Principles of Mathematical Analysis

I. THE REAL AND COMPLEX NUMBER SYSTEM

Definition 1 *Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:*

1. *If $x, y \in S$ then one and only one of the statements*

$$x < y, \quad x = y, \quad x > y \tag{1}$$

   is true.

2. *if $x, y, z \in S,\ x < y,\ y < z$ then $x < z$.*

Definition 2 *An **ordered set** is a set S in which an order is defined.*

Definition 3 *Suppose $E \subset S$. If $\exists \beta \in S\ \forall x \in E\ x \leq \beta$, E is said to be bounded above, and β is called an **upper bound** of E.*

Definition 4 *α is the **least upper bound** of E ($\alpha = \sup E$) if*

1. *α is an upper bound of E.*
2. *if $\gamma < \alpha$ then γ is not an upper bound of E.*

Definition 5 *An ordered set S is said to have the **least-upper-bound property** if the following is true:*

$$\begin{cases} E \subset S, E \neq \emptyset \\ E \text{ is bounded above} \end{cases} \implies \sup E \in S$$

Theorem 1 *Every ordered set with the least-upper-bound property also has the **greatest-lower-bound property**.*

Definition 6 *A **field** is a set F with two operations, called addition and multiplication, which satisfy the following so-called "field axioms" (A), (M) and (D).*

A1: $\forall x, y \in F,\ x + y \in F$.

A2: $\forall x, y \in F,\ x + y = y + x$.

A3: $\forall x, y, z \in F,\ (x + y) + z = x + (y + z)$.

A4: *F contains an element 0 such that $\forall x \in F,\ 0 + x = x$.*

A5: *To $\forall x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.*

M1: $\forall x, y \in F,\ xy \in F$.

M2: $\forall x, y \in F,\ xy = yx$.

M3: $\forall x, y, z \in F,\ (xy)z = x(yz)$.

M4: *F contains an element $1 \neq 0$ such that $\forall x \in F,\ 1x = x$.*

M5: *To $\forall x \in F (x \neq 0)$ corresponds an element $1/x \in F$ such that $x(1/x) = 1$.*

D: $\forall x, y, z \in F,\ x(y + z) = xy + xz$.

Proposition 1 *Properties of fields.*

1. $x + y = x + z \implies y = z$
2. $x + y = x \implies y = 0$
3. $x + y = 0 \implies y = -x$

4. $-(-x) = x$
5. $x \neq 0, xy = xz \implies y = z$
6. $x \neq 0, xy = x \implies y = 1$
7. $x \neq 0, xy = 1 \implies y = 1/x$
8. $x \neq 0, 1/(1/x) = x$
9. $0x = 0$
10. $x \neq 0, y \neq 0 \implies xy \neq 0$
11. $(-x)y = -(xy) = x(-y)$
12. $(-x)(-y) = xy$

Definition 7 *An* ***ordered field*** *is a field F which is also an ordered set, such that*

1. $\forall x, y, z \in F, y < z \implies x + y < x + z$
2. $\forall x, y \in F, x > 0, y > 0 \implies xy > 0$

Proposition 2 *Properties of ordered fields*

1. $x > 0 \iff -x < 0$
2. $x > 0, y < z \implies xy < xz$
3. $x < 0, y < z \implies xy > xz$
4. $x \neq 0 \implies x^2 > 0$, *in particular* $1 > 0$
5. $0 < x < y \implies 0 < 1/y < 1/x$

Theorem 2 *There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property and contains $\mathbb{Q}$ as a subfield.*

Proof.

Step 1: We define the cut as any set $\alpha \subset \mathbb{Q}$ with the following 3 properties:

1. α is not empty, and $\alpha \neq \mathbb{Q}$.
2. If $p \in \alpha, q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.
3. If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

We assume that R consists of cuts. (Note that the elements themselves are not important. The operations on the elements are what we should focus on. It is OK to associate a number with a cut.)

$p, q, r, \cdots$ will denote rational numbers, and $\alpha, \beta, \gamma, \cdots$ will denote cuts.

Define $r^* = \{p \in \mathbb{Q} : p < r\}$. It is clear that r^* is a cut.

Step 2: Define "$\alpha < \beta$" to mean: α is a proper subset of β. Thus R is now an ordered set.

Step 3: The ordered set R has the least-upper-bound property. Let A be a nonempty subset of R, and assume that A is above bounded. Define

$$\gamma = \bigcup A \tag{2}$$

It is not difficult to prove that $\gamma \in R$ and $\gamma = \sup A$.

Step 4: Axioms of addition. Define $\alpha + \beta = \{r + s : r \in \alpha, s \in \beta\}$. 0^* plays the role of 0.

Step 5: Axioms of multiplication. Define $R^+ = \{\alpha \in R : \alpha > 0^*\}$. If $\alpha, \beta \in R^+$, define $\alpha\beta = \{p : p \leq rs, r \in \alpha, s \in \beta, r > 0, s > 0\}$.

Step 6: The distribution law.

We have now completed the proof that R is an ordered field with the least-upper-bound property.

Define $Q = \{r^* : r \in \mathbb{Q}\}$. It is easy to find that the ordered field Q is isomorphic to the ordered field $\mathbb{Q}$. This identification allows us to regard $\mathbb{Q}$ as a subfield of R.

Theorem 3 *(Archimedean property) If $x, y \in \mathbb{R}$, and $x > 0$, then there exists a positive integer n such that*

$$nx > y \tag{3}$$

Theorem 4 *($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x, y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.*

Definition 8 *A **complex number** is an ordered pair (a, b) of real numbers.*

Definition 9 *The **complex field** is the set of complex numbers with the following definitions of addition and multiplication*

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc)$$

Definition 10 *The unit of imaginary part is denoted as*

$$i = (0, 1) \tag{4}$$

II. BASIC TOPOLOGY

Definition 11 *A **metric space** is a set X with a distance function (metric)*

$$d : X \times X \to \mathbb{R} \tag{5}$$

defined on it such that $\forall p, q, r \in X$

1. *$d(p, q) \geq 0$ with equality iff $p = q$*
2. *$d(p, q) = d(q, p)$*
3. *$d(p, q) \leq d(p, r) + d(r, p)$*

Definition 12 *Let X be a metric space and E be a subset of X.*

- *Neighborhood: The neighborhood of a point $p \in X$ is defined as $N_r(p) = \{q \in X : d(p, q) < r, r > 0\}$*
- *Limit point (wikipedia): A point $p \in X$ is a limit point of the set E if every neighborhood of p contains at least one point $q \neq p$ such that $q \in E$. (An equivalent definition: every neighborhood of p contains infinitely many points of E. A third definition: there is a sequence of points in $E \backslash \{p\}$ whose limit is p.)*
- *Isolated point: A point $p \in X$ is a isolated point of the set E if $p \in E$ and p is not a limit point of E.*
- *Interior point: A point $p \in X$ is a interior point of the set E if there exists a neighborhood $N(p)$ such that $N(p) \subset E$.*
- *Closed: E is closed if every limit point of E is a point of E. (Both $\emptyset$ and X are closed.)*
- *Open: E is open if every point of E is an interior point. (Both $\emptyset$ and X are open.)*
- *Complement: $E^c = \{p \in X : p \notin E\}$.*
- *Perfect: E is perfect if E is closed and every point of E is a limit point of E.*
- *Bounded: E is bounded if it is contained in a neighborhood $N_r(q)$ of a point $q \in X$.*
- *Dense: E is dense in X if $\forall p \notin E$ is a limit point of E.*

Theorem 5 *A set E is **open** iff its complement is closed.*

Theorem 6 *A set E is **closed** iff its complement is open.*

Definition 13 *X is a metric space, $E \subset X$, E' denotes the set of all limit points of E in X. The closure of E is defined as*

$$\overline{E} = E \cup E' \tag{6}$$

This closure property can be described as a relation $R \subset X^\infty \times X$ on X, where R is the set of tuples consisting of a convergent sequence $\{p_i\}_{i=1}^\infty$ of points on X and its limit point p. For a closed set E, if $p_1, p_2, \cdots, p_\infty \in E$ and $(p_1, p_2, \cdots, p_\infty, p) \in R$, then $p \in E$.

Thus $\overline{E}$ is the smallest closed set which contains E, and $\overline{E}$ is the intersection of all closed sets which contains E.

Theorem 7 *$\{G_i\}$ are open sets, $\{F_i\}$ are closed sets, $n \in \mathbb{N}$.*

1. *$\bigcup_i G_i$ is open.*
2. *$\bigcap_i F_i$ is closed.*
3. *$\bigcap_{i=1}^n G_i$ is open.*
4. *$\bigcup_{i=1}^n F_i$ is closed.*

Remark : $\bigcap_{i=1}^\infty G_i$ may not be open, for instance, $G_i = (-\frac{1}{i}, \frac{1}{i})$, $\bigcap_{i=1}^\infty G_i = \{0\}$. Similarily, $\bigcup_{i=1}^\infty F_i$ may not be closed, for instance, $F_i = (-\infty, -\frac{1}{i}] \cup [\frac{1}{i}, \infty)$, $\bigcup_{i=1}^\infty F_i = (-\infty, 0) \cup (0, +\infty)$.

Theorem 8 *Suppose $E \subset Y \subset X$, E is open relative to Y iff $E = Y \cap G$ for some open subset G of X.*

Proof.
(1) If $E = Y \cap G$ for some open subset $G \subset X$, then $\forall p \in E$, there exists an $r_p > 0$ such that

$$\{q \in X : d(p,q) < r_p\} \subset G \tag{7}$$

It follows that

$$\{q \in Y : d(p,q) < r_p\} = \{q \in X : d(p,q) < r_p\} \cap Y \subset (G \cap Y) = E \tag{8}$$

(2) If E is open relative to Y, then

$$\forall p \in E \ \exists r_p > 0 \ \ (\{q \in Y : d(p,q) < r_p\} \subset E) \tag{9}$$

Define $V_p = \{q \in X : d(p,q) < r_p\}$ and let $G = \bigcup_{p \in E} V_p$. It is clear that G is an open set and $p \in G$. Thus $E \subset Y \cap G$. And

$$Y \cap G = \bigcup_{p \in E} \{q \in Y : d(p,q) < r_p\} \subset E \tag{10}$$

Thus $E = Y \cap G$.

Remark 1 *It is interesting that E is open relative to Y without being open relative to X. The property of being open thus depends on the space in which E is embedded.*

Definition 14 ***Open cover*** *of a set E: $\{G_i\}$ are open sets, $E \subset \bigcup_i G_i$.*

Definition 15 *A subset K of a metric space is* ***compact*** *iff every open cover of K contains a finite subcover.*

Theorem 9 *Suppose $K \subset Y \subset X$, then K is compact relative to X iff K is compact relative to Y.*

Proof.
(1) Let $\{V_i\}$ be a open cover of K relative to Y. We know that $V_i = Y \cap G_i$ for some open set relative to X. Then we have a finite subcover of K, $\{G_i\}_{i=1}^n$, relative to X, that is $K \subset \bigcup_{i=1}^n G_i$. Thus, noticing that $K \subset Y$, $K \subset \bigcup_{i=1}^n V_i$.
(2) Obviously.

Theorem 10 *Compact subsets of a metric space are closed.*

Proof. Let X be a compact subset of metric space X and let $p \in K^c$, $q \in K$. Suppose V_q and W_q are neighborhoods of p and q of radius less that $\frac{1}{2}d(p,q)$.

$\{W_q\}_{q\in K}$ is an open cover of $K \implies$ there are a finitely many points $q_1, \cdots, q_n$ such that $K \subset W_{q_1} \cup \cdots \cup W_{q_n} = W$. $V = V_{q_1} \cap \cdots \cap V_{q_n}$ is a neighborhood of p and $V \cap W = \emptyset \implies p$ is an interior point $\implies K^c$ is open thus K is closed.

Theorem 11 *Closed subsets of compact sets are compact.*

Proof. $F \subset K \subset X$. F is closed relative to X, K is compact. Let $\{V_i\}$ be a open cover of F, then $\{V_i\} \cup \{F^c\}$ is an open cover of $K \implies$ a finite subset Ω of $\{V_i\} \cup \{F^c\}$ is an open cover of K and thus F. $\Omega \backslash \{F^c\}$ is still an open cover of $F \implies$ a finite subset of $\{V_i\}$ is an open cover of K.

III. NUMERICAL SEQUENCES AND SERIES

A. Numerical Sequences

Definition 16 *A sequence $\{p_n\}$ in a metric space X is said to* ***converge*** *(in X) if there is a point $p \in X$ such that*

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z} \; (n \geq N \implies d(p_n, p) < \epsilon) \tag{11}$$

Theorem 12 *Let $\{p_n\}$ be a sequence in metric space X.*

1. *$p_n \to p$ iff every neighborhood of p contains p_n for all but finitely many n.*
2. *$p_n \to p, \; p_n \to p' \implies p = p'$.*
3. *converge $\implies$ bounded*
4. *If $E \subset X$, p is a limit point of $E \iff$ there is a sequence $\{p_n\} \subset E$ such that $p_n \to p$.*
5. *$\{d(p_n, p)\}$ is a sequence in $\mathbb{R}$. $p_n \to p$ iff $d(p_n, p) \to 0$ with the metric defined as $d(a, b) = |a - b|$.*

Theorem 13 *Suppose $\{s_n\}$ and $\{t_n\}$ are complex sequence, and $s_n \to s$, $t_n \to t$. Then*

1. *$s_n + t_n \to s + t$*
2. *$s_n t_n \to st$*
3. *if $s_n \neq 0, \; n = 0, 1, \cdots$, then $\frac{1}{s_n} \to \frac{1}{s}$*

Definition 17 *Let $\{n_i\}$ be a sequence of positive integers such that $n_1 < n_2 < \cdots$. Then $\{p_{n_i}\}$ is a* ***subsequence*** *of $\{p_n\}$.*

Theorem 14 *The subsequential limits of a sequence $\{p_n\}$ in a metric space X forms a closed subset of X.*

Proof. Let E^* be the set of the subsequential limits of $\{p_n\}$ and let q be a limit point of E^*. We need to show that there is a subsequence of $\{p_n\}$ which converges to q.

1. Choose $p_{n_1} \neq q$, and put $\delta = d(q, p_{n_1})$.
2. Suppose $p_{n_1}, \cdots, p_{n_i}$ are chosen. Then
   - q is a limit point of $E^* \implies$ there exists a point $x \in E^*$ such that $d(x, q) < 2^{-1-i}\delta$.
   - $x \in E^* \implies$ there exists $n_{i+1} > n_i$ such that $d(p_{n_{i+1}}, x) < 2^{-1-i}\delta$.

   Thus $d(q, p_{n_{i+1}}) \leq d(x, q) + d(p_{n_{i+1}}, x) < 2^{-i}\delta$
3. By induction, we construct a sequence $\{p_{n_i}\}$ that converges to q. Thus $q \in E^*$.

Definition 18 *A sequence $\{p_n\}$ in a metric space X is said to be a* ***Cauchy sequence*** *if*

$$\forall \epsilon > 0 \ \exists N \in \mathbb{Z} \ (m, n \geq N \implies d(p_m, p_n) < \epsilon) \tag{12}$$

Obviously, every convergent sequence in a metrix space is Cauchy.

Definition 19 *Let E be a non-empty subset of a metric space X, the* ***diameter*** *of E is defined as*

$$\operatorname{diam} E = \sup\{d(p, q) : p, q \in E\} \tag{13}$$

Theorem 15 *Let E be a non-empty subset of a metric space X, then $\operatorname{diam} \overline{E} = \operatorname{diam} E$.*

Theorem 16 *A sequence $\{p_n\}$ in a metric space X is a Cauchy sequence iff*

$$\lim_{N \to \infty} \operatorname{diam}\{p_N, p_{N+1}, \cdots\} = 0 \tag{14}$$

Theorem 17 *If $\{K_n\}$ is a sequence of non-empty compact sets in X such that*

$$\begin{cases} K_n \supset K_{n+1} (n = 1, 2, \cdots) \\ \lim_{n \to \infty} \operatorname{diam} K_n = 0 \end{cases} \implies \bigcap_1^\infty K_n \textit{ consists of exactly one point.} \tag{15}$$

Definition 20 *A metric space in which every Cauchy sequence converges is said to be* ***complete****.*

Theorem 18 *Suppose $\{s_n\}$ is monotonic, then $\{s_n\}$ converges iff it is bounded.*

Definition 21 *Let $\{s_n\}$ be a sequence of real numbers.*
$s_n \to +\infty$ if

$$\forall M \in \mathbb{R} \ \exists N \in \mathbb{Z} \ (n \geq N \implies s_n \geq M) \tag{16}$$

$s_n \to -\infty$ if

$$\forall M \in \mathbb{R} \ \exists N \in \mathbb{Z} \ (n \geq N \implies s_n \leq M) \tag{17}$$

Let $E = \{x \in [-\infty, \infty] : s_{n_k} \to x\}$, upper and lower limits of $\{s_n\}$ is defined as

$$\limsup_{n \to \infty} s_n = \sup E \tag{18}$$

$$\liminf_{n \to \infty} s_n = \inf E \tag{19}$$

Theorem 19 *Let $E = \{x \in [-\infty, \infty] : s_{n_k} \to x\}$.*

1. *$\limsup_{n \to \infty} s_n \in E$.*

2. *If $x > \limsup_{n \to \infty} s_n$, then $\exists N \in \mathbb{Z} \ (n \geq N \implies s_n < x)$.*

Theorem 20 *If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n \tag{20}$$

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n \tag{21}$$

B. Series

Definition 22 *With a sequence $\{a_n\} \subset \mathbb{C}$, we call $\sum a_n$ a **series**.*

$$\sum a_n = a_1 + a_2 + \cdots \tag{22}$$

Remark 2 *If $\{a_n\}$ start from a_0, then $\sum a_n = a_0 + a_1 + a_2 + \cdots$. We often associate a partial sum sequence $\{A_n\}$ to a series $\sum a_n$ where $A_n = \sum_{i=1}^n a_n$. We say $\sum a_n$ converges if the corresponding partial sum sequence $\{A_n\}$ converges.*

Theorem 21 *Some theorems about series.*

1. *(Cauchy criterion) $\sum a_n$ converges iff*

$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}\ \forall m \geq n \geq N \quad \left| \sum_{k=n}^{m} a_k \right| \leq \epsilon \tag{23}$$

2. *$\sum a_n$ converges $\implies$ $a_n \to 0$*
3. *If $\forall n \in \mathbb{N}^*,\ a_n \in \mathbb{R}^+$ then $\sum a_n$ converges iff $\{A_n\}$ is bounded.*
4. *If $|a_n| \leq c_n$ for $n \geq N_0$, then $\sum c_n$ converges $\implies$ $\sum a_n$ converges.*
5. *If $a_n \geq d_n \geq 0$ for $n \geq N_0$, then $\sum d_n$ diverges $\implies$ $\sum a_n$ diverges.*

Definition 23 *The number e is defined to be*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{24}$$

$s_n = \sum_{i=0}^n \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 3 \implies \{s_n\}$ converges. Thus the definition makes sense.

Theorem 22 *Another definition of e.*

$$(1 + \frac{1}{n})^n \to e \tag{25}$$

Proof. Let $s_n = \sum_{k=0}^n \frac{1}{n!}$, $t_n = (1 + \frac{1}{n})^n$.
Using binomial theorem, we get

$$\begin{cases} \forall n \in \mathbb{N}^* \quad t_n \leq s_n \\ \forall m \in \mathbb{N}^* \quad \liminf_{k\to\infty} t_k \geq s_m \end{cases} \implies e \leq \liminf_{k\to\infty} t_k \leq \limsup_{k\to\infty} t_k \leq e \implies \lim_{k\to\infty} t_k = e \tag{26}$$

Theorem 23 *If a series converges absolutely, then it converges. i.e.*

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges} \tag{27}$$

Theorem 24 *(Root test)*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \tag{28}$$

1. *$\alpha < 1 \implies \sum a_n$ converges.*
2. *$\alpha > 1 \implies \sum a_n$ diverges.*

Consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, both have $\alpha = 1$ but the first diverges and the second converges.

Theorem 25 *(Ratio test)*

1. $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \sum a_n$ *converges.*

2. $\exists N_0 \in \mathbb{Z}\ \forall n \geq N_0\ \left|\frac{a_{n+1}}{a_n}\right| \geq 1 \implies \sum a_n$ *diverges.*

Theorem 26 *Let $\{c_n\} \subset \mathbb{R}^+$, then*

$$\liminf_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \leq \liminf_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \tag{29}$$

Remark 3 *Theorem 26 indicates that root test is more powerful than ratio test. However, using ratio test is generally easier than using root test. Besides, root test and ratio test both deal with series converges absolutely.*

Definition 24 *Given $\{c_n\} \in \mathbb{C}$, $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n z^n$ is called a **power series**.*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \tag{30}$$

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = \frac{|z|}{R} \tag{31}$$

We call R the radius of convergence of $\sum c_n z^n$.

Theorem 27 *Multiplication of power series*

$$\begin{aligned}\sum c_n z^n &= \left(\sum a_n z^n\right)\left(\sum b_n z^n\right)\\ &= a_0b_0 + (a_0b_1 + a_1b_0)z + (a_0b_2 + a_1b_1 + a_2b_0)z^2 + \cdots\end{aligned}$$

Thus, we define the product of two series as

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{32}$$

Theorem 28 *Let $\{a_n\} \subset \mathbb{C}$, $A_n = \sum_{i=1}^{n} a_n$, and $\{b_n\} \subset \mathbb{R}$.*

$$\begin{cases}\{A_n\} \text{ is bounded}\\ b_0 \geq b_1 \geq ...\\ b_n \to 0\end{cases} \implies \sum a_n b_n \text{ converges.} \tag{33}$$

Corollary 1 *(Alternating series) Let $\{a_n\} \subset \mathbb{R}$.*

$$\begin{cases}a_{2m-1} \geq 0,\ a_{2m} \leq 0, \quad m = 1, 2, \cdots\\ |a_1| \geq |a_2| \geq ...\\ a_n \to 0\end{cases} \implies \sum a_n \text{ converges.} \tag{34}$$

Theorem 29

$$\begin{cases}\sum a_n \text{ converges absolutely}\\ \sum a_n = A\\ \sum b_n = B\\ c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{cases} \implies \sum c_n = AB \tag{35}$$

Theorem 30

$$\begin{cases}\sum a_n = A\\ \sum b_n = B\\ \sum c_n = C\\ c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{cases} \implies C = AB \tag{36}$$

Definition 25 *Let $\{k_n\}$ be a 1-to-1 function from N^* to N^*. Then $\sum a_{k_n}$ is called a rearrangement of a_n.*

Theorem 31 *Let $\sum a_n$ be a series of real numbers which converges non-absolutely. Suppose $-\infty \le \alpha \le \beta \le \infty$. Then there exists a rearrangement $\sum a'_n$ with partial sum A'_n such that*

$$\liminf_{n\to\infty} A'_n = \alpha \qquad \limsup_{n\to\infty} A'_n = \beta \tag{37}$$

Theorem 32 *absolutely converge $\implies$ every rearrangement converges to the same sum*

IV. CONTINUITY

Definition 26 *Let X and Y be metric spaces; suppose $E \subset X$, $f : E \to Y$, and p is a limit point of E. We write*

$$f(x) \to q \text{ as } x \to p \text{ or } \lim_{x\to p} f(x) = q \tag{38}$$

if there is a point $q \in Y$ such that

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in E\backslash\{p\}\ \ (d_X(x,p) < \delta \implies d_Y(f(x), q) < \epsilon) \tag{39}$$

Remark 4 *In the definition, f may not have definition at p. Using a subset E of a metric space X loses nothing of interest. This simplifies statements and proofs of some theorems.*

Theorem 33 *$f(x) \to q$ as $x \to p$ iff*

$$\forall \{p_n\} \subset E\backslash\{p\}\ (p_n \to p \implies f(p_n) \to q) \tag{40}$$

Corollary 2 *As $x \to p$, $f(x) \to q$, $f(x) \to q' \implies q = q'$.*

Theorem 34 *As $x \to p$, $f(x) \to A$ and $g(x) \to B$. Then*

1. $(f+g)(x) \to A + B$
2. $(fg)(x) \to AB$
3. $(\frac{f}{g})(x) \to \frac{A}{B}\ \ (B \neq 0)$

Definition 27 *Let X and Y be metric spaces; suppose $E \subset X$, $f : E \to Y$, and $p \in E$. f is said to be continuous at p if*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in E\ \ (d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \epsilon) \tag{41}$$

Theorem 35 *f is continuous at p iff*

$$\forall \{p_n\} \subset E\ (p_n \to p \implies f(p_n) \to f(p)) \tag{42}$$

Remark 5 *If p is an isolated point of E, then by definition f is continuous at p.*

Theorem 36 *If p is a limit point of E, then f is continuous at p iff $f(x) \to f(p)$ as $x \to p$.*

Theorem 37 *$f : X \to Y$ is continuous on X iff $\forall$ open set $V \subset Y$, $f^{-1}(V) \subset X$ is open.*

Corollary 3 *$f : X \to Y$ is continuous on X iff $\forall$ closed set $C \subset Y$, $f^{-1}(C) \subset X$ is closed.*

Definition 28 *$f : E \to \mathbb{R}^k$ is said to be **bounded** if*

$$\forall x \in E\ \exists M \in \mathbb{R}\ (|f(x)| < M) \tag{43}$$

Theorem 38 *Let X be a compact metric space and Y be a metric space; suppose $f : X \to Y$. Then $f(X)$ is compact.*

Theorem 39 *$f : E \to \mathbb{R}^k$ is continuous on E, and E is compact. Then $f(X)$ is closed and bounded.*

Theorem 40 *$f : E \to \mathbb{R}$ is continuous on E, and E is compact. Then there exists points $p, q \in E$ such that $f(p) = \sup f(X)$ and $f(q) = \inf f(X)$.*

Definition 29 ***Uniformly continuous***

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall p, q \in E \ \ (d_X(p,q) < \delta \implies d_Y(f(p), f(q)) < \epsilon) \tag{44}$$

Theorem 41 *Uniformly continuous*

$$\forall \{p_n\}, \{q_n\} \subset E \ \ (d_X(p_n, q_n) \to 0 \implies d_Y(f(p_n), f(q_n)) \to 0) \tag{45}$$

Proof.
(1) Suppose f is uniformly continuous, then by definition

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall p, q \in E \ \ (d_X(p,q) < \delta \implies d_Y(f(p), f(q)) < \epsilon) \tag{46}$$

Since $d_X(p_n, q_n) \to 0$, by definition

$$\forall \delta > 0 \ \exists N \in \mathbb{Z} \ \ (n \geq N \implies d_X(p_n, q_n) < \delta) \tag{47}$$

Thus

$$\forall \epsilon > 0 \ \exists N \in \mathbb{Z} \ \ (n \geq N \implies d_Y(f(p_n) - f(q_n)) < \epsilon) \tag{48}$$

Hence, $d_Y(f(p_n) - f(q_n)) \to 0$.
(2) Conversely, suppose f is not uniformly continuous, then

$$\exists \epsilon > 0 \ \forall \delta > 0 \ \exists p, q \in E \ \ (d_X(p,q) < \delta \ \wedge \ d_Y(f(p), f(q)) \geq \epsilon) \tag{49}$$

Taking $\delta_n = 1/n$, we thus find a pair of sequences $\{p_n\}$ and $\{q_n\}$ such that

$$\forall n \in \mathbb{Z} \ (d_X(p_n, q_n) < 1/n) \text{ and } \exists \epsilon > 0 \ \forall n \in \mathbb{Z} \ [d_Y(f(p_n), f(q_n)) \geq \epsilon] \tag{50}$$

In other words, $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n, q_n)) \not\to 0$.

Theorem 42 *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Theorem 43 *Let f be a continuous mapping of a compact metric space X into a metric space Y, and if E is a connected subset of X. Then $f(E)$ is connected.*

Theorem 44 *Let f be a continuous real function on the interval $[a, b]$.*

$$f(a) < c < f(b) \implies \exists x \in (a, b) \ \ (f(x) = c) \tag{51}$$

V. DIFFERENTIATION OF REAL FUNCTIONS

Definition 30 *$f : [a, b] \to \mathbb{R}$. For a fixed $x \in [a, b]$, form the quotient*

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b, \ t \neq x) \tag{52}$$

and define

$$f'(x) = \lim_{t \to x} \phi(t) \tag{53}$$

Theorem 45 *f is* ***differentiable*** *at $x \implies f$ is continuous at x.*

Examples

1. $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

2. $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0 but not differentiable at 0.

3. $f(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous and differentiable at 0.

Theorem 46 *Suppose f, g are differentiable. Then $f+g$, fg and f/g are differentiable.*

1. $(f+g)' = f' + g'$
2. $(fg)' = f'g + fg'$
3. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2} \qquad (g \neq 0)$

Theorem 47 *If $h(x) = g(f(x))$, then $h'(x) = g'(f(x)) \cdot f'(x)$.*

Theorem 48 *f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

Theorem 49 *(Generalized mean value theorem) If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x) \tag{54}$$

Theorem 50 *(Taylor's theorem) Suppose $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}^*$, $f^{(n-1)}$ is continuous on $[a,b]$, and $f^{(n)}$ exists on (a,b). Let $\alpha, \beta \in [a,b]$ be two distinct points and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \tag{55}$$

Then $\exists x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n \tag{56}$$

VI. THE RIEMANN-STIELTJES INTEGRAL

Definition 31 *Let $[a,b]$ be a given interval. A **partition** P of $[a,b]$ is defined as a finite set of points $\{x_0, \cdots, x_n\}$ where $a = x_0 \leq \cdots \leq x_n = b$. Suppose $f : [a,b] \to \mathbb{R}$ is bounded.*

For a partition P, the upper and lower sum of f over $[a,b]$ are defined as

$$U(P,f) = \sum_{i=1}^{n} \sup f([x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \tag{57}$$

$$L(P,f) = \sum_{i=1}^{n} \inf f([x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \tag{58}$$

And the upper and lower Riemann integral are defined to be

$$\overline{\int_a^b} f(x)\mathrm{d}x = \inf_P U(P,f) \tag{59}$$

$$\underline{\int_a^b} f(x)\mathrm{d}x = \sup_P L(P,f) \tag{60}$$

*If $\overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$, we say that f is **Riemann-integrable** on $[a,b]$, and write $f \in \mathscr{R}$.*

Consider a more general situation.

Definition 32 *Let α be a monotonically increasing function on $[a,b]$.*

$$U(P,f,\alpha)=\sum_{i=1}^{n}\sup f([x_{i-1},x_i])\cdot[\alpha(x_i)-\alpha(x_{i-1})] \tag{61}$$

$$L(P,f,\alpha)=\sum_{i=1}^{n}\inf f([x_{i-1},x_i])\cdot[\alpha(x_i)-\alpha(x_{i-1})] \tag{62}$$

$$\overline{\int_a^b} f(x)\mathrm{d}\alpha(x)=\inf_P U(P,f,\alpha) \tag{63}$$

$$\underline{\int_a^b} f(x)\mathrm{d}\alpha(x)=\sup_P L(P,f,\alpha) \tag{64}$$

If $\overline{\int_a^b} f(x)\mathrm{d}\alpha(x)=\underline{\int_a^b} f(x)\mathrm{d}\alpha(x)$, we say that f is ***Stieltjes-integrable*** *with respect to α on $[a,b]$, and write $f\in\mathscr{R}(\alpha)$.*

Definition 33 *We say partition P^* is a* ***refinement*** *of P if $P^*\supset P$.*

Theorem 51 *If P^* is a refinement of P, then*

$$L(P,f,\alpha)\le L(P^*,f,\alpha) \tag{65}$$

$$U(P,f,\alpha)\ge U(P^*,f,\alpha) \tag{66}$$

Theorem 52

$$\underline{\int_a^b} f(x)\mathrm{d}\alpha(x)\le \overline{\int_a^b} f(x)\mathrm{d}\alpha(x) \tag{67}$$

Theorem 53 *$f\in\mathscr{R}(\alpha)$ on $[a,b]$ iff*

$$\forall\epsilon>0\ \exists P\ \ (U(P,f,\alpha)-L(P,f,\alpha)<\epsilon) \tag{68}$$

Theorem 54 *Suppose $U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$*

1. *$\forall P^*\supset P$*

$$U(P^*,f,\alpha)-L(P^*,f,\alpha)\le U(P,f,\alpha)-L(P,f,\alpha)<\epsilon \tag{69}$$

2. *If s_i,t_i are arbitrary points in $[x_{i-1},x_i]$, then*

$$\sum_{i=1}^{n}|f(s_i)-f(t_i)|\cdot[\alpha(x_i)-\alpha(x_{i-1})]\le U(P,f,\alpha)-L(P,f,\alpha)<\epsilon \tag{70}$$

3. *If $f\in\mathscr{R}(\alpha)$ and t_i is a arbitrary point in $[x_{i-1},x_i]$*

$$\left|\sum_{i=1}^{n}f(t_i)\cdot[\alpha(x_i)-\alpha(x_{i-1})]-\int_a^b f\mathrm{d}\alpha\right|<\epsilon \tag{71}$$

Theorem 55 *Some types of Stieltjes-integrable functions on $[a,b]$.*

1. *f is continuous on $[a,b]$.*

2. *f is monotonic on $[a,b]$ and α is continuous on $[a,b]$.*
3. *f is bounded on $[a,b]$ with finite many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous.*
4. *$f(x) = \phi(g(x))$ where $g \in \mathscr{R}(\alpha)$, $m \le g \le M$ and ϕ is continuous on $[m, M]$.*

Theorem 56 *Let $f \in \mathscr{R}$ on $[a,b]$, put*

$$F(x) = \int_a^x f(t)\mathrm{d}t \qquad x \in [a,b] \tag{72}$$

Then

1. *F is continuous.*
2. *f is continuous at $x_0 \implies F$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.*

Theorem 57 *(Fundamental theorem of calculus) Let $f \in \mathscr{R}$ on $[a,b]$ and $F' = f$, then*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \tag{73}$$

Theorem 58 *(Integration by parts) Suppose $F' = f \in \mathscr{R}$ and $G' = g \in \mathscr{R}$, then*

$$\int_a^b F(x)g(x)\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\mathrm{d}x \tag{74}$$

VII. SEQUENCES AND SERIES OF FUNCTIONS

Definition 34 *Suppose $\{f_n\}$ is a sequence of functions defined on a set E.*
*If $\forall x \in E$, $f_n(x)$ converges, we can define a **limit function***

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad x \in E \tag{75}$$

*If $\forall x \in E$, $\sum f_n(x)$ converges, we can define a **sum function***

$$F(x) = \sum f_n(x) \qquad x \in E \tag{76}$$

The main problem is that if the important properties are preserved when under those limit operations.
Some examples

1. Double sequence

$$s_{m,n} = \frac{m}{m+n} \tag{77}$$

$$\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1 \tag{78}$$

$$\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0 \tag{79}$$

2. Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \tag{80}$$

$$F(x) = \sum f_n(x) = \begin{cases} 0 & x = 0 \\ 1 + x^2 & x \neq 0 \end{cases} \tag{81}$$

We can see that $\{f_n\}$ are continuous functions, but F is not continuous at 0.

Definition 35 ***Uniformly convergence:***

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z} \; \forall x \in E \; (n \geq N \implies d(f_n(x), f(x)) < \epsilon) \tag{82}$$

Definition 36 *Suppose $f_n \to f$, then $f_n \to f$ uniformly iff*

$$\sup_{x \in E} d(f_n(x), f(x)) \to 0 \tag{83}$$

Theorem 59 *Some properties of uniform convergence*

1. *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, then*

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) \tag{84}$$

2. $\begin{cases} \{f_n\} \text{ are continuous} \\ f_n \to f \text{ uniformly} \end{cases} \implies f$ *is continuous*

VIII. SOME SPECIAL FUNCTIONS

A. Power series

Theorem 60 *Suppose the series $\sum_{n=0}^{\infty} c_n x^n (x \in \mathbb{R})$ converges for $|x| < R$ and define*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad |x| < R \tag{85}$$

Then

1. $\sum_{n=0}^{\infty} c_n x^n (x \in \mathbb{R})$ *converges uniformly on* $[-R+\epsilon, R-\epsilon] \;\; \forall \epsilon > 0$.
2. *f is differentiable.*

$$f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \tag{86}$$

Corollary 4

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n x^{n-k} \tag{87}$$

In particular, $f^{(k)}(0) = k! c_k$.

Theorem 61 *If $\sum_{n=0}^{\infty} c_n$ converges, put*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad -1 < x < 1 \tag{88}$$

Then

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n \tag{89}$$

Theorem 62 *Extenstion of Taylor's theorem*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad |x-a| < R-a \tag{90}$$

Theorem 63 *Suppose*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \qquad x \in E \tag{91}$$

If E has a limit point, then $a_n = b_n \; \forall n \in \mathbb{N}$.

B. Elementary Functions

Definition 37 *Define*

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C} \tag{92}$$

Theorem 64 *Some properties of $E(z)$.*

1. $E(z+w) = E(z)E(w)$
2. $E'(z) = E(z)$

Suppose we have defined function $x^n (n \in \mathbb{N}^*, x \in \mathbb{R})$ and $\sqrt[n]{x} (n \in \mathbb{N}^*, x \in \mathbb{R}^+)$. Next we define the exponential function $e^x (x \in \mathbb{R})$ naturally with the function $E(x)$.

- Since $E(0) = 1$, define $e^0 = 1$.
- Let $n, m \in \mathbb{N}^*$

$$\left[E\left(\frac{n}{m}\right)\right]^m = E(n) = e^n \tag{93}$$

Thus define

$$e^{\frac{n}{m}} = \sqrt[m]{e^n} \tag{94}$$

- Let $p \in \mathbb{Q}$

$$E(-p) = \frac{1}{E(p)} = \frac{1}{e^p} \tag{95}$$

Thus define

$$e^{-p} = \frac{1}{e^p} \tag{96}$$

- Let $x \in \mathbb{R} \backslash \mathbb{Q}$

$$E(x) = \sup\{E(p) : p < x, p \in \mathbb{Q}\} = \sup\{e^p : p < x, p \in \mathbb{Q}\} \tag{97}$$

Thus define

$$e^x = \sup\{e^p : p < x, p \in \mathbb{Q}\} \tag{98}$$

Since $E \in \mathscr{C}'(\mathbb{R})$ and $\forall x \in \mathbb{R}\ E(x) \neq 0$, E has an inverse function $L \in \mathscr{C}'(\mathbb{R}^+) : \mathbb{R}^+ \to \mathbb{R}$. We use $\ln(x)$ to denote this function.

Definition 38 *Exponential function with arbitrary base*

$$a^x = e^{x \ln a} \qquad a \in \mathbb{R}^+, x \in \mathbb{R} \tag{99}$$

Definition 39 *(Trigonometric Functions)*

$$\cos(x) = \frac{1}{2}[E(ix) + E(-ix)] \tag{100}$$

$$\sin(x) = \frac{1}{2i}[E(ix) - E(-ix)] \tag{101}$$

Theorem 65 *Derivatives elementary functions*

1. *Let $a \in \mathbb{R}^+, x \in \mathbb{R}$*

$$(a^x)' = \left[e^{x \ln a}\right]' = a^x \ln a \tag{102}$$

2. *Let $x \in \mathbb{R}^+, \alpha \in \mathbb{R}$,*

$$(x^\alpha)' = \left[e^{\alpha \ln x}\right]' = \alpha x^{\alpha - 1} \tag{103}$$

3. *Let $x \in \mathbb{R}$*

$$\sin'(x) = \cos(x) \qquad \cos'(x) = -\sin(x) \tag{104}$$

C. Fourier Series

Definition 40 *Let $\{\phi_n\}$ be a sequence of complex functions on $[a,b]$. $\{\phi_n\}$ is said to be* ***orthonormal*** *if*

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}\mathrm{d}x = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{105}$$

If $\{\phi_n\}$ is orthonormal on $[a,b]$, and

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}\mathrm{d}t \tag{106}$$

We call the following series the Fourier series of f relative to $\{\phi_n\}$

$$f(x) \sim \sum c_n\phi_n(x) \tag{107}$$

Theorem 66 *Let $\{\phi_n\}$ is orthonormal on $[a,b]$. $f(x) \sim \sum c_n\phi_n(x)$.*

$$s_n(x) = \sum_{m=1}^{n} c_m\phi_m(x) \tag{108}$$

$$t_n(x) = \sum_{m=1}^{n} \gamma_m\phi_m(x) \tag{109}$$

Then

$$\int_a^b |f - s_n|^2\mathrm{d}x \leq \int_a^b |f - t_n|^2\mathrm{d}x \tag{110}$$

with equality iff $\gamma_m = c_m$ $(m = 1, \cdots, n)$.

Theorem 67 *(Bessel inequality) Let $\{\phi_n\}$ is orthonormal on $[a,b]$. $f(x) \sim \sum c_n\phi_n(x)$.*

$$\sum |c_n|^2 \leq \int_a^b |f(x)|^2\mathrm{d}x \tag{111}$$

IX. FUNCTIONS ON SEVERAL VARIABLES

Definition 41 *A* ***vector space*** *over a field F is a set V together with two operations that satisfy the eight axioms listed below. The first operation, called addition, is $+ : V \times V \to V$; the second operation, called scalar multiplication, is $\cdot : F \times V \to V$.*

1. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.
2. $\forall \mathbf{u}, \mathbf{v} \in V,\ \mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.
3. *V contains an element $\mathbf{0}$ such that $\forall \mathbf{u} \in V,\ \mathbf{0}+\mathbf{u} = \mathbf{u}$.*
4. *To $\forall \mathbf{u} \in V$ corresponds an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.*
5. $\forall \mathbf{u} \in V,\ 1\mathbf{u} = \mathbf{u}$.
6. $\forall \mathbf{u} \in V,\ a, b \in F,\ a(b\mathbf{u}) = (ab)\mathbf{u}$.
7. $\forall \mathbf{u} \in V,\ a, b \in F,\ (a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
8. $\forall \mathbf{u}, \mathbf{v} \in V,\ a \in F,\ a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

Definition 42 *Let X and Y be vector spaces. A mapping $\mathbf{A} : X \to Y$ is said to be a **linear transformation** if $\forall \mathbf{x}_1, \mathbf{x}_2 \in X, c_1, c_2 \in F$*

$$\mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{A}(\mathbf{x}_1) + c_2\mathbf{A}(\mathbf{x}_2) \tag{112}$$

Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_n\}$ are bases of vector spaces X and Y. Then every $\mathbf{A} \in L(X,Y)$ determines an $n \times p$ matrix $A = [a_{jk}]_{n\times p}$ such that

$$\mathbf{A}(\mathbf{x}_k) = \sum_{j=1}^{n} a_{jk}\mathbf{y}_j \tag{113}$$

Let $\mathbf{x} = \sum_{k=1}^{p} \alpha_k \mathbf{x}_k$, $\mathbf{y} = \mathbf{A}(\mathbf{x}) = \sum_{j=1}^{n} \beta_j \mathbf{y}_j$, then

$$\mathbf{A}(\mathbf{x}) = \sum_{k=1}^{p} \alpha_k \sum_{j=1}^{n} a_{jk}\mathbf{y}_j \tag{114}$$

$$\mathbf{A}\left(\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_p\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix}\right) = \begin{bmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_n\end{bmatrix} A \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \tag{115}$$

$$\begin{bmatrix}\beta_1\\ \vdots \\ \beta_n\end{bmatrix} = A \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \tag{116}$$

Therefore, once the bases are determined, we can study the properties of linear space with matrices and coodinates.

Let $\mathbf{B} \in L(Y,Z)$ and $\{\mathbf{z}_1, \cdots, \mathbf{z}_m\}$ be a basis of Z, and the matric corresponds to $\mathbf{B}$ is denoted as $B = [b_{ij}]_{m\times n}$. Then

$$\mathbf{B}(\mathbf{y}_j) = \sum_{i=1}^{m} b_{ij}\mathbf{z}_i \tag{117}$$

Consider the composite map $\mathbf{C} = \mathbf{BA}$

$$\mathbf{C}(\mathbf{x}_k) = \mathbf{B}(\mathbf{A}(\mathbf{x}_k)) = \mathbf{B}(\sum_{j=1}^{n} a_{jk}\mathbf{y}_j) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} b_{ij}a_{jk}\right)\mathbf{z}_i \tag{118}$$

Thus the matrix $C = [c_{ik}]_{m\times p}$ corresponds to the map $\mathbf{C}$ satisfies

$$c_{ik} = \sum_{j=1}^{n} b_{ij}a_{jk} \tag{119}$$

Definition 43 *The product of $A = [a_{ij}]_{p\times m}$ and $B = [b_{ij}]_{m\times n}$ is defined to be a matrix $C = [c_{ij}]_{p\times n}$*

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} \tag{120}$$

Definition 44 *Suppose $E \subset \mathbb{R}^n$ is open, $f : E \to \mathbb{R}^m$, and $x \in E$. If there exists $A : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0 \tag{121}$$

*then we say that f is **differentiable** at x.*

Theorem 68 *Suppose $E_1 \subset \mathbb{R}^n$ is a open set, $f : E_1 \to \mathbb{R}^m$, and f is differentiable at $x_0 \in E_1$. Suppose $f(E_1) \subset E_2 \subset \mathbb{R}^m$, E_2 is an open set, $g : E_2 \to \mathbb{R}^k$, and f is differentiable at $f(x_0) \in E_2$. Then $h(x) = f(g(x))$ is differentiable at x_0, and*

$$h'(x_0) = g'(f(x_0))f'(x_0) \tag{122}$$

Definition 45 *Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^m$. Let $\{e_1, \cdots, e_n\}$ and $\{u_1, \cdots, u_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Decompose f in standard basis of $\mathbb{R}^m$:*

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \tag{123}$$

We define

$$\frac{\partial f_i}{\partial x_j} = \lim_{t\to 0} \frac{f_i(x + te_j) - f_i(x)}{t} \tag{124}$$

Definition 46 *Let u be a unit vector in $\mathbb{R}^n$, define **directional derivative**:*

$$\frac{\partial f_i}{\partial u} = \lim_{t\to 0} \frac{f_i(x + tu) - f_i(x)}{t} \tag{125}$$

Theorem 69 *Suppose f is differentiable at $x \in E$, then the partial derivatives exists, and*

$$f' = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{126}$$

Remark 6 *The existence of partial derivatives do not imply differentiability. For example,*

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & x_1^2 + x_2^2 \neq 0 \\ 0 & x_1^2 + x_2^2 = 0 \end{cases} \tag{127}$$

$$\frac{\partial f}{\partial x_1}(0,0) = \frac{\partial f}{\partial x_2}(0,0) = 0 \tag{128}$$

However, f is not differentiable at $(0, 0)$.

Definition 47 *Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^m$. **Gradient** of f is defined as*

$$(\nabla f)(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \cdots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix} \tag{129}$$

Definition 48 *Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^m$is differentiable. If $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then f is said to be **continuously differentiable**, and write $f \in \mathscr{C}'(E)$.*

Theorem 70 *Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^m$. $f \in \mathscr{C}'(E)$ iff its partial derivatives $\frac{\partial f_i}{x_j}$ are continuous.*

Theorem 71 *(The inverse function theorem) Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^n$, and $f \in \mathscr{C}'(E)$. For some $a \in E$, $f'(a)$ is invertible. Then*

1. *$\exists$open sets $U, V \subset \mathbb{R}^n$ such that $a \in U$, $f(a) \in V$, and f is 1-to-1 from U to V.*
2. *$f^{-1} \in \mathscr{C}'(V)$.*

Theorem 72 *(The implicit function theorem) If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_m) \in \mathbb{R}^m$, we write (x, y) for $(x_1, \cdots, x_n, y_1, \cdots, y_m)$. For $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$, split it into $A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$*

$$A(h,k) = A_x h + A_y k \qquad \forall h \in \mathbb{R}^n, k \in \mathbb{R}^m \tag{130}$$

If the following conditions are satisfied

$$\begin{cases} E \in \mathbb{R}^{n+m} \text{ is an open set} \\ (a,b) \in E \\ f \in \mathscr{C}'(E) : E \to \mathbb{R}^n \\ f(a,b) = 0 \\ A = f'(a,b) \text{ is invertible} \end{cases}$$

Then there exists open sets $U \subset \mathbb{R}^{n+m}, W \subset \mathbb{R}^m$ and a mapping $g \in \mathscr{C}'(W) : W \to \mathbb{R}^n$ such that

$$\begin{cases} (a,b) \in U \\ b \in W \\ f(g(y), y) = 0, \ \forall y \in W \\ g'(b) = -(A_x)^{-1} A_y \end{cases}$$

Definition 49 *Suppose $E \subset \mathbb{R}^n$ is an open set, $f : E \to \mathbb{R}^n$ is differentiable at point $x \in E$. The **Jacobian** of f at x is defined as*

$$J_f(x) = \det f'(x) \tag{131}$$

Definition 50 ***Second-order partial derivatives**. We write*

$$D_{ij} f = D_i D_j f = \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) = \frac{\partial^2 f}{\partial x_j \partial x_i} \tag{132}$$

Theorem 73 *$E \subset \mathbb{R}^2$ is an open set and $f : E \to \mathbb{R}$.*

$$\begin{cases} D_1 f, D_2 f, D_{21} f \text{ exists } \forall x \in E \\ D_{21} \text{ is continuous at } (a,b) \in E \end{cases} \implies (D_{12} f)(a,b) = (D_{21} f)(a,b) \tag{133}$$