CHAPTER 4

Matrices

Exercise 4.1

Show that for two matrices A and B:

1. $\tilde{AB} = \tilde{B}\tilde{A}$,
2. $(AB)^\dagger = B^\dagger A^\dagger$, and
3. $(AB)^{-1} = B^{-1}A^{-1}$.

Solution 4.1

Exercise 4.2

Prove that

1. the product of two unitary matrices is also unitary and
2. the inverse of a unitary matrix is unitary.

Solution 4.2

Exercise 4.3

Show that if four matrices obey the equation $D = ABC$, then

$$D_{ij} = \sum_k \sum_l A_{ik} B_{kl} C_{lj}.$$

Solution 4.3

Exercise 4.4

1. Show that $\text{Trace}(AB) = \sum_i \sum_j A_{ij} B_{ji}$.
2. Given two matrices A and B of demensions $n \times m$ and $m \times n$ respectively, prove $\text{Trace}(AB) = \text{Trace}(BA)$.

Solution 4.4

Exercise 4.5

Prove that for any matrix A:

1. $AA^\dagger$ and $A^\dagger A$ are Hermitian;

2. $(A + A^\dagger)$ and $i(A - A^\dagger)$ are Hermitian.

Solution 4.5

Exercise 4.6

If $AB = BA$, show that $QA\tilde{Q}$ and $QB\tilde{Q}$ commute if Q is orthogonal.

Solution 4.6

Exercise 4.7

Find the inverse of:

1. $\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$,
2. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,
3. $\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$,
4. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$,
5. $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and
6. $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution 4.7

Exercise 4.8

Show that the matrices:

1. $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$,
2. $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, and
3. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

are orthogonal.

Solution 4.8

Exercise 4.9

Show that if λ_1, λ_2, $\ldots$, λ_n are the eigenvalues of A, then $\lambda_1 - k$, $\lambda_2 - k$, $\ldots$, $\lambda_n - k$ are the eigenvalues of $A - kE$.

Solution 4.9

Exercise 4.10

Obtain the eigenvalues and normalized eigenvectors of:

1. $\begin{pmatrix} 1 & -8 \\ 2 & 11 \end{pmatrix}$,
2. $\begin{pmatrix} 5 & 10 & 8 \\ 10 & 2 & -2 \\ 8 & -2 & 11 \end{pmatrix}$,
3. $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution 4.10

Exercise 4.11

If $A = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, show that $Q^{-1}AQ$ is diagonal.

Solution 4.11

Exercise 4.12

Diagonalize the following matrices:

1. $\begin{pmatrix} 5 & 10 & 8 \\ 10 & 2 & -2 \\ 8 & -2 & 11 \end{pmatrix}$,
2. $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and
3. $\begin{pmatrix} 2 & 4-i \\ 4+i & -14 \end{pmatrix}$.

Solution 4.12