Gaussian random vectors and processes

in Stochastic Process

V2 A normalized Gaussian rv:

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) \qquad W \sim N(0,1)$$

MGF: $Z \sim N(0, \sigma^2)$

$$g_Z(r) = E[\exp(rZ)] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp(rz) \exp\left[\frac{-z^2}{2\sigma^2}\right] dz$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[\frac{-z^2 + 2\sigma^2 rz - r^2\sigma^4}{2\sigma^2} + \frac{r^2\sigma^2}{2}\right] dz$$

$$= \exp\left[\frac{r^2\sigma^2}{2}\right] \left\{ \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[\frac{-(z - r\sigma^2)^2}{2\sigma^2}\right] dz \right\}$$

$$= \exp\left[\frac{r^2\sigma^2}{2}\right]$$

If $U \sim N(u, \sigma^2)$, $Z = U - \mu$.

$$g_U(r) = E[\exp(r(u + Z))] = e^{ru} E[e^{rZ}]$$

$$= \exp\left(ru + \frac{r^2\sigma^2}{2}\right)$$

An IID n-rv $\bar{W}$ where each component W_j, $1 \le j \le n$

$W_j \sim N(0, 1)$, joint density.

$$f_{\bar{W}}(w) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(\frac{-w_1^2 - w_2^2 - \cdots - w_n^2}{2}\right)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(\frac{-\bar{w}^T \bar{w}}{2}\right)$$

$$g_{\bar{W}}(\bar{r}) = E[\exp(\bar{r}^T \bar{W})] = E\left[\prod_j \exp(r_j W_j)\right]$$

$$= \prod_j E[\exp(r_j W_j)] = \prod_j \exp\left(\frac{r_j^2}{2}\right)$$

$$= \exp\left(\frac{\bar{r}^T \bar{r}}{2}\right)$$

Joint-Gaussian random vectors

D1: $\{Z_1, Z_2, \cdots Z_n\}$ is a set of $\boxed{\text{jointly}}$ Gaussian zero mean rvs. $\bar{Z} = (Z_1, \cdots, Z_n)^T$ is a Gaussian zero-mean n-rv, if, for some finite set of IID $N(0,1)$ rvs, W_1, W_m. each Z_j can be express as,

$$Z_j = \sum_{l=1}^{m} a_{jl} W_l \quad \text{i.e. } \bar{Z} = [A]\bar{W}$$

More generally, $\bar{U} = (U_1, \ldots, U_n)^T$ is a Gaussian n-rv, if $\bar{U} = \bar{Z} + \bar{u}$, where Z is a zero-mean Gaussian n-rv and $\bar{u}$ is a real n vector.

T1: Let $\bar{Z} = (Z_1, \ldots Z_n)^T$ be a zero-mean Gaussian n-rv. Let $\bar{Y} = (Y_1, \ldots Y_k)^T$ be a k-rv satisfying $\bar{Y} = [B]\bar{Z}$, Then $\bar{Y}$ is a zero-mean Gaussian k-rv.

T2: Let $\bar{Z} = (Z_1, \ldots Z_n)^T$ be a zero-mean Gaussian n-rv. Then for any real n-vector $\bar{a} = (a_1, \ldots, a_n)^T$, the linear combination $a^T\bar{Z}$ is a zero-mean Gaussian k-rv.

Condition: $\bar{Z}$ must be jointly Gaussian

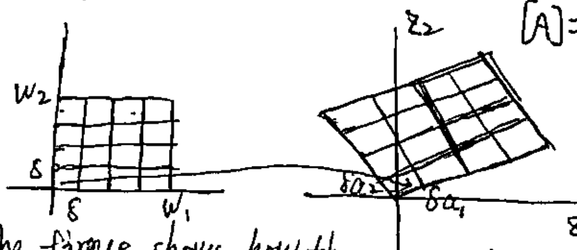
Joint probability density for Gaussian n-rvs

Covariance of Z:

$$[K] = E[\bar{Z}\bar{Z}^T] = E[[A]\bar{W}\bar{W}^T[A]^T] = [A]E[\bar{W}\bar{W}^T][A]^T$$

find $f_Z(z)$: $z = [A]w$ $\qquad = [A][A]^T$

example($n=2$)



$$[A] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$Z_1 = 2W_1 - W_2$

$Z_2 = W_1 + W_2$

The figure shows how the $w_1 w_2$ space can be quantized into adjoining squares.

独立一定不相关，不相关不一定独立

uncorrelated Gaussian rvs are independent

$$f_{\bar{Z}}(\bar{z})\,|dz| = f_{\bar{W}}(\bar{w})\,|d\bar{w}| \qquad \because \bar{w} = [A]^{-1} \bar{z}$$

$\because |d\bar{w}| = \delta^n$. $|dz| = \delta^n |\det[A]|$

$$\Rightarrow \frac{d|\bar{z}|}{d|\bar{w}|} = |\det[A]|$$

And $\because \bar{Z} = [A]\bar{W}$

$$\therefore f_{\bar{Z}}(\bar{z}) = \frac{\exp(-\frac{1}{2}\bar{z}^T [A^{-1}]^T [A^{-1}]\bar{z})}{(2\pi)^{\frac{n}{2}} |\det[A]|}$$

$\because [K] = [AA^T]$, $\therefore [K^{-1}] = [A^{-1}]^T[A^{-1}]$

$\because \det[AB] = \det[A]\det[B]$, $\det[A] = \det[A^T]$

$\therefore \det[K] = \det[A]\det[A^T] = (\det[A])^2 > 0$

$$\therefore f_{\bar{Z}}(\bar{z}) = \frac{\exp(-\frac{1}{2}\bar{z}^T[K^{-1}]\bar{z})}{(2\pi)^{\frac{n}{2}}\sqrt{\det[K]}}$$

Consider above for two-dimentional case Let $E[Z_1^2] = \sigma_1^2$, $E[Z_2^2] = \sigma_2^2$ and $E[Z_1Z_2] = K_{12}$, Define the normalized covariance, ρ, as $K_{12}/(\sigma_1\sigma_2)$.

$\det[K] = \sigma_1^2\sigma_2^2 - K_{12}^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$. For $[A]$ to be non-singular, we need $\det[K] = (\det[A])^2 > 0$. so we need $|\rho| < 1$, We then have.

$$[K]^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - K_{12}^2}\begin{bmatrix} \sigma_2^2 & -K_{12} \\ -K_{12} & \sigma_1^2 \end{bmatrix} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

$$f_{\bar{Z}}(\bar{z}) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 - K_{12}^2}} \exp\left(\frac{-z_1^2\sigma_2^2 + 2z_1z_2K_{12} - z_2^2\sigma_1^2}{2(\sigma_1^2\sigma_2^2 - K_{12}^2)}\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(\frac{-\frac{z_1^2}{\sigma_1^2} + \frac{2\rho z_1 z_2}{\sigma_1\sigma_2} - \frac{z_2^2}{\sigma_2^2}}{2(1-\rho^2)}\right)$$

Let $\bar{U} = \bar{u} + \bar{Z}$, where $\bar{u} = E[\bar{U}]$ and $\bar{Z}$ is zero-mean Gaussian n-rv with density above, the density of $\bar{U}$ is given by

$$f_{\bar{U}}(\bar{w}) = \frac{\exp\left(-\frac{1}{2}(\bar{u}-\bar{\mu})[K^{-1}](\bar{u}-\bar{\mu})\right)}{(2\pi)^{\frac{n}{2}}\sqrt{\det[K]}}$$

where $[K]$ is the covariance matrix of both $\bar{U}$ and $\bar{Z}$

T3: Conditions for a zero-mean n-rv $\bar{Z}$ to be a zero-mean Gaussian n-rv. i.e. for the components $Z_1, \dots, Z_n$ of $\bar{Z}$ to be jointly Gaussian:

(1) $\bar{Z}$ can be expressed as $\bar{Z} = [A]\bar{W}$, where $[A]$ is real and $\bar{W}$ is $N(0, [I])$,

(2) For all real n-vectors $\bar{a}$. the rv $\bar{a}^T\bar{Z}$ is zero-mean Gaussian.

(3) The linearly independent components of $\bar{Z}$ has the probability density above.

(4) The characteristic function of $\bar{Z}$ is given above.

Conditional PDFs for Gaussian random vector.

The conditional probability $f_{X|Y}(x|y)$ for two zero mean jointly-Gaussian rvs X and Y with non-singular covariance matrix,

$$f_{X,Y}(x|y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[\frac{-\left(\frac{x}{\sigma_X}\right)^2 + 2\rho\left(\frac{x}{\sigma_X}\right)\left(\frac{y}{\sigma_Y}\right) - \left(\frac{y}{\sigma_Y}\right)^2}{2(1-\rho^2)}\right.$$

$\because \rho = \frac{E[XY]}{\sigma_X\sigma_Y}$ and $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left(\frac{-y^2}{2\sigma_Y^2}\right)$

$\therefore$

Note: For $[A]\bar{w}$, which means the transformation $[A]\bar{w}$ carries the cube into a parallelepiped. If determinant $\det[A] = 0$, ($[A]$ is singular), then the n-dimensional unit cube in the $\bar{w}$ space is transformed into a lower-dimentional parallelepiped whose volume (as a regin of n-dimentional space) is 0

$$f_{X|Y}(x|y) = \frac{1}{\sigma_X\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-\left(\frac{x}{\sigma_X}\right)^2 + 2\rho\left(\frac{x}{\sigma_X}\right)\left(\frac{y}{\sigma_Y}\right) - \rho^2\left(\frac{y}{\sigma_Y}\right)^2}{2(1-\rho^2)}\right]$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{1}{\sigma_X\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-[x-\rho(\frac{\sigma_X}{\sigma_Y})y]^2}{2\sigma_X^2(1-\rho^2)}\right]$$

$\therefore$ Given $Y=y$, we can view X as a rv in the restricted sample space where $Y=y$. In that restricted sample space, X is $N(\rho\frac{\sigma_X}{\sigma_Y}y,\ \sigma_X^2(1-\rho^2))$.

We see ~~that the~~ the variance of X, given $Y=y$, has been reduced by a factor of $1-\rho^2$ from the variance before the observation.

① The reduction is large when $|\rho|$ is close to 1 and negligible when ρ is close to 0.

② This conditional variance is the same for all values of y.

③ The conditional mean of X is linear in y and that the conditional distribution is Gaussian with a variance constant in y.

Let $\bar{X} = (X_1, \ldots, X_n)^T$ and $\bar{Y} = (Y_1, \ldots, Y_m)^T$ be zero-mean jointly-Gaussian rvs and length n and m (i.e., $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are jointly Gaussian). Let their covariance matrices be $[K_X]$ and $[K_Y]$ respectively. Let $[K]$ be the covariance matrix of the $(n+m)$-rv $(X_1, \ldots, X_n, Y_1, \ldots Y_m)^T$.

The $(n+m)\times(n+m)$ covariance matrix $[K]$ can be partitioned into n rows on top and m rows on bottom, and then further partitioned into n and m columns, yielding

$$[K] = \begin{bmatrix} [K_X] & [K_{X,Y}] \\ [K_{X,Y}^T] & [K_Y] \end{bmatrix}$$

Here $[K_X] = E[\bar{X}\bar{X}^T]$, $[K_{X,Y}] = E[\bar{X}\bar{Y}^T]$, and $[K_Y] = E[\bar{Y}\bar{Y}^T]$

Note that if $\bar{X}$ and $\bar{Y}$ have means, then

$[K_X] = E[(\bar{X}-\bar{u}_X)(\bar{X}-\bar{u}_X)^T]$, $[K_{X,Y}] = E[(\bar{X}-\bar{X})(\bar{Y}-\bar{u}_Y)^T]$

Assume $[K]$ is non-singular,

$$[K^{-1}] = \begin{bmatrix} [B] & [C] \\ [C^T] & [D] \end{bmatrix}$$

The blocks $[B]$, $[C]$, $[D]$ can be calculated directly from $[KK^{-1}] = I$.

T4: Let $\bar{X}$ and $\bar{Y}$ be zero-mean, jointly-Gaussian, jointly non-singular rvs. Then $\bar{X}$, conditioned on $\bar{Y}=\bar{y}$, is $N(-[B^{-1}C]\bar{y}, [B^{-1}])$.

$$f_{\bar{X}|\bar{Y}}(\bar{x}|\bar{y}) = \frac{\exp\{-\frac{1}{2}(\bar{x}+[B^{-1}C]\bar{y})^T[B](\bar{x}+[B^{-1}C]\bar{y})\}}{(2\pi)^{\frac{n}{2}}\sqrt{\det[B^{-1}]}}$$

Proof: Express $f_{X|Y}(\bar{x}|\bar{y})$ as $f_{XY}(\bar{x},\bar{y})/f_Y(\bar{y})$,

$$f_{XY}(\bar{x},\bar{y}) = \frac{\exp\{-\frac{1}{2}(\bar{x}^T,\bar{y}^T)[K^{-1}](\bar{x}^T,\bar{y}^T)^T}{(2\pi)^{(n+m)/2}\sqrt{\det[K^{-1}]}}$$

$$= \frac{\exp\{-\frac{1}{2}(\bar{x}^T[B]\bar{x}+\bar{x}^T[C]\bar{y}+\bar{y}^T[C^T]\bar{x}+\bar{y}^T[D]\bar{y})\}}{(2\pi)^{(n+m)/2}\sqrt{\det[K^{-1}]}}$$

Note that $\bar{x}$ appears only in the first three terms of exponent above, and that $\bar{x}$ does not appear all in $f_Y(\bar{y})$. Thus, we can express the dependence on $\bar{x}$ in $f_{X|Y}(\bar{x}|\bar{y})$ by

$$f_{X|Y}(\bar{x}|\bar{y}) = \phi(\bar{y})\exp\left\{-\frac{1}{2}\left[\bar{x}^T[B]\bar{x} + \bar{x}^T[C]\bar{y} + \bar{y}^T[C^T]\bar{x}\right]\right\}$$

where $\phi(\bar{y})$ is some function of $\bar{y}$. We now complete the square around $[B]$ in the exponent above getting,

$$f_{X|Y}(\bar{x}|\bar{y}) = \phi(\bar{y})\exp\left\{-\frac{1}{2}\left[(\bar{x}+[B^{-1}C]\bar{y})^T[B](\bar{x}+[B^{-1}C]\bar{y}) - \bar{y}^T[C^TB^{-1}C]\bar{y}\right]\right\}$$

since the last term in the exponent doesn't depend on $\bar{x}$, we can absorb it into $\phi(\bar{y})$.

$$\therefore f_Y(\bar{y}) = \frac{\exp(-\frac{1}{2}\bar{y}^T[K_Y^{-1}]\bar{y})}{(2\pi)^{\frac{n}{2}}\sqrt{\det[K_Y]}}$$

$\therefore$ $\phi(\bar{y})$ must be $\dfrac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det[B^{-1}]}}$