Some proofs of Fourier transform theorems

Each theorem will be stated along with a proof in most cases. Several examples giving applications will be given after the statements of all the theorems. In the statements of the theorems, $x(t)$,$x_1(t)$, and $x_2(t)$ denote signals with $X(f), X_1(f)$, and $X_2(f)$ denoting their respective Fourier transforms. Constants are denoted by a, a_1, a_2, t_0, and f_0.

Superposition Theorem

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(f) + a_2 X_2(f) \tag{2.100}$$

Proof: By the defining integral for the Fourier transform,

$$\begin{aligned}
\Im\{a_1 x_1(t) + a_2 x_2(t)\} &= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-j2\pi f t}\, dt \\
&= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi f t} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi f t}\, dt \\
&= a_1 X_1(f) + a_2 X_2(f)
\end{aligned} \tag{2.101}$$

Time-Delay Theorem

$$x(t - t_0) \longleftrightarrow X(f) e^{-j2\pi f t_0} \tag{2.102}$$

Proof: Using the defining integral for the Fourier transform, we have

$$\begin{aligned}
\Im\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(t - t_0) e^{-j2\pi f t}\, dt \\
&= \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f(\lambda + t_0)}\, d\lambda \\
&= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda}\, d\lambda \\
&= X(f) e^{-j2\pi f t_0}
\end{aligned} \tag{2.103}$$

where the substitution $\lambda = t - t_0$ was used in the first integral.

Scale-Change Theorem

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right) \tag{2.104}$$

Proof: First, assume that $a > 0$. Then

$$\begin{aligned}\Im\{x(at)\} &= \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft}\, dt \\ &= \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda/a} \frac{d\lambda}{a} = \frac{1}{a} X\left(\frac{f}{a}\right)\end{aligned} \tag{2.105}$$

where the substitution $\lambda = at$ has been used. Next, considering $a < 0$, we write

$$\begin{aligned}\Im\{x(at)\} &= \int_{-\infty}^{\infty} x(-|a|t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(\lambda) e^{+j2\pi f\lambda/|a|} \frac{d\lambda}{|a|} \\ &= \frac{1}{|a|} X\left(-\frac{f}{|a|}\right) = \frac{1}{|a|} X\left(\frac{f}{a}\right)\end{aligned} \tag{2.106}$$

where use has been made of the relation $-|a| = a$ if $a < 0$.

Duality Theorem

$$X(t) \longleftrightarrow x(-f) \tag{2.107}$$

That is, if the Fourier transform of $x(t)$ is $X(f)$, then the Fourier transform of $X(f)$ with f replaced by t is the original time-domain signal with t replaced by $-f$.

Proof: The proof of this theorem follows by virtue of the fact that the only difference between the Fourier transform integral and the inverse Fourier transform integral is a minus sign in the exponent of the integrand.

Frequency Translation Theorem

$$x(t) e^{j2\pi f_0 t} \longleftrightarrow X(f - f_0) \tag{2.108}$$

Proof: To prove the frequency translation theorem, note that

$$\int_{-\infty}^{\infty} x(t) e^{j2\pi f_0 t} e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi(f-f_0)t}\, dt = X(f - f_0) \tag{2.109}$$

Modulation Theorem

$$x(t)\cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0) \tag{2.110}$$

Proof: The proof of this theorem follows by writing $\cos(2\pi f_0 t)$ in exponential form as $\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$ and applying the superposition and frequency translation theorems.

Differentiation Theorem

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j2\pi f)^n X(f) \tag{2.111}$$

Proof: We prove the theorem for $n = 1$ by using integration by parts on the defining Fourier transform integral as follows:

$$\begin{aligned}\Im\left\{\frac{dx}{dt}\right\} &= \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j2\pi ft} dt \\ &= x(t)e^{-j2\pi ft}|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt \\ &= j2\pi f\, X(f)\end{aligned} \tag{2.112}$$

where $u = e^{-j2\pi ft}$ and $dv = (dx/dt)\, dt$ have been used in the integration-by-parts formula, and the first term of the middle equation vanishes at each end point by virtue of $x(t)$ being an energy signal. The proof for values of $n > 1$ follows by induction.

Integration Theorem

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \longleftrightarrow (j2\pi f)^{-1} X(f) + \frac{1}{2} X(0)\delta(f) \tag{2.113}$$

Proof: If $X(0) = 0$ the proof of the integration theorem can be carried out by using integration by parts as in the case of the differentiation theorem. We obtain

$$\Im\left\{\int_{-\infty}^{t} x(\lambda)\, d(\lambda)\right\} = \left\{\int_{-\infty}^{t} x(\lambda)\, d(\lambda)\right\}\left(-\frac{1}{j2\pi f} e^{-j2\pi ft}\right)\Big|_{-\infty}^{\infty} + \frac{1}{j2\pi f}\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt \tag{2.114}$$

The first term vanishes if $X(0) = \int_{-\infty}^{\infty} x(t)\, dt = 0$, and the second term is just $X(f)/(j2\pi f)$. For $X(0) \neq 0$, a limiting argument must be used to account for the Fourier transform of the nonzero average value of $x(t)$.

Convolution Theorem

$$\int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)\,d\lambda \triangleq \int_{-\infty}^{\infty} x_1(t-\lambda)x_2(\lambda)d\lambda \longleftrightarrow X_1(f)X_2(f) \tag{2.115}$$

Proof: To prove the convolution theorem of Fourier transforms, we represent $x_2(t-\lambda)$ in terms of the inverse Fourier transform integral as

$$x_2(t-\lambda) = \int_{-\infty}^{\infty} X_2(f)e^{j2\pi f(t-\lambda)}\,df \tag{2.116}$$

Denoting the convolution operation as $x_1(t)*x_2(t)$, we have

$$\begin{aligned} x_1(t)*x_2(t) &= \int_{-\infty}^{\infty} x_1(\lambda)\left[\int_{-\infty}^{\infty} X_2(f)e^{j2\pi f(t-\lambda)}\,df\right]d\lambda \\ &= \int_{-\infty}^{\infty} X_2(f)\left[\int_{-\infty}^{\infty} x_1(\lambda)e^{-j2\pi f\lambda}\,d\lambda\right]e^{j2\pi ft}\,df \end{aligned} \tag{2.117}$$

where the last step results from reversing the orders of integration. The bracketed term inside the integral is $X_1(f)$, the Fourier transform of $x_1(t)$. Thus

$$x_1 * x_2 = \int_{-\infty}^{\infty} X_1(f)X_2(f)e^{j2\pi ft}\,df \tag{2.118}$$

which is the inverse Fourier transform of $X_1(f)X_2(f)$. Taking the Fourier transform of this result yields the desired transform pair.

Multiplication Theorem

$$x_1(t)x_2(t) \longleftrightarrow X_1(f)*X_2(f) = \int_{-\infty}^{\infty} X_1(\lambda)X_2(f-\lambda)\,d\lambda \tag{2.119}$$

Proof: The proof of the multiplication theorem proceeds in a manner analogous to the proof of the convolution theorem.