that $A = \{A_1, A_2, \ldots, A_n\}$ is the finite set of n alternatives, and $C = \{C_1, C_2, \ldots, C_m\}$ is the set of criteria with which the elements of A are compared in the hierarchical structure.

B. Comparative Judgments With Intuitionistic Fuzzy Values

After decomposing the complex multi-criteria decision-making problem into different levels, we need to set up a preference relation to carry out pairwise comparisons of the relative importance of the elements in a level with respect to the elements in the level immediately above it. Employing (binary) relations to express preferences is arguably convenient from a knowledge acquisition point of view, especially since people often find it easier to compare two alternatives than to assess single alternatives in terms of numerical utility degrees [38]. In order to represent the relative importance between the pairwise compared elements, Saaty [39] gave the definition of fundamental scale as follows.

Definition 1 [39]: Let $A = \{A_1, A_2, \ldots, A_n\}$ be a finite set of alternatives, and $C = \{C_1, C_2, \ldots, C_m\}$ be a set of criteria to compare the alternatives. A fundamental scale for the criteria $C_j \in C (j = 1, 2, \ldots, m)$ is a mapping P_{C_j}, which assigns to every pair $(A_i, A_k) \in A \times A$ a positive real number $P_{C_j}(A_i, A_k) = a_{ik}$ that denotes the relative intensity with which an individual perceives the criterion $C_j \in C$ in an element $A_i \in A$ in relation to the other $A_k \in A$.

Simultaneously, Saaty also developed the 1–9 scale to describe the preferences between alternatives as being either equally, moderately, strongly, very strongly, or extremely preferred. These preferences are translated into pairwise weights of one, three, five, seven, or nine, respectively, with two, four, six, eight as the intermediate values (see Table I for more details).

The 1–9 scale satisfies the reciprocal condition, which is the intensity of preference of A_i over A_k is inversely related to the intensity of preference of A_k over A_i, i.e.,

$$a_{ik} = 1/a_{ki}, \forall A_i, A_k \in A, \;\; C_j \in C, \;\; j = 1, 2, \ldots, m. \quad (5)$$

TABLE I
COMPARISON BETWEEN THE 1–9 SCALE AND THE 0.1–0.9 SCALE

1-9 scale	0.1-0.9 scale	Meaning
1/9	0.1	Extremely not preferred
1/7	0.2	Very strongly not preferred
1/5	0.3	Strongly not preferred
1/3	0.4	Moderately not preferred
1	0.5	Equally preferred
3	0.6	Moderately preferred
5	0.7	Strongly preferred
7	0.8	Very strongly preferred
9	0.9	Extremely preferred
other values between 1/9 and 9	other values between 0 and 1	Intermediate values used to present compromise

With the 1–9 scale, in general, Saaty pointed out that $A_i \succ_{C_j} A_k$ if and only if $a_{ik} > 1$, where the binary relation "$\succ_{C_j}$" represents "be preferred to" according to the criterion C_j; $A_i \sim_{C_j} A_k$ if and only if $a_{ik} = 1$, where the binary relation "$\sim_{C_j}$" represents "be indifferent to" according to the criterion C_j.

The use of the 1–9 scale is central to the conventional AHP. To obtain the ratio scale, the decision makers must elicit the ratio ω_i/ω_k measuring the relative dominance of the alternative i over the alternative k, which satisfies

$$a_{ik} = \frac{\omega_i}{\omega_k} \cdot e_{ik} \tag{6}$$

where w_i and w_k are the underlying subjective priority weights belonging to a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_i > 0$ $(i = 1, 2, \ldots, n)$, and $\sum_{i=1}^{n} \omega_i = 1$ (n is the number of alternatives), and e_{ik} is a multiplicative variable accounting for random errors and inconsistency observed in practice. It is assumed that e_{ik} is close to 1 and is reciprocally symmetric, i.e., $e_{ik} = 1/e_{ki}$ with $e_{ii} = 1, \ \ i = 1, 2, \ldots, n$. All the judgments are stored in an $n \times n$ matrix $\Omega = (a_{ik})_{n\times n}$, or called a multiplicative preference relation.

Definition 2 [1]: A multiplicative preference relation Ω on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $\Omega = (a_{ik})_{n\times n}$, where a_{ik} is the intensity of preference of x_i over x_k measured using the 1–9 scale, and satisfies the reciprocal condition, i.e., $a_{ik} = 1/a_{ki} \forall x_i, x_k \in X$.

Note that all judgments in the multiplicative preference relation that corresponds to Saaty's 1–9 scale are crisp values, which are hard to be exactly determined due to the complexity and uncertainty that are involved in the real-world decision-making problems and incomplete information or knowledge. Nonetheless, it is suitable to represent these judgments by fuzzy numbers, each of which is characterized by a membership function, denoting the relative intensity with which the alternative A_i is preferred to A_k over the criterion $C_j \in C$. If all the pairwise comparison values are represented by fuzzy numbers within the interval $[0, 1]$ and are stored in a matrix, a fuzzy preference relation $\Delta = (b_{ik})_{n\times n}$ can be generated, where $0 \le b_{ik} \le 1$, and $b_{ik} + b_{ki} = 1$. b_{ik} indicates the degree that the alternative A_i is preferred to A_k. Concretely speaking, the case $b_{ik} = 0.5$ indicates that there is indifference between the alternatives A_i and A_k; $b_{ik} > 0.5$ indicates that the alternative A_i is preferred to A_k, especially, $b_{ik} = 1$ means that the alternative A_i is absolutely preferred to A_k; $b_{ik} < 0.5$ indicates that the alternative A_k is preferred to A_i, especially, $b_{ik} = 0$ means that the alternative A_k is absolutely preferred to A_i.

Definition 3 [23], [40]–[42]: A fuzzy preference relation Δ on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $\Delta = (b_{ik})_{n\times n}$, where b_{ik} is the intensity of preference of x_i over x_k, and satisfies $b_{ij} \in [0, 1]$, and $b_{ik} + b_{ki} = 1$, $\forall x_i, x_k \in X$.

The substantial difference between the multiplicative preference relation and the fuzzy preference relation are the scales they used: the former utilizes the unbalance distribution scale to depict the judgments, while the latter assumes the grades between "extremely not preferred" and "extremely preferred" are distributed uniformly and symmetrically (see Table I for more details). The former scale satisfies the reciprocal condition, shown in (5), while the latter satisfies the additional condition, i.e., $b_{ik} + b_{ki} = 1$.

Analogous to 1–9 scale, the relationship between the fuzzy judgments b_{ik} and the underlying subjective priority weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ can be obtained, which is

$$b_{ij} = \frac{\omega_i}{\omega_i + \omega_j}, \quad \text{for all } i, j = 1, 2, \ldots, n \tag{7}$$

Although the FAHP can do better than the classic AHP in capturing and representing a human's appraisal of ambiguity for a comprehensive multi-criteria decision-making problem, the single membership function of fuzzy set also limits the application of the FAHP because it cannot be used to express the support and objection evidences simultaneously. As presented in the introduction, the IFS, which is characterized by a membership function, a nonmembership function, and a hesitancy function, is suitable to model these situations and then the scale of the pairwise comparison can be represented by an intuitionistic fuzzy set.

Definition 4 [19]–[21]: Let a crisp set X be fixed, and let $A \subset X$ be a fixed set. An intuitionistic fuzzy set (IFS) $\tilde{A}$ in X is an object of the following form:

$$\tilde{A} = \{(x, \mu_A(x), v_A(x)) | x \in X\} \tag{8}$$

where the functions $\mu_A : E \to [0, 1]$ and $v_A : E \to [0, 1]$ define the degree of membership and the degree of nonmembership of the element $x \in X$ to the set A, respectively, and for every $x \in X, 0 \le \mu_A + v_A \le 1$ holds.

Definition 5 [19]–[21]: For each IFS $\tilde{A}$ on X, then

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x) \tag{9}$$

is called the degree of nondeterminacy (uncertainty) of the membership of the element $x \in X$ to the set A. In the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.

Szmidt and Kacprzyk [24] also pointed out that $\pi_A(x)$ is a hesitancy degree of x to A, and justified that $\pi_A(x)$ cannot be omitted when calculating the distance between two IFSs. Obviously, $\pi_A(x) \in [0, 1]$ for all $x \in X$. For convenience, Xu [43] called $\alpha = (\mu_\alpha, v_\alpha, \pi_\alpha)$ an intuitionistic fuzzy value

(IFV), where $\mu_\alpha \in [0,1]$, $v_\alpha \in [0,1]$, $\pi_\alpha \in [0,1]$, $\mu_\alpha + v_\alpha \leq 1$, and let Ψ be the set of all IFVs.

In order to rank the IFVs, Szmidt and Kacprzyk [44] proposed a function, which is in the mathematic form:

$$\rho(\alpha) = 0.5(1+\pi_\alpha)(1-\mu_\alpha) \tag{10}$$

The smaller the value of $\rho(\alpha)$, the greater the IFV α in the sense of the amount of positive information included and reliability of information.

Example 1: Consider three IFVs $\alpha_1 = (0.9, 0)$, $\alpha_2 = (0.0001, 0.9)$, and $\alpha_3 = (0, 0.9)$. With the formula (10), we can get $\rho(\alpha_1) = 0.055$, $\rho(\alpha_2) = 0.5499$, $\rho(\alpha_3) = 0.55$, and thus, $\alpha_1 > \alpha_2 > \alpha_3$, which is consistent with our intuition.

C. Intuitionistic Preference Relation

If all the pairwise comparison judgments are represented by IFVs, an intuitionistic preference relation can be obtained in nature. Xu [26] introduced the concept of intuitionistic preference relation.

Definition 6 [26]: An intuitionistic preference relation R on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $R = (r_{ik})_{n\times n}$, where $r_{ik} = \langle (x_i, x_k), \mu(x_i, x_k), v(x_i, x_k)\rangle$ for all $i, k = 1, 2, \ldots, n$. For convenience, we let $r_{ik} = (\mu_{ik}, v_{ik})$, where μ_{ik} denotes the degree to which the object x_i is preferred to the object x_k, v_{ik} indicates the degree to which the object x_i is not preferred to the object x_k, and $\pi(x_i, x_k) = 1 - \mu(x_i, x_k) - v(x_i, x_k)$ is interpreted as an indeterminacy degree or a hesitancy degree, with the condition

$$\begin{aligned}&\mu_{ik},\ v_{ik} \in [0,1],\ \mu_{ik} + V_{ik} \leq 1,\ \mu_{ik} = v_{ki},\ \mu_{ki} = v_{ik}\\ &\mu_{ii} = v_{ii} = 0.5,\ \pi_{ik} = 1 - \mu_{ik} - v_{ik}\\ &\qquad\qquad\qquad\qquad \text{for all } i, k = 1, 2, \ldots, n\end{aligned} \tag{11}$$

For any two IFVs $r_{ik} = (\mu_{ik}, v_{ik})$ and $r_{tl} = (\mu_{tl}, v_{tl})$ in R, Deschrijver *et al.* [45] introduced the notion of intuitionistic fuzzy t-norm and t-conorm and investigated under what conditions a representation theorem can be obtained. Motivated by the intuitionistic fuzzy triangular norms and conorms, Xu [26] introduced the following operations:

1) $r_{ik} \oplus r_{tl} = (\mu_{ik} + \mu_{tl} - \mu_{ik}\mu_{tl},\ v_{ik}v_{tl})$;
2) $r_{ik} \otimes r_{tl} = (\mu_{ik}\mu_{tl},\ v_{ik} + v_{tl} - v_{ik}v_{tl})$;
3) $\lambda r_{ik} = (1 - (1-\mu_{ik})^\lambda,\ v_{ik}^\lambda),\ \lambda > 0$;
4) $r_{ik}^\lambda = (\mu_{ik}^\lambda,\ 1 - (1-v_{ik})^\lambda),\ \lambda > 0$.

The intuitionistic preference relation R may have several properties [26]:

Let $R = (r_{ik})_{n\times n}$ be an intuitionistic preference relation, where $r_{ik} = (\mu_{ik}, v_{ik}), i, k = 1, 2, \ldots, n$. By using the comparison law of IFVs, we have

1) if $r_{it} \oplus r_{tk} \geq r_{ik}$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the triangle condition;
2) if $r_{it} \geq (0.5, 0.5), r_{tk} \geq (0.5, 0.5) \Rightarrow r_{ik} \geq (0.5, 0.5)$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the weak transitivity property;
3) if $r_{ik} \geq \min\{r_{it}, r_{tk}\}$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the max–min transitivity property;
4) if $r_{ik} \geq \max\{r_{it}, r_{tk}\}$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the max–max transitivity property;
5) if $r_{it} \geq (0.5, 0.5), r_{tk} \geq (0.5, 0.5) \Rightarrow r_{ik} \succeq \min\{r_{it}, r_{tk}\}$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the restricted max–min transitivity property;
6) if $r_{it} \geq (0.5, 0.5), r_{tk} \geq (0.5, 0.5) \Rightarrow r_{ik} \geq \max\{r_{it}, r_{tk}\}$, for all $i, t, k = 1, 2, \ldots, n$, then we say R satisfies the restricted max–max transitivity property.

III. Consistency Checking

In the IFAHP, in order to get a reasonable solution, before deriving the priorities of the alternatives and criteria, we need to check whether the intuitionistic preference relation is consistent or not. Consistency is an important topic in preference relations and the lack of consistency of preference relations may lead to misleading solutions. In the conventional AHP, Saaty [1] provided a consistency index CI and a consistency ratio CR to measure the degree of consistency for a multiplicative preference relation, whose mathematical forms were given as:

$$\mathrm{CI} = (\lambda_{\max} - n)/(n-1) \tag{12}$$

$$\mathrm{CR} = \mathrm{CI}/\mathrm{RI}\,(n) \tag{13}$$

where $\lambda_{\max}$ is the largest eigenvalue of the multiplicative preference relation, n is the dimension of the multiplicative preference relation and $\mathrm{RI}\,(n)$ is a random index which depends on n. Saaty [1] pointed out that if the consistency ratio CR is less than 0.1, the multiplicative preference relation is of acceptable consistency; otherwise, the multiplicative preference relation is inconsistent and has to be returned to the decision maker for reevaluation until acceptable. As to the FAHP, many researchers, such as Chang [10], Kulak and Kahraman [13], Erensal *et al*. [14], Bozbura *et al*. [16], and Dağdeviren and Yüksel [17], did not take the consistency checking process of the fuzzy preference relation into account. This is also the flaw of the FAHP that is developed or applied by them. Although Kwong and Bai [12] and Chan and Kumar [15] realized this drawback of the FAHP, they all tackled the consistency of fuzzy preference relations by converting the fuzzy reference relations into its corresponding crisp multiplicative preference relations, and then using Saaty's method to check the consistency. In addition, they didn't discuss how to repair the inconsistent fuzzy preference relation into a consistent one. Saaty's method is powerful in checking the consistency but cannot repair or improve the inconsistent preference relation automatically except returning the inconsistent preference relation to the decision maker for reevaluation. Note that the interactive process is time consuming, and sometimes, the decision maker does not want to participate in this repairing process because of a lack of interest and motivation to continue with the tedious supervision. In the following, we will use another method to check the consistency of an intuitionistic preference relation and then repair the inconsistent intuitionistic preference relation into a consistent one automatically.

Multiplicative consistency is a very important property of preference relations. Genc *et al.* [46] employed this property to discover whether an interval fuzzy preference relation is consistent or not, as well as to derive the priority vector of a consistent interval fuzzy preference relation. Xu *et al*. [47] gave the definition of multiplicative consistent intuitionistic preference relation as follows:

Definition 7 [47]: An intuitionistic preference relation $R = (r_{ik})_{n\times n}$ with $r_{ik} = (\mu_{ik}, v_{ik})(i, k = 1, 2, \ldots, n)$ is multiplicative consistent if

$$\mu_{ik} = \begin{cases} 0, & \text{if } (\mu_{it}, \mu_{tk}) \in \{(0,1),(1,0)\} \\ \dfrac{\mu_{it}\mu_{tk}}{\mu_{it}\mu_{tk} + (1-\mu_{it})(1-\mu_{tk})}, & \text{otherwise} \end{cases} \quad \text{for all } i \le t \le k \tag{14}$$

$$v_{ik} = \begin{cases} 0, & \text{if } (v_{it}, v_{tk}) \in \{(0,1),(1,0)\} \\ \dfrac{v_{it}v_{tk}}{v_{it}v_{tk} + (1-v_{it})(1-v_{tk})}, & \text{otherwise} \end{cases} \quad \text{for all } i \le t \le k \tag{15}$$

Note that if $(\mu_{it}, \mu_{tk}) \in \{(0,1),(1,0)\}$, which means $(\mu_{it}, \mu_{tk}) = (0,1)$ or $(\mu_{it}, \mu_{tk}) = (1,0)$ or both of them hold, the denominator in (14) will be equal to 0, i.e., $\mu_{it}\mu_{tk} + (1-\mu_{it})(1-\mu_{tk}) = 0$; thus, $\mu_{ik} = \frac{\mu_{it}\mu_{tk}}{\mu_{it}\mu_{tk}+(1-\mu_{it})(1-\mu_{tk})}$ makes no sense. Hence, we let $\mu_{ik} = 0$, when $(\mu_{it}, \mu_{tk}) \in \{(0,1),(1,0)\}$. Similarly, in (11), we let $v_{ik} = 0$, when $(v_{it}, v_{tk}) \in \{(0,1), (1,0)\}$.

As to the fuzzy preference relation, Xia and Xu [48] have proven the following theorem:

Theorem 1 [48]: For a fuzzy preference relation $\Delta = (b_{ik})_{n\times n}$, the following statements are equivalent:

1) $b_{ik} = \dfrac{b_{it}b_{tk}}{b_{it}b_{tk} + (1-b_{it})(1-b_{tk})}$, $i, t, k = 1, 2, \ldots, n$;

2) $b_{ik} = \dfrac{\sqrt[n]{\prod_{s=1}^{n} b_{is}b_{sk}}}{\sqrt[n]{\prod_{s=1}^{n} b_{is}b_{sk}} + \sqrt[n]{\prod_{s=1}^{n}(1-b_{is})(1-b_{sk})}}$, i $k = 1, 2, \ldots, n$.

Motivated by Theorem 1, we can develop an algorithm to construct a perfect multiplicative consistent intuitionistic preference relation $\bar{R} = (\bar{r}_{ik})_{n\times n}$:

Algorithm I

Step 1: For $k > i+1$, let $\bar{r}_{ik} = (\bar{\mu}_{ik}, \bar{v}_{ik})$, where

$$\bar{\mu}_{ik} = \frac{\sqrt[k-i-1]{\prod_{t=i+1}^{k-1} \mu_{it}\mu_{tk}}}{\sqrt[k-i-1]{\prod_{t=i+1}^{k-1} \mu_{it}\mu_{tk}} + \sqrt[k-i-1]{\prod_{t=i+1}^{k-1}(1-\mu_{it})(1-\mu_{tk})}} \quad k > i+1 \tag{16}$$

$$\bar{v}_{ik} = \frac{\sqrt[k-i-1]{\prod_{t=i+1}^{k-1} v_{it}v_{tk}}}{\sqrt[k-i-1]{\prod_{t=i+1}^{k-1} v_{it}v_{tk}} + \sqrt[k-i-1]{\prod_{t=i+1}^{k-1}(1-v_{it})(1-v_{tk})}} \quad k > i+1. \tag{17}$$

Step 2: For $k = i+1$, let $\bar{r}_{ik} = r_{ik}$.

Step 3: For $k < i$, let $\bar{r}_{ik} = (\bar{v}_{ki}, \bar{\mu}_{ki})$.

Example 2: Suppose that a decision maker provides his/her preference information over a set of alternatives x_1, x_2, x_3, x_4 in IFVs and thus constructs the following intuitionistic preference relation:

$$R = \begin{pmatrix} (0.5, 0.5) & (0.2, 0.6) & (0.3, 0.4) & (0.6, 0.2) \\ (0.6, 0.2) & (0.5, 0.5) & (0.5, 0.4) & (0.6, 0.4) \\ (0.4, 0.3) & (0.4, 0.5) & (0.5, 0.5) & (0.3, 0.2) \\ (0.2, 0.6) & (0.4, 0.6) & (0.2, 0.3) & (0.5, 0.5) \end{pmatrix}$$

We use Algorithm I to construct the perfect multiplicative consistent intuitionistic preference relation $\bar{R} = (\bar{r}_{ik})_{n\times n}$ for R.

Step 1: Use (16) and (17) to derive $\bar{r}_{13}, \bar{r}_{14}$, and $\bar{r}_{24}$. Taking $\bar{r}_{14}$ as an example, we have $i = 1, k = 4$, and

$$\bar{\mu}_{14} = \frac{\sqrt{\prod_{t=2}^{3} \mu_{1t}\mu_{t4}}}{\sqrt{\prod_{t=2}^{3} \mu_{1t}\mu_{t4}} + \sqrt{\prod_{t=2}^{3}(1-\mu_{1t})(1-\mu_{t4})}}$$

$$= \frac{\sqrt{0.2\times 0.6\times 0.3\times 0.3}}{\sqrt{0.2\times 0.6\times 0.3\times 0.3} + \sqrt{(1-0.2)\times(1-0.6)\times(1-0.3)\times(1-0.3)}}$$
$$= 0.2079$$

$$\bar{v}_{14} = \frac{\sqrt{\prod_{t=2}^{3} v_{1t}v_{t4}}}{\sqrt{\prod_{t=2}^{3} v_{1t}v_{t4}} + \sqrt[k-i-1]{\prod_{t=2}^{3}(1-v_{1t})(1-v_{t4})}}$$

$$= \frac{\sqrt{0.2\times 0.6\times 0.3\times 0.3}}{\sqrt{0.2\times 0.6\times 0.3\times 0.3} + \sqrt{(1-0.2)\times(1-0.6)\times(1-0.3)\times(1-0.3)}}$$
$$= 0.2899.$$

Similarly, we can calculate that $\bar{\mu}_{13} = 0.3333, \bar{v}_{13} = 0.5$, $\bar{\mu}_{24} = 0.3956$, and $\bar{v}_{24} = 0.2899$.

Step 2: We have $\bar{r}_{12} = r_{12} = (0.2, 0.6), \bar{r}_{23} = r_{23} = (0.5, 0.4)$, and $\bar{r}_{34} = r_{34} = (0.3, 0.2)$.

Step 3: We can get the lower triangular elements of the matrix, and the corresponding perfect multiplicative consistent intuitionistic preference relation is obtained as

$$\bar{R} = \begin{pmatrix} (0.5, 0.5) & (0.2, 0.6) & (0.3333, 0.5) & (0.2079, 0.2899) \\ (0.6, 0.2) & (0.5, 0.5) & (0.5, 0.4) & (0.3956, 0.2899) \\ (0.5, 0.3333) & (0.4, 0.5) & (0.5, 0.5) & (0.3, 0.2) \\ (0.2899, 0.2079) & (0.2899, 0.3956) & (0.2, 0.3) & (0.5, 0.5) \end{pmatrix}$$

From this example, we can see that we only update less than half of the elements in the original intuitionistic preference relation to construct the perfect multiplicative consistent intuitionistic preference relation $\bar{R} = (\bar{r}_{ik})_{n\times n}$ for R.

Definition 8: Let R be an intuitionistic preference relation, then we call R an acceptable multiplicative consistent intuitionistic preference relation, if

$$d(R, \bar{R}) < \tau \tag{18}$$

where $d(R, \bar{R})$ is the distance measure [44] between the given intuitionistic preference relation R and its corresponding perfect multiplicative consistent intuitionistic preference relation $\bar{R}$, which can be calculated by

$$d(\bar{R}, R) = \frac{1}{2(n-1)(n-2)} \sum_{i=1}^{n} \sum_{k=1}^{n} (|\bar{\mu}_{ik} - \mu_{ik}| + |\bar{v}_{ik} - v_{ik}| + |\bar{\pi}_{ik} - \pi_{ik}|) \quad (19)$$

and τ is the consistency threshold.

Consider that Saaty [1] derived a consistency ratio from the maximum eigenvalue of a multiplicative preference relation and a randomization process, and pointed out that the multiplicative preference relation is of acceptable consistency if its consistency ratio is less than 0.1. Without loss of generality, we also let $\tau = 0.1$ as the consistency threshold. Note that in (19), we take the denominator as $2(n-1)(n-2)$, but not $2n^2$, although the summation goes over n^2 elements of two compared matrices. The reason for this is, when $k = i+1, \bar{r}_{ik} = r_{ik}$, and when $k < i, \bar{r}_{ik} = (\bar{v}_{ki}, \bar{\mu}_{ki})$, which means that we only need to calculate the differences over the upper triangular elements, and the number of the upper triangular elements is $(n-1)(n-2)$.

Very often, the intuitionistic preference relation constructed by the expert is always with unacceptable multiplicative consistency due to the lack of knowledge or the hardness of discriminating the degree to which some alternatives are better than the others. Therefore, in order to get a reasonable solution, it is important to help the decision maker to obtain a consistent intuitionistic preference relation through adjusting or repairing the inconsistent one until it is with acceptable consistency. We certainly can return the inconsistent intuitionistic preference relation to the decision maker for reevaluation, but as mentioned previously, it is time consuming, and sometimes, the decision maker does not want to participate in this repairing process because of the lack of interest and motivation to continue with the tedious supervision. Hence, we need to develop some automatic algorithm to solve this problem.

For any inconsistent intuitionistic preference relation $R = (r_{ik})_{n\times n}$, we can use Algorithm I to transform it into its corresponding perfect multiplicative consistent intuitionistic preference relation $\bar{R} = (\bar{r}_{ik})_{n\times n}$. Meanwhile, the deviation $d(\bar{R}, R)$ between the initial intuitionistic preference relation and the transformed one can be calculated by using (19). If the deviation $d(\bar{R}, R)$ is too large, then we may think that the transformed intuitionistic preference relation $\bar{R}$ can not represent the initial preferences of the decision maker. It is desirable that the modified intuitionistic preference relation should not only have acceptable multiplicative consistency but maintain the original preference information of the decision maker as much as possible as well. Hence, it is proper to fuse the initial intuitionistic preference relation R and its corresponding perfect multiplicative consistent intuitionistic preference relation $\bar{R}$ into a new intuitionistic preference relation $\tilde{R} = (\tilde{r}_{ik})_{n\times n}$, where each element is defined as

$$\tilde{\mu}_{ik} = \frac{(\mu_{ik})^{1-\sigma}(\bar{\mu}_{ik})^{\sigma}}{(\mu_{ik})^{1-\sigma}(\bar{\mu}_{ik})^{\sigma} + (1-\mu_{ik})^{1-\sigma}(1-\bar{\mu}_{ik})^{\sigma}} \quad i,k = 1,2,\ldots,n \quad (20)$$

$$\tilde{v}_{ik} = \frac{(v_{ik})^{1-\sigma}(\bar{v}_{ik})^{\sigma}}{(v_{ik})^{1-\sigma}(\bar{v}_{ik})^{\sigma} + (1-v_{ik})^{1-\sigma}(1-\bar{v}_{ik})^{\sigma}} \quad i,k = 1,2,\ldots,n \quad (21)$$

where σ is a controlling parameter that is determined by the decision maker, the small the value of σ, the closer $\tilde{R}$ is to R. Especially if $\sigma = 0, \tilde{R} = R$; if $\sigma = 1, \tilde{R} = \bar{R}$. Obviously, $\tilde{R}$ is also an intuitionistic preference relation. Generally, the fused intuitionistic preference relation $\tilde{R}$ contains not only the preference information of the initial intuitionistic preference relation R but the preference information of its corresponding perfect multiplicative consistent intuitionistic preference relation $\bar{R}$ as well. The controlling parameter σ also represents the preference of the decision maker to some extent.

Based on the aforementioned analysis, an automatic algorithm to repair the inconsistent intuitionistic preference relation can be developed.

Algorithm II

Step 1: Suppose that p is the number of iterations. Let $p = 1$, and construct the perfect multiplicative consistent intuitionistic preference relation $\bar{R}$ from $R^{(p)}$ by Algorithm I.

Step 2: Calculate the distance $d(\bar{R}, R^{(p)})$ between R and $R^{(p)}$, where

$$d(\bar{R}, R^{(p)}) = \frac{1}{2(n-1)(n-2)} \sum_{i=1}^{n} \sum_{k=1}^{n} (|\bar{\mu}_{ik} - \mu_{ik}^{(p)}| + |\bar{v}_{ik} - v_{ik}^{(p)}| + |\bar{\pi}_{ik} - \pi_{ik}^{(p)}|) \quad (22)$$

If $d(\bar{R}, R^{(p)}) < \tau$, then output $R^{(p)}$; otherwise, go to the next step.

Step 3: Construct the fused intuitionistic preference relation $\tilde{R}^{(p)} = (\tilde{r}_{ik}^{(p)})_{n\times n}, (\tilde{r}_{ik}^{(p)} = (\tilde{\mu}_{ik}^{(p)}, \tilde{v}_{ik}^{(p)}))$ by using

$$\tilde{\mu}_{ik}^{(p)} = \frac{(\mu_{ik}^{(p)})^{1-\sigma}(\bar{\mu}_{ik})^{\sigma}}{(\mu_{ik}^{(p)})^{1-\sigma}(\bar{\mu}_{ik})^{\sigma} + (1-\mu_{ik}^{(p)})^{1-\sigma}(1-\bar{\mu}_{ik})^{\sigma}} \quad i,k = 1,2,\ldots,n \quad (23)$$

$$\tilde{v}_{ik}^{(p)} = \frac{(v_{ik}^{(p)})^{1-\sigma}(\bar{v}_{ik})^{\sigma}}{(v_{ik}^{(p)})^{1-\sigma}(\bar{v}_{ik})^{\sigma} + (1-v_{ik}^{(p)})^{1-\sigma}(1-\bar{v}_{ik})^{\sigma}} \quad i,k = 1,2,\ldots,n \quad (24)$$

where σ is a controlling parameter determined by the decision maker: The smaller the value of σ, the closer $\tilde{R}^{(p)}$ is to $R^{(p)}$. Let $R^{(p+1)} = \tilde{R}^{(p)}$, i.e., $\mu_{ik}^{(p+1)} = \tilde{\mu}_{ik}^{(p)}$, and $v_{ik}^{(p+1)} = \tilde{v}_{ik}^{(p)}$. Let $p = p+1$, and then, go to Step 2.

Through this algorithm, we can improve the consistency level of any intuitionistic preference relation automatically without losing much original information. Comparing this algorithm with the interactive method, our procedure can save a lot of time for the decision maker. It shows many advantages in helping the decision maker to reach a quick decision. This procedure is convergent, and the derived intuitionistic preference relation has weak transitivity.

IV. Priority Method

A fundamental scale stored in the preference relation does not directly give a scale of priorities. A scale of priorities is, according to Saaty's [39] idea, an n-dimensional vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ obtained from the multiplicative preference relation, and ω_i is a weight which accurately represents the relative dominance of the alternative A_i among the alternatives in A. One question that has triggered off a lot of debate over the classic AHP concerns the best prioritization method, that is, the method to derive the vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ for the multiplicative preference relation [49]. The aim of this section is to investigate a method to generate the global or composite priorities of the elements at the lowest level of the hierarchy.

Considering an intuitionistic preference relation $R = (r_{ik})_{n\times n}$, where $r_{ik} = (\mu_{ik}, v_{ik})$, since $\mu_{ik},\ v_{ik} \in [0,1]$, $\mu_{ik} + v_{ik} \le 1$, we have $\mu_{ik} \le 1 - v_{ik}$. Then, we can transform the pair (μ_{ik}, v_{ik}) into the interval $[\mu_{ik}, 1 - v_{ik}]$. Hence, the intuitionistic preference relation $R = ((\mu_{ik}, v_{ik}))_{n\times n}$ can be transformed into an interval-valued preference relation $R' = ([\mu_{ik}, 1 - v_{ik}])_{n\times n}$. If we want to derive the priority vector of the intuitionistic preference relation $R = (r_{ik})_{n\times n}$, we can accomplish it via analyzing the interval-valued preference relation $R' = (r'_{ik})_{n\times n} = ([\mu_{ik}, 1 - v_{ik}])_{n\times n}$. Based on the operational laws of intervals [50], we develop a new normalizing rank summation method to derive the priority weights as follows:

$$
\begin{aligned}
\omega_i &= \frac{\sum_{k=1}^{n} r'_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n} r'_{ik}} = \frac{\sum_{k=1}^{n}[\mu_{ik}, 1 - v_{ik}]}{\sum_{i=1}^{n}\sum_{k=1}^{n}[\mu_{ik}, 1 - v_{ik}]} \\
&= \frac{[\sum_{k=1}^{n}\mu_{ik}, \sum_{k=1}^{n}(1 - v_{ik})]}{[\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}, \sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})]} \\
&= \left[\frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})}, \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}}\right] \\
&\qquad i = 1, 2, \ldots, n \qquad (25)
\end{aligned}
$$

Then, we can transform each interval

$$
\left[\frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})}, \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}}\right]
$$

into a corresponding IFV

$$
\left(\frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})},\ 1 - \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}}\right).
$$

Thus

$$
\omega_i = \left(\frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})},\ 1 - \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}}\right) \qquad i = 1, 2, \ldots, n \qquad (26)
$$

from which we get the priority vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ of the intuitionistic preference relation $R = (r_{ik})_{n\times n}$, where each weight ω_i is an IFV. In fact, since $\mu_{ik} \le 1 - v_{ik}$, then, we get

$$
\begin{aligned}
&\frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})} + 1 - \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} \\
&= 1 + \frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}(1 - v_{ik})} - \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} \\
&\le 1 + \frac{\sum_{k=1}^{n}\mu_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} - \frac{\sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} \\
&= 1 + \frac{\sum_{k=1}^{n}\mu_{ik} - \sum_{k=1}^{n}(1 - v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} \\
&= 1 + \frac{\sum_{k=1}^{n}(\mu_{ik} - 1 + v_{ik})}{\sum_{i=1}^{n}\sum_{k=1}^{n}\mu_{ik}} \le 1 \qquad (27)
\end{aligned}
$$

Note that we have developed a new way of determining priorities of the compared elements, which is quite different from the methods that are used in the AHP and the FAHP. However, in our point of view, this does not influence the extension of the original AHP methodology. In fact, in the proposed FAHP (See [7]–[17]), the methods of priorities are also quite different from the classical AHP. Some other methods also can be used to derive the priorities [51].

V. Intuitionistic Fuzzy Analytic Hierarchy Process

Now, we give the procedure of intuitionistic fuzzy analytic hierarchy process, which is as follows.

Step 1: Identify the objective, criteria, subcriteria, and alternatives of the decision-making problem, and then, construct the hierarchy of the considered problem. Then, go to the next step.

Step 2: Determine the intuitionistic preference relations via the pairwise comparison between each criterion and subcriterion. Simultaneously, the alternatives are compared under each criterion or subcriterion, and then, the intuitionistic preference relations are constructed. The scale regarding the relative importance degrees is denoted as IFVs. Go to the next step.

Step 3: Check the consistency of each intuitionistic preference relation according to (18). If all of the intuitionistic preference relations are of acceptable consistency, go to Step 5; otherwise, go to Step 4.

Step 4: Repair the inconsistent intuitionistic preference relations according to Algorithm II (or return the inconsistent intuitionistic preference relations to the decision makers for reevaluation until they are acceptable). Then, go to the next step.

Step 5: Calculate the priority vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ of each intuitionistic preference relation through (26).

Step 6: Fuse all the weights from the lowest level to the highest level by the operations of IFVs introduced in Section II, rank the overall weights using the formula (10), and then, choose the best alternative.

Step 7: End.

Schematic diagram of the developed intuitionistic fuzzy AHP method for determining the priorities of alternatives is provided in Fig. 1.

VI. Illustrative Examples

We now consider a multi-criteria decision-making problem that concerns the global supplier development [15] to illustrate our procedure of the IFAHP.

The current globalized market trend identifies the necessity of the establishment of long-term business relationship with competitive global suppliers spread around the world. How to

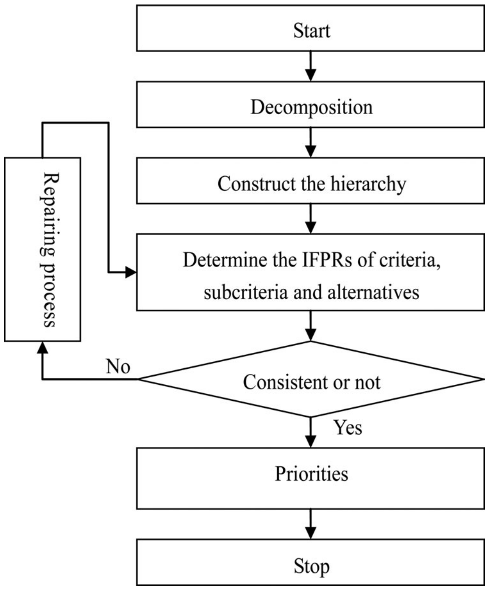

Fig. 1. Schematic diagram of the IFAHP.

select different unfamiliar international suppliers according to the broad comparison is very critical and has a direct impact on the performance of an organization. Global supplier development is a multi-criteria decision-making problem which includes both qualitative and quantitative factors. It is more complex than domestic one and it needs more critical analysis. Since most decision makers cannot handle more than nine factors when making decision [52], it is necessary to break down this complex problem into more manageable subproblems and thus the hierarchy can be constructed. Chan and Kumar [15] decomposed this problem into four levels, where the main objective is the selection of the best global supplier for manufacturing firm and the criteria considered in achieving the objective are as follows.

1) C_1: overall cost of the product, which consists of three subcriteria—S_1: product price, S_2: freight cost, and S_3: tariff;
2) C_2: quality of the product, which consists of four subcriteria—S_4: rejection rate of the product, S_5: increased lead time, S_6: quality assessment, and S_7: remedy for quality problems;
3) C_3: service performance of supplier, which consists of four subcriterion—S_8: delivery schedule, S_9: technological and R&D support, S_{10}: response to changes, and S_{11}: ease of communication;
4) C_4: supplier's profile, which consists of four subcriteria—S_{12}: financial status, S_{13}: customer base, S_{14}: performance history, S_{15}: production facility and capacity;
5) C_5: risk factor, which consists of four subcriteria—S_{16}: geographical location, S_{17}: political stability, S_{18}: economy, and S_{19}: terrorism.

Suppose there are three suppliers under consideration. Then, the decomposed hierarchy can be shown as in Fig. 2. The hierarchy consists of four levels. The overall objective is placed at Level 1, criteria at Level 2, subcriteria at Level 3, and alternatives at Level 4. After building the hierarchy, the pairwise comparison of the importance of one criterion, subcriterion, or alternative over another can be done with the help of the questionnaire. It can be determined by the available research, the current business scenario or by the experience of the experts. To simplify the presentation, in this paper, we do not want to pay attention to the comparison of the subcriteria but take them as a whole. In practice, we can do it in detail, but the method remains the same. Suppose that the comparison judgments are represented in IFVs and shown as Tables II–VII.

In the following, we use our proposed IFAHP method to solve this problem:

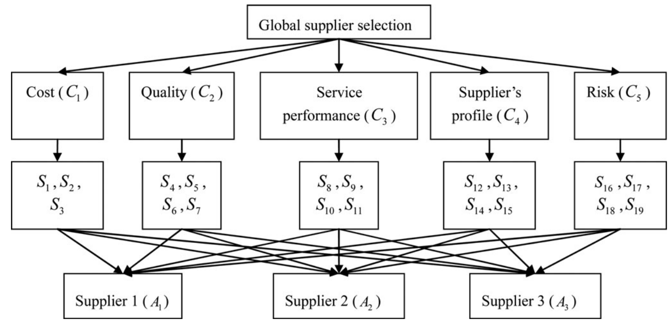

Fig. 2. Hierarchy for the global supplier selection.

TABLE II
INTUITIONISTIC PREFERENCE RELATION OF CRITERIA WITH RESPECT TO THE OVERALL OBJECTIVE

R	C_1	C_2	C_3	C_4	C_5
C_1	(0.5,0.5)	(0.6,0.2)	(0.6,0.2)	(0.65,0.25)	(0.65,0.25)
C_2	(02.,0.6)	(0.5,0.5)	(0.6,0.2)	(0.65,0.25)	(0.65,0.25)
C_3	(0.2,0.6)	(0.2,0.6)	(0.5,0.5)	(0.6,0.2)	(0.6,0.2)
C_4	(0.25,0.65)	(0.25,0.65)	(0.2,0.6)	(0.5,0.5)	(0.6,0.2)
C_5	(0.25,0.65)	(0.25,0.65)	(0.2,0.6)	(0.2,0.6)	(0.5,0.5)

TABLE III
INTUITIONISTIC PREFERENCE RELATION OF ALTERNATIVES WITH RESPECT TO THE CRITERION C_1

R_1	A_1	A_2	A_3
A_1	(0.5,0.5)	(0.6,0.2)	(0.4,0.5)
A_2	(0.2,0.6)	(0.5,0.5)	(0.3,0.4)
A_3	(0.5,0.4)	(0.4,0.3)	(0.5,0.5)

TABLE IV
INTUITIONISTIC PREFERENCE RELATION OF ALTERNATIVES WITH RESPECT TO THE CRITERION C_2

R_2	A_1	A_2	A_3
A_1	(0.5,0.5)	(0.6,0.2)	(0.45,0.4)
A_2	(0.2,0.6)	(0.5,0.5)	(0.35,0.2)
A_3	(0.4,0.45)	(0.2,0.35)	(0.5,0.5)

TABLE V
INTUITIONISTIC PREFERENCE RELATION OF ALTERNATIVES WITH RESPECT TO THE CRITERION C_3

R_3	A_1	A_2	A_3
A_1	(0.5,0.5)	(0.55,0.25)	(0.65,0.1)
A_2	(0.25,0.55)	(0.5,0.5)	(0.6,0.3)
A_3	(0.1,0.65)	(0.3,0.6)	(0.5,0.5)

TABLE VI
INTUITIONISTIC PREFERENCE RELATION OF ALTERNATIVES WITH RESPECT TO THE CRITERION C_4

R_4	A_1	A_2	A_3
A_1	(0.5,0.5)	(0.8,0.1)	(0.7,0.2)
A_2	(0.1,0.8)	(0.5,0.5)	(0.55,0.4)
A_3	(0.2,0.7)	(0.4,0.55)	(0.5,0.5)

TABLE VII
INTUITIONISTIC PREFERENCE RELATION OF ALTERNATIVES WITH RESPECT TO THE CRITERION C_5

R_5	A_1	A_2	A_3
A_1	(0.5,0.5)	(0.7,0.2)	(0.65,0.2)
A_2	(0.2,0.7)	(0.5,0.5)	(0.55,0.25)
A_3	(0.2,0.65)	(0.25,0.55)	(0.5,0.5)

First, we check the consistency of the pairwise judgments for each intuitionistic preference relation and repair the inconsistent one(s) via Algorithm II until it is acceptable. Let us illustrate the process of consistency checking for the intuitionistic preference relation R of criteria as an example:

According to Algorithm I, we can construct the perfect multiplicative consistent intuitionistic preference relation $\bar{R} = (\bar{r}_{ik})_{5\times 5}$ of the intuitionistic preference relation R of criteria:

$$\bar{R} = \begin{pmatrix} (0.5, 0.5) & (0.6, 0.2) & (0.6923, 0.0588) & (0.7145, 0.0673) & (0.8069, 0.0204) \\ (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) & (0.6923, 0.0588) & (0.7145, 0.0673) \\ (0.0588, 0.6923) & (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) & (0.6923, 0.0588) \\ (0.0673, 0.7145) & (0.0588, 0.6923) & (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) \\ (0.0204, 0.8069) & (0.0673, 0.7145) & (0.0588, 0.6923) & (0.2, 0.6) & (0.5, 0.5) \end{pmatrix}$$

Here, we take $\bar{r}_{14}$ as an example:

$$\begin{aligned}\bar{\mu}_{14} &= \frac{\sqrt{\mu_{12}\mu_{24} \cdot \mu_{13}\mu_{34}}}{\sqrt{\mu_{12}\mu_{24} \cdot \mu_{13}\mu_{34}} + \sqrt{(1-\mu_{12})(1-\mu_{24}) \cdot (1-\mu_{13})(1-\mu_{34})}} \\ &= \frac{\sqrt{0.6 \times 0.65 \times 0.6 \times 0.6}}{\sqrt{0.6 \times 0.65 \times 0.6 \times 0.6} + \sqrt{0.4 \times 0.35 \times 0.4 \times 0.4}} = 0.7145\end{aligned}$$

$$\begin{aligned}\bar{\nu}_{14} &= \frac{\sqrt{v_{12}v_{24} \cdot v_{13}v_{34}}}{\sqrt{v_{12}v_{24} \cdot v_{13}v_{34}} + \sqrt{(1-v_{12})(1-v_{24}) \cdot (1-v_{13})(1-v_{34})}} \\ &= \frac{\sqrt{0.2 \times 0.25 \times 0.2 \times 0.2}}{\sqrt{0.2 \times 0.25 \times 0.2 \times 0.2} + \sqrt{0.8 \times 0.75 \times 0.8 \times 0.8}} = 0.0673\end{aligned}$$

By calculating the distance between R and $\bar{R}$ via (19), we get $d(\bar{R}, R) = 0.1698 > 0.1$, which means the intuitionistic preference relation is of unacceptable consistency. Hence, we need to repair it. Considering the interactive process is time consuming, we use our automatic algorithm to improve it. With (23) and (24) of Algorithm II, we can get the fused intuitionistic preference relation $\tilde{\bar{R}}$. Here, we let $\sigma = 0.8$:

$$\tilde{\bar{R}} = \begin{pmatrix} (0.5, 0.5) & (0.6, 0.2) & (0.6748, 0.0762) & (0.7022, 0.0892) & (0.7804, 0.035) \\ (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) & (0.6748, 0.0762) & (0.7022, 0.0892) \\ (0.0762, 0.6748) & (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) & (0.6748, 0.0762) \\ (0.0892, 0.7022) & (0.0762, 0.6748) & (0.2, 0.6) & (0.5, 0.5) & (0.6, 0.2) \\ (0.035, 0.7804) & (0.0892, 0.7022) & (0.0762, 0.6748) & (0.2, 0.6) & (0.5, 0.5) \end{pmatrix}$$

With (22), we can calculate the distance between $\tilde{\bar{R}}$ and $\bar{R}$, and get $d(\bar{R}, \tilde{\bar{R}}) = 0.0205 < 0.1$, which means $\tilde{\bar{R}}$ is of acceptable multiplicative consistency.

The consistency checking for the other intuitionistic preference relations $R_i (i = 1, 2, 3, 4, 5)$ of alternatives with respect to the criteria $C_i (i = 1, 2, 3, 4, 5)$ can be done following the same process, and we no longer illustrate it. After checking these comparison matrices, we can see all the other preference relations are consistent, and we do not need to repair them.

In the following, we go to the process of deriving the priorities of each acceptable consistent intuitionistic preference relation. In addition, we take $\tilde{\bar{R}}$ as an example. By using (26), the priority vector of the intuitionistic preference relation $\tilde{\bar{R}}$ can be calculated as

$$\omega_1 = (0.2172, 0.6039), \quad \omega_2 = (0.1785, 0.6585)$$
$$\omega_3 = (0.1367, 0.7151), \quad \omega_4 = (0.0977, 0.7418)$$
$$\omega_5 = (0.06, 0.8317)$$

Similarly, we can calculate the weight vectors of the other intuitionistic preference relations $R_i (i = 1, 2, 3, 4, 5)$ of alternatives with respect to the criteria $C_i (i = 1, 2, 3, 4, 5)$, and then, we obtain the different weights of the alternatives over the criteria $C_i (i = 1, 2, 3, 4, 5)$, which are all IFVs and can be shown in Table VIII.

Finally, we aggregate all the weights by using the operations introduced in Section II with respect to each alternative (see

TABLE VIII
WEIGHTS OF THE ALTERNATIVES OVER THE CRITERIA $C_i\,(i = 1, 2, 3, 4, 5)$ IN THE IFAHP

R	C_1 (0.2172, 0.6039)	C_2 (0.1785,0 .6585)	C_3 (0.1367,0 .7151)	C_4 (0.0977,0 .7418)	C_5 (0.06,0.831 7)	W_j
A_1	(0.2941, 0.5385)	(0.2925,0 .4865)	(0.3366,0 .4557)	(0.4211,0 .4824)	(0.3737,0.4 815)	(0.2172, 0.6039)
A_2	(0.1961, 0.6154)	(0.1981,0 .5405)	(0.2673,0 .5823)	(0.2421,0 .6941)	(0.2525,0.6 173)	(0.1444, 0.5425)
A_3	(0.2745, 0.5385)	(0.2075,0 .5405)	(0.1782,0 .6835)	(0.2316,0 .7059)	(0.1919,0.6 79)	(0.1464, 0.5479)

TABLE IX
WEIGHTS OF THE ALTERNATIVES OVER THE CRITERIA $C_i\,(i = 1, 2, 3, 4, 5)$ IN THE AHP

R	C_1 (0.2891)	C_2 (0.2347)	C_3 (0.1837)	C_4 (0.1614)	C_5 (0.1311)	W_j'
A_1	0.3825	0.4206	0.4545	0.4935	0.4767	0.4349
A_2	0.2439	0.2692	0.3376	0.2487	0.2960	0.2747
A_3	0.3736	0.3102	0.2079	0.2578	0.2274	0.2904

Table VIII). Taking W_1 as an example, we have

$$\begin{aligned} W_1 = \overset{5}{\underset{j=1}{\oplus}} (\omega_j \otimes w_{1j}) &= (0.2172, 0.6039) \otimes (0.2941, 0.5385) \\ &\oplus (0.1785, 0.6585) \otimes (0.2925, 0.4865) \\ &\oplus (0.1367, 0.7151) \otimes (0.3366, 0.4557) \\ &\oplus (0.0977, 0.7418) \otimes (0.4211, 0.4824) \oplus (0.06, 0.8317) \\ &\otimes (0.3737, 0.4815) = (0.2172, 0.6039) \end{aligned}$$

Using (10), we obtain

$$\begin{aligned} \rho(W_1) &= 0.5 \times (1 + 1 - 0.2172 - 0.6039) \times (1 - 0.2172) \\ &= 0.4614 \\ \rho(W_2) &= 0.5 \times (1 + 1 - 0.1444 - 0.5425) \times (1 - 0.1444) \\ &= 0.5617 \\ \rho(W_3) &= 0.5 \times (1 + 1 - 0.1464 - 0.5479) \times (1 - 0.1464) \\ &= 0.5573. \end{aligned}$$

Since $\rho(W_2) > \rho(W_3) > \rho(W_1)$, then the ranking of the suppliers is $A_2 \succ A_3 \succ A_1$, i.e., A_2 is the best supplier needed to develop.

If we only consider the membership degrees of the comparison judgments, i.e., the first part of the IFSs, then the intuitionistic preference relation downgrades to a real number matrix. We can use the FAHP method to solve it in such a case. Harker and Vargas [3] proved that the eigenvector method is the only method which is correct when the comparison matrix is an inconsistent matrix, that is, when the comparison matrix does not satisfy the relation $a_{ik}a_{kj} = a_{ij}$, for all i, j and k. Since the matrices with only the membership degrees of the intuitionistic preference relations are not consistent, we use the eigenvector method to derive the priorities. The results are shown in Table IX.

Since $W_1' > W_3' > W_2'$, in such a case, we can obtain the ranking of the suppliers as $A_1 \succ A_3 \succ A_2$, which is different from the result of the IFAHP. However, in the IFAHP, if we do not consider the nonmembership degree of $W_j\ (j = 1, 2, 3)$ when ranking the alternatives, the result of the IFAHP method will be $A_1 \succ A_3 \succ A_2$ as well because $0.2172 > 0.1464 > 0.1444$, which is consistent with the result of the FAHP method. The different results between the AHP method and the IFAHP method imply that the IFVs can represent the preferences of the pairwise comparison more comprehensive, and thus, the IFAHP method is more powerful in reflecting the vagueness and uncertainty due to that the nonmembership degrees and the hesitancy degrees, to some extent, also imply the preferences of the decision makers.

From the aforementioned example, we can see that both the FAHP and our proposed IFAHP are the extensions of the classical AHP. The FAHP uses triangular fuzzy numbers or trapezoidal fuzzy numbers to express the pairwise comparisons, while the IFAHP use IFVs to denote the preferences. An IFS can represent membership degree, nonmembership degree, and hesitancy degree by the three grades of membership function, respectively, and thus, it can be seen as a particular case of type 2 fuzzy set. However, the triangular fuzzy numbers and the trapezoidal fuzzy numbers do not have this property and each of them only can represent one grade of membership that is crisp in the unit interval $[0, 1]$.

Comparing our developed IFAHP with Chang's FAHP [10], we can see that the IFAHP show many advantages over the FAHP.

1) Our IFAHP method represents the fundamental scale of pairwise comparison over different criteria or alternatives in IFVs, while in Chang's FAHP, triangular fuzzy numbers are used to describe the preference judgment values of the decision maker. As triangular fuzzy numbers cannot be used to express support and objection evidences simultaneously, the scale used in the FAHP cannot reflect the perceptions of the decision maker accurately. While, the IFAHP uses IFVs to represent the preferences of the decision maker, each of which is characterized by a membership degree, a nonmembership degree, and a hesitancy degree, to represent the preferences of the decision maker. It can depict the decision maker's preferences from different aspects exhaustively. The IFS can denote the vagueness and uncertainty from three different "grades" (membership degree, nonmembership degree, and hesitancy degree), while the triangular fuzzy numbers only can represent one "grade" (membership degree). From this point of view, the IFS provides the user with a richer structure to express the fuzziness and uncertainty, and the IFAHP proposed in this paper can provide a richer framework than the traditional AHP and the FAHP.
2) In the original proposal of Chang [10], the consistency checking procedure of the preference relation was not considered. Later, although Kwong and Bai [12] complemented this step for the IFAHP, they did it by transforming triangular fuzzy numbers into crisp numbers and then used Saaty's [1] consistency index and consistency ratio to measure the consistency. One drawback of their method

is that the transformation from triangular fuzzy numbers to crisp numbers may lose some information and, hence, may distort the final result. In addition, they did not point out how to improve the inconsistent judgments. Perhaps in their opinion, the inconsistent evaluation values should be returned to the decision maker for reevaluation, but they proposed no methods to detect which assessment is inconsistent. In such a case, the decision maker is hard to repair the inconsistent preference relation until it is acceptable, which would waste a lot time for making a decision. Needless to say, the decision maker is often reluctant to participate in the repairing process because of the lack of interest. However, our developed IFAHP proposes a new way to check the consistency which is different from the FAHP, and meanwhile, we have also introduced an automatic scheme to repair the inconsistent intuitionistic preference relation, which do not need much participation from the decision maker.

3) In Chang's method, a crisp priority vector was derived by using the comparison law for triangular fuzzy numbers. However, as pointed by Wang *et al*. [11], Chang's method cannot estimate the true weights and has led to quite a number of misapplications. In our paper, a normalizing rank summation method to derive the priority vector of an intuitionistic preference relation has been developed, through which the priorities of the hierarchy in the IFAHP can be obtained. Our method is simple and easy to use in the actual applications.

VII. Conclusion

In this paper, we have extended the classical AHP and the FAHP to the context of IFS, where the scale of pairwise comparisons of the decision maker is represented by IFVs (the basic elements of IFS). Since the IFS is powerful in describing vagueness and uncertainty, the IFAHP allows a more accurate description of the decision-making process, which makes the IFAHP have much more advantages than the AHP and the FAHP. We have also proposed a method to check whether the preference relation is consistent or not, and then developed an automatic algorithm to repair the inconsistent one till acceptable. Since this procedure can be done without much participation of the decision maker, it can save a lot of time and reach a quick decision, especially at the beginning of the decision-making process. The priority vector of the intuitionistic preference relation has been derived through a normalizing rank summation method, it needs pointing out that the derived priority weights are also IFVs, and in our method, some basic operations on IFVs are used to aggregate the priority weights from the lowest level to the highest level in the constructed hierarchy structure, and based on the overall priority weights, the considered alternatives are then ranked. A numerical example has been utilized to illustrate the computational process of our proposed IFAHP. Considering that the IFAHP is a newly developed method, in the future, we can further investigate the IFAHP in group decision-making environments, as well as the hierarchy with feedback.

Since IFS is a particular case of type 2 fuzzy set, and the general type 2 fuzzy set provides for a much richer structure to represent uncertainty in a user's preferences, some readers may arise a question that why not to investigate the analytical hierarchy process within the context of the general type 2 fuzzy sets. As a matter of fact, we certainly acknowledge the power of the general type 2 fuzzy sets in depicting the vagueness and uncertainty, but from our point of view, there is not a lot of substantial significance to use the more complex type 2 fuzzy sets to represent the decision makers' pairwise comparison judgments in many practical cases. When comparing two different alternatives with respect to one criterion, the grades "preferred," "not preferred," and "indeterminate" are enough in general situations of our daily life, and there is not much need to add other grades to make the simple problem complex. However, as a theoretical investigation, to study the analytical hierarchy process within the context of general type 2 fuzzy sets is also very interesting, and it may be done in the future. Some other structures, such as the intuitionistic multiplicative set [37], the hesitant fuzzy set [53], and the hesitant fuzzy linguistic term set [54], are also powerful tools in modeling vagueness and uncertainty. Therefore, in the future, we can investigate the analytical hierarchy process with these techniques as well.

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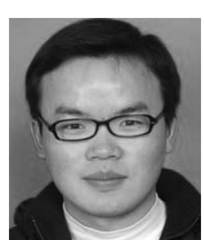

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