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1、Let $P \in \mathbb{R}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector. (Hint: Use the SVD.)

Proof:

since P is a nonzero projector, we have $P = P^2$ and $\|P\|_2 \neq 0$. Then, based on Cauchy-Schwarz inequality, we have

$$\|P\|_2 = \left\|P^2\right\|_2 \leq \|P\|_2^2$$

Hence, $\|P\|_2 \geq 1$ If P is an orthogonal projector, then $P^* = P$. Suppose P has the SVD of the form $P = U\Sigma V^*$, where $UU^* = VV^* = I$ Hence,

$$\|P\|_2 = \left\|P^2\right\|_2 = \|PP^*\|_2 = \|\Sigma\Sigma^*\|_2 = \sigma_1^2$$

where σ_1 is the largest singular value of Σ. since $\|P\|_2 = \|\Sigma\|_2 = \sigma_1 > 0$. We have $\sigma_1^2 = \sigma_1$. Therefore, $\sigma_1 = 1$. i.e. $\|P\|_2 = 1$ Assume that the projector P is not orthogonal. i.e., $\operatorname{range}(P)$ is not perpendicular to $\operatorname{range}(I-P)$. Then, we can find a vector a such that $Pa \neq a$ and $a \perp range(I-P)$. Hence

$$\|Pa\|_2 = \|a + (P-I)a\|_2 > \|a\|_2$$

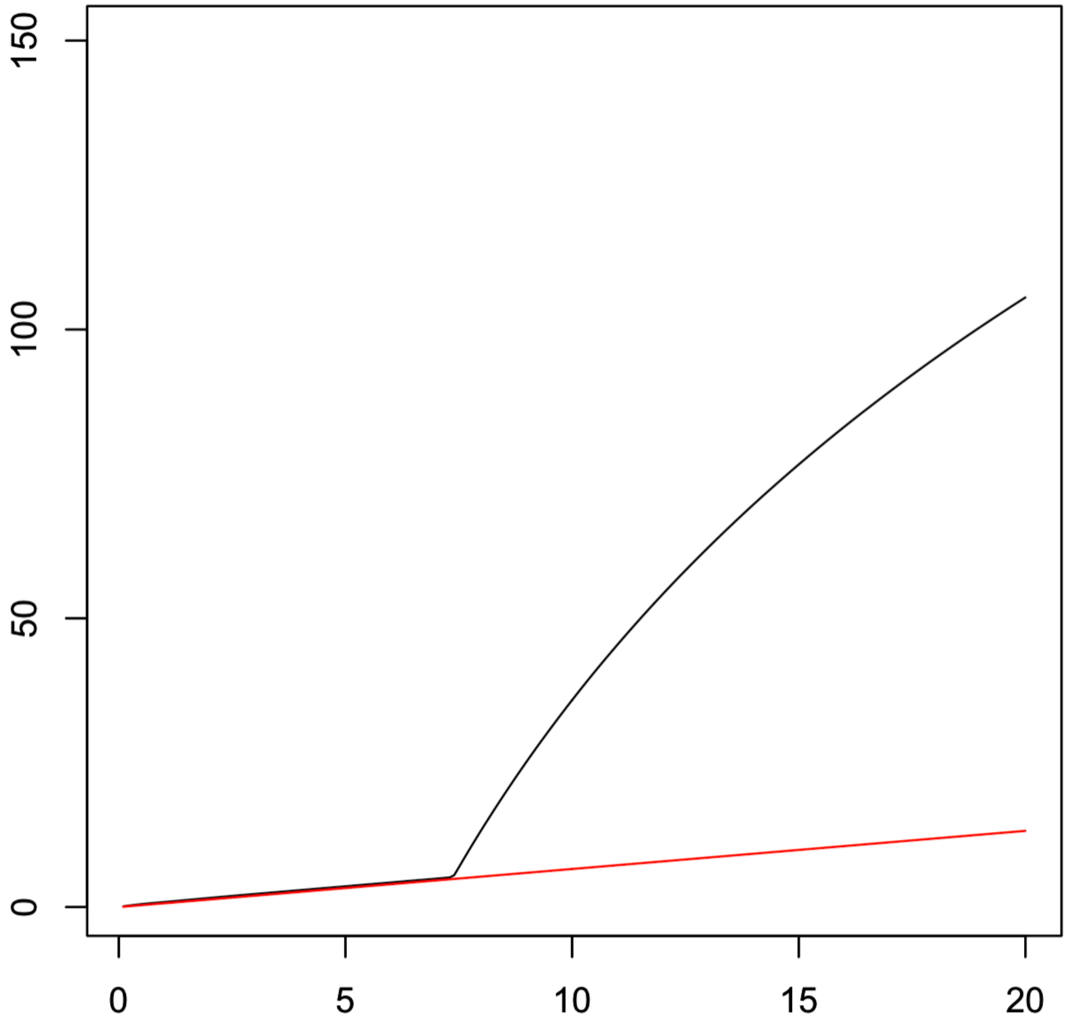
Therefore,

$$\|P\|_2 = \sup_{\|a\|_2=1} \|Pa\|_2 > \sup_{\|a\|_2=1} \|a\|_2 = 1$$

2、Let A be a 10×10 random matrix with entries from the standard normal distribution, minus twice the identity. Write a program to plot $\left\|e^{tA}\right\|_2$ against t for $0 \leq t \leq 20$ on a log scale, comparing the result to the straight line $e^{t\alpha(A)}$, where $\alpha(A) = \max_j \operatorname{Re}(\lambda_j)$ is the spectral abscissa of A. Run the program for ten random matrices A and comment on the results. What property of a matrix leads to a $\left\|e^{tA}\right\|_2$ curve that remains oscillatory as $t \to \infty$?

$A = U^T\Sigma V$ $\left\|e^{tA}\right\|_2 = e^t\left\|U^T e^{\Sigma} V\right\|_2$ where $e^{\sum} = \operatorname{diag}\left(e^{\lambda_1}, e^{\lambda_2}\cdots e^{\lambda_n}\right)$

the black curve is $\left\|e^{tA}\right\|_2$.the red curve is $e^{t\alpha(A)}$.



Code:

```
n=200
eta = c()
etalpha =c()
a = matrix(rnorm(100),10,10)-2*diag(1,10)
alpha = max(Re(eigen(a)$values))
for(i in 1:n){
  sum = diag(1,10)
   t = i/10
  for(i in 1:100){
    temp = diag(1,10)
    for(j in 1:i){
      temp = (a*t)%*%temp/j
    }
    sum=sum+temp
  }
  eta = cbind(eta,norm(sum,'2'))
  etalpha = cbind(etalpha,exp(t*alpha))
}
plot(1:n/10,log(eta),type='l',ylim=c(1,150))
par(new=TRUE)
```

```
plot(1:n/10,log(etalpha),type='l',col='red',ylim=c(1,150))
```

3、 Suppose the $m \times m$ real matrix A has an SVD$A = U\Sigma V^T$. Find an eigenvalue decomposition of the matrix $B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$

Proof:

We note

$$\begin{aligned}
\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & V\Sigma^* U^* \\ U\Sigma V^* & 0 \end{bmatrix} &= \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U\Sigma V^* & 0 \\ 0 & V\Sigma^* U^* \end{bmatrix} \\
&= \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}
\end{aligned}$$

The inverse of $\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix}$ is itself and the inverse of $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$ is $\begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}$. So the inverse of

$$\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix}$$

$$\text{Note } \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}
\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} &= \left(\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right) \left(\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix} \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix}\right) \left(\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)^{-1} \\
&= \left(\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right) \begin{bmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\right)^{-1}
\end{aligned}$$

Note Σ is a diagonal matrix with nonnegative diagonal elements, we must have $\Sigma = \Sigma^*$. It's easy to see

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix} = \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}, \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix} \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix} = 2I_{2m\times 2m}$$

So the formula

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix} \cdot \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix}$$

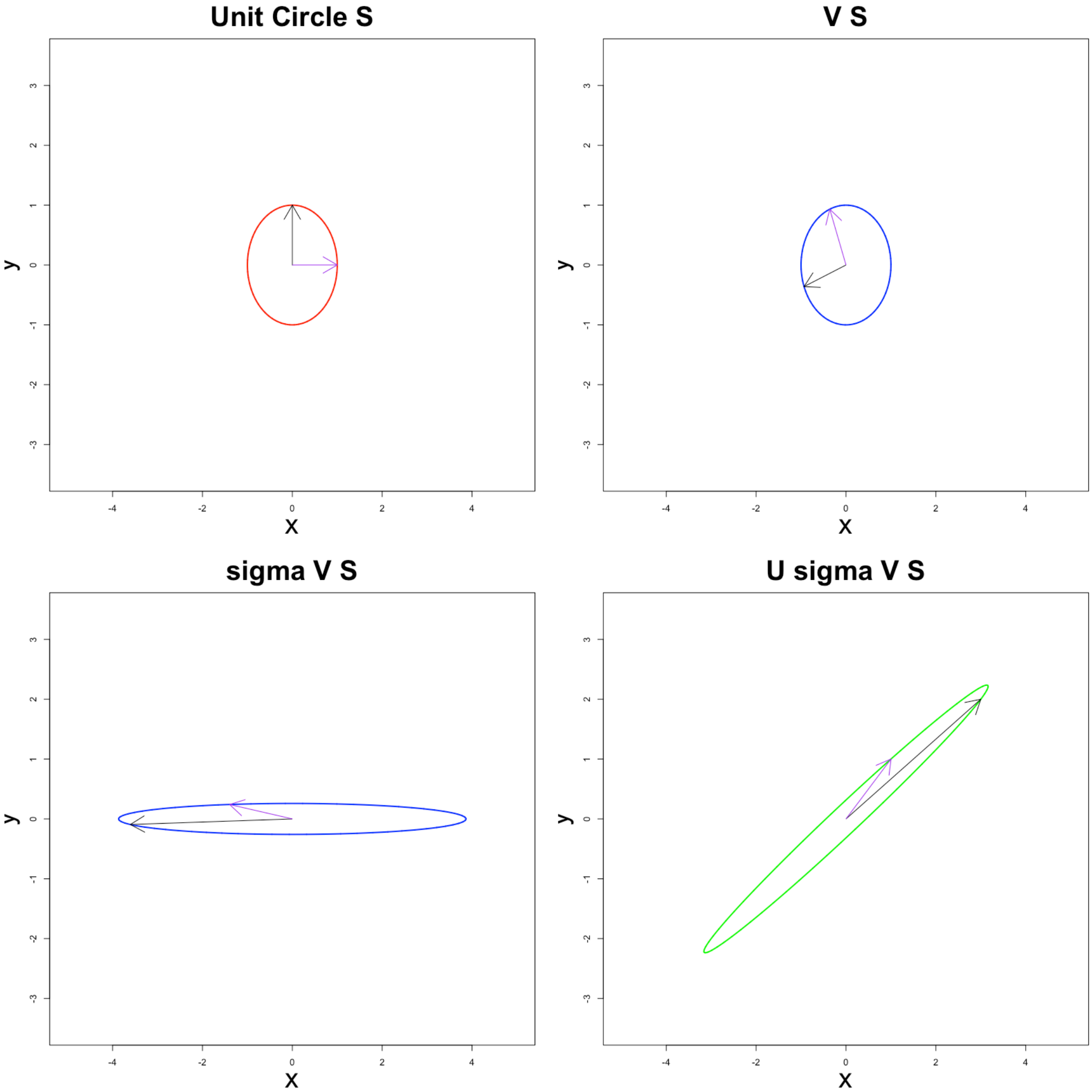
gives the eigenvalue decomposition of $\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}$. Combined, if we set

$$X = \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} I_{m\times m} & I_{m\times m} \\ I_{m\times m} & -I_{m\times m} \end{bmatrix}$$

then $X \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} X^{-1}$ gives the eigenvalue decomposition of $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$

4、Plot the unit circle S. Then take a matrix and plot the ellipse AS.

Modeled on ppt：

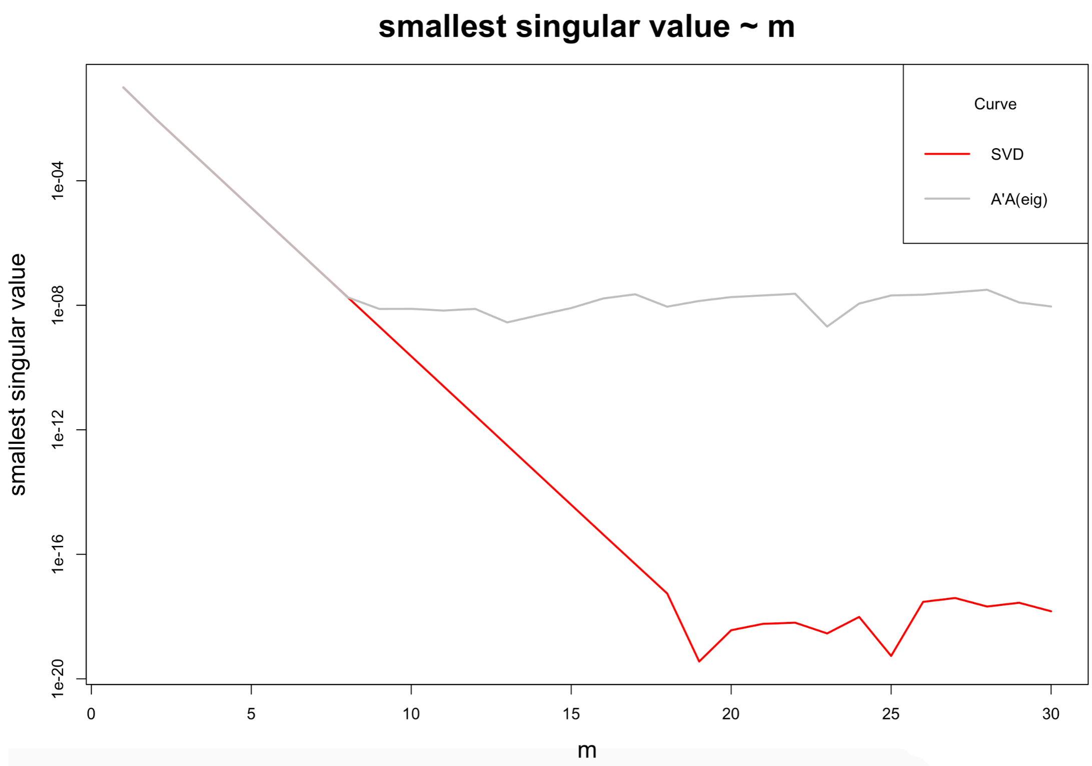


Code:

```
f= seq(from=0, to=2*pi, 0.001)
length_f = length(f)
x = sin(f)
y = cos(f)
A = matrix(c(1,1,3,2), ncol=2)
par(mfrow=c(2,2))
plot(x,y, type='l', col='red', lwd=2, main='Unit Circle S',cex.main=3,
xlab='x',ylab='y',cex.lab=3, xlim=c(-5,5), ylim=c(-3.5,3.5))
arrows(x0 = 0, y0 = 0, x1 = x[1], y1 = y[1], col='black', lwd=1)
arrows(x0 = 0, y0 = 0, x1 = x[1571], y1 = y[1571], col='purple', lwd=1)
A_svd = svd(A)
xy1 = t(A_svd$v) %*% rbind(t(x),t(y))
plot(xy1[1,], xy[2,], type='l', col='blue', lwd=2, main='V S',cex.main=3,
xlab='x', ylab='y',cex.lab=3, xlim=c(-5,5), ylim=c(-3.5,3.5))
arrows(x0 = 0, y0 = 0, x1 = xy[1,1], y1 = xy[2,1], col='black', lwd=1)
```

```
arrows(x0 = 0, y0 = 0, x1 = xy[1,1571], y1 =xy[2,1571], col='purple', lwd=1)
xy2 = diag(A_svd$d) %*% xy
plot(xy2[1,], xy[2,], type='l', col='blue', lwd=2, main='sigma V S',cex.main=3,
xlab='x', ylab='y',cex.lab=3, xlim=c(-5,5), ylim=c(-3.5,3.5))
arrows(x0 = 0, y0 = 0, x1 = xy[1,1], y1 = xy[2,1], col='black', lwd=1)
arrows(x0 = 0, y0 = 0, x1 = xy[1,1571], y1 = xy[2,1571], col='purple', lwd=1)
xy3 = A_svd$u %*% xy
plot(xy3[1,], xy[2,], type='l', col='green', lwd=2, main='U sigma V
S',cex.main=3, xlab='x', ylab='y',cex.lab=3, xlim=c(-5,5), ylim=c(-3.5,3.5))
arrows(x0 = 0, y0 = 0, x1 = xy[1,1], y1 = xy[2,1], col='black', lwd=1)
arrows(x0 = 0, y0 = 0, x1 = xy[1,1571], y1 = xy[2,1571], col='purple', lwd=1)
```

5、 Let A be the $m \times m$ upper-triangular matrix with 0.1 on the main diagonal and 1 everywhere above the diagonal. Write a program to compute the smallest singular value of A in two ways: by calling a standard svd function, and by forming $A^T A$ and computing the square root of its smallest eigenvalue. Run your program for $1 \leq m \leq 30$ and plot the results as two curves on a log scale.



Code:

```
smallest_singular_1 = list(30)
smallest_singular_2 = list(30)
for (i in 1:30){
  A = diag(rep(0.1, time=i))
  if(i>=2){
    for (j in 2:i){
      A[1:(j-1),j] = matrix(rep(1, time=j-1), ncol=1)
```

```r
    }
  }
  if(i==1){
    A = matrix(c(0.1),ncol=1)
  }

  A_SVD = svd(A)
  smallest_singular_1[i] = min(A_SVD$d)

  A_TA = t(A) %*% A
  tmp = min(abs(eigen(A_TA)$values))
  smallest_singular_2[i] = sqrt(tmp)
}
X = 1:30
plot(X, smallest_singular_1, type='l', col='red', lwd=2, log='y', xlab='m',
ylab='smallest singular value', cex.lab=1.5,main='smallest singular value ~ m',
cex.main=2)
lines(X, smallest_singular_2, type='l', col='gray', lwd=2, log='y')
legend('topright', title='Curve', c('SVD', "A'A(eig)"), col=c('red', 'gray'),
lty=1, lwd=2)
```