Homework 1

Nov. 12, 2020

NOTE: Homework 1 is due next Thursday (Nov. 19, 2020)

1. Denote the population by a random variable (r.v.) $X \sim B(1,p)$ (i.e., $P(X=1)=p,\ P(X=0)=1-p$, where p is an unknown parameter). Let $\boldsymbol{X}=(X_1,X_2,\cdots,X_6)$ be a random sample from the population X,
   - Write out the sample space and the probability distribution of $\boldsymbol{X}$;
   - Point out which of the followings are statistics: $X_1,\ X_1X_2,\ \min_{1\le i\le 5} X_i,\ X_6+p^2,\ X_6-E(X_2),\ (X_6-X_1)^2/Var(X_2)$;
   - If there are m out of the observations of $X_1,X_2,\cdots,X_6$ taking values 1 and $6-m$ out of them taking values 0, derive the empirical distribution function.
2. Let the r.v. X be distributed as uniform distribution $U(0,1)$ and set $Y=-\log X$.
   - Determine the d.f. of Y and then its p.d.f.
   - If $X_1,\cdots,X_n$ is a random sample from the $U(0,1)$ distribution and $Y_i=-2\log X_i$, using the characteristic function approach to show that $\sum_{i=1}^n Y_i$ is distributed as χ^2_{2n}.
3. For any r.v.'s $X_1,\cdots,X_n$, let

$$\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad S^2=\frac{1}{n-1}\sum_{i=1}^n\left(X_i-\bar{X}\right)^2,$$

   and show that:
   -
$$(n-1)S^2=\sum_{i=1}^n X_i^2-n\bar{X}^2;$$
   - If the r.v.'s have common (finite) expectation μ, then
$$\sum_{i=1}^n\left(X_i-\mu\right)^2=(n-1)S^2+n\left(\bar{X}-\mu\right)^2.$$
4. Let $\boldsymbol{X}=(X_1,X_2,\cdots,X_n)$ be a random sample from the the population $X\sim N(\mu,2)$, determine the sample size n such that $P(|\bar{X}-\mu|<0.1)\ge 0.997$.
5. Let r.v.'s $X_1,\ldots,X_n$ *i.i.d.* $\sim N(\mu,\sigma^2)$, $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i$, $S_n^2=\frac{1}{n}\sum_{i=1}^n(X_i-\bar{X})^2$. Let r.v. $X_{n+1}\sim N(\mu,\sigma^2)$ and suppose that X_{n+1} is independent of $X_1,\ldots,X_n$. Determine the distribution of the r.v. $\frac{X_{n+1}-\bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}}$.
6. Let r.v. $X\sim\chi^2_n$.
   - Prove that $E(X)=n$, $\mathrm{Var}(X)=2n$;

- Calculate the population kurtosis $\beta_2 = \{E(X - EX)^4\}/\{E(X - EX)^2\}^2 - 3$;
- Show that $(X - n)/\sqrt{2n} \xrightarrow{d} N(0, 1)$ as $n \to \infty$;
- If $X_i \sim \chi^2_{n_i}$, $i = 1, 2, \cdots, k$ and $X_1, \cdots, X_k$ are independent, let $m = \sum_{i=1}^k n_i$, then using characteristic function method to show that $\sum_{i=1}^k X_i \sim \chi^2_m$.

7. The p.d.f. of the Weibull distribution with two unknown parameters α and β is

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, \quad x > 0 \quad (\alpha, \beta > 0).$$

Let $X_1, \ldots, X_n$ be a random sample from this distribution. Let $Y = X_{(1)}$ be the smallest order statistic.

- Is the family of Weibull distributions an exponential family?
- Obtain the p.d.f. of Y.
- Calculate the expectation $E(Y)$ and the variance $Var(Y)$.