Financial Statistics
Homework 3

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1 3.2

For the GARCH(1,2) model

$$X_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2 + b_2 \sigma_{t-2}^2$$

(a) Express X_t^2 as an ARMA $(2,2)$ process.

(b) Express the GARCH(1,2) as an ARCH(∞) model. Here for simplicity, you may assume that $1 - b_1 B - b_2 B^2 = (1 - c_1 B)(1 - c_2 B)$ for two real constants $c_1 \neq c_2$.

(c) Derive the recursive formula for the multi-step ahead forecasts of volatility.

Solution:

(a)

Let $\eta_t = \sigma_t^2 (\varepsilon_t^2 - 1) = X_t^2 - \sigma_t^2$. Then, the $\{\eta_t\}$ is a sequence of martingale differences and hence is a white noise process. We have

$$\begin{aligned} X_t^2 =& \sigma_t^2 + \eta_t = a_0 + a_1 X_{t-1}^2 + b_1 (X_{t-1}^2 - \eta_{t-1}) + b_2 (X_{t-2}^2 - \eta_{t-2}) + \eta_t \\ =& a_0 + (a_1 + b_1) X_{t-1}^2 + b_2 X_{t-2}^2 + \eta_t - b_1 \eta_{t-1} - b_2 \eta_{t-2} \end{aligned}$$

which implies $\{X_t^2\} \sim \text{ARMA}(2,2)$

(b)

$$\begin{aligned} (1 - c_1 B)(1 - c_2 B)\sigma_t^2 &= \sigma_t^2 - b_1 \sigma_{t-1}^2 - b_2 \sigma_{t-2}^2 = a_0 + a_1 X_{t-1}^2 \\ \Rightarrow \sigma_t^2 &= \left(\sum_{i=0}^{\infty} c_1^j B^j \right) \left(\sum_{i=0}^{\infty} c_2^j B^j \right) \left(a_0 + a_1 X_{t-1}^2 \right) \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} c_1^j c_2^{i-j} \right) B^i \left(a_0 + a_1 X_{t-1}^2 \right) \end{aligned}$$

which implies $\{\sigma_t^2\} \sim \text{ARCH}(\infty)$

(c)

Beacuse of $X_t^2 \sim \text{ARMA}(2,2)$

$$\begin{aligned}
\operatorname{Var}_t\left(X_{t+k}\right) &= \mathrm{E}_t X_{t+k}^2 \\
&= a_0 + \left(a_1+b_1\right) \mathrm{E}_t X_{t+k-1}^2 + b_2 \mathrm{E}_t X_{t+k-2}^2
\end{aligned}$$

Let k be 1,2,k, we have the formula of one step ahead forecasts of volatility

$$\begin{aligned}
&\operatorname{Var}_t\left(X_{t+1}\right)=a_0+\left(a_1+b_1\right) X_t^2+b_2 X_{t-1}^2 \\
&\operatorname{Var}_t\left(X_{t+2}\right)=a_0+\left(a_1+b_1\right) \operatorname{Var}_t\left(X_{t+1}\right)+b_2 X_t^2 \\
&\operatorname{Var}_t\left(X_{t+k}\right)=a_0+\left(a_1+b_1\right) \operatorname{Var}_t\left(X_{t+k-1}\right)+b_2 \operatorname{Var}_t\left(X_{t+k-2}\right)
\end{aligned}$$

which can iteratively calculate multi-step case.

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2 3.4

Suppose that the volatilities of the daily log-returns of the Coco-Cola company follow the GARCH(1,1) model

$$X_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2$$

with $a_1 + b_1 < 1$ and $\varepsilon_t \sim N(0,1)$.

(a) If $a_0 = 0.006, a_1 = 0.05$ and $b_1 = 0.55$, is the tail of the distribution lighter than that of t_4 in terms of kurtosis?

(b) What is the auto-correlation function of the series $\{X_t^2\}$?

(c) If a_0, a_1 and b_1 are estimated as $0.006, 0.1$ and 0.4 respectively with associated covariance matrix

$$10^{-4}\begin{pmatrix} 15 & 5 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$$

what is the estimated long-run variance (unconditional variance)? What is the associated standard error?

(d) With the parameters in (a), if $X_T^2 = 0.02$ and $\sigma_T^2 = 0.03$, give the one-step and two-step forecast of the volatility.

Solution:

(a)

The kurtosis of X_t is

$$\kappa_x = \frac{\mathrm{E}X_t^4}{\left(\mathrm{E}X_t^2\right)^2} = \frac{3\left(1-(a_1+b_1)^2\right)}{1-2a_1^2-(a_1+b_1)^2} = 3.024$$

For student t distribution,kurtosis equal to $\frac{6}{\nu-2}$ when $\nu > 4$, equal to ∞ when $2 < \nu \leq 4$. So the tail of GARCH is lighter than t_4.

(b)

Let $\eta_t = \sigma_t^2\left(\varepsilon_t^2-1\right) = X_t^2 - \sigma_t^2$. Then, the $\{\eta_t\}$ is a sequence of martingale differences and hence is a white noise process. We have

$$\begin{aligned} X_t^2 =& \sigma_t^2 + \eta_t = a_0 + a_1 X_{t-1}^2 + b_1(X_{t-1}^2 - \eta_{t-1}) + \eta_t \\ =& a_0 + (a_1+b_1)X_{t-1}^2 + \eta_t - b_1\eta_{t-1} \end{aligned}$$

which implies $\{X_t^2\} \sim \mathrm{ARMA}(1,1)$.Then we can easlily get the ACF

$$\rho_X(h) = \begin{cases} 1 \text{ if } h=0 \\ \frac{(\phi+\theta)(1+\phi\theta)}{1+2\phi\theta+\theta^2} \text{ if } |h|=1 \\ \phi\rho_X(h-1) \text{ if } |h|>1. \end{cases}$$

where $\phi = a_1 + b_1$ and $\theta = -b_1$.

(c)

Firstly, we get the estimated long-run variance

$$\sigma_t^2 = \frac{a_0}{1-a_1-b_1} = 0.012$$

Using the Delta method, we have

$$\mathrm{se}(\hat{\sigma_t^2}) \approx \sqrt{\left[f(\boldsymbol{\theta})'\right]^T \boldsymbol{S} f(\boldsymbol{\theta})'} = 0.078$$

where $f(\hat{\boldsymbol{\theta}})' = \left(\frac{1}{1-a_1-b_1}, \frac{a_0}{(1-a_1-b_1)^2}, \frac{a_0}{(1-a_1-b_1)^2}\right)^T$

(d)

$$\begin{aligned} \mathrm{Var}_T\left(X_{T+1}\right) &= a_0 + a_1 X_T^2 + b_1\sigma_T^2 = 0.0235 \\ \mathrm{Var}_T\left(X_{T+2}\right) &= a_0 + (a_1+b_1)\,\mathrm{Var}_T\left(X_{T+1}\right) = 0.0201 \end{aligned}$$

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3 5.4

Assets 1 and 2 have the following covariance matrix:

$$\Sigma = \begin{pmatrix} 0.01 & -0.02 \\ -0.02 & 0.05 \end{pmatrix}$$

(a) What is the covariance between portfolio A, which has 10% in asset 1 and 90% in asset 2, and portfolio B, which has 60% in asset 1 and 40% in asset 2?

(b) Suppose that the excess returns of Assets 1 and 2 are 7% and 10%, respectively, what are the Sharpe ratios of portfolios A and B ?

Solution:

(a)

$$\begin{aligned}
\operatorname{Cov}(p_A, p_B) &= \operatorname{Cov}(0.1a_1 + 0.9a_2, 0.6a_1 + 0.4a_2) \\
&= 0.06 \times 0.01 + (0.54 + 0.04) \times (-0.02) + 0.36 \times 0.05 \\
&= 0.007
\end{aligned}$$

(b)

$$\begin{aligned}
\operatorname{Var}(p_A) &= 0.01 \operatorname{Var}(a_1) + 0.81 \operatorname{Var}(a_2) + 0.18 \operatorname{Cov}(a_1, a_2) = 0.037 \\
S_A &= \frac{7\% \times 0.1 + 10\% \times 0.9}{\sqrt{0.037}} = 0.504 \\
\operatorname{Var}(p_B) &= 0.36 \operatorname{Var}(a_1) + 0.16 \operatorname{Var}(a_2) + 0.48 \operatorname{Cov}(a_1, a_2) = 0.002 \\
S_B &= \frac{7\% \times 0.6 + 10\% \times 0.4}{\sqrt{0.002}} = 1.834
\end{aligned}$$

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4 5.10

Let Y be the excess returns of risky assets. Let $X = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{Y}$ be a portfolio with allocation vector a. Denote by $\boldsymbol{\Sigma} = \operatorname{var}(\boldsymbol{Y})$ and $\boldsymbol{\mu} = E\boldsymbol{Y}$. Consider the following decomposition (regression)

$$\boldsymbol{Y} = \boldsymbol{\alpha} + \boldsymbol{\beta}X + \varepsilon, \quad E\varepsilon = 0, \quad \operatorname{cov}(\varepsilon, X) = 0$$

(a) Show that if $a = c\Sigma^{-1}\mu$ (optimal portfolio in the mean-variance efficiency), then $\alpha = 0$

(b) Conversely, if $\alpha = 0$, there exists a constant c such that $a = c\Sigma^{-1}\mu_0$.

Solution:

(a)

We have known that the market beta's given by

$$\beta = \frac{\mathrm{cov}(Y, X)}{\mathrm{var}(X)} = \frac{\Sigma a}{a^T \Sigma a}$$

According to $a = c\Sigma^{-1}\mu$,we have $\mu = \frac{1}{c}\Sigma a$. Now from the decomposition,we have

$$\boldsymbol{\alpha} = \mathrm{E}Y - \beta \mathrm{E}X = \mu - \beta a^T \mu = \frac{1}{c}\Sigma a - \frac{\Sigma a}{a^T \Sigma a} a^T \frac{1}{c}\Sigma a = \frac{1}{c}\Sigma a - \frac{1}{c}\Sigma a = 0$$

(b)

Note that $\alpha = 0$, then

$$0 = \boldsymbol{\alpha} = \mathrm{E}Y - \beta \mathrm{E}X = \mu - \beta a^T \mu = \mu - \frac{\Sigma a}{a^T \Sigma a} a^T \mu$$

To keep the equation hold ,we need $\mu \propto \Sigma a$. Obviously, there exit a constant c subject to $\mu = \frac{1}{c}\Sigma a$ which leads to $a = c\Sigma^{-1}\mu$.

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