Financial Statistics
Homework 1

Chenghua Liu
liuch18@mails.tsinghua.edu.cn
Department of Computer Science
Tsinghua University

1 2.2

Consider a path dependent payoff function $Y_t = a_1 r_{t+1} + \cdots + a_k r_{t+k}$ where $\{a_i\}_{i=1}^k$ are given weights. If the return time series is weakly stationary in the sense that $\operatorname{cov}(r_t, r_{t+j}) = \gamma(j)$. Show that

$$\operatorname{var}(Y_t) = \sum_{i=1}^k \sum_{j=1}^k a_i a_j \gamma(i-j)$$

A natural estimate of this variance is the following substitution estimator:

$$\widehat{\operatorname{var}}(Y_t) = \sum_{i=1}^k \sum_{j=1}^k a_i a_j \hat{\gamma}(i-j)$$

where $\hat{\gamma}(i-j)$ is defined by (2.4). Show that $\widehat{\operatorname{var}}(Y_t) \geqslant 0$

solution:

$$\begin{aligned}\operatorname{var}(Y_t) &= cov(Y_t, Y_t) = cov(a_1 r_{t+1} + \cdots + a_k r_{t+k}, a_1 r_{t+1} + \cdots + a_k r_{t+k}) \\ &= \sum_{i=1}^k \sum_{j=1}^k a_i a_j cov(r_{t+i}, r_{t+j}) = \sum_{i=1}^k \sum_{j=1}^k a_i a_j \gamma(i-j)\end{aligned}$$

Review the definition (2.4)

$$\widehat{\gamma}(k) = \frac{1}{T} \sum_{t=k+1}^T \left(X_t - \bar{X}\right)\left(X_{t-k} - \bar{X}\right), \quad \widehat{\rho}(k) = \widehat{\gamma}(k)/\widehat{\gamma}(0)$$

where $\bar{X} = T^{-1} \sum_{1 \leq t \leq T} X_t$. We have the k-dimensional sample covariance matrix

$$\widehat{\boldsymbol{V}}_k = \begin{pmatrix} \widehat{\gamma}(0) & \widehat{\gamma}(1) & \ldots & \widehat{\gamma}(k-1) \\ \widehat{\gamma}(1) & \widehat{\gamma}(0) & \ldots & \widehat{\gamma}(k-2) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\gamma}(k-1) & \widehat{\gamma}(k-2) & \ldots & \widehat{\gamma}(0) \end{pmatrix}$$

To show $\widehat{\operatorname{var}}(Y_t) \geqslant 0$ means to show that

$$\boldsymbol{a}^{\mathrm{T}} \widehat{\boldsymbol{V}}_k \boldsymbol{a} \geq 0$$

for any k -dimensional real vector a.This is equivalent to saying that $\widehat{\boldsymbol{V}}_k$ is a non-negative definite matrix. This can be easily obtained if we can express the matrix $\widehat{\boldsymbol{V}}$ as the following product

$$\widehat{V}_k = \frac{1}{k} C C^{\mathrm{T}}$$

for some matrix $\boldsymbol{C}$. Take vector $\boldsymbol{Y} = (X_1 - \ \bar{X}_1, \ldots, X_k - \bar{X}_k)^{\mathrm{T}}$. Then

$$\boldsymbol{C} = \begin{pmatrix} 0 & \ldots & 0 & 0 & Y_1 & Y_2 & \ldots & Y_k \\ 0 & \ldots & 0 & Y_1 & Y_2 & \ldots & Y_k & 0 \\ \vdots & & & & & & & \vdots \\ 0 & Y_1 & Y_n & \ldots & Y_k & 0 & \ldots & 0 \end{pmatrix}$$

It is easy to see that multiplying C by C^T we obtain a matrix of sums of squares and products of Y_i which when divided by k is the $\widehat{\boldsymbol{V}}_{\boldsymbol{k}}$ matrix. Hence,

$$\begin{aligned} \boldsymbol{a}^{\mathrm{T}} \widehat{\boldsymbol{V}}_{\boldsymbol{k}} \boldsymbol{a} &= \boldsymbol{a}^{\mathrm{T}} \frac{1}{k} \boldsymbol{C} \boldsymbol{C}^{\mathrm{T}} \boldsymbol{a} \\ &= \frac{1}{k} \left(\boldsymbol{a}^{\mathrm{T}} \boldsymbol{C}\right) \left(\boldsymbol{C}^{\mathrm{T}} \boldsymbol{a}\right) \geq 0 \end{aligned}$$

■

2 2.5

Consider the Yule-Walker equation (2.27). Show that the solution to the difference equation (2.27) admits the form

$$\gamma(k) = c_1 \alpha_1^{-k} + \cdots + c_p \alpha_p^{-k}$$

for sufficiently large k, where $\alpha_1, \cdots, \alpha_p$ are the roots of the characteristic function $b(x)$ and are assumed to be distinct. Hint: use $b(B)\gamma(k) = (1 - \alpha_1 B) \cdots (1 - \alpha_p B) \gamma(k) = 0$ and hence one of $(1 - \alpha_j B) \gamma(k) = 0$.

solution:

the Yule-Walker equation:

$$\gamma(k) = b_1\gamma(k-1) + \cdots + b_p\gamma(k-p), \quad k \geqslant 1$$

Now suppose that this equation is written as

$$\phi(B)\gamma_k = 0$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and B now operates on k and not t. Then, writing

$$\phi(B) = \prod_{i=1}^{p}(1-\alpha_i B)$$

hence we get

$$(1-\alpha_j B)\,\gamma(k) = 0$$

Therefore, the fundamental system of solutions of the linear difference equation are $\alpha_1^{-k}, \alpha_2^{-k}, \ldots, \alpha_p^{-k}$. According to the superposition principle, the general solution for γ_k is

$$\rho_k = c_1\alpha_1^{-k} + c_2\alpha_2^{-k} + \cdots + c_p\alpha_p^{-k}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the roots of the characteristic equation

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p = 0$$

$\blacksquare$

3 2.6

For an $AR(p)$ process

$$X_t = b_0 + b_1 X_{t-1} + \cdots + b_p X_{t-p} + \varepsilon_t$$

show that (a) $\pi(k) = 0$ for $k > p$ and (b) it is stationary when $\sum_{j=1}^{p}|b_j| < 1$

solution:

$\pi(k) =$ the last regression coefficient of

$$\min_{\beta} \mathrm{E}\left(X_{t+k+1} - \beta_0 - \beta_1 X_{t+k} - \cdots - \beta_k X_{t+1}\right)^2$$

When $k > p$, The above formula can be rewritten as

$$\min_{\beta} \mathrm{E}\left(\varepsilon_t + (b_0-\beta_0) + (b_1-\beta_1)X_{t+k} + \cdots + (b_p-\beta_p)X_{t+k+1-p} - \beta_{p+1}X_{t+k-p} - \cdots - \beta_k X_{t+1}\right)^2$$

$$\geq \sigma_t^2 + (b_0-\beta_0)^2 + (b_1-\beta_1)^2\mathrm{E}X_{t+k}^2 + \cdots + (b_p-\beta_p)^2\mathrm{E}X_{t+k+1-p}^2 + \beta_{p+1}^2\mathrm{E}X_{t+k-p}^2 + \cdots + \beta_k^2\mathrm{E}X_{t+1}^2$$

Obviously, $\begin{cases}\beta_i = b_i & 0 \leq i \leq p \\ \beta_i = 0 & p < i \leq k\end{cases}$ minimizes the formula.This illustrates $\pi(k) = 0$.

Lemma 3.0.1. *the* AR(p) *is weakly stationary provided all roots of the characteristic equation* $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0$ *lie outside the complex unit circle.*

Proof: Keep writing habits, we use ϕ_i instead of b_i. For convenience, we assume that the mean is 0. Then the AR(p)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t$$

can be recast as the AR(1) model: $\xi_t = F\xi_{t-1} + \varepsilon_t$

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then we can write the AR model recursively:

$$\xi_t = \varepsilon_t + F\varepsilon_{t-1} + F^2\varepsilon_{t-2} + \cdots + F^{k-1}\varepsilon_{t-k+1} + F^k\xi_{t-k}$$

Multiply both sides by $F^k\xi_{t-k}$ and take the expectation:

$$cov(\xi_{t-k}, \xi_t) = F^k cov(\xi_{t-k}, \xi_{t-k})$$

To be weakly stationary,we need $\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{cov(\xi_{t-k},\xi_t)}{cov(\xi_{t-k},\xi_{t-k})} = F^k$ only related to k.It's also imply that F^k satisfies certain conditions (a special Toeplitz matrix). Remember the eigenvalue decomposition: $F = T\Lambda T^{-1}$ and the propriety that: $F^k = T\Lambda^k T^{-1}$ with

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_p^k \end{bmatrix}$$

So the AR(1) model is stable if $|\lambda_i| < 1$ The condition on F is that all λ from $|F - \lambda I| = 0$ are < 1. Easily, we can find that the eigenvalues of F are:

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \ldots - \phi_{p-1}\lambda - \phi_p = 0$$

But the λ are the reciprocal of the values z that solve the characteristic polynomial of the AR(p) $(1 - \varphi_1 z - \varphi_2 z^2 - \ldots - \phi_p z^p) = 0$. So the roots of the polynomial should be > 1, or, with complex values, outside the unit circle.

$\square$

Now, let's prove (b). Consider the characteristic equation modulo on both sides.

$$1 = |b_1 z + \cdots + b_p z^p| \leq |b_1||z| + \cdots + |b_p||z^p|$$

if $\exists z$ satisfies $|z| \leq 1$,then $|z^k| \leq 1 \quad \forall k \in \mathrm{N}^+$. So we have the contradictory.

$$|b_1||z| + \cdots + |b_p||z^p| \leq |b_1| + \cdots + |b_p| < 1$$

Therefore, $\forall z$, $|z| > 1$ which means the AR(p) is weakly stationary by Lemma.

$\blacksquare$