example 6 decoding a Message.

encoded transmission

13	-26	21	|	33	-53	-12
18	-23	-42	|	5	-20	56
-24	23	77	|			

encoding Matrix

$$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = A$$

decoding Matrix requires

A^{-1} = inverse encoding Matrix

$\cancel{A}$ & $[A \;\; I] = [I \;\; A^{-1}]$

inverse of encoding
Matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & -9 & 0 & 0 & 1 \end{array}\right]$$

Column 1 clearing

$R_1 + R_2 \rightarrow R_{2\,new}$

$$\begin{array}{rcrrrrrr} R_1 & = & 1 & -2 & 2 & 1 & 0 & 0 \\ R_2 & = & -1 & 1 & 3 & 0 & 1 & 0 \\ R_{2\,new} & = & 0 & -1 & 5 & 1 & 1 & 0 \end{array}$$

1st new matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 & 0 \\ 1 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$-R_1 + R_3 \rightarrow R_{3_{new}}$

$R_3 = 1 \quad -1 \quad -4 \quad 0 \quad 0 \quad 1$

$-R_1 = -1 \quad +2 \quad -2 \quad -1 \quad 0 \quad 0$

$R_{3_{new}} = 0 \quad 1 \quad -6 \quad -1 \quad 0 \quad 1$

~~2nd new matrix~~

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 & 0 \\ 0 & 1 & -4 & 0 & & \end{array}\right]$$

2nd new matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 & 0 \\ 0 & 1 & -6 & -1 & 0 & 1 \end{array}\right]$$

$R_3 + R_2 \rightarrow R_{2new}$

$R_3 = 0 \quad 1 \quad -6 \qquad -1 \quad 0 \quad 1$

$R_2 = 0 \quad -1 \quad 5 \qquad 1 \quad 1 \quad 0$

$R_{2new} = 0 \quad 0 \quad -1 \qquad 0 \quad 1 \quad 1$

3rd new matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -6 & -1 & 0 & 1 \end{array}\right]$$

$2R_3 + R_1 \rightarrow R_{1_{new}}$

$2R_3 = 0 \quad 2 \quad -12 \quad -2 \quad 0 \quad 2$

$R_1 = 1 \quad -2 \quad 2 \quad 1 \quad 0 \quad 0$

$R_{1_{new}} = 1 \quad 0 \quad -10 \quad -1 \quad 0 \quad 2$

4th new matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -6 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

5th new matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & -1 & 0 & 2 \\ 0 & 1 & -6 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{array}\right]$$

$-R_3 \to R_{3\,new}$

$R_{3\,new} = 0\ 0\ 1 \quad 0\ -1\ -1$

6 th new matrix

$$\begin{bmatrix} 1 & 0 & -10 & -1 & 0 & 2 \\ 0 & 1 & -6 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

$6R_3 + R_2 \to R_{2\,new}$

$6R_3 = 0\ 0\ 6 \quad 0\ -6\ -6$

$R_2 = 0\ 1\ -6 \quad -1\ 0\ 1$

$R_{2\,new} = 0\ 1\ 0 \quad -1\ -6\ -5$

7 th new matrix

$$\begin{bmatrix} 1 & 0 & -10 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & -6 & -5 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

$10 R_3 + R_1 \rightarrow R_1$ new

$10 R_3 = 0 \; 0 \; 10 \; 0 \; -10 \; -10$

$R_1 = 1 \; 0 \; -10 \; -1 \; 0 \; 2$

$R_{1_{new}} = 1 \; 0 \; 0 \; -1 \; -10 \; -8$

8 th new matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -10 & -8 \\ 0 & 1 & 0 & -1 & -6 & -5 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

inverse verification

$$[A^{-1}][A] = [I]$$

$$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & +2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

$$(-1)(1) + (-10)(-1) + (-8)(1) =$$

$$-1 + 10 + -8 = 1$$

$$(-1)(-2) + (-10)(1) + (-8)(-1) =$$

$$2 + -10 + \cancel{-24}\ 8 = 0$$

$$(-1)(2) + (-10)(3) + (-8)(-4) =$$

$$-2 + -30 + 32 = 0$$

$(-1)(1) + (-6)(-1) + (-5)(1) =$

$-1 + 6 + -5 = 0$

$(-1)(-2) + (-6)(1) + (-5)(-1) =$

$2 + -6 + 5 = 1$

$(-1)(2) + (-6)(3) + (-5)(-4) =$

$-2 + -18 + 20 = 0$

$$(0)(1) + (-1)(-1) + (-1)(1) =$$

$$0 + 1 + -1 = 0$$

$$(0)(-2) + (-1)(1) + (-1)(-1)$$

$$0 + -1 + 1 = 0$$

$$0(2) + (-1)(3) + (-1)(-4) =$$

$$0 + -3 + 4 = 1$$

output

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

verifed.

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$ = decoding Matrix candidate

$(3 \times 1 \quad \text{coded})(A^{-1}) = \begin{bmatrix} \text{decoded} \\ \text{matrix} \end{bmatrix}$

$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$

$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$

$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$

$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$

$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$

decoding calculation

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$(13)(-1) + (-26)(-1) + (21)(0) =$

$-13 + 26 \quad 0 = 13$

$(13)(-10) + (-26)(-6) + (21)(-1)$

$-130 + 156 + -21 = 5$

$(13)(-8) + (-26)(-5) + (21)(-1)$

$-104 + 130 + -21 = 5$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$$(33)(-1) + (-53)(-1) + (-12)(0) =$$
$$-33 + 53 + 0 = 20$$

$$(33)(-10) + (-53)(-6) + (-12)(-1) =$$
$$-330 + \overset{318}{\cancel{336}} + 12 = \cancel{18}\,0$$

$$(33)(-8) + (-53)(-5) + (-12)(-1) =$$
$$-264 + 265 + 12 = 13$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$(18)(-1) + (-23)(-1) + (0)(-42) =$

$-18 + 23 + 0 = 5$

$(18)(-10) + (-23)(-6) + (-42)(-1) =$

$-180 + 138 + 42 = 0$

$(18)(-8) + (-23)(-5) + (-42)(-1)$

$-144 + 115 + 42 = 13$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$(5)(-1) + (-20)(-1) + (56)(0) =$

$-5 + 20 + 0 = 15$

$(5)(-10) + (-20)(-6) + (56)(-1) =$

$-50 + 120 + -56$

$-106 + 120 = 14$

$(5)(-8) + (-20)(-5) + (56)(-1) =$

$-40 + 100 + -56 = \cancel{6}4$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$(-24)(-1) + (23)(-1) + (77)(0) = 0$

$24 + -23 + 0 = 1$

$-24(-10) + (23)(-6) + (77)(-1)$

$240 + -138 + -77 = 25$

decoded string

13	m
5	e
5	e
20	T
~~18~~0	_
13	m
5	~~#~~ E
0	_
13	m
15	o
14	n
~~[illegible]~~4	D
1	A
25	Y