Lemma 3.1 (odd powers)

Claim: For any $k \geq 0$, odd s, and $\varepsilon > 0$, there exists a single shallow tanh network of width $(s+1)/2$ that simulataneously approximates all odd monomials y^p with $p \leq s$ to error $\leq \varepsilon$ in $W^{k,\infty}$.

Scales: Weights can be chosen with size polynomial in M, s and like $O(\varepsilon^{-\frac{s}{2}})$ in ε.

Idea: A central finite-difference construction on tanh synthesizes $y, y^3, \ldots$

Lemma 3.2 (odd powers)

Claim: Under the same k, s, M, there exists a shallow tanh network of width $3(s+1)/2$ that simultaneously approximates every monomial y^p with $p \leq s$ (odd and even) to error $\leq \varepsilon$ in $W^{k,\infty}$.

Scales: The width is $O(s)$ and independant of ε; weights grow like $O(\varepsilon^{-\frac{s}{2}})$.

Idea: Combine three shifted copies of the odd-power network (inputs $y-\alpha$, y, $y+\alpha$) via an algebraic identity to get even powers.

Example: If $s=5$: width $=3$ for odd powers (y, y^3, y^5); width $=9$ for all powers up to 5.