Unital channels and majorization

Definition 1 $\Phi \in \mathsf{Chan}(\mathcal{X})$ *is a unital channel if* $\Phi(\mathbb{1}_{\mathcal{X}}) = \mathbb{1}_{\mathcal{X}}$.

I. SUBCLASSES OF UNITAL CHANNELS

Definition 2 $\Phi \in \mathsf{Chan}(\mathcal{X})$ *is mixed-unitary channel if*

$$\Phi(X) = \sum_{a\in\Sigma} p(a) U_a X U_a^* \tag{1}$$

where $p \in \mathcal{P}(\Sigma)$ *and* $\{U_a : a \in \Sigma\} \subset \mathsf{U}(\mathcal{X})$.

Remark 1 *Unital channels may not be mixed-unitary. Let* $\mathcal{X} = \mathbb{C}^3$*, then*

$$\Phi(X) = \frac{1}{2}\operatorname{Tr}(X)\mathbb{1} - \frac{1}{2}X^T \tag{2}$$

is unital but not mixed-unitary. We can prove that this channel is a extreme point of $\mathsf{Chan}(\mathcal{X})$ *but* Φ *is not unitary. So it can not be represented as the convex combination of unitary channels.*

Definition 3 $\Phi \in \mathsf{Chan}(\mathcal{X})$ *is a pinching if*

$$\Phi(X) = \sum_{a\in\Sigma} \Pi_a X \Pi_a \qquad \sum_{a\in\Sigma} \Pi_a = \mathbb{1}_{\mathcal{X}} \tag{3}$$

Theorem 1 *Every pinching channel is a mixed-unitary channel.*

Proof. Define

$$U_w = \sum_{a\in\Sigma} w(a)\Pi_a \qquad w \in \{-1,1\}^{\Sigma} \tag{4}$$

Then

$$\frac{1}{2^{|\Sigma|}} \sum_{w\in\{-1,1\}^{\Sigma}} U_w X U_w^* = \sum_{a\in\Sigma} \Pi_a X \Pi_a = \Phi(X) \tag{5}$$

Theorem 2 *Let* $A \in \mathsf{U}(\mathcal{X}, \mathcal{X}\otimes\mathcal{Z})$ *be an isometry. Let* $\Phi \in \mathsf{Chan}(\mathcal{X})$ *such that*

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXA^*) \tag{6}$$

The following two statements are equivalent

1. *Φ is a mixed-unitary channels.*
2. *There exists a collection of channels $\{\Psi_a : a \in \Sigma\} \subset \mathsf{Chan}(\mathcal{X})$ and a measurement $\mu : \Sigma \to \mathsf{Pos}(\mathcal{Z})$*

$$X = \sum_{a\in\Sigma} \Psi_a\left(\operatorname{Tr}_{\mathcal{Z}}[(\mathbb{1}_{\mathcal{X}} \otimes \mu(a))AXA^*]\right) \tag{7}$$

Proof.

1. Assume 1 holds.

$$\begin{cases} \Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXA^*) = \sum_{a\in\Sigma}(\mathbb{1}_{\mathcal{X}} \otimes v_a^*)AXA^*(\mathbb{1}_{\mathcal{X}} \otimes v_a) \\ \Phi(X) = \sum_{a\in\Sigma} p(a)U_a X U_a^* \end{cases} \tag{8}$$

where $\sum_{a\in\Sigma} v_a v_a^* = \mathbb{1}_{\mathcal{Z}}$. Then there is a unitary matrix connecting them

$$\exists W \in \mathsf{U}(\mathbb{C}^\Sigma)\ \sqrt{p(a)}U_a = \sum_{b\in\Sigma} W(a,b)(\mathbb{1}_{\mathcal{X}} \otimes v_b^*)A = \left(\mathbb{1}_{\mathcal{X}} \otimes \sum_{b\in\Sigma} W(a,b)v_b^*\right)A \tag{9}$$

Define

$$u_a = \sum_{b\in\Sigma} \overline{W(a,b)} v_b$$

We can see that

$$\sum_{a\in\Sigma} u_a u_a^* = \left(\sum_{b\in\Sigma} \overline{W(a,b)} v_b\right)\left(\sum_{c\in\Sigma} W(a,c)v_c^*\right) = \sum_{a\in\Sigma} v_a v_a^* = \mathbb{1}_{\mathcal{Z}}$$
$$\sqrt{p(a)}U_a = (\mathbb{1}_{\mathcal{X}} \otimes u_a^*)A$$

Define

$$\mu(a) = u_a u_a^* \tag{10}$$

Then

$$\mathrm{Tr}_{\mathcal{Z}}[(\mathbb{1}_{\mathcal{X}} \otimes \mu(a))AXA^*] = (\mathbb{1}_{\mathcal{X}} \otimes u_a^*)AXA^*(\mathbb{1}_{\mathcal{X}} \otimes u_a) = p(a)U_a X U_a^* \tag{11}$$

Obviously, the following channels work.

$$\Psi_a(X) = U_a^* X U_a$$

2. If statement 2 holds, define

$$\Phi_a(X) = \mathrm{Tr}_{\mathcal{Z}}((\mathbb{1}_{\mathcal{X}} \otimes \mu(a))AXA^*) \tag{12}$$

Let

$$\Psi_a(X) = \sum_{b\in\Gamma} A_{a,b} X A_{a,b}^* \qquad \Phi_a(X) = \sum_{c\in\Gamma} B_{a,c} X B_{a,c}^* \tag{13}$$

$$\sum_{a\in\Sigma} \Psi_a \Phi_a = \mathbb{1}_{\mathsf{Lin}(\mathcal{X})}$$
$$\implies \sum_{a\in\Sigma} \sum_{b,c\in\Gamma} \mathrm{vec}(A_{a,b}B_{a,c})\,\mathrm{vec}(A_{a,b}B_{a,c})^* = \mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*$$

Since $\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*$ is a extreme point, we have

$$\forall a,b,c\ A_{a,b}B_{a,c} = \alpha_{a,b,c}\mathbb{1}_{\mathcal{X}} \qquad \sum_{a,b,c} |\alpha_{a,b,c}|^2 = 1 \tag{14}$$

Since Ψ_a are channels

$$\forall b\ \sum_b A_{a,b}^* A_{a,b} = \mathbb{1}$$

This implies

$$\sum_b |\alpha_{a,b,c}|^2 \mathbb{1} = \sum_b (A_{a,b}B_{a,c})^*(A_{a,b}B_{a,c}) = B_{a,c}^* B_{a,c} \tag{15}$$

Thus $\Phi_a(X) = \sum_{c\in\Gamma} B_{a,c} X B_{a,c}$ is mixed-unitary.

A. Weyl-covariant channels

The set $\mathbb{Z}_n$ is defined as

$$\mathbb{Z}_n = \{0, \cdots, n-1\} \tag{16}$$

This set forms a ring, with respect to addition and multiplication modulo n.

The discrete Weyl operators are a collection of unitary operators acting on $\mathcal{X} = \mathbb{C}^{\mathbb{Z}_n}$, for a given positive integer n, defined in the following way. One first defines a scalar value

$$\zeta = \exp\left(\frac{2\pi i}{n}\right) \tag{17}$$

along with unitary operators

$$U = \sum_{c\in\mathbb{Z}_n} E_{c+1,c} \qquad V = \sum_{c\in\mathbb{Z}_n} \zeta^c E_{c,c} \tag{18}$$

The discrete Weyl operator $W_{a,b}$ is defined as

$$W_{a,b} = U^a V^b = \sum_{c\in\mathbb{Z}_n} \zeta^{bc} E_{a+c,c} \tag{19}$$

Property 1 *Properties of the discrete Weyl operator*

$$\mathbb{1} = W_{0,0} \quad \sigma_z = W_{0,1} \quad \sigma_x = W_{1,0} \quad -i\sigma_y = W_{1,1} \tag{20}$$

$$UV = \sum_{c\in\mathbb{Z}_n} \zeta^c E_{c+1,c} \qquad VU = \sum_{c\in\mathbb{Z}_n} \zeta^{c+1} E_{c+1,c} \qquad VU = \zeta UV \tag{21}$$

$$\overline{W_{a,b}} = W_{a,-b} \qquad W_{a,b}^T = \zeta^{-ab} W_{-a,b} \qquad W_{a,b}^* = \zeta^{ab} W_{-a,-b} \tag{22}$$

$$W_{a,b}W_{c,d} = \zeta^{bc} W_{a+c,b+d} = \zeta^{bc-ad} W_{c,d} W_{a,b} \tag{23}$$

$$\sum_{c\in\mathbb{Z}_n} \zeta^{ac} = \begin{cases} n & a = 0 \\ 0 & a \in \{1, \cdots, n-1\} \end{cases} \tag{24}$$

$$\mathrm{Tr}(W_{a,b}) = \begin{cases} n & (a,b) = (0,0) \\ 0 & (a,b) \neq (0,0) \end{cases} \tag{25}$$

$$\langle W_{a,b}, W_{c,d} \rangle = \begin{cases} n & (a,b) = (c,d) \\ 0 & (a,b) \neq (c,d) \end{cases} \tag{26}$$

$$\left\{ \frac{1}{\sqrt{n}} W_{a,b} : (a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n \right\} \ \textit{is an ONB.} \tag{27}$$

$$\textit{discrete Fourier transform operator } F = \frac{1}{\sqrt{n}} \sum_{a,b\in\mathbb{Z}_n} \zeta^{ab} E_{a,b} \tag{28}$$

$$F \in \mathsf{U} \tag{29}$$

Definition 4 *Let $X = \mathbb{C}^{\mathbb{Z}_n}$. A map $\Phi \in \mathsf{T}(\mathcal{X})$ is a Weyl-covariant map if*

$$\Phi(W_{a,b} X W_{a,b}^*) = W_{a,b} \Phi(X) W_{a,b}^* \tag{30}$$

Theorem 3 $X = \mathbb{C}^{\mathbb{Z}_n}, \ \Phi \in \mathsf{T}(\mathcal{X})$

1. *Φ is a Weyl-covariant map*
2. *$\exists A \in \mathsf{Lin}(\mathcal{X})$ such that*

$$\Phi(W_{a,b}) = A(a,b) W_{a,b} \tag{31}$$

3. $\exists B \in \mathsf{Lin}(\mathcal{X})$ *such that*

$$\Phi(X) = \sum_{a,b\in\mathbb{Z}_n} B(a,b) W_{a,b} X W_{a,b}^* \tag{32}$$

Under the assumption that these three statements hold, the operators A and B in statements 2 and 3 are related by the equation

$$A^T = nF^* B F \tag{33}$$

B. Completely depolarizing and dephasing channels

$$\Omega(X) = \frac{\mathrm{Tr}(X)}{\dim(\mathcal{X})} \mathbb{1}_{\mathcal{X}} = \frac{1}{n^2} \sum_{a,b\in\mathbb{Z}_n} W_{a,b} X W_{a,b}^*$$
$$\Delta(X) = \sum_{a\in\Sigma} X(a,a) E_{a,a} = \frac{1}{n} \sum_{c\in\mathbb{Z}_n} W_{0,c} X W_{0,c}^*$$

C. Schur channels

Definition 5 *Schur map:*

$$\Phi(X) = A \odot X \tag{34}$$

where $\odot$ is the entry-wise product of A and X

$$(A \odot X)(a,b) = A(a,b) X(a,b) \tag{35}$$

Theorem 4 $A \in \mathsf{Lin}(\mathcal{X})$, $\Phi(X) = A \odot X$ *is completely positive iff A is positive iff Kraus representations consisting only of equal pairs of diagonal operators.*

Theorem 5 $A \in \mathsf{Lin}(\mathcal{X})$, $\Phi(X) = A \odot X$ *is trace-preserving iff $\forall a \in \Sigma$ $A(a,a) = 1$ iff Φ is unital.*

II. GENERAL PROPERTIES OF UNITAL CHANNELS

A. Extreme points of the set of unital channels

Define an operator $V \in \mathsf{Lin}(\mathcal{X} \oplus \mathcal{X}, (\mathcal{X} \oplus \mathcal{X}) \otimes (\mathcal{X} \oplus \mathcal{X}))$

$$V \operatorname{vec}(X) = \operatorname{vec} \begin{bmatrix} X & 0 \\ 0 & X^T \end{bmatrix} \tag{36}$$

Define $\phi(\Phi) \in \mathsf{T}(\mathcal{X} \oplus \mathcal{X})$

$$J(\phi(\Phi)) = V J(\Phi) V^* \tag{37}$$

Theorem 6 $\Phi \in \mathsf{T}(\mathcal{X})$

$$\Phi(X) = \sum_{a\in\Sigma} A_a X B_a^* \tag{38}$$

$$\phi(\Phi) \begin{bmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{bmatrix} = \sum_{a\in\Sigma} \begin{bmatrix} A_a & 0 \\ 0 & A_a \end{bmatrix} \begin{bmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{bmatrix} \begin{bmatrix} B_a & 0 \\ 0 & B_a \end{bmatrix}^* \tag{39}$$

Theorem 7 *1. $\Phi \in \mathrm{CP}(\mathcal{X})$ iff $\phi(\Phi) \in \mathrm{CP}(\mathcal{X} \oplus \mathcal{X})$.*

2. $\Phi \in \mathsf{T}(\mathcal{X})$ is trace preserving and unital iff $\phi(\Phi)$ is trace preserving.

3. Φ is an extreme point of the set of all unital channels in $\mathsf{Chan}(\mathcal{X})$ iff $\phi(\Phi)$ is an extreme point of the set of channels $\mathsf{Chan}(\mathcal{X} \oplus \mathcal{X})$.

Theorem 8 *$\Phi \in \mathsf{Chan}(\mathcal{X})$ is a unital channel and*

$$\Phi(X) = \sum_{a \in \Sigma} A_a X A_a^* \tag{40}$$

where A_a is linearly independent.

Φ is an extreme point of the set of all unital channels in $\mathsf{Chan}(\mathcal{X})$ iff

$$\left\{ \begin{bmatrix} A_b^* A_a & 0 \\ 0 & A_a A_b^* \end{bmatrix} : (a,b) \in \Sigma \times \Sigma \right\} \tag{41}$$

is linearly independent.

Lemma 1 $A_0, A_1 \in \mathsf{Lin}(\mathcal{X})$

$$A_0^* A_0 + A_1^* A_1 = \mathbb{1}_{\mathcal{X}} = A_0 A_0^* + A_1 A_1^* \implies \exists U, V \in \mathsf{U}(\mathcal{X}) \; V A_0 U^*, V A_1 U^* \in \mathsf{diag}(\mathcal{X}) \tag{42}$$

Proof. It suffices to prove that there exists a unitary operator $W \in \mathsf{U}(\mathcal{X})$ such that the operators WA_0 and WA_1 are both normal and satisfy

$$[WA_0, WA_1] = 0 \tag{43}$$

Polar decompositions: $U_0, U_1 \in \mathsf{U}(\mathcal{X}), P_0, P_1 \in \mathsf{Pos}(\mathcal{X})$

$$A_0 = U_0 P_0 \qquad A_1 = U_1 P_1 \tag{44}$$

Let $W = U_0^*$, then $WA_0 = P_0$ is normal.

$$A_0^* A_0 + A_1^* A_1 = \mathbb{1}_{\mathcal{X}} \implies P_0^2 + P_1^2 = \mathbb{1}_{\mathcal{X}} \tag{45}$$

$$A_0 A_0^* + A_1 A_1^* = \mathbb{1}_{\mathcal{X}} \implies U_0 P_0^2 U_0^* + U_1 P_1^2 U_1^* = \mathbb{1}_{\mathcal{X}} \tag{46}$$

Then

$$\begin{aligned}
(WA_1)(WA_1)^* &= U_0^* A_1 A_1^* U_0 \\
&= U_0^* U_1 P_1^2 U_1^* U_0 \\
&= U_0^* (\mathbb{1}_{\mathcal{X}} - U_0 P_0^2 U_0^*) U_0 \\
&= \mathbb{1}_{\mathcal{X}} - P_0^2 \\
&= P_1^2 \\
&= P_1 U_1^* U_0 U_0^* U_1 P_1 \\
&= (WA_1)^* (WA_1)
\end{aligned}$$

Thus WA_1 is normal.

$$\begin{aligned}
&\begin{cases} P_0^2 + P_1^2 = \mathbb{1}_{\mathcal{X}} \\ U_0 P_0^2 U_0^* + U_1 P_1^2 U_1^* = \mathbb{1}_{\mathcal{X}} \end{cases} \\
\implies & U_0 P_0^2 U_0^* = U_1 P_0^2 U_1^* \\
\implies & U_0 P_0 U_0^* = U_1 P_0 U_1^* \\
\implies & P_0 (U_0^* U_1) = (U_0^* U_1) P_0
\end{aligned}$$

$$P_0^2 P_1^2 = P_0^2 (\mathbb{1}_{\mathcal{X}} - P_0^2) = (\mathbb{1}_{\mathcal{X}} - P_0^2) P_0^2 = P_1^2 P_0^2 \implies P_0 P_1 = P_1 P_0 \tag{47}$$

Then

$$(WA_0)(WA_1) = P_0 U_0^* U_1 P_1 = U_0^* U_1 P_1 P_0 = (WA_1)(WA_0) \tag{48}$$

Theorem 9 *Every unital qubit channel is mixed unitary.*

Proof. It suffices to establish that every unital channel $\Phi \in \mathsf{Chan}(\mathcal{X})$ that is not a unitary channel is not an extreme point.