Bipartite entanglement

Definition 1 $R \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y})$ *if*

$$R = \sum_{a \in \Sigma} P_a \otimes Q_a \tag{1}$$

where $\{P_a : a \in \Sigma\} \subset \mathsf{Pos}(\mathcal{X})$ *and* $\{Q_a : a \in \Sigma\} \subset \mathsf{Pos}(\mathcal{Y})$.

Definition 2

$$\mathsf{SepD}(\mathcal{X} : \mathcal{Y}) = \mathsf{Sep}(\mathcal{X} : \mathcal{Y}) \cap \mathsf{D}(\mathcal{X} \otimes \mathcal{Y}) \tag{2}$$

Theorem 1 $\mathsf{Sep}(\mathcal{X} : \mathcal{Y})$ *is a convex cone and* $\mathsf{SepD}(\mathcal{X} : \mathcal{Y})$ *is convex.*

Proof. It suffices to prove that $Sep(X : Y)$ is closed under addition as well as multiplication by any nonnegative real number.

$$R_0 = \sum_{a \in \Sigma_0} P_a \otimes Q_a \qquad R_1 = \sum_{a \in \Sigma_1} P_a \otimes Q_a \tag{3}$$

Then

$$R_0 + R_1 = \sum_{a \in \Sigma_0 \cup \Sigma_1} P_a \otimes Q_a \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y}) \qquad \lambda R_0 = \sum_{a \in \Sigma_0} \lambda P_a \otimes Q_a \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y}) \tag{4}$$

$\mathsf{SepD}(\mathcal{X} : \mathcal{Y})$ is convex because it is the intersection of two convex sets. ■

Lemma 1 $\mathcal{A} \subset \mathsf{Pos}(\mathcal{Z})$ *is a cone.* $\emptyset \neq \mathcal{B} = \mathcal{A} \cap \mathsf{D}(\mathcal{Z})$

$$\mathcal{A} = \mathrm{cone}(\mathcal{B}) \tag{5}$$

Proof.

1. $\mathrm{cone}(\mathcal{B}) \subset \mathcal{A}$ is obvious.
2. Assume $P \in \mathcal{A}$, then

$$\frac{P}{\mathrm{Tr}(P)} \in \mathcal{A} \cap \mathsf{D}(\mathcal{Z}) = \mathcal{B} \tag{6}$$

■

Theorem 2 *Let* $\xi \in \mathsf{D}(\mathcal{X} \otimes \mathcal{Y})$ *be a density operator. The following statements are equivalent:*

1. $\xi \in \mathsf{SepD}(\mathcal{X} : \mathcal{Y})$
2. *There exists an alphabet* Σ

$$\xi = \sum_{a \in \Sigma} p(a) \rho_a \otimes \sigma_a \tag{7}$$

3. *There exists an alphabet* Σ

$$\xi = \sum_{a \in \Sigma} p(a) x_a x_a^* \otimes y_a y_a^* \tag{8}$$

Theorem 3 *If $\xi \in \mathsf{SepD}(\mathcal{X} : \mathcal{Y})$, then there exists an alphabet Σ with $|\Sigma| \leq \mathsf{rank}(\xi)^2$, $\{x_a : a \in \Sigma\} \subset \mathsf{S}(\mathcal{X})$ and $\{y_a : a \in \Sigma\} \subset \mathsf{S}(\mathcal{Y})$ such that*

$$\xi = \sum_{a\in\Sigma} p(a) x_a x_a^* \otimes y_a y_a^* \tag{9}$$

Proof. It holds that

$$\mathsf{SepD}(\mathcal{X} : \mathcal{Y}) = \mathrm{conv}\{xx^* \otimes yy^* : x \in \mathcal{S}(\mathcal{X}), y \in \mathcal{S}(\mathcal{Y})\} \tag{10}$$

And it is easy to see that

$$\xi \in \mathrm{conv}\{xx^* \otimes yy^* : x \in \mathcal{S}(\mathcal{X}), y \in \mathcal{S}(\mathcal{Y}), \mathsf{im}(xx^* \otimes yy^*) \subset \mathsf{im}(\xi)\} \tag{11}$$

Notice that the following is a real affine space satisfying of dimension $\mathsf{rank}(\xi)^2 - 1$

$$\{H \in \mathsf{Herm}(\mathcal{X} \otimes \mathcal{Y}) : \mathsf{im}(H) \subset \mathsf{im}(\xi), \mathrm{Tr}(H) = 1\} \tag{12}$$

Thus ξ is contained in a affine space of dimension $\mathsf{rank}(\xi)^2 - 1$. This completes the proof. ■

Corollary 1 *If $R \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y})$ and $R \neq 0$, then there exists an alphabet Σ with $|\Sigma| \leq \mathsf{rank}(R)^2$, $\{x_a : a \in \Sigma\} \subset \mathcal{X}$ and $\{y_a : a \in \Sigma\} \subset \mathcal{Y}$ such that*

$$R = \sum_{a\in\Sigma} x_a x_a^* \otimes y_a y_a^* \tag{13}$$

Theorem 4 *For every choice of complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, the set* $\mathrm{SepD}(\mathcal{X} : \mathcal{Y})$ *is compact and the set* $\mathrm{Sep}(\mathcal{X} : \mathcal{Y})$ *is closed.*

Theorem 5 *(Horodecki criterion) $R \in \mathsf{Pos}(\mathcal{X} \otimes \mathcal{Y})$. The following are equivalent:*

1. *$R \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y})$*
2. *If $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Z})$ is a positive map, then*

$$(\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})(R) \in \mathsf{Pos}(\mathcal{Z} \otimes \mathcal{Y}) \tag{14}$$

3. *If $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$ is a positive unital map, then*

$$(\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})(R) \in \mathsf{Pos}(\mathcal{Y} \otimes \mathcal{Y}) \tag{15}$$

Proof. It is easy to see $1 \implies 2$ and $2 \implies 3$. Thus we focus on $3 \implies 1$.

Suppose $R \notin \mathsf{Sep}(\mathcal{X} : \mathcal{Y})$. As $\mathsf{Sep}(\mathcal{X} : \mathcal{Y})$ is a closed, convex cone within the real vector space $\mathsf{Herm}(\mathcal{X} \otimes \mathcal{Y})$, the hyperplane separation theorem implies that

$$\exists H \in \mathsf{Herm}(\mathcal{X} \otimes \mathcal{Y})\ \forall S \in \mathsf{Sep}(\mathcal{X} : \mathcal{Y})\ \begin{cases} \langle H, R\rangle < 0 \\ \langle H, S\rangle \geq 0 \end{cases} \tag{16}$$

Such an H will be used to construct a positive and unital map.

Define $\Psi \in \mathsf{T}(\mathcal{Y}, \mathcal{X})$ as

$$J(\Psi) = H \tag{17}$$

Let $P \in \mathsf{Pos}(\mathcal{X})$, $Q \in \mathsf{Pos}(\mathcal{Y})$. Then

$$\langle P, \Psi(Q)\rangle = \langle P \otimes \overline{Q}, J(\Psi)\rangle = \langle H, P \otimes \overline{Q}\rangle \geq 0 \tag{18}$$

Thus Ψ is a positive map. It follows that Ψ^* is also a positive map.

Define Φ

$$\Phi(X) = A^{-\frac{1}{2}} \Psi^*(X) A^{-\frac{1}{2}} \tag{19}$$

where $A = \Psi^*(\mathbb{1}_{\mathcal{X}})$.

Now we get a positive and unital map Φ. Then we show that $(\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})(R)$ is not positive

$$\begin{aligned} 0 &> \langle H, R\rangle \\ &= \langle J(\Psi), R\rangle \\ &= \langle \mathrm{vec}(\mathbb{1}_{\mathcal{Y}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{Y}})^*, (\Psi^* \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})(R)\rangle \\ &= \langle \mathrm{vec}(\sqrt{A})\,\mathrm{vec}(\sqrt{A})^*, (\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})(R)\rangle \end{aligned}$$

Definition 3 *Define the following projectors*

$$\Delta_0 = \frac{1}{n}\sum_{a,b\in\Sigma} E_{a,b} \otimes E_{a,b} \qquad \Pi_0 = \frac{1}{2}\mathbb{1}\otimes\mathbb{1} + \frac{1}{2}W$$

$$\Delta_1 = \mathbb{1}\otimes\mathbb{1} - \Delta_0 \qquad \Pi_1 = \mathbb{1}\otimes\mathbb{1} - \Pi_1$$

Isotropic states

$$\lambda\Delta_0 + (1-\lambda)\frac{\Delta_1}{n^2-1} \tag{20}$$

Werner states

$$\lambda\frac{\Pi_0}{\binom{n+1}{2}} + (1-\lambda)\frac{\Pi_1}{\binom{n}{2}} \tag{21}$$

Theorem 6 *The isotropic state is entangled for $\lambda \in (1/n, 1]$, while the Werner state is entangled for $\lambda \in [0, 1/2)$.*

Proof. Let T denote the transpose map

$$\mathrm{T}(X) = X^T \tag{22}$$

We have

$$(\mathrm{T}\otimes\mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})\left(\lambda\Delta_0 + (1-\lambda)\frac{\Delta_1}{n^2-1}\right) = \frac{1+\lambda n}{2}\frac{\Pi_0}{\binom{n+1}{2}} + \frac{1-\lambda n}{2}\frac{\Pi_1}{\binom{n}{2}} \tag{23}$$

$$(\mathrm{T}\otimes\mathbb{1}_{\mathsf{Lin}(\mathcal{Y})})\left(\lambda\frac{\Pi_0}{\binom{n+1}{2}} + (1-\lambda)\frac{\Pi_1}{\binom{n}{2}}\right) = \frac{2\lambda-1}{n}\Delta_0 + \left(1 - \frac{2\lambda-1}{n}\right)\Delta_1 \tag{24}$$

This completes the proof. ■

Lemma 2

$$X = \sum_{a,b\in\Sigma} X_{a,b} \otimes E_{a,b} \in \mathsf{Lin}(\mathcal{X}\otimes\mathcal{Y}) \tag{25}$$

We have

$$\|X\|^2 \leq \sum_{a,b\in\Sigma} \|X_{a,b}\|^2 \tag{26}$$

Proof. Define

$$Y_a = \sum_{b\in\Sigma} X_{a,b} \otimes E_{a,b} \tag{27}$$

$$\|Y_a^* Y_a\| = \left\|\sum_{b\in\Sigma} X_{a,b}X_{a,b}^* \otimes E_{a,a}\right\|$$

$$
\begin{aligned}
\|X\|^2 &= \|XX^*\| \\
&\leq \sum_{a\in\Sigma} \|Y_a^* Y_a\| \\
&\leq \sum_{a,b\in\Sigma} \|X_{a,b} X_{a,b}^*\| \\
&= \sum_{a,b\in\Sigma} \|X_{a,b}\|^2
\end{aligned}
$$

Lemma 3 $\Phi \in \mathsf{T}(\mathcal{X},\mathcal{Y})$ *is a positive and unital map. It holds that*

$$\|\Phi(X)\| \leq \|X\| \tag{28}$$

Theorem 7 $H \in \mathsf{Herm}(\mathcal{X}\otimes\mathcal{Y})$ *and* $\|H\|_2 \leq 1$*, it holds that*

$$\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - H \in \mathsf{Sep}(\mathcal{X}:\mathcal{Y}) \tag{29}$$