可靠性数据与生存分析作业

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1 第一次作业

1.1 Prove: If $\Lambda(t)$ is cumulative hazard function of T, then $\Lambda(\mathrm{T}) \sim EXP(1)$, the unit exponential distribution.

We have known that Cumulative hazard function $\Lambda(t)$

$$\Lambda(t)=\int_0^t \lambda(u)du=-\log\{S(t)\}$$

Where $S(t)=P[T>t]=1-F(t)$ is survival function. Since $S(0)=1$,

$$S(t)=\exp\{-\Lambda(t)\}=\exp\left\{-\int_0^t \lambda(u)du\right\}$$

So we have

$$P(\Lambda(T)\geq t)=P\left(T\geq \Lambda^{-1}(t)\right)=S\left(\Lambda^{-1}(t)\right)=\exp\left\{-\Lambda\left(\Lambda^{-1}(t)\right)\right\}=e^{-t}$$

Q.E.D.

1.2 Prove: If T_1 and T_2 are two independent survival times with hazard function $\lambda_1(t)$ and $\lambda_2(t)$, respectively, then $T=\min(T_1,T_2)$ has a hazard function $\lambda_T(t)=\lambda_1(t)+\lambda_2(t)$

We have survival function

$$S_T(t)=P\left(\min(T_1,T_2)\geq t\right)=P\left(T_1\geq t\right)P\left(T_2\geq t\right)=S_1(t)S_2(t)$$

Then we have

$$\lambda_T(t)=-\frac{\mathrm{d}\log\{S_T(t)\}}{\mathrm{d}t}=-\frac{\mathrm{d}\log\{S_1(t)\}}{\mathrm{d}t}-\frac{\mathrm{d}\log\{S_2(t)\}}{\mathrm{d}t}=\lambda_1(t)+\lambda_2(t)$$

Q.E.D.

1.3 Prove the formula from the lecture notes

$$mrl\left(t_0\right)=E\left[T-t_0\mid T\geq t_0\right]=\frac{\int_{t_0}^{\infty}S(t)dt}{S\left(t_0\right)}$$

We remark the nonnegative random variable T with distribution function F. The mean residual life is defined as

$$mrl\left(t_0\right)=E\left[T-t_0\mid T\geq t_0\right]=\frac{E\left[(T-t_0)I_{\{T>t_0\}}\right]}{P\{T>t_0\}}=\frac{1}{1-F(t_0)}\int_{t_0}^{\infty}(t-t_0)dF(t)$$

for $t_0 > 0$, according to the Fubini's theorem

$$\int_{t_0}^{\infty}(t-t_0)dF(t) = \int_{t_0}^{\infty}\left(\int_{t_0}^{t}du\right)dF(t) = \int_{t_0}^{\infty}\left(\int_{u}^{\infty}dF(t)\right)du = \int_{t_0}^{\infty}P\{X > u\}du = \int_{t_0}^{\infty}(1-F(u))du$$

Therefore,

$$mrl\,(t_0) = \frac{1}{1-F(t_0)}\int_{t_0}^{\infty}(1-F(u))du = \frac{\int_{t_0}^{\infty}S(t)dt}{S\,(t_0)}$$

Q.E.D.

1.4 The time in days to development of a tumor for rats exposed to a carcinogen follows a Weibull distribution with $\alpha = 2$ and $\lambda = 0.01$

$$f(t) = \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}}$$

(a) Find the probabilities that a (random) rat will be tumor free at 50 days. (b) What is the average time to tumor development? (Hint $\Gamma(0.5) = \sqrt{\pi}$) (c) Find the hazard rate of time to tumor development at 50 days. (d) Find the median time to tumor development.

(a) according to $f(t) = \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}}$, we have known that $S(t) = e^{-\lambda\times t^{\alpha}}$.So,

$$P(T \geq 50) = S(50) = e^{-0.01\times(50)^2} = 0.082085$$

(b) the average time to tumor development is

$$\mathrm{E}(T) = \int_0^{\infty}S(t)dt = \int_0^{\infty}e^{-0.01\times(t)^2}dt = 5\sqrt{\pi} = 8.86227$$

(c) the hazard rate of time to tumor development is

$$\lambda(t) = -\frac{\mathrm{d}\log\{S(t)\}}{\mathrm{d}t} = \alpha\lambda t^{\alpha-1}$$

the hazard rate of time to tumor development at 50 days is

$$\lambda(50) = 2\times 0.01\times 50 = 1$$

(d) the median time to tumor development is

$$t_{0.5} = \left[\frac{\log 2}{\lambda}\right]^{\frac{1}{\alpha}} = 8.325546$$

1.5 Suppose we have a small data set with different kinds of censoring: $2+, 3, 4, 5-, 6, 7+, [8, 10]$, **where $+(-)$ means right (left) censored observations and $[a, b]$ means an interval censored observation. Suppose the distribution of the underlying survival time is an exponential distribution with a constant hazard λ. Write down the likelihood function of λ for this given data set.**

Obviously $\lambda > 0$.And the likelihood function is

$$\begin{aligned}L(\lambda) &= S(2)\cdot f(3)\cdot f(4)\cdot F(5)\cdot f(6)\cdot S(7)\cdot[F(10)-F(8)]\\ &= \lambda^3 e^{-30\lambda}\left(1-e^{-5\lambda}\right)\left(1-e^{-2\lambda}\right)\end{aligned}$$

1.6 Derive the following properties for a) exponential distribution and b) Weibull distribution. (1) Survival function (2) Hazard function (3) Cumulative hazard function (4) Mean survival time

Weibull distribution with the parametrization $W(\lambda, p), \lambda > 0$ and $p > 0$:

(1) $S(t) = e^{-(\lambda t)^p}$

(2) $\lambda(t) = p\lambda(\lambda t)^{p-1}$

(3) $\Lambda(t) = (\lambda t)^p$

(4) $E(t) = \frac{\Gamma(1+1/p)}{\lambda}$

in the case of exponential distribution, just set $p = 1$ in the above formula.