CHAPTER 1

Mathematical Review

1.1 Linear Algebra

1.1.1 Three-Dimensional Vector Algebra

Exercise 1.1

a) Show that $O_{ij} = \hat{e}_i \cdot \mathscr{O}\hat{e}_j$. b) If $\mathscr{O}\vec{a} = \vec{b}$ show that $b_i = \sum_j O_{ij} a_j$.

Solution 1.1

1. Using (1.7) and (1.13), we get that

$$\hat{e}_i \cdot \mathscr{O}\hat{e}_j = \hat{e}_i \cdot \sum_{k=1}^{3} \hat{e}_k O_{kj} = \sum_{k=1}^{3} \hat{e}_i \cdot \hat{e}_k O_{kj} = \sum_{k=1}^{3} \delta_{ik} O_{kj} = O_{ij}. \tag{1.1-1}$$

2. Similarly,

$$\vec{b} = \sum_{i=1}^{3} b_i \hat{e}_i = \mathscr{O}\vec{a} = \mathscr{O}\sum_{j=1}^{3} a_j \hat{e}_j = \sum_{j=1}^{3} a_j \mathscr{O}\hat{e}_j = \sum_{j=1}^{3} a_j \sum_{i=1}^{3} \hat{e}_i O_{ij} = \sum_{i=1}^{3} \Big(\sum_{j=1}^{3} O_{ij} a_j \Big) \hat{e}_i.$$

From the uniqueness of linear expression by a basis, we arrive at

$$b_i = \sum_{j=1}^{3} O_{ij} a_j. \tag{1.1-2}$$

Exercise 1.2

Calculate $[\mathbf{A}, \mathbf{B}]$ and $\{\mathbf{A}, \mathbf{B}\}$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution 1.2

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 3 \\ -3 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 2 & 0 & 3 \\ -4 & -3 & 0 \end{pmatrix},$$

$$\{\mathbf{A}, \mathbf{B}\} \equiv \mathbf{AB} + \mathbf{BA} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 3 \\ -3 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 3 \\ -2 & 3 & -2 \end{pmatrix}.$$

1.1.2 Matrices

Exercise 1.3

If $\mathbf{A}$ is an $N \times M$ matrix and $\mathbf{B}$ is a $M \times K$ matrix show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$.

Solution 1.3

It is obvious that

$$(\mathbf{B}^\dagger\mathbf{A}^\dagger)_{ij} = \sum_{k=1}^{M}(\mathbf{B}^\dagger)_{ik}(\mathbf{A}^\dagger)_{kj} = \sum_{k=1}^{M} B_{ki}^* A_{jk}^* = \left(\sum_{k=1}^{M} A_{jk}B_{ki}\right)^* = [(\mathbf{AB})^*]_{ji} = [(\mathbf{AB})^\dagger]_{ij}, \qquad (1.3\text{-}1)$$

which means that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$.

Exercise 1.4

Show that

a. $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.

b. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

c. If $\mathbf{U}$ is unitary and $\mathbf{B} = \mathbf{U}^\dagger\mathbf{AU}$, then $\mathbf{A} = \mathbf{UBU}^\dagger$.

d. If the product $\mathbf{C} = \mathbf{AB}$ of two Hermitian matrices is also Hermitian, then $\mathbf{A}$ and $\mathbf{B}$ commute.

e. If $\mathbf{A}$ is Hermitian then $\mathbf{A}^{-1}$, if it exists, is also Hermitian.

f. If $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{(A_{11}A_{22}-A_{12}A_{21})}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$.

Solution 1.4

a. At this time, we assume that $\mathbf{A}$ is an $N \times M$ matrix while $\mathbf{B}$ is a $M \times N$ matrix. Then,

$$\mathrm{tr}(\mathbf{AB}) = \sum_{i=1}^{N}(\mathbf{AB})_{ii} = \sum_{i=1}^{N}\sum_{k=1}^{M} A_{ik}B_{ki} = \sum_{k=1}^{M}\sum_{i=1}^{N} B_{ki}A_{ik} = \sum_{k=1}^{M}(\mathbf{BA})_{kk} = \mathrm{tr}(\mathbf{BA}). \qquad (1.4\text{-}1)$$

From this issue, we assume that both $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices.

b. We find that

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{1}.$$

Since the inverse of a matrix is unique, we immediately get that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \qquad (1.4\text{-}2)$$

c. Due to $\mathbf{B} = \mathbf{U}^\dagger\mathbf{AU}$, we can find

$$\mathbf{A} = \mathbf{1A1} = (\mathbf{UU}^\dagger)\mathbf{A}(\mathbf{UU}^\dagger) = \mathbf{U}(\mathbf{U}^\dagger\mathbf{AU})\mathbf{U}^\dagger = \mathbf{UBU}^\dagger. \qquad (1.4\text{-}3)$$

d. Because $\mathbf{C} = \mathbf{AB}$ of two Hermitian matrices is also Hermitian, we know that

$$\mathbf{C}^\dagger = (\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger = \mathbf{C} = \mathbf{AB}.$$

With $\mathbf{A}^\dagger = \mathbf{A}$, $\mathbf{B}^\dagger = \mathbf{B}$, we find

$$\mathbf{B}^\dagger\mathbf{A}^\dagger = \mathbf{BA} = \mathbf{AB}. \qquad (1.4\text{-}4)$$

In other words, $\mathbf{A}$ and $\mathbf{B}$ commute.

e. It is obvious that if $\mathbf{A}^{-1}$ exists,

$$(\mathbf{A}^{-1})^\dagger\mathbf{A}^\dagger = (\mathbf{AA}^{-1})^\dagger = \mathbf{1}^\dagger = \mathbf{1}.$$

We know $(\mathbf{A}^\dagger)^{-1} = (\mathbf{A}^{-1})^\dagger$. Then, with $\mathbf{A} = \mathbf{A}^\dagger$, we find that

$$(\mathbf{A}^{-1})^\dagger = (\mathbf{A}^\dagger)^{-1} = \mathbf{A}^{-1}. \qquad (1.4\text{-}5)$$

Namely, $\mathbf{A}^{-1}$ is also Hermitian if it exists.

f. If $A_{11}A_{22} - A_{12}A_{21} \neq 0$, we can find

$$\mathbf{A} \times \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which means that if $A_{11}A_{22} - A_{12}A_{21} \neq 0$,

$$\mathbf{A}^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}. \tag{1.4-6}$$

1.1.3 Determinants

Exercise 1.5

Verify the above properties for 2×2 determinants.

Solution 1.5

Exercise 1.6

Using properties (1)-(5) prove that in general

6. If any two rows (or columns) of a determinant are equal, the value of the determinant is zero.
7. $|\mathbf{A}^{-1}| = (|\mathbf{A}|)^{-1}$.
8. If $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{1}$, then $|\mathbf{A}|(|\mathbf{A}|)^{*} = 1$.
9. If $\mathbf{U}^{\dagger}\mathbf{O}\mathbf{U} = \mathbf{\Omega}$ and $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{1}$, then $|\mathbf{O}| = |\mathbf{\Omega}|$.

Solution 1.6

Exercise 1.7

Using Eq.(1.39), note that the inverse of a 2×2 matrix $\mathbf{A}$ obtained in Exercise 1.4f can be written as

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

and thus $\mathbf{A}^{-1}$ does not exist when $|\mathbf{A}| = 0$. This result holds in general for $N \times N$ matrices. Show that the equation

$$\mathbf{A}\mathbf{c} = \mathbf{0}$$

where $\mathbf{A}$ is an $N \times N$ matrix and $\mathbf{c}$ is a column matrix with elements c_i, $i = 1, 2, \ldots, N$ can have a nontrivial solution ($\mathbf{c} \neq 0$) only when $|\mathbf{A}| = 0$.

Solution 1.7