Homework 2

Nov. 19, 2020

NOTE: Homework 2 is due next Thursday (Nov. 26, 2020). The questions started with * are for Exercise only, and you are not required to submit the answers.

1. Let r.v.'s $X_1, \cdots, X_n$ *i.i.d.* $\sim N(\theta+1, \theta^2)$, is $\bar{X}$ a sufficient statistic of θ?

2. Let $X_1, \cdots, X_n$ be a random sample from the distribution with p.d.f.

$$f(x;\theta) = \frac{1}{2\theta} \exp\left\{-\frac{|x|}{\theta}\right\}, \quad -\infty < x < +\infty,\ \theta > 0.$$

   Show that $T = \sum_{i=1}^{n} |X_i|$ is a sufficient and complete statistic of θ.

3. Let $X_1, \cdots, X_m$ *i.i.d.* $\sim N(a, \sigma^2)$, $Y_1, \cdots, Y_n$ *i.i.d.* $\sim N(b, \sigma^2)$, and X_i's and Y_j's are independent. Derive the sufficient and complete statistic of (a, b, σ^2).

4. Let r.v.'s $X_1, \cdots, X_n$ *i.i.d.* $\sim U(-\theta/2, \theta/2)$, $\theta > 0$, show that $(X_{(1)}, X_{(n)})$ is sufficient but not complete.

5. Let $X_1, \cdots, X_n$ be a random sample from $N(a, \sigma^2)$, show that $\bar{X}$ is independent of $X_{(n)} - X_{(1)}$.

6. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from exponential distribution with p.d.f.

$$f(x;\theta) = \exp\{-(x-\theta)\} I_{\{x>\theta\}}, \quad -\infty < \theta < +\infty$$

   - Derive the moment estimator of θ and show that it is unbiased;

7. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from Normal distribution $N(0, \sigma^2)$, derive the moment estimators of σ and $P(X > 1)$.

8. Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ be a random sample from Uniform distribution $U(0, 2\theta)$.

   - Show that $\hat{\theta}_1 = \bar{X}$ and $\hat{\theta}_2 = (n+1)X_{(n)}/(2n)$ are unbiased estimators of θ;
   - Show that $\hat{\theta}_1$ is a strongly consistent estimator of θ, and $\theta_2^* = X_{(n)}/2$ is a weakly consistent estimator of θ;
   - Which one of $\hat{\theta}_1$ and $\hat{\theta}_2$ is more efficient?

9. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from the Hong Kong Children height data. Let $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ be the population means, variances, and correlation of height and weight. The observed values of the sample $\boldsymbol{X}$ with sample size $n = 200$ are given in the attached file "Homework_HongKong_Height_Weight.csv".

   - Compute the moment estimators of $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ (denoted as $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\rho})$).
   - Estimate the biases and variances of $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1^2, \hat{\sigma}_2^2)$ (suppose X_i and Y_i are normally distributed).
   - How to decrease the MSE of the above estimators?

*1. Show that the n-dimensional normal family $\{f(\boldsymbol{x};\boldsymbol{\mu},\Sigma);\boldsymbol{\mu}\in R^n,\ \Sigma\in\mathcal{M}_n\}$ is an exponential family, where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are n-dimensional column vector, $\mathcal{M}_n$ is a collection of $n\times n$ symmetric positive definite matrices and

$$f(\boldsymbol{x};\boldsymbol{\mu},\Sigma)=\frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right],\quad \boldsymbol{x}\in R^n.$$

*2. Let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample from Poisson distribution $P(\lambda)$, derive the limiting distribution of $(\bar{X}-\lambda)/\sqrt{\bar{X}/n}$.

*3. Let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample from Normal distribution $N(a,\sigma^2)$, let $S^2=\sum_{i=1}^n(X_i-\bar{X})^2/(n-1)$ be the sample variance, show that S^2 is an unbiased, weakly consistent, and consistent in quadratic mean estimator of σ^2.

*4. Let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample from exponential distribution with p.d.f.

$$f(x;\theta)=\theta e^{-\theta x},\ x>0,\ \theta\in\Theta=(0,\infty).$$

- Show that both $\bar{X}$ and $nX_{(1)}$ are unbiased estimators of $1/\theta$;
- Which of these two estimators would you prefer?

*5. Let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample from Gamma distribution with parameters α, β and p.d.f.

$$f(x;\alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta},\quad x>0,\ \alpha>0,\ \beta>0.$$

Derive the moment estimators of α and β.

*6 Let $X_1,\cdots,X_n$ be a random sample from two parameter exponential distribution with p.d.f.

$$f(x;\lambda,\mu)=\lambda^{-1}\exp\left\{-\frac{x-\mu}{\lambda}\right\}I_{\{x>\mu\}},$$

where $0<\lambda<+\infty$, $-\infty<\mu<+\infty$ are two unknown parameters. Show that

- $\left(X_{(1)},\sum_{i=1}^n X_{(i)}\right)$ is sufficient for (λ,μ);
- $X_{(1)}$ is independent of $\sum_{i=1}^n(X_i-X_{(1)})$.