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Abstract. A crucial problem in the analytic hierarchy process (or its extension analytic network process) deriving priorities form pairwise comparison matrices. The most popular methods for deriving priorities are eigenvector method, which is originally proposed by Saaty [5], logarithmic least square method, and least square method. The paper deals with other alternative approaches using goal programming methodology - one of them is based on minimization of sum of absolute or relative deviations and the other one on minimization of maximum deviation. The results of methods are compared on a set of randomly generated matrices of different sizes and consistency levels.

1. Introduction

The Analytic Hierarchy Process (AHP) was formulated by T. Saaty in 1977 – more detailed information about it can be found e.g. in Saaty [6] – and this method has become one of the most popular tools for analysis of complex decision making problems. Even though the AHP was proposed many years ago it is still subject for research and a powerful tool for applications in various fields of human being. The Analytic Network Process (ANP) is a generalization of the AHP methodology that was introduced by Saaty [7]. ANP splits the decision making problem into clusters and allows interdependencies among clusters and within clusters.

AHP (and ANP) is still theoretically discussed and developed, and used as a main or secondary modelling tool in variety of real-world case studies. Among others, in the theoretical area the review paper [4] contains main developments of the AHP and ANP in the last decade. A great attention is paid to prioritization methods in the AHP/ANP. Their recent review is published in [2]. An interesting approach to this problem based on combination of current methods is presented in [9]. A wide review of the AHP/ANP applications can be found in [8].

One of the main features of both methods consists in expression of priorities by decision maker among evaluated elements using pairwise comparisons and then deriving priorities from pairwise comparison matrices. The priorities can be derived using several methodological approaches. Saaty [6] proposes eigenvalue method which has several advantages but is quite computationally uncomfortable. Other commonly used methods for deriving priorities are least square method, logarithmic least square method, modified least square method and others. We propose method that are based on goal programming methodology and either minimizes the sum of deviations or maximum deviation of theoretical and empirical pairwise comparisons.

The paper is organized as follows. Section 2 summarizes main principles of AHP and ANP. Next section formulates models for deriving priorities from pairwise comparison matrices. Comparison of results on the set of randomly generated matrices of different sizes and consistency levels is given in Section 4. The results and possible directions for a future research are discussed in the final section of the paper.

2. Analytic hierarchy/network process

AHP is a powerful and popular tool for analysis of general decision making problems. This approach organizes the decision problem as a hierarchical structure containing always several levels. The first level defines the main goal of the decision problem and the last one usually describes the decision alternatives or scenarios. The levels in between can contain secondary goals, criteria and subcriteria of the decision problem. The number of the levels is not limited, but in the typical case it does not exceed four or five.

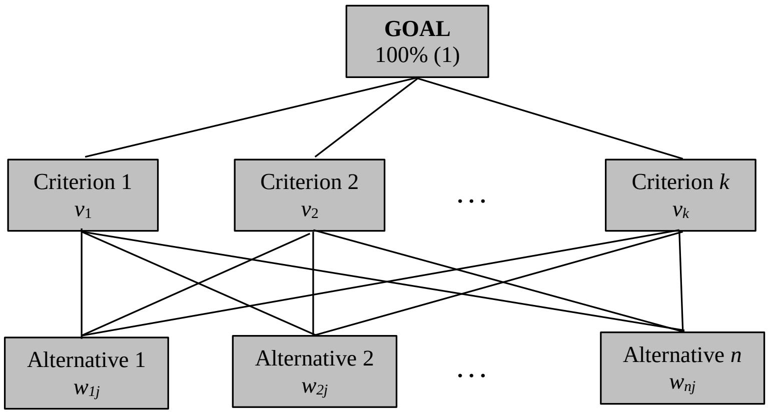

Figure 1. AHP hierarchy with three levels.

Let us consider a simple three-level hierarchy that represents a standard decision problem with a finite set of alternatives - n alternatives $X_1, X_2, \ldots, X_n$, are evaluated by k-criteria $Y_1, Y_2, \ldots, Y_k$. The hierarchy corresponding to this problem is presented on Figure 1 – this figure is a modified version of the one presented in Jablonsky [5]. The decision maker expresses his preferences by comparing the importance of the elements on the given level with respect to an element of the preceding level. The decision maker's judgements with respect to a given element are synthesised into the local priorities. They can express, e.g. the relative importance of criteria (weight coefficients - in Figure 1 denoted by v_j, $i = 1, 2,..., k$) or preference indices of the units with respect to the given criterion (w_{ij}, $i = 1, 2..., n$, $j = 1, 2,..., k$). In a conventional AHP model the decision maker's judgements on elements in a given level with respect to an element in the level above are organised into pairwise comparison matrices. The judgements are estimates of the preference between two elements of the lower level with respect to the element in the level above. Let us denote the pairwise comparison matrix $\mathbf{A} = \{a_{ij} \mid a_{ji} = 1/a_{ij}, a_{ij} > 0, i, j = 1, 2,..., k\}$, where k is the number of elements in the particular comparison set of the lower level. Elements of pairwise comparison matrix a_{ij} express how many times is the i-th criterion (alternative) more (or less) important than the j-th criterion (alternative). Saaty [6] proposes to use a_{ij} integers in the range 1 through 9 to express preference, where 1 means that the i-th and the j-th element are equally important and 9 means that the i-th element is absolutely more important than the j-th element. The local priorities from matrix $\mathbf{A}$ can be derived either by Saaty's original method, which is solving the following eigenvector problem (1), or by several alternative approaches that are formulated in the next section of the paper:

$$\mathbf{A}.\mathbf{w} = \lambda_{\max}\mathbf{w}, \quad \sum_{i=1}^{n} w_i = 1, \tag{1}$$

where λ_{max} is the largest eigenvalue of matrix **A** and **w** is the normalised right eigenvector belonging to λ_{max}. The vector **w** can be computed using various algorithms. One of them is based on the following relation:

$$c\mathbf{w} = \lim_{p\to\infty} \frac{\mathbf{A}^p\mathbf{e}}{\mathbf{e}^{\mathbf{T}}\mathbf{A}^p\mathbf{e}}, \tag{2}$$

where $\mathbf{e}^{\mathbf{T}} = (1, 1,\ldots, 1)$ and c is a scalar. Calculation of the right eigenvector w using formula (2) is not complex but it is quite time demanding and that is why other approximation methods are used as proposed in Saaty [6].

In order to get appropriate results the inconsistency level of matrix **A** must not exceed given threshold. Measuring of consistency (or inconsistency) is based on the property of positive reciprocal matrices. For fully consistent positive reciprocal (n, n) matrices $\lambda_{max} = n$, otherwise $\lambda_{max} > n$. Saaty defines so called consistency index C.I. = $(\lambda_{max} - n)/(n - 1)$ and pairwise comparison matrix is considered to be consistent if C.I. < 0.1.

Taking into account the hierarchy given on Figure 1, the decision maker must build one pairwise comparison matrix of order k on second level of hierarchy (comparison of criteria with respect to the main goal). Priorities derived from this matrix are weights of the criteria v_j, $j = 1, 2,\ldots, k$. Then, he/she must build k comparison matrices of order n on the third level of hierarchy (comparison of alternatives with respect to particular criteria). Priorities from these matrices (w_{ij}, $i = 1, 2,\ldots, n$, $j = 1, 2,\ldots, k$) can be explained as preference indices of alternative i with respect to given criterion j. Global priorities of alternatives are given as a simple weighted sum, i.e.

$$u(\mathrm{X}_i) = \sum_{j=1}^{k} v_j w_{ij}, \quad i = 1,2,\ldots,n. \tag{3}$$

The aim of this paper is not discuss in detail the theory of the AHP or ANP. ANP is a generalization of the AHP that takes into account dependencies among the elements of the hierarchy. The motivation for working with ANP models rather than AHP models consists in fact that many decision making problems cannot be structured hierarchically because they contain dependencies that can be expressed within the hierarchy but must be constructed as a network with clusters and their elements. Detailed description of this methodology can be found in Saaty's original book [7].

3. Prioritization methods in the AHP

Since formulation of principles of the AHP (and later ANP) several prioritization methods for deriving priorities from pairwise comparison matrices have been proposed. The original Saaty's procedure, as presented above, computes the prioritization vector as the right eigenvector **w** belonging to the largest eigenvalue λ_{max} of the pairwise comparison matrix **A**. Due to computational problems (solving using (2) may be quite time demanding) with solving of problem (1) some other prioritization methods have been formulated by Saaty and later by other researchers. All of them are based on minimization of a metric (a deviation function) between elements of pairwise comparison matrices a_{ij} on one side and ratios of estimated priorities w_i/w_j on the other side.

Least square method (*LSM*) constructs the deviation function as the sum of squares of deviations between elements a_{ij} and ratios w_i/w_j, i.e. the model is as follows:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}-\frac{w_i}{w_j}\right)^2 \tag{4}$$

subject to $$\sum_{i=1}^{n} w_i = 1, \tag{5}$$

$$w_i \geq 0,\ i = 1, 2, \ldots, n.$$

The problem (4)-(5) is a difficult non-linear problem with non-unique solutions that are hardly computable. That is why the *LSM* cannot be used for practical purposes. A modification of the *LSM* is the *logarithmic least square method* (*LLSM*) that minimizes the objective function

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\ln a_{ij}-\ln\left(\frac{w_i}{w_j}\right)\right)^2 \tag{6}$$

with respect to constraints (5). The solution of the problem (5)-(6) can be simply given as the geometric mean of the elements of each row of matrix A that is normalized to unit sum. That is why this method (originally proposed by Saaty) is often called *geometric mean method*. Solution of this problem is identical to the eigenvector problem (1) in case the matrix **A** is fully consistent and it is close to this solution when the consistency measure is on a satisfactory level.

Because of computational problems with *LSM* method, its modification that minimizes the following metric has been proposed

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}w_j - v_i\right)^2 \tag{7}$$

The objective function (7) is not linear but it can be easily transformed into a system of linear equations – see e.g. Bozoki [1] or Gao [3]. Let us denote this method as *modified LSM* (*MLSM*).

The models with objective functions (4), (6) and (7) minimize the sum of squares. They need not be (and usually are not) easily solvable. Instead of the minimization of squares it is possible to apply a goal programming methodology that allows to minimize the sum of positive and negative deviations or to minimize the maximum deviation. In both cases the deviations can be measured either as their absolute values or as relative deviations (in %). The optimization problem for minimization of the sum of relative deviations can be written as follows – let us denote this problem as *RSUM*:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d_{ij}^{-}+d_{ij}^{+}}{a_{ij}},$$

subject to $$a_{ij}+d_{ij}^{-}-d_{ij}^{+}=\frac{w_i}{w_j},\qquad i=1,2,\ldots,n,\ j=1,2,\ldots,n, \tag{8}$$

and constraints (4).

A solution that minimizes the maximum relative deviation can be given by solving the following optimization problem – let us denote this problem as *RMAX*:

Minimize D,

subject to $$a_{ij} + d_{ij}^{-} - d_{ij}^{+} = \frac{w_i}{w_j}, \qquad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{9}$$

$$\frac{d_{ij}^{-} - d_{ij}^{+}}{a_{ij}} \le D, \qquad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n,$$

and constraints (4).

A significant part of constraints in both problems (8) and (9) are not linear but their solution can be given quite easily by any non-linear solver, e.g. included in modeling and optimization system LINGO (www.lindo.com).
To avoid non-linearity in models (7) and (8) their simplified version can be formulated and solved. The model for minimization of deviations is as follows:

Minimize $$\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}w_j - w_i\right| \tag{10}$$

subject to constraints (4). The model that minimizes maximum deviation is

Minimize $$\max_{i,j}\left|a_{ij}w_j - w_i\right| \tag{11}$$

subject to constraints (4). The models (10) and (11) are not linear but it is possible to re-formulate them using deviational variables into linear models very easily. The solution of models (8) and (9) on one hand and models (10) and (11) on the other hand is identical only for fully consistent matrices.

Formulation of all models in this section assumes that all elements of matrix **A** are taken into account either in constraints or the objective function. Due to the reciprocal nature of the pairwise comparison matrix **A** it is questionable whether to consider all elements or the elements greater or equal 1 only, i.e. $a_{ij} \ge 1$. All the models presented here can be modified accordingly.

4. Computational experiments

The models for deriving priorities presented in the previous section were tested on randomly generated matrices of different sizes and different consistency levels. We are going to present the results for (4, 4) pairwise comparison matrices. Consistency indices (C.I.) of generated matrices are from very small values (less than 0.01) until values that indicate inconsistent matrices (more than 0.1; the largest value was approx. 0.2). Table 1 presents priorities derived by six methods – eigenvector method, *LLSM*, minimization of the sum of absolute and relative deviations (*ASUM* and *RSUM*) and minimization of the maximum deviation (absolute *AMAX* and relative *RMAX*) – for one of the almost consistent matrices (C.I. approx. 0.015).

Table 1. Results for an almost fully consistent matrix (C.I. = 0.015)

	Eigenval	***LLSM***	***ASUM***	***RSUM***	***AMAX***	***RMAX***
w_1	0.5052	0.5047	0.4706	0.4706	0.4807	0.5068
w_2	0.3328	0.3322	0.3529	0.3529	0.3503	0.3245
w_3	0.1057	0.1061	0.1177	0.1177	0.1038	0.1039
w_4	0.0563	0.0570	0.0588	0.0588	0.0652	0.0648
ASUM	3.30	3.27	**1.67**	**1.67**	3.55	3.75
***RSUM* [%]**	84.62	85.19	**58.33**	**58.33**	101.43	104.19
AMAX	0.97	0.87	1.00	1.00	**0.63**	0.99
***RMAX* [%]**	24.08	24.02	33.33	33.33	31.39	**21.92**

All optimization problems are solved for elements of the generated matrix greater or equal 1 only and each method is described in Table 1 by four characteristics – the best (optimal) values for all characteristics are given in bold:

- Sum of absolute deviations of original elements a_{ij} and ratios w_i/w_j (*ASUM*).
- Sum of relative deviations (*RSUM*), i.e. the objective function of model (8).
- Maximum absolute deviation (*AMAX*).
- Maximum relative deviation (*RMAX*).

The results show that the original eigenvector procedure has average values of all four characteristics and is not the best in any of them. Differences in priorities are quite high with regard to quite low C.I. value. It is clear that application of other methods than the first two are can lead to quite different final results in evaluation of alternatives, scenarios, etc.

Table 2 contains the same information as Table 1 but for a matrix that is inconsistent and its C.I. slightly exceeds the recommended threshold 0.1. It is clear and understandable that all optimization criteria are much worse for matrices with higher C.I. The derived priorities are very different for all methods except the first two ones. It is quite surprising that eigenvector method and *LLSM* is worse in all four criteria than almost all other methods. Maximum difference in *LLSM* is nearly 7 and it is really the extremely high value. This fact leads to question whether the commonly used methods are acceptable for inconsistent or nearly inconsistent matrices. This question is difficult to answer but all computational experiments show that the other methods are much better. On the other hand they are much computationally demanding and that is why their real applications depend on availability of appropriate powerful solvers.

Table 2. Results for an inconsistent matrix (C.I. = 0.108)

	Eigenval	***LLSM***	***ASUM***	***RSUM***	***AMAX***	***RMAX***
w_1	0.5634	0.5604	0.4038	0.5526	0.4480	0.5085
w_2	0.2636	0.2721	0.4038	0.2763	0.3639	0.2832
w_3	0.1309	0.1271	0.1346	0.0921	0.1427	0.1587
w_4	0.0421	0.0404	0.0578	0.0790	0.0454	0.0496
ASUM	13.55	13.38	**8.67**	9.33	11.81	12.56
***RSUM* [%]**	239.74	236.03	177.78	**163.89**	224.55	238.84
AMAX	6.37	6.87	3.67	4.83	**2.86**	3.26
***RMAX* [%]**	90.98	98.19	66.67	80.56	58.97	**46.61**

5. Conclusions

AHP (and its generalization ANP) belongs to one of the most popular methods for structuring and analysis of complex decision making problems. Both the methods are based on deriving priorities for the elements on each level of hierarchy by pairwise comparisons. The paper presents a survey of possible methods for deriving priorities. The standard methods as *eigenvector method* or *LLSM* are both computationally simple but they does not reach acceptable values of the optimization criteria as maximum deviation, sum of deviations, etc. The alternative procedures seem to be better with respect to all optimization criteria but they are computationally more demanding and their wider using is questionable for users without advanced background in optimization. Future research can be focused on analysis of rank reversals in case different prioritization methods are used or are combined in solving a decision problem.

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