The strong hull property for affine irreducible Coxeter groups of rank 3

Proof of the strong hull conjecture for an interesting class

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May 20, 2025

1 Background and preliminaries

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Basic notation

- G: a connected undirected graph
- $V(G)$: vertices of G
- $d : V(G) \times V(G) \to \mathbb{Z}_+$: distance function forms *hull metric*
- $d(x, y)$: the shortest path between vertices x and y in $V(G)$
- $C \subset V(G)$ is *convex*: $\forall u, v \in V(G)$

$$d(u, w) + d(w, v) = d(u, v) \Rightarrow w \in C$$

- $\mathrm{Conv}(X)$: intersection of all convex sets containing X.

Basic notation

- *Coxeter group*: together with a generating set $S = \{s_1, \cdots, s_r\}$ subject to the relations $s_i^2 = \mathbf{1}$ for $i = 1, \cdots, r$ and $(s_i s_j)^{m_{ij}} = \mathbf{1}$ for $i \neq j \in \{1, \cdots, r\}$ for some numbers $m_{ij} \in \{2, 3, \cdots, \infty\}$
- *Rank*: cardinality of the generating set S of a Coxeter group
- *Coxeter graph*:
  - vertices are the elements of S;
  - attach s_i and s_j to form an edge if $m_{ij} \geq 3$;
  - label the edges with m_{ij} where $m_{ij} \geq 4$
- *Irreducible* Coxeter group: its Coxeter graph is connected

Basic notation

- Cayley graph $\mathrm{Cay}(\Gamma, S)$:
  - vertices are the elements of a group Γ;
  - at each vertex $g \in \Gamma$, insert an edge joining g to gs_α for each of the chosen generators $s_\alpha \in S$.

Example (Cayley graph of infinite dihedral group)

$I_2(\infty)$ has the presentation $\langle s_0, s_1 \mid s_0^2 = s_1^2 = \mathbf{1} \rangle \Rightarrow$ Coxeter group

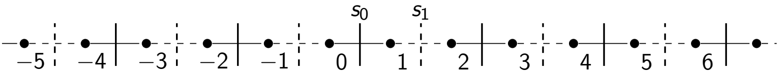

Figure 1: The Cayley graph of $I_2(\infty)$

Strong hull conjecture

A graph G has the *strong hull property* if for every u, v, $w \in V(G)$,

$$|\mathrm{Conv}(u, v)| \cdot |\mathrm{Conv}(v, w)| \geq |\mathrm{Conv}(u, v, w)|. \tag{1}$$

Conjecture (Strong hull conjecture [GG22])

Let W be any Coxeter group. Then $\mathrm{Cay}(W)$ has the strong hull property.

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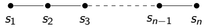
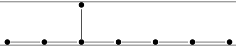
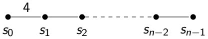
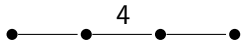
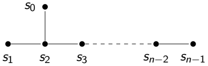
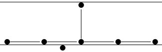
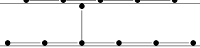
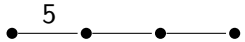
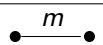
Finite irreducible Coxeter groups

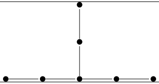
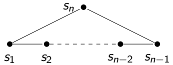
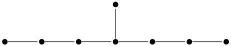
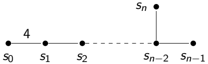
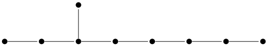
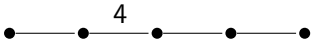
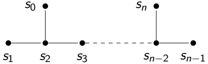
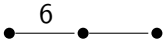
Coxeter groups were classified in 1935 for the finite case [Cox35].

Type	Graph	Type	Graph
$A_n,\ n \geq 1$		E_8	
$B_n,\ n \geq 2$		F_4	
$D_n\ n \geq 4$		G_2	
E_6		H_3	
E_7		H_4	
		$I_2(m),\ m \geq 3$	

Progress

1. In [GG22], Gaetz-Gao proved that the strong hull conjecture holds when
   - W is a symmetric group (type A);
   - W is a hyperoctahedral group (type B);
   - W is any right-angled Coxeter group;
   - W is types D_4, F_4, G_2, or H_3 (verified by computer).
2. It can be easily verified that the Cayley graph of $I_2(\infty)$ has the strong hull property.
3. This thesis investigates the affine types $\widetilde{A}_2$, $\widetilde{C}_2$, and $\widetilde{G}_2$. By employing key concepts from building theory, we develop novel techniques:
   - Reducing and classifying the convex hull;
   - Combinatorial computations.

Affine irreducible Coxeter groups

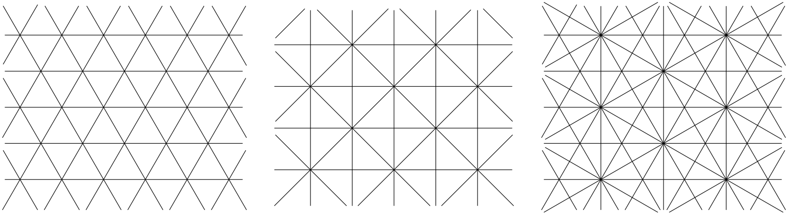
Type	Graph	Type	Graph
$\widetilde{A}_1 = I_2(\infty)$		$\widetilde{E}_6$	
$\widetilde{A}_{n-1},\ n \geq 3$		$\widetilde{E}_7$	
$\widetilde{B}_n,\ n \geq 3$		$\widetilde{E}_8$	
$\widetilde{C}_n,\ n \geq 2$		$\widetilde{F}_4$	
$\widetilde{D}_n,\ n \geq 4$		$\widetilde{G}_2$	

Prop. A. 17 of [MT11] and Section 6.7 in [Hum90] imply that the three types of affine irreducible groups of rank 3 are $\widetilde{A}_2$, $\widetilde{C}_2$, $\widetilde{G}_2$.

Main result

Theorem (Main result)

The strong hull conjecture holds for affine irreducible Coxeter groups of rank 3.

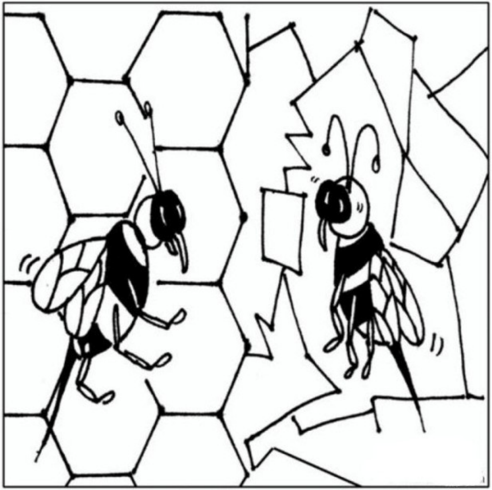


(a) Reflection hyperplanes of type $\widetilde{A}_2$

(b) Reflection hyperplanes of type $\widetilde{C}_2$

(c) Reflection hyperplanes of type $\widetilde{G}_2$

Figure 2: The triangulation of Euclidean space for affine irreducible Coxeter groups of rank 3



"That's what you get for skipping geometry class."

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[Cox35] Harold SM Coxeter, *The complete enumeration of finite groups of the form* $r_i^2 = (r_i r_j)^{k_{ij}} = 1$, Journal of the London Mathematical Society **1** (1935), no. 1, 21–25.

[GG22] Christian Gaetz and Yibo Gao, *The hull metric on Coxeter groups*, Comb. Theory **2** (2022), no. 2, Paper No. 7, 15. MR 4449815

[Hum90] James E Humphreys, *Reflection groups and coxeter groups*, Cambridge studies in advanced mathematics, no. 29, Cambridge university press, 1990.

[MT11] Gunter Malle and Donna Testerman, *Linear algebraic groups and finite groups of lie type*, vol. 133, Cambridge university press, 2011.

Thanks!