Gibbs Sampling of 2D Gaussian Distribution

I. Multivariate Gaussian Distribution

Suppose $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and that we partition $\boldsymbol{x}$ into two disjoint subsets $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$, which means $\boldsymbol{x} = (\boldsymbol{x}_a, \boldsymbol{x}_b)^T$. We also define corresponding partitions of the mean vector $\boldsymbol{\mu}$ given by $\boldsymbol{\mu} = (\boldsymbol{\mu}_a, \boldsymbol{\mu}_b)^T$ and of the covariance matrix $\boldsymbol{\Sigma}$ by

$$\begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix} \tag{1}$$

The corresponding conditional distributions $p(\boldsymbol{x}_a|\boldsymbol{x}_b)$ and $p(\boldsymbol{x}_b|\boldsymbol{x}_a)$ are Gaussian distributions as well, defined as

$$p(\boldsymbol{x}_a|\boldsymbol{x}_b) = \mathcal{N}(\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b}), \quad \boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}(\boldsymbol{x}_b - \boldsymbol{\mu}_b), \quad \boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba}$$
$$p(\boldsymbol{x}_b|\boldsymbol{x}_a) = \mathcal{N}(\boldsymbol{\mu}_{b|a}, \boldsymbol{\Sigma}_{b|a}), \quad \boldsymbol{\mu}_{b|a} = \boldsymbol{\mu}_b + \boldsymbol{\Sigma}_{ba}\boldsymbol{\Sigma}_{aa}^{-1}(\boldsymbol{x}_a - \boldsymbol{\mu}_a), \quad \boldsymbol{\Sigma}_{b|a} = \boldsymbol{\Sigma}_{bb} - \boldsymbol{\Sigma}_{ba}\boldsymbol{\Sigma}_{aa}^{-1}\boldsymbol{\Sigma}_{ab}$$

II. Gibbs Sampling

The Gibbs sampler works in much the same way as the component-wise Metropolis-Hastings algorithms except that instead drawing from a proposal distribution for each dimension, then accepting or rejecting the proposed sample, we simply draw a value for that dimension according to the variables corresponding conditional distribution, which is actually a Gaussian distribution in case of a multivariate Gaussian distribution. We also accept all values that are drawn. Similar to the component-wise Metropolis-Hastings algorithm, we step through each variable sequentially, sampling it while keeping all other variables fixed. The Gibbs sampling procedure is outlined below

Algorithm 1 Gibbs Sampling

1: Set $t = 1$ and generate a initial value $\boldsymbol{\theta}^{(t)} = (\theta_1^{(t)}, \ldots, \theta_n^{(t)})$.
2: **repeat**
3: $\quad t = t + 1$
4: $\quad$ **for** i=1 **to** n **do**
5: $\quad\quad$ set $\boldsymbol{\theta}_i^* = (\theta_1^{(t)}, \cdots, \theta_{i-1}^{(t)}, \theta_{i+1}^{(t-1)}, \cdots \theta_n^{(t-1)})$
6: $\quad\quad$ sample $\theta_i^{(t)}$ from $p(\theta_i|\boldsymbol{\theta}_i^*)$
7: $\quad$ **end for**
8: **until** $t = T$

III. Instructions on experiments

Given a 2D Gaussian distribution, defined by a 2D mean vector and a 2×2 covariance matrix, you should generate a set of random samples from this distribution, which can be verified by plotting the samples together with the contours of this 2D Gaussian distributions in one figure, as follows. Part of the code will be provided and you need to fill in the missing part, as indicated in the code. Please use the MATLAB built-in function normrnd to sample a univariate Gaussian distribution. if you do not know how to use it, please refer the help documentation.

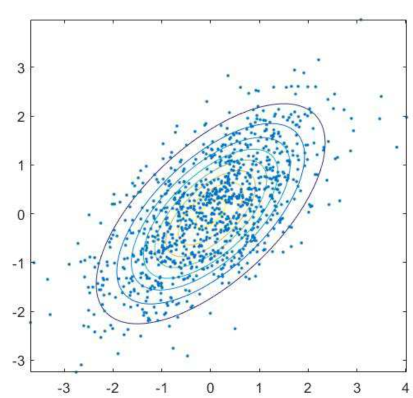

Fig. 1. Expected results.