Basic notions of quantum information

I. REGISTERS AND STATES

Definition 1 *A* ***register*** X *is either one of the following two objects:*

1. *An alphabet* Σ.
2. *An* n*-tuple* $\mathsf{X} = (\mathsf{Y}_1, \cdots, \mathsf{Y}_n)$*, where* n *is a positive integer and* $\mathsf{Y}_1, \cdots, \mathsf{Y}_n$ *are registers.*

The classical state set of a register X *is determined as follows:*

1. *If* $\mathsf{X} = \Sigma$ *is a simple register, the classical state set of* X *is* Σ.
2. *If* $\mathsf{X} = (\mathsf{Y}_1, \cdots, \mathsf{Y}_n)$ *is a compound register, the classical state set of* X *is the Cartesian product*

$$\Sigma = \Gamma_1 \times \cdots \times \Gamma_n \tag{1}$$

where Γ_k *denotes the classical state set associated with the register* Y_k *for each* $k \in \{1, \cdots, n\}$.

Elements of a register's classical state set are called ***classical states*** *of that register.*

Definition 2 *A* ***probabilistic state*** *of a register* X *refers to a probability distribution, or random mixture, over the classical states of that register.*

Assuming the classical state set of X is Σ, a probabilistic state of X is identified with a probability vector $p \in \mathcal{P}(\Sigma)$; the value $p(a)$ represents the probability associated with a given classical state $a \in \Sigma$.

Definition 3 *The complex Euclidean space (****quantum system****) associated with a register* X *is defined to be* $\mathbb{C}^\Sigma$*, for* Σ *being the classical state set of* X.

The complex Euclidean space $\mathcal{X}$ associated with a compound register $\mathsf{X} = (\mathsf{Y}_1, \cdots, \mathsf{Y}_n)$ is given by the tensor product

$$\mathcal{X} = \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_n \tag{2}$$

Definition 4 *A quantum state is a density operator of the form* $\rho \in \mathsf{D}(\mathcal{X})$ *for some choice of a complex Euclidean space* $\mathcal{X}$.

Convex combinations of quantum states

$$\rho = \sum_{a \in \Gamma} p(a) \rho_a \tag{3}$$

is an valid state.

Definition 5 ***Ensembles*** *of quantum states*

$$\eta : \Gamma \to \mathsf{Pos}(\mathcal{X}) \tag{4}$$

$$\mathrm{Tr}\left(\sum_{a \in \Gamma} \eta(a)\right) = 1 \tag{5}$$

The operator $\eta(a)$ *represents a state together with the probability associated with that state: the probability is* $\mathrm{Tr}(\eta(a))$*, while the state is*

$$\frac{\eta(a)}{\mathrm{Tr}(\eta(a))} \tag{6}$$

A quantum state $\rho \in \mathsf{D}(\mathcal{X})$ *is said to be a* ***pure state*** *if it has rank equal to 1. Equivalently,* ρ *is a pure state if there exists a unit vector* $u \in \mathcal{X}$ *such that*

$$\rho = uu^* \tag{7}$$

Theorem 1 *It follows from the spectral theorem that every quantum state is a mixture of pure quantum states, and moreover that a state $\rho \in \mathsf{D}(\mathcal{X})$ is pure if and only if it is an extreme point of the set $\mathsf{D}(\mathcal{X})$.*

Proof. If ρ is pure, then $\rho = uu^*$ for some unit vector u. Let

$$\rho = \lambda\rho_0 + (1-\lambda)\rho_1 \quad \rho_0, \rho_1 \in \mathsf{D}(\mathcal{X}) \ \lambda \in (0,1) \tag{8}$$

Then

$$u^*\rho_0 u = u^*\rho_1 u = 1 \tag{9}$$

This implies

$$\rho_0 = \rho_1 = \rho \tag{10}$$

If $\mathsf{rank}(\rho) > 1$, then $\rho = \sum_{a\in\Sigma} p(a)u_a u_a^*$ for some alphabet Σ with $|\{p(a) > 0 : a \in \Sigma\}| \geq 2$. Let

$$\rho_0 = \rho - \delta u_a u_a^* + \delta u_b u_b^* \quad \rho_1 = \rho + \delta u_a u_a^* - \delta u_b u_b^* \tag{11}$$

For small enough δ, we get

$$\rho_0, \rho_1 \in \mathsf{D}(\mathcal{X}) \quad \rho_0 \neq \rho_1 \quad \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1 = \rho \tag{12}$$

Definition 6 *A quantum state $\rho \in \mathsf{D}(\mathcal{X})$ is said to be a flat state if it holds that*

$$\rho = \frac{\Pi}{\mathrm{Tr}(\Pi)} \tag{13}$$

for a nonzero projection operator $\Pi \in \mathsf{Proj}(\mathcal{X})$.

Theorem 2 *Bases of density operators: for every complex Euclidean space $\mathcal{X}$ there exist spanning sets of the space $\mathsf{Lin}(\mathcal{X})$ consisting only of density operators.*

Let Σ be an alphabet, and assume that a total ordering has been defined on Σ. For every pair $(a, b) \in \Sigma \times \Sigma$, define a density operator

$$\rho_{a,b} = \begin{cases} E_{a,a} & a = b \\ \frac{1}{2}(e_a + e_b)(e_a + e_b)^* & a < b \\ \frac{1}{2}(e_a + ie_b)(e_a + ie_b)^* & a > b \end{cases} \tag{14}$$

It follows that

$$\mathsf{Span}\{\rho_{a,b} : (a, b) \in \Sigma \times \Sigma\} = \mathsf{Lin}(\mathbb{C}^\Sigma) \tag{15}$$

Classical states and probabilistic states as quantum states:

- The operator $E_{a,a} \in \mathsf{D}(\mathcal{X})$ is taken as a representation of the register X being in the classical state a, for each $a \in \Sigma$.
- Probabilistic states of registers correspond to diagonal density operators, with each probabilistic state $p \in \mathcal{P}(\Sigma)$ being represented by the density operator.

Definition 7 *The partial trace and reductions of quantum states. Let*

$$\mathsf{X} = (\mathsf{Y}_1, \cdots, \mathsf{Y}_n) \tag{16}$$

be a compound register. By removing the register Y_k from X and leaving the remaining registers untouched

$$\mathrm{Tr}_{\mathcal{Y}_k}(\rho) = \rho[\mathsf{Y}_1, \cdots, \mathsf{Y}_{k-1}, \mathsf{Y}_{k+1}, \cdots, \mathsf{Y}_n] \tag{17}$$

The definition above may be generalized in a natural way so that it allows one to specify the states that result from removing an arbitrary collection of subregisters from a given compound register, assuming that this removal results in a valid register.

Definition 8 *$P \in \mathsf{Pos}(\mathcal{X})$, $u \in \mathcal{X} \otimes \mathcal{Y}$ is said to be a purification of P if*

$$\mathrm{Tr}_{\mathcal{Y}}(uu^*) = P \tag{18}$$

Theorem 3 *The following statements are equivalent:*

1. *There exists a purification $u \in \mathcal{X} \otimes \mathcal{Y}$ of P.*
2. *There exists an operator $A \in \mathsf{Lin}(\mathcal{Y}, \mathcal{X})$ such that $P = AA^*$.*

(Hint: $\mathrm{Tr}_{\mathcal{Y}}(\mathrm{vec}(A)\,\mathrm{vec}(A)^* = AA^*)$

Theorem 4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, and let $P \in \mathsf{Pos}(\mathcal{X})$ be a positive semidefinite operator.*

$$\exists u \in \mathcal{X} \otimes \mathcal{Y} \;\; \mathrm{Tr}_{\mathcal{Y}}(uu^*) = P \Longleftrightarrow \dim(\mathcal{Y}) \geq \mathsf{rank}(P) \tag{19}$$

Theorem 5 *Unitary equivalence of purifications: Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $u, v \in \mathcal{X} \otimes \mathcal{Y}$ be vectors, and assume that*

$$\mathrm{Tr}_{\mathcal{Y}}(uu^*) = \mathrm{Tr}_{\mathcal{Y}}(vv^*) \tag{20}$$

There exists a unitary operator $U \in \mathsf{U}(\mathcal{Y})$ such that $v = (\mathbb{1}_{\mathcal{X}} \otimes U)u$.

II. CHANNELS

Definition 9 *A quantum channel is a linear map $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$, satisfying two properties:*

1. *Φ is completely positive.*
2. *Φ is trace preserving.*

The collection of all channels $\mathcal{X} \to \mathcal{Y}$ is denoted $\mathsf{Chan}(\mathcal{X}, \mathcal{Y})$.

Definition 10 *Representations and characterizations of channels*

1. ***The natural representation.***

$$X \mapsto \Phi(X) \tag{21}$$

$$\mathrm{vec}(X) \mapsto \mathrm{vec}(\Phi(X)) \tag{22}$$

Natural representation is not convenient for expressing complete positivity and trace preservation.

2. ***The Choi representation***

One may define a mapping $J : \mathsf{T}(\mathcal{X}, \mathcal{Y}) \to \mathsf{Lin}(\mathcal{Y} \otimes \mathcal{X})$ as

$$J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{X})})(\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*) = \sum_{a,b \in \Sigma} \Phi(E_{a,b}) \otimes E_{a,b} \tag{23}$$

The action of the mapping Φ can be recovered from the operator $J(\Phi)$ by means of the equation

$$\Phi(X) = \mathrm{Tr}_{\mathcal{X}}(J(\Phi)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)) \tag{24}$$

3. ***Kraus representations***

$$\Phi(X) = \sum_{a \in \Sigma} A_a X B_a^* \tag{25}$$

where $\{A_a : a \in \Sigma\}$, $\{B_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$.

Kraus representations are not unique.

4. ***Stinespring representations***

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*) \tag{26}$$

where $A, B \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$.

Stinespring representations are not unique.

Theorem 6 *Relationships among the representations.*
Let $\{A_a : a \in \Sigma\}$, $\{B_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$, $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$, *and* $A, B \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ $A = \sum_{a\in\Sigma} A_a \otimes e_a, B = \sum_{a\in\Sigma} B_a \otimes e_a$

$$\begin{aligned} & K(\Phi) = \sum_{a\in\Sigma} A_a \otimes \overline{B_a} \\ \iff & J(\Phi) = \sum_{a\in\Sigma} \mathrm{vec}(A_a)\, \mathrm{vec}(B_a)^* \\ \iff & \Phi(X) = \sum_{a\in\Sigma} A_a X B_a^* \qquad \forall X \in \mathsf{Lin}(\mathcal{X}) \\ \iff & \Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*) \qquad \forall X \in \mathsf{Lin}(\mathcal{X}) \end{aligned}$$

Theorem 7 *Let* $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$ *be a non-zero map. The following statements are equivalent:*

1. Φ *is completely positive*
2. $\Phi \otimes \mathbb{1}_{\mathsf{Lin}(\mathcal{X})}$ *is positive*
3. $J(\Phi) \in \mathsf{Pos}(\mathcal{Y} \otimes \mathcal{X})$
4. $\exists \{A_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$
$$\Phi(X) = \sum_{a\in\Sigma} A_a X A_a^* \qquad \forall X \in \mathsf{Lin}(\mathcal{X}) \tag{27}$$
5. *Statement 4 holds for an alphabet* Σ *satisfying* $|\Sigma| = \mathsf{rank}(J(\Phi))$.
6. $\exists A \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$
$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXA^*) \tag{28}$$
7. *Statement 6 holds for* $\mathcal{Z}$ *having dimension equal to* $\mathsf{rank}(J(\Phi))$.

Corollary 1 *"Uniqueness" of representation*

1. $\{A_a : a \in \Sigma\}, \{B_b : b \in \Gamma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$
$$\forall X \in \mathsf{Lin}(\mathcal{X}) \ \sum_{a\in\Sigma} A_a X A_a^* = \sum_{b\in\Gamma} B_b X B_b^* \tag{29}$$
$$\implies \forall b \in \Gamma \ (B_b = \sum_{a\in\Sigma} W(b,a) A_a) \wedge (WW^* \in \mathsf{Proj}(\mathbb{C}^\Gamma), W^*W \in \mathsf{Proj}(\mathbb{C}^\Sigma)) \tag{30}$$
2. $\{A_a : a \in \Sigma\}, \{B_b : b \in \Gamma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$ *and* $|\Sigma| \leq |\Gamma|$
$$\forall X \in \mathsf{Lin}(\mathcal{X}) \ \sum_{a\in\Sigma} A_a X A_a^* = \sum_{b\in\Gamma} B_b X B_b^* \tag{31}$$
$$\implies \forall b \in \Gamma \ (B_b = \sum_{a\in\Sigma} W(b,a) A_a) \wedge (W \in \mathsf{U}(\mathbb{C}^\Sigma, \mathbb{C}^\Gamma)) \tag{32}$$

3. $\{A_a : a \in \Sigma\}, \{B_b : b \in \Gamma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$ *and* $\{A_a : a \in \Sigma\}$ *is an orthogonal set.*

$$\forall X \in \mathsf{Lin}(\mathcal{X}) \ \sum_{a\in\Sigma} A_a X A_a^* = \sum_{b\in\Gamma} B_b X B_b^* \implies (\forall b \in \Gamma \ B_b = \sum_{a\in\Sigma} W(b,a) A_a) \wedge (W \in \mathsf{U}(\mathbb{C}^\Sigma, \mathbb{C}^\Gamma)) \tag{33}$$

4. $\{A_a : a \in \Sigma\}, \{B_b : b \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$

$$\forall X \in \mathsf{Lin}(\mathcal{X}) \ \sum_{a\in\Sigma} A_a X A_a^* = \sum_{b\in\Sigma} B_b X B_b^* \implies (\forall b \in \Sigma \ B_b = \sum_{a\in\Sigma} U(b,a) A_a) \wedge (U \in \mathsf{U}(\mathbb{C}^\Sigma)) \tag{34}$$

5. $A, B \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$

$$\forall X \in \mathsf{Lin}(\mathcal{X}) \ \operatorname{Tr}_{\mathcal{Z}}(AXA^*) = \operatorname{Tr}_{\mathcal{Z}}(BXB^*) \implies (B = (\mathbb{1}_{\mathcal{Y}} \otimes U)A) \wedge (U \in \mathsf{U}(\mathcal{Z})) \tag{35}$$

Theorem 8 *Let* $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$ *be a non-zero map. The following statements are equivalent:*

1. Φ *is a Hermitian preserving.*
2. *It holds that* $(\Phi(X))^* = \Phi(X^*)$ *for every* $X \in \mathsf{Lin}(\mathcal{X})$*.*
3. *It holds that* $J(\Phi) \in \mathsf{Herm}(\mathcal{Y} \otimes \mathcal{X})$
4. $\exists \Phi_0, \Phi_1 \in \mathrm{CP}(\mathcal{X}, \mathcal{Y}) \ \Phi = \Phi_0 - \Phi_1$

Theorem 9 *Let* $\Phi \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$ *be a non-zero map. The following statements are equivalent:*

1. Φ *is a trace-preserving.*
2. Φ^* *is a unital map.*
3. $\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$
4. *There exist collections* $\{A_a : a \in \Sigma\}$, $\{B_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$ *of operators such that*

$$\Phi(X) = \sum_{a\in\Sigma} A_a X B_a^* \qquad \sum_{a\in\Sigma} A_a^* B_a = \mathbb{1}_{\mathcal{X}} \tag{36}$$

5. *There exist operators* $A, B \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$*, for some complex Euclidean space* $\mathcal{Z}$*, such that*

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXB^*) \qquad A^*B = \mathbb{1} \tag{37}$$

Corollary 2 *Let* $\Phi \in \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$ *be a map. The following statements are equivalent:*

1. Φ *is a channel.*
2. $J(\Phi) \in \mathsf{Pos}(\mathcal{Y} \otimes \mathcal{X})$ *and* $\operatorname{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$*.*
3. *There exists an alphabet* Σ *and a collection* $\{A_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}, \mathcal{Y})$ *satisfying*

$$\sum_{a\in\Sigma} A_a^* A_a = \mathbb{1}_{\mathcal{X}} \text{ and } \forall X \in \mathsf{Lin}(\mathcal{X}) \ \Phi(X) = \sum_{a\in\Sigma} A_a X A_a^* \tag{38}$$

4. *Statement 3 holds for* $|\Sigma| = \mathsf{rank}(J(\Phi))$
5. *There exists an isometry* $A \in \mathsf{U}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$*, for some choice of a complex Euclidean space* $\mathcal{Z}$*, such that*

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}(AXA^*) \ \forall X \in \mathsf{Lin}(\mathcal{X}) \tag{39}$$

6. *Statement 5 holds under the requirement* $\dim(\mathcal{Z}) = \mathsf{rank}(J(\Phi))$

Theorem 10 *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be complex Euclidean spaces. The set* $\mathsf{Chan}(\mathcal{X}, \mathcal{Y})$ *is compact and convex.*

Examples of channels

1. Unitary channels
2. Replacement channels. Let $A \in \mathsf{Lin}(\mathcal{X})$ and $B \in \mathsf{Lin}(\mathcal{Y})$

$$\Phi(X) = \langle A, X \rangle B \tag{40}$$

$$\begin{aligned} K(\Phi) &= \mathrm{vec}(B)\,\mathrm{vec}(A)^* \\ J(\Phi) &= B \otimes \overline{A} \\ \Phi(X) &= \sum_{(a,b)\in\Sigma\times\Gamma} C_{a,b} X D_{a,b} \quad \Big(A = \sum_{a\in\Sigma} u_a x_a^*, B = \sum_{b\in\Gamma} v_b y_b^*, C_{a,b} = v_b u_a^*, D_{a,b} = y_b x_a^*\Big) \\ \Phi(X) &= \mathrm{Tr}_{\mathcal{Z}}(CXD^*) \quad \Big(C = \sum_{(a,b)\in\Sigma\times\Gamma} C_{a,b} \otimes e_{(a,b)}, D = \sum_{(a,b)\in\Sigma\times\Gamma} D_{a,b} \otimes e_{(a,b)}, \mathcal{Z} = \mathbb{C}^{\Sigma\times\Gamma}\Big) \end{aligned}$$

For the completely depolarizing channel

$$\Omega(X) = \mathrm{Tr}(X) \frac{\mathbb{1}_{\mathcal{X}}}{\dim(\mathcal{X})} \tag{41}$$

$$\begin{aligned} K(\Omega) &= \frac{\mathrm{vec}(\mathbb{1}_{\mathcal{X}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*}{\dim(\mathcal{X})} \\ J(\Omega) &= \frac{\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{X}}}{\dim(\mathcal{X})} \\ \Omega(X) &= \frac{1}{\dim(\mathcal{X})} \sum_{a,b\in\Sigma\times\Sigma} v_b u_a^* X u_a v_b^* \end{aligned}$$

3. Product channels
4. State preparations
5. Trace map
6. Transpose map

$$\mathrm{T}(X) = X^T \tag{42}$$

$$\begin{aligned} K(\mathrm{T})(\mathrm{vec}(X)) &= \mathrm{vec}(X^T) \\ K(\mathrm{T})(u \otimes v) &= \mathrm{vec}(v \otimes u) \\ J(\mathrm{T}) &= \sum_{a,b\in\Sigma} E_{b,a} \otimes E_{a,b} \\ \mathrm{T}(X) &= \sum_{a,b\in\Sigma} E_{a,b} X E_{b,a}^* \end{aligned}$$

7. The completely dephasing channel.

$$\Delta(X) = \sum_{a\in\Sigma} X(a,a) E_{a,a} \tag{43}$$

$$\begin{aligned} K(\Delta)(e_a \otimes e_b) &= \begin{cases} e_a \otimes e_b & a = b \\ 0 & a \neq b \end{cases} \\ J(\Delta) &= \sum_{a\in\Sigma} E_{a,a} \otimes E_{a,a} \end{aligned}$$

Theorem 11 *Let $A \in \mathsf{Lin}(\mathcal{Y}, \mathcal{X})$ be an operator.*

$$\{P \in \mathsf{Pos}(\mathcal{X}) : \mathsf{im}(P) \subset \mathsf{im}(A)\} = \{AQA^* : Q \in \mathsf{Pos}(\mathcal{Y})\} \tag{44}$$

Proof.

$$X = AQA^* \implies \begin{cases} X \in \mathsf{Pos}(\mathcal{X}) \\ \mathsf{im}(X) \subset \mathsf{im}(A) \end{cases} \tag{45}$$

Let $Q = A^+P(A^+)^*$, Then

$$\begin{cases} P \in \mathsf{Pos}(\mathcal{X}) \\ \mathsf{im}(P) \subset \mathsf{im}(A) \end{cases} \implies AQA^* = (AA^+)P(AA^+)^* = \Pi_{\mathsf{im}(A)} P \Pi_{\mathsf{im}(A)} = P \tag{46}$$

Theorem 12 *$\Phi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y})$ and $\Phi(X) = \sum_{a\in\Sigma} A_a X A_a^*$ with $\{A_a : a \in \Sigma\}$ being a linearly independent set. Then Φ is a extreme point of $\mathsf{Chan}(\mathcal{X}, \mathcal{Y})$ iff*

$$\{A_b^* A_a : (a, b) \in \Sigma \times \Sigma\} \subset \mathsf{Lin}(\mathcal{X}) \tag{47}$$

is linearly independent.

Proof. Let $\mathcal{Z} = \mathbb{C}^\Sigma$, define an operator $M \in \mathsf{Lin}(\mathcal{Z}, \mathcal{Y} \otimes \mathcal{X})$ as

$$M = \sum_{a\in\Sigma} \mathrm{vec}(A_a) e_a^* \tag{48}$$

and observe that

$$\begin{cases} MM^* = J(\Phi) \\ \{A_a : a \in \Sigma\} \text{ is a linearly independent set} \end{cases} \implies \mathsf{ker}(M) = \{0\} \tag{49}$$

1. Assume that Φ is not a extreme point. Then

$$\Phi = \lambda\Psi_0 + (1-\lambda)\Psi_1 \qquad \lambda \in (0,1)\ \Psi_0 \neq \Psi_1 \tag{50}$$
$$J(\Phi) = \lambda J(\Psi_0) + (1-\lambda) J(\Psi_1) \tag{51}$$

Since $\lambda J(\Psi_0), (1-\lambda)J(\Psi_1) \in \mathsf{Pos}(\mathcal{Y} \otimes \mathcal{X})$

$$\mathsf{im}(J(\Psi_0)) \subset \mathsf{im}(J(\Phi)) = \mathsf{im}(M) \tag{52}$$
$$\mathsf{im}(J(\Psi_1)) \subset \mathsf{im}(J(\Phi)) = \mathsf{im}(M) \tag{53}$$

Then

$$J(\Psi_0) = MR_0M^* \qquad J(\Psi_1) = MR_1M^*$$

where $R_0, R_1 \in \mathsf{Pos}(\mathcal{Z})$. Let $H = R_0 - R_1$

$$0 = \mathrm{Tr}_{\mathcal{Y}}(J(\Psi_0)) - \mathrm{Tr}_{\mathcal{Y}}(J(\Psi_1)) = \mathrm{Tr}_{\mathcal{Y}}(MHM^*) = \sum_{a,b\in\Sigma} H(a,b)(A_b^*A_a)^T \tag{54}$$

$$H \neq 0 \implies \{A_b^* A_a : (a,b) \in \Sigma \times \Sigma\} \text{ is not an independent set.} \tag{55}$$

2. Assume the set is not linearly independent. Then

$$\sum_{a,b\in\Sigma} Z(a,b) A_b^* A_a = 0 \qquad Z \neq 0 \tag{56}$$

Take the adjoint of both sides, we get

$$\sum_{a,b\in\Sigma} Z^*(a,b)A_b^*A_a = 0 \qquad Z \neq 0 \tag{57}$$

It follows that

$$\sum_{a,b\in\Sigma} H(a,b)A_b^*A_a = 0 \qquad H = \frac{Z+Z^*}{2} \text{ or } \frac{Z-Z^*}{2} \tag{58}$$

Choose the non-zero H. Define $\Psi_0, \Psi_1 \in \mathsf{T}(\mathcal{X}, \mathcal{Y})$

$$J(\Psi_0) = M(\mathbb{1} + \delta H)M^* \qquad J(\Psi_1) = M(\mathbb{1} - \delta H)M^* \tag{59}$$

For small enough δ, $J(\Psi_0)$ and $J(\Psi_1)$ are positive.

$$\mathrm{Tr}_{\mathcal{Y}}(MHM^*) = \sum_{a,b\in\Sigma} H(a,b)(A_b^*A_a)^T = 0 \tag{60}$$

Then

$$\mathrm{Tr}_{\mathcal{Y}}(J(\Psi_0)) = \mathrm{Tr}_{\mathcal{Y}}(J(\Psi_1)) = \mathrm{Tr}_{\mathcal{Y}}(MM^*) = \mathbb{1}_{\mathcal{X}} \tag{61}$$

Thus, Ψ_0 and Ψ_1 are channels.

Finally, we get $\Psi_0 \neq \Psi_1$.

$$\frac{1}{2}J(\Psi_0) + \frac{1}{2}J(\Psi_1) = MM^* = J(\Phi) \implies \Phi = \frac{1}{2}\Psi_0 + \frac{1}{2}\Psi_1 \tag{62}$$

III. MEASUREMENTS

Definition 11 *Measurements defined by measurement operators.*

$$\begin{cases} \mu : \Sigma \to \mathsf{Pos}(\mathcal{X}) \\ \sum_{a\in\Sigma} \mu(a) = \mathbb{1}_{\mathcal{X}} \end{cases} \tag{63}$$

Born's rule:

$$p(a) = \langle \mu(a), \rho \rangle \tag{64}$$

Definition 12 $\Phi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y})$ *is a quantum-to-classical channel if*

$$\Phi = \Delta\Phi \tag{65}$$

for Δ denoting the completely dephasing channel.

Theorem 13 *Measurements as channels. Let $\mathcal{Y} = \mathbb{C}^\Sigma$*

$$\Phi \textit{ is quantum-to-classical} \implies \exists \textit{ unique } \mu \;\; \Phi(X) = \sum_{a\in\Sigma} \langle \mu(a), X \rangle E_{a,a} \tag{66}$$

$$\forall \mu \;\; \Phi(X) = \sum_{a\in\Sigma} \langle \mu(a), X \rangle E_{a,a} \textit{ is quantum-to-classical} \tag{67}$$

Theorem 14 *The set of quantum-to-classical channels $\{\Delta\Psi : \Psi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y})\}$ is compact and convex.*

Proof. $\mathsf{Chan}(\mathcal{X}, \mathcal{Y})$ is compact and convex. The mapping $\Psi \to \Delta\Psi$ is continuous.

Definition 13 *1. Product measurements. Suppose $X = (Y_1, \cdots, Y_n)$ is a compound register.*

$$\mu : \Sigma_1 \times \cdots \times \Sigma_n \to \mathsf{Pos}(\mathcal{X}) \tag{68}$$
$$\mu(a_1, \cdots, a_n) = \mu_1(a_1) \otimes \cdots \otimes \mu_n(a_n) \tag{69}$$

It may be verified that when a product measurement is performed on a product state, the measurement outcomes resulting from the individual measurements are independently distributed.

2. Partial measurements. Suppose $X = (Y_1, \cdots, Y_n)$ is a compound register.

$$\mu : \Sigma \to \mathsf{Pos}(\mathcal{Y}_k) \tag{70}$$

Consider the quantum-to-classical channel that corresponds to the measurement μ.

$$\Phi(Y) = \sum_{a\in\Sigma} \langle \mu(a), Y \rangle E_{a,a} \tag{71}$$

Applying the channel Φ to Y_k, followed by the application of a channel that performs the permutation of registers.

$$\sum_{a\in\Sigma} E_{a,a} \otimes \mathrm{Tr}_{\mathcal{Y}_k}[(\mathbb{1}_{\mathcal{Y}_1\otimes\cdots\otimes\mathcal{Y}_{k-1}} \otimes \mu(a) \otimes \mathbb{1}_{\mathcal{Y}_{k+1}\otimes\cdots\otimes\mathcal{Y}_n})\rho] \tag{72}$$

The state is a classical-quantum state, and is naturally associated with the ensemble

$$\eta : \Sigma \to \mathsf{Pos}(\mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_{k-1} \otimes \mathcal{Y}_{k+1} \otimes \cdots \otimes \mathcal{Y}_n) \tag{73}$$
$$\eta(a) = \mathrm{Tr}_{\mathcal{Y}_k}[(\mathbb{1}_{\mathcal{Y}_1\otimes\cdots\otimes\mathcal{Y}_{k-1}} \otimes \mu(a) \otimes \mathbb{1}_{\mathcal{Y}_{k+1}\otimes\cdots\otimes\mathcal{Y}_n})\rho] \tag{74}$$

Definition 14 *Let $\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ be a projective measurement. The set $\{\mu(a) : a \in \Sigma\}$ is an orthogonal set.*

Theorem 15 *Let $\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ be a measurement, and let $\mathcal{Y} = \mathbb{C}^\Sigma$. There exists a isometry $A \in \mathsf{U}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ such that*

$$\mu(a) = A^*(\mathbb{1}_\mathcal{X} \otimes E_{a,a})A \tag{75}$$

Proof. Define

$$A = \sum_{a\in\Sigma} \sqrt{\mu(a)} \otimes e_a \tag{76}$$

Then $\mu(a) = A^*(\mathbb{1}_\mathcal{X} \otimes E_{a,a})A$ and $A^*A = \sum_{a\in\Sigma} \mu(a) = \mathbb{1}_\mathcal{X}$.

Corollary 3 *Let $\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ be a measurement, $\mathcal{Y} = \mathbb{C}^\Sigma$ and $u \in \mathcal{Y}$ be a unit vector. There exists a projective measurement $\nu : \Sigma \to \mathsf{Pos}(\mathcal{X} \otimes \mathcal{Y})$ such that*

$$\langle \nu(a), X \otimes uu^* \rangle = \langle \mu(a), X \rangle \tag{77}$$

Proof. Choose $U \in \mathsf{U}(\mathcal{X} \otimes \mathcal{Y})$ such that

$$U(\mathbb{1}_\mathcal{X} \otimes u) = A \tag{78}$$

Define

$$\nu(a) = U^*(\mathbb{1}_\mathcal{X} \otimes E_{a,a})U \tag{79}$$

Theorem 16 *Let $\{A_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X})$ be a collection of operators for which*

$$\mathsf{Span}\{A_a : a \in \Sigma\} = \mathsf{Lin}(\mathcal{X}) \tag{80}$$

The mapping $\phi : \mathsf{Lin}(\mathcal{X}) \to \mathbb{C}^\Sigma$ defined by

$$(\phi(X))(a) = \langle A_a, X \rangle \tag{81}$$

is an injective mapping.

Proof.

$$\begin{aligned} &\phi(X) = \phi(Y) \\ \implies &\langle A_a, X - Y\rangle = 0 \qquad \forall a \in \Sigma \\ \implies &\langle Z, X - Y\rangle = 0 \\ \implies &X - Y = 0 \end{aligned}$$

One way to construct an information complete measurement. Let $\{\rho_{a,b} : a \times b \in \Sigma \times \Sigma\}$ be a collection of density operators that spans all of $\mathsf{Lin}(\mathcal{X})$. Define

$$Q = \sum_{(a,b)\in\Sigma\times\Sigma} \rho_{a,b} \tag{82}$$

Then

$$\mu(a, b) = Q^{-\frac{1}{2}} \rho_{a,b} Q^{-\frac{1}{2}} \tag{83}$$

is an information-complete measurement.

Definition 15 *Nondestructive measurement*

$$\{M_a : a \in \Sigma\} \subset \mathsf{Lin}(\mathcal{X}) \qquad \sum_{a\in\Sigma} M_a^* M_a = \mathbb{1}_{\mathcal{X}} \tag{84}$$

When measurement is applied, two things happens

1. *An element of Σ is selected at random, with each outcome $a \in \Sigma$ being obtained with probability $\langle M_a^* M_a, \rho\rangle$.*
2. *Conditioned on the measurement outcome $a \in \Sigma$ having been obtained, the state of the register X becomes*

$$\frac{M_a \rho M_a^*}{\langle M_a^* M_a, \rho\rangle} \tag{85}$$

Definition 16 *Instruments.*

$$\{\Phi_a : a \in \Sigma\} \subset \mathrm{CP}(\mathcal{X}, \mathcal{Y}) \qquad \sum_{a\in\Sigma} \Phi_a \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y}) \tag{86}$$

When measurement is applied, two things happens

1. *An element of Σ is selected at random, with each outcome $a \in \Sigma$ being obtained with probability $\mathrm{Tr}(\Phi_a(\rho))$.*
2. *Conditioned on the measurement outcome $a \in \Sigma$ having been obtained, the state of the register X becomes*

$$\frac{\Phi_a(\rho)}{\mathrm{Tr}(\Phi_a(\rho))} \tag{87}$$

It is easy to see that indirect measurement is a special case of instrument.

Processes that are expressible as instruments, including nondestructive measurements, can alternatively be described as compositions of channels and (ordinary) measurements. Introduce a (classical) register Z having classical state set Σ, and define a channel $\Phi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Z} \otimes \mathcal{Y})$ as

$$\Phi(X) = \sum_{a\in\Sigma} E_{a,a} \otimes \Phi_a(X) \tag{88}$$

Definition 17 *Convex combinations of measurements*

$$\mu_b : \Sigma \to \mathsf{Pos}(\mathcal{X}) \qquad b \in \Gamma \tag{89}$$

$$\mu = \sum_{b\in\Gamma} p(b)\mu_b \qquad p \in \mathcal{P}(\Gamma) \tag{90}$$

Consider the vector space consists of the form of functions

$$\theta : \Sigma \to \mathsf{Herm}(\mathcal{X}) \tag{91}$$

Definition 18 *$\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ is an extremal measurement if*

$$\mu = \lambda\mu_0 + (1-\lambda)\mu_1 \;\; \lambda \in (0,1) \implies \mu_0 = \mu_1 \tag{92}$$

Theorem 17 *$\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ is an extremal measurement iff*

$$\forall \theta : \Sigma \to \mathsf{Herm}(\mathcal{X}) \quad \left(\begin{cases} \sum_{a\in\Sigma} \theta(a) = 0 \\ \forall a \in \Sigma \;\, \mathsf{im}(\theta(a)) \subset \mathsf{im}(\mu(a)) \end{cases} \implies \forall a \in \Sigma \;\, \theta(a) = 0 \right) \tag{93}$$

Proof.

1. Assume μ is not extremal. Then

$$\mu = \lambda\mu_0 + (1-\lambda)\mu_1 \;\; \mu_0 \neq \mu_1 \tag{94}$$

One may construct ν_0, ν_1 such that

$$\mu = \frac{\nu_0 + \nu_1}{2} \tag{95}$$

by

$$\begin{cases} \nu_0 = 2\lambda\mu_0 + (1-2\lambda)\mu_1, \;\; \nu_1 = \mu_1 & \lambda \leq \frac{1}{2} \\ \nu_0 = \mu_0, \;\; \nu_1 = (2\lambda-1)\mu_0 + (2-2\lambda)\mu_1 & \lambda \geq \frac{1}{2} \end{cases} \tag{96}$$

So we get

$$\theta(a) = \nu_0(a) - \nu_1(a) \qquad \forall a \in \Sigma \tag{97}$$

It holds that $\sum_{a\in\Sigma} \theta(a) = 0$ and $\mathsf{im}(\theta(a)) \subset \mathsf{im}(\nu_0(a)) + \mathsf{im}(\nu_1(a)) = \mathsf{im}(\mu(a))$. However, $\theta \neq 0$.

2. Assume

$$\begin{cases} \theta \neq 0 \\ \sum_{a\in\Sigma} \theta(a) = 0 \\ \forall a \in \Sigma \;\, \mathsf{im}(\theta(a)) \subset \mathsf{im}(\mu(a)) \end{cases} \tag{98}$$

Define

$$\mu_0 = \mu - \delta\theta \qquad \mu_1 = \mu + \delta\theta \tag{99}$$

By virtue of the fact that $\mu(a)$ is positive semidefinite and $\theta(a)$ is a Hermitian operator with $\mathsf{im}(\theta(a)) \subset \mathsf{im}(\mu(a))$, for small enough δ, $\mu_0, \mu_1 \in \mathsf{Pos}(\mathcal{X})$.
Thus $\mu = \frac{\mu_0+\mu_1}{2}$ and $\mu_0 \neq \mu_1$. μ is not extremal.

Corollary 4 *1. If $\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ is an extremal measurement, then*

$$|\{a \in \Sigma : \mu(a) \neq 0\}| \leq \dim(\mathcal{X})^2 \tag{100}$$

Proof. *Consider the measurement $\mu : \Gamma \to \mathsf{Pos}(\mathcal{X})$ such that $|\Gamma| > \dim(\mathcal{X})^2$. The measurement vectors are in the vector space $\mathsf{Herm}(\mathcal{X})$ so they are linearly dependent. We have*

$$\sum_{\alpha\in\Gamma} \alpha_a \mu(a) = 0 \tag{101}$$

Define

$$\theta(a) = \begin{cases} \alpha_a \mu(a) & a \in \Gamma \\ 0 & a \notin \Gamma \end{cases} \tag{102}$$

It holds that $\sum_{a\in\Sigma} \theta(a) = 0$ and $\mathsf{im}(\theta(a)) \subset \mathsf{im}(\mu(a))$ $\forall a \in \Sigma$. However, $\theta \neq 0$.

2. *Let* $\mu : \Sigma \to \mathsf{Pos}(\mathcal{X})$ *be a measurement. There exists a collection of measurements* $\{\mu_b : b \in \Gamma, |\{a \in \Sigma : \mu_b(a) \neq 0\}| \leq \dim(\mathcal{X})^2\}$ *such that*

$$\mu = \sum_{b\in\Gamma} p(b)\mu_b \tag{103}$$

(Hint: extremal points convex span)

3. *Let* $\{x_a : a \in \Sigma\} \subset \mathcal{X}$ *be nonzero vectors satisfying*

$$\sum_{a\in\Sigma} x_a x_a^* = \mathbb{1}_{\mathcal{X}} \tag{104}$$

Then $\mu(a) = x_a x_a^*$ *is extremal iff* $\{x_a x_a^* : a \in \Sigma\}$ *is a linearly independent set.*

(Hint : $\begin{cases} H \in \mathsf{Herm}(\mathcal{X}) \\ \mathsf{im}(H) \subset \mathsf{im}(uu^*) \end{cases} \Leftrightarrow H = \alpha uu^*$*)*

4. *Projective measurements are extremal.*

Theorem 18 *The convex combination of ensembles is also an ensemble.*

$$\eta_b : \Sigma \to \mathsf{Pos}(\mathcal{X}) \qquad \mathrm{Tr}\left(\sum_{a\in\Sigma} \eta_b(a)\right) = 1 \tag{105}$$

$$\rho_b = \sum_{a\in\Sigma} \eta_b(a) \tag{106}$$

$$\eta = \sum_{b\in\Gamma} p(b)\eta_b \tag{107}$$

$$\rho = \sum_{a\in\Sigma} \eta(a) = \sum_{b\in\Gamma} p(b)\rho_b \tag{108}$$

The extremal points of ensembles

$$\eta(a) = \begin{cases} uu^* & a = b \\ 0 & a \neq b \end{cases} \tag{109}$$

for some choice of a unit vector $u \in \mathcal{X}$ *and a symbol* $b \in \Sigma$.

Theorem 19 *Let* $\rho = \sum_{a\in\Sigma} \eta(a)$. *There exists a collection of ensembles* $\{\eta_b : b \in \Gamma\}$ *such that*

1. $\forall b \in \Gamma \;\; \sum_{a\in\Sigma} \eta_b(a) = \rho$
2. $\forall b \in \Gamma \;\; |\{a \in \Sigma : \eta_b(a) \neq 0\}| \leq \mathsf{rank}(\rho)^2$
3. $\exists p \in \mathcal{P}(\Gamma) \;\; \eta = \sum_{b\in\Gamma} p(b)\eta_b$

IV. QUANTUM SUPERMAP

Definition 19 *Deterministic quantum supermaps are higher-order maps whose input and output are both quantum channels.*

$$\begin{gathered} S : \mathsf{Chan}(\mathcal{X}_1, \mathcal{Y}_1) \to \mathsf{Chan}(\mathcal{X}_2, \mathcal{Y}_2) \\ \Phi_1 \mapsto \Phi_2 \end{gathered}$$

Using Choi isomophism, to each deterministic supermap $S : \mathsf{Chan}(\mathcal{X}_1, \mathcal{Y}_1) \to \mathsf{Chan}(\mathcal{X}_2, \mathcal{Y}_2)$ *corresponds a map* $\boldsymbol{S} \in \mathrm{CP}(\mathcal{Y}_1 \otimes \mathcal{X}_1, \mathcal{Y}_2 \otimes \mathcal{X}_2)$.

$$\begin{gathered} \boldsymbol{S} : \mathsf{Pos}(\mathcal{Y}_1 \otimes \mathcal{X}_1) \to \mathsf{Pos}(\mathcal{Y}_2 \otimes \mathcal{X}_2) \\ J(\Phi_1) \mapsto J(\Phi_2) \end{gathered}$$

Lemma 1 $C \in \mathsf{Lin}(\mathcal{Y} \otimes \mathcal{X})$

$$\forall \Phi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y}) \;\; \mathrm{Tr}(CJ(\Phi)) = 1 \iff \exists \rho \in \mathsf{Lin}(\mathcal{X}) \;\; \begin{cases} C = \mathbb{1}_{\mathcal{Y}} \otimes \rho \\ \mathrm{Tr}(\rho) = 1 \end{cases} \tag{110}$$

Proof.

1. Let $\Psi \in \mathrm{CP}(\mathcal{X}, \mathcal{Y})$ such that

$$P = \mathrm{Tr}_{\mathcal{Y}}(J(\Psi)) \leq \mathbb{1}_{\mathcal{X}} \tag{111}$$

   Let $\sigma \in \mathsf{D}(\mathcal{Y})$, then $J(\Psi) + \sigma \otimes (\mathbb{1}_{\mathcal{X}} - P)$ is the Choi operator for some channel. Notice that $\sigma \otimes \mathbb{1}_{\mathcal{X}}$ is also the Choi operator for some channel. Thus

$$\mathrm{Tr}(CJ(\Psi)) = \mathrm{Tr}(C(\sigma \otimes P)) = \mathrm{Tr}((\mathbb{1}_{\mathcal{Y}} \otimes \rho)J(\Psi)) \tag{112}$$

   where $\rho = \mathrm{Tr}_{\mathcal{Y}}(C(\sigma \otimes \mathbb{1}_{\mathcal{X}}))$

   Since ρ is independent of $J(\Psi)$

$$\forall A \in \mathsf{Pos}(\mathcal{Y} \otimes \mathcal{X}) \;\; \mathrm{Tr}(CA) = \mathrm{Tr}((\mathbb{1}_{\mathcal{Y}} \otimes \rho)A) \tag{113}$$

   Thus

$$C = \mathbb{1}_{\mathcal{Y}} \otimes \rho \tag{114}$$

2. Suppose $C = \mathbb{1}_{\mathcal{Y}} \otimes \rho$ and $\mathrm{Tr}(\rho) = 1$, then

$$\forall \Phi \in \mathsf{Chan}(\mathcal{X}, \mathcal{Y}) \;\; \mathrm{Tr}(CJ(\Phi)) = \mathrm{Tr}((\mathbb{1}_{\mathcal{Y}} \otimes \rho)J(\Phi)) = \mathrm{Tr}(\rho \, \mathrm{Tr}_{\mathcal{Y}}(J(\Phi))) = \mathrm{Tr}(\rho) = 1 \tag{115}$$

■

Lemma 2 $\boldsymbol{S} \in \mathrm{CP}(\mathcal{Y}_1 \otimes \mathcal{X}_1, \mathcal{Y}_2 \otimes \mathcal{X}_2)$ *is a deterministic supermap iff*

$$\exists \Psi \in \mathsf{Chan}(\mathcal{X}_2, \mathcal{X}_1) \;\; \forall \rho \in \mathsf{D}(\mathcal{X}_2) \;\; \boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho) = \mathbb{1}_{\mathcal{Y}_1} \otimes \Psi(\rho) \tag{116}$$

Proof.

1. Let $\boldsymbol{S}$ be a deterministic supermap and $\rho \in \mathsf{D}(\mathcal{X}_2)$. Then

$$\forall \Phi \in \mathsf{Chan}(\mathcal{X}_1, \mathcal{Y}_1) \;\; \langle \boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho), J(\Phi) \rangle = \langle \mathbb{1}_{\mathcal{Y}_2} \otimes \rho, \boldsymbol{S}(J(\Phi)) \rangle = 1 \tag{117}$$

   According to Lemma 1

$$\boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho) = \mathbb{1}_{\mathcal{Y}_1} \otimes \sigma \tag{118}$$

   where $\sigma \in \mathsf{D}(\mathcal{X}_1)$.

   Since the maps $\rho \mapsto \mathbb{1}_{\mathcal{Y}_2} \otimes \rho$, $\boldsymbol{S}^*$ and $\mathbb{1}_{\mathcal{Y}_1} \otimes \sigma \mapsto \sigma$ are all CP, we have $\sigma = \Psi(\rho)$, $\Psi \in \mathsf{Chan}(\mathcal{X}_2, \mathcal{X}_1)$.

2. Suppose

$$\exists \Psi \in \mathsf{Chan}(\mathcal{X}_2, \mathcal{X}_1) \;\; \forall \rho \in \mathsf{D}(\mathcal{X}_2) \;\; \boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho) = \mathbb{1}_{\mathcal{Y}_1} \otimes \Psi(\rho) \tag{119}$$

   Let $\Phi \in \mathsf{Chan}(\mathcal{Y}_1 \otimes \mathcal{X}_1)$, then

$$\langle \mathbb{1}_{\mathcal{Y}_2} \otimes \rho, \boldsymbol{S}(J(\Phi)) \rangle = \langle \boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho), J(\Phi) \rangle = \langle \mathbb{1}_{\mathcal{Y}_1} \otimes \Psi(\rho), J(\Phi) \rangle = 1 \tag{120}$$

   This means

$$\begin{aligned} & \forall \rho \in \mathsf{D}(\mathcal{X}_2) \;\;\; \mathrm{Tr}(\rho \, \mathrm{Tr}_{\mathcal{Y}_2} \boldsymbol{S}(J(\Phi))) = 1 && (121) \\ \implies & \forall u \in \mathcal{X}_2 \;\; u^* \, \mathrm{Tr}_{\mathcal{Y}_2} \boldsymbol{S}(J(\Phi)) u = 1 && (122) \\ \implies & \mathrm{Tr}_{\mathcal{Y}_2} \boldsymbol{S}(J(\Phi)) = \mathbb{1}_{\mathcal{X}_2} && (123) \end{aligned}$$

   Thus $\boldsymbol{S}$ maps Choi operators on $\mathsf{Lin}(\mathcal{Y}_1 \otimes \mathcal{X}_1)$ to Choi operators on $\mathsf{Lin}(\mathcal{Y}_2 \otimes \mathcal{X}_2)$.

■

Lemma 3 *$\boldsymbol{S} \in \mathrm{CP}(\mathcal{Y}_1 \otimes \mathcal{X}_1, \mathcal{Y}_2 \otimes \mathcal{X}_2)$ is a deterministic supermap iff there exists a unital, completely positive map $\Psi^* \in \mathrm{CP}(\mathcal{X}_1, \mathcal{X}_2)$ such that*

$$\forall A \in \mathsf{Lin}(\mathcal{Y}_1 \otimes \mathcal{X}_1) \quad \mathrm{Tr}_{\mathcal{Y}_2}(\boldsymbol{S}(A)) = \Psi^*(\mathrm{Tr}_{\mathcal{Y}_1}(A)) \tag{124}$$

Proof. Let $\rho \in \mathsf{D}(\mathcal{X}_2), A \in \mathsf{Lin}(\mathcal{Y}_1 \otimes \mathcal{X}_1)$

$$\begin{aligned}
\mathrm{Tr}[\rho \,\mathrm{Tr}_{\mathcal{Y}_2} \boldsymbol{S}(A)] &= \mathrm{Tr}[(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho)\boldsymbol{S}(A)] \\
&= \mathrm{Tr}[\boldsymbol{S}^*(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho)A] \\
&= \mathrm{Tr}[(\mathbb{1}_{\mathcal{Y}_1} \otimes \Psi(\rho))A] \\
&= \mathrm{Tr}[(\mathbb{1}_{\mathcal{Y}_2} \otimes \rho)(\mathbb{1}_{\mathsf{Lin}(\mathcal{Y}_1)} \otimes \Psi^*)A] \\
&= \mathrm{Tr}[\rho\Psi^*(\mathrm{Tr}_{\mathcal{Y}_1}(A))]
\end{aligned}$$

■

Theorem 20 *Every deterministic supermap can be realized by a four-port quantum circuit where the input operation Φ_1 is inserted between two isometries V and W and a final ancilla is discarded. The output operation $\Phi_2 = S(\Phi_1)$ is given by*

$$S(\Phi_1)(K) = \mathrm{Tr}_{\mathcal{A}}(W(\Phi_1 \otimes \mathbb{1}_{\mathcal{B}})(VKV^*)W^*) \qquad K \in \mathsf{Lin}(\mathcal{X}_2) \tag{125}$$

Proof. Consider the following Kraus representations

$$\Psi(K) = \sum_{b\in\Gamma} B_b K B_b^* \qquad K \in \mathsf{Lin}(\mathcal{X}_2) \tag{126}$$

$$\boldsymbol{S}(A) = \sum_{a\in\Sigma} S_a A S_a^* \qquad A \in \mathsf{Lin}(\mathcal{Y}_1 \otimes \mathcal{X}_1) \tag{127}$$

Let $\{u_c : c \in \Lambda\}$ be an ONB of $\mathcal{Y}_1 = \mathbb{C}^\Lambda$ and $\{v_d : d \in M\}$ be an ONB of $\mathcal{Y}_2 = \mathbb{C}^M$

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{Y}_2}(\boldsymbol{S}(A)) &= \sum_{a\in\Sigma} \mathrm{Tr}_{\mathcal{Y}_2}(S_a A S_a^*) = \sum_{a\in\Sigma}\sum_{d\in M} (v_d^* \otimes \mathbb{1}_{\mathcal{X}_2}) S_a A S_a^* (v_d \otimes \mathbb{1}_{\mathcal{X}_2}) \\
\Psi^*(\mathrm{Tr}_{\mathcal{Y}_1}(A)) &= \sum_{b\in\Gamma} B_b^* \,\mathrm{Tr}_{\mathcal{Y}_1}(A) B_b = \sum_{b\in\Gamma}\sum_{c\in\Lambda} (u_c^* \otimes B_b^*) A (u_c \otimes B_b)
\end{aligned}$$

Thus $\{(v_d^* \otimes \mathbb{1}_{\mathcal{X}_2})S_a : a \in \Sigma, d \in M\}$ and $\{u_c^* \otimes B_b^* : b \in \Gamma, c \in \Lambda\}$ are Kraus representations of the same CP. And the second one is canonical. So there exists an isometry $\widetilde{W} : (\Sigma \times M) \times (\Gamma \times \Lambda) \to \mathbb{C}$ connecting them

$$(v_d^* \otimes \mathbb{1}_{\mathcal{X}_2}) S_a = \sum_{(b,c)\in\Gamma\times\Lambda} \widetilde{W}((a,d),(b,c)) u_c^* \otimes B_b^* \tag{128}$$

Let $\{x_a : a \in \Sigma\}$ be an ONB of $\mathcal{A} = \mathbb{C}^\Sigma$ and $\{y_b : b \in \Gamma\}$ be an ONB of $\mathcal{B} = \mathbb{C}^\Gamma$. Define the operator $W : \mathsf{Lin}(\mathcal{Y}_1 \otimes \mathcal{B}, \mathcal{Y}_2 \otimes \mathcal{A})$

$$\widetilde{W}((a,d),(b,c)) = (v_d^* \otimes x_a^*) W (u_c \otimes y_b) \tag{129}$$

Then

$$\begin{aligned}
S_a &= \sum_{d\in M,(b,c)\in\Gamma\times\Lambda} \widetilde{W}((a,d),(b,c))(v_d \otimes \mathbb{1}_{\mathcal{X}_2})(u_c^* \otimes B_b^*) \\
&= \sum_{d\in M, b\in\Gamma, c\in\Lambda} (v_d^* \otimes x_a^*) W (u_c \otimes y_b)(v_d \otimes \mathbb{1}_{\mathcal{X}_2})(u_c^* \otimes B_b^*) \\
&= \left(\sum_{d\in M} (v_d \otimes \mathbb{1}_{\mathcal{X}_2})(v_d^* \otimes x_a^* \otimes \mathbb{1}_{\mathcal{X}_2})\right)(W \otimes \mathbb{1}_{\mathcal{X}_2})\left(\sum_{(b,c)\in\Gamma\times\Lambda} (u_c \otimes y_b \otimes \mathbb{1}_{\mathcal{X}_2})(u_c^* \otimes B_b^*)\right) \\
&= (\mathbb{1}_{\mathcal{Y}_2} \otimes x_a^* \otimes \mathbb{1}_{\mathcal{X}_2})(W \otimes \mathbb{1}_{\mathcal{X}_2})\left(\sum_{b\in\Lambda} \mathbb{1}_{\mathcal{Y}_1} \otimes y_b \otimes B_b^*\right) \\
&= (\mathbb{1}_{\mathcal{Y}_2} \otimes x_a^* \otimes \mathbb{1}_{\mathcal{X}_2})(W \otimes \mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_1} \otimes Z)
\end{aligned}$$

where $Z = \sum_{b\in\Lambda} y_b \otimes B_b^* \in \mathsf{Lin}(\mathcal{X}_1, \mathcal{B}\otimes\mathcal{X}_2)$.

So we come to

$$\begin{aligned}\boldsymbol{S}(A) &= \sum_{a\in\Sigma} S_a A S_a^* \\ &= \sum_{a\in\Sigma} (\mathbb{1}_{\mathcal{Y}_2}\otimes x_a^*\otimes\mathbb{1}_{\mathcal{X}_2})(W\otimes\mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_1}\otimes Z)A(\mathbb{1}_{\mathcal{Y}_1}\otimes Z^*)(W^*\otimes\mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_2}\otimes x_a\otimes\mathbb{1}_{\mathcal{X}_2}) \\ &= \mathrm{Tr}_{\mathcal{A}}((W\otimes\mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_1}\otimes Z)A(\mathbb{1}_{\mathcal{Y}_1}\otimes Z^*)(W^*\otimes\mathbb{1}_{\mathcal{X}_2}))\end{aligned}$$

Then for the original supermap,

$$\begin{aligned}S(\Phi_1)(K) &= \mathrm{Tr}_{\mathcal{X}_2}((\mathbb{1}_{\mathcal{Y}_2}\otimes K^T)\boldsymbol{S}(J(\Phi_1))) \\ &= \mathrm{Tr}_{\mathcal{X}_2}(\mathbb{1}_{\mathcal{Y}_2}\otimes K^T)\,\mathrm{Tr}_{\mathcal{A}}((W\otimes\mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_1}\otimes Z)J(\Phi_1)(\mathbb{1}_{\mathcal{Y}_1}\otimes Z^*)(W^*\otimes\mathbb{1}_{\mathcal{X}_2})) \\ &= \mathrm{Tr}_{\mathcal{A}}\,\mathrm{Tr}_{\mathcal{X}_2}(W\otimes\mathbb{1}_{\mathcal{X}_2})(\mathbb{1}_{\mathcal{Y}_1}\otimes\mathbb{1}_{\mathcal{B}}\otimes K^T)(\mathbb{1}_{\mathcal{Y}_1}\otimes Z)J(\Phi_1)(\mathbb{1}_{\mathcal{Y}_1}\otimes Z^*)(W^*\otimes\mathbb{1}_{\mathcal{X}_2}) \\ &= \mathrm{Tr}_{\mathcal{A}}[W[\mathrm{Tr}_{\mathcal{X}_2}(\mathbb{1}_{\mathcal{Y}_1}\otimes\mathbb{1}_{\mathcal{B}}\otimes K^T)(\mathbb{1}_{\mathcal{Y}_1}\otimes Z)J(\Phi_1)(\mathbb{1}_{\mathcal{Y}_1}\otimes Z^*)]W^*] \\ &= \mathrm{Tr}_{\mathcal{A}}(W(\Phi_1\otimes\mathbb{1}_{\mathcal{B}})(VKV^*)W^*)\end{aligned}$$

where $V = \sum_{b\in\Lambda} \overline{B_b}\otimes y_b \in \mathsf{Lin}(\mathcal{X}_2, \mathcal{X}_1\otimes\mathcal{B})$ is an isometry. ■

Remark 1 *In particular, a channel can be viewed as a supermap, mapping the state preparation channel to another state preparation channel.*

$$\rho_1 = \Phi_1(1) \mapsto \rho_2 = \Phi_2(1) \tag{130}$$

$$\Phi_2(1) = S(\Phi_1)(1) = \mathrm{Tr}_{\mathcal{A}}(W\Phi_1(1)W^*) \tag{131}$$