Solution of Homework 3

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Problem 1

The likelihood function is

$$L(\mu,\sigma|\mathbf{x})=\prod_{i=1}^{n}f(x_i;\mu,\sigma)=\frac{1}{\sigma^n}exp\{-\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma}\}$$

The log-likelihood function is

$$l(\mu,\sigma|\mathbf{x})=-nlog\sigma-\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma}$$

(Hint: translation of exponential distribution)

$$\begin{aligned}EX&=\int_{\mu}^{\infty}\frac{x}{\sigma}e^{-\frac{x-\mu}{\sigma}}dx\\&=\int_{0}^{\infty}\frac{x}{\sigma}e^{-\frac{x}{\sigma}}dx+\mu\\&=\sigma+\mu\end{aligned}$$

$$EX^2=Var(X)+(EX)^2=\sigma^2+(\sigma+\mu)^2$$

(1)MME:

$$\hat{\sigma}_{MME}=\bar{X}-\mu$$

MLE:

$$\frac{\partial l(\mu,\sigma|\mathbf{x})}{\partial\sigma}=-\frac{n}{\sigma}+\frac{n(\bar{X}-\mu)}{\sigma^2}=0\Rightarrow\hat{\sigma}_{MLE}=\bar{X}-\mu$$

(2)MME:

$$\hat{\mu}_{MME}=\bar{X}-\sigma$$

MLE: Because as μ increasing, the value of likelihood function increases. So $\hat{\mu}_{MLE}=X_{(1)}$.

(3)MME:

$$\begin{cases}\sigma+\mu=\bar{X}\\\sigma^2+(\sigma+\mu)^2=\bar{X^2}\end{cases}\Rightarrow\begin{cases}\hat{\sigma}_{MME}=\sqrt{\bar{X^2}-\bar{X}^2}\\\hat{\mu}_{MME}=\bar{X}-\sqrt{\bar{X^2}-\bar{X}^2}\end{cases}$$

MLE:

$$\begin{cases}\frac{\partial l(\mu,\sigma|\mathbf{x})}{\partial\sigma}=0\\\hat{\mu}_{MLE}=X_{(1)}\end{cases}\Rightarrow\begin{cases}\hat{\sigma}_{MLE}=\bar{X}-X_{(1)}\\\hat{\mu}_{MLE}=X_{(1)}\end{cases}$$

$$P(X_1\geq t)=\int_{t}^{\infty}\frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}}dx=e^{-\frac{t-\mu}{\sigma}}$$

Put the moment estimators and MLEs of μ and σ in it.

Problem 2

(1) The likelihood function is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{(\theta/2)^n} I_{\frac{2}{\theta} \le x_i \le \theta} = \frac{2^n}{\theta^n} I_{x_{(n)} \le \theta \le 2x_{(1)}},$$

hence,

$$\hat{\theta} = X_{(n)}.$$

(2) The MLE is biased since

$$E\hat{\theta} = \frac{2n+1}{2n+2}\theta \ne \theta.$$

An unbiased estimate based on the MLE is

$$\hat{\theta}^* = \frac{2n+2}{2n+1}\hat{\theta} = \frac{2n+2}{2n+1}X_{(n)}.$$

(3) MLE is weakly consistent because applying Markov's inequality, we have

$$\begin{aligned} P(|\hat{\theta} - \theta| > \epsilon) &\le \frac{E|\hat{\theta} - \theta|}{\epsilon} \\ &= \frac{\theta - E\hat{\theta}}{\epsilon} \\ &= \frac{\theta}{(2n+2)\epsilon} \to 0 \end{aligned}$$

Problem 3

The likelihood function is

$$\begin{aligned} L(\theta|\mathbf{x}) &= \prod_{i=1}^{n} f(x_i;\theta) = \theta^n (1-\theta)^{\sum_{i=1}^n x_i - n} \\ &= \left(\frac{\theta}{1-\theta}\right)^n \exp[\log(1-\theta) \cdot \sum_{i=1}^{n} x_i] \end{aligned}$$

from the properties in an exponential family,$T(X) = \sum_{i=1}^n x_i$ is sufficient and complete statistic for θ.

$$E(T(X)) = E(x_1) = \frac{n}{\theta}$$

Then $E[\frac{1}{n}\sum_{i=1}^n x_i] = 1/\theta$ Then the UMVUE of $\frac{1}{\theta}$ is $\frac{1}{n}\sum_{i=1}^n x_i$

Let $\psi(x_1) = I_{x_1=1}$, and $E[\psi(x_1)] = \theta$

$$\begin{aligned} E[\psi(x_1)|\sum_{i=1}^{n} x_i = t] &= \frac{\theta\binom{t-2}{n-2}\theta^{n-1}(1-\theta)^{t-n+1}}{\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}} \\ &= \frac{n-1}{t-1} \end{aligned}$$

Then the UMVUE of θ is $\hat{\theta} = \frac{n-1}{\sum_{i=1}^n x_i - 1}$

Problem 4

Denote $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$, then $(\bar{X}, S^2)$ is sufficient and complete. Note that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, i.e., $\Gamma(\frac{n-1}{2}, 2)$.

$$E(\bar{X} + S^2) = \mu + \sigma^2.$$

$$E[\frac{\bar{X}^2}{(n-1)S^2}] = E(\bar{X}^2)E[\frac{1}{(n-1)S^2}] = (\frac{\sigma^2}{n} + \mu^2)(\frac{1}{(n-3)\sigma^2}) = \frac{1}{n(n-3)} + \frac{1}{(n-3)}\frac{\mu^2}{\sigma^2}.$$

By L-S theorem,

$$\bar{X} + S^2$$

and

$$(n-3)(\frac{\bar{X}^2}{(n-1)S^2} - \frac{1}{n(n-3)}) = \frac{(n-3)\bar{X}^2}{(n-1)S^2} - \frac{1}{n}$$

are the UMVUE of $\mu + \sigma^2$ and $\mu^2/(\sigma^2)$.

Problem 5

$$\hat{\sigma} = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}\left(\sum_{i=1}^n X_i^2\right)^{1/2}$$

$\sum_{i=1}^n (X_i - 1)^2$ is sufficient and complete according to the property of exponential family. Note that $Y \overset{\triangle}{=} \sum_{i=1}^n (X_i-1)^2/\sigma^2 \sim \chi^2(n)$,i.e.,$\Gamma(\frac{n}{2}, 2)$. Then $\hat{\sigma} = c(n)\sigma Y^{\frac{1}{2}}$, where $c(n) = \frac{\Gamma(\frac{n}{2})}{\sqrt{2}\Gamma(\frac{n+1}{2})}$.

$$E(Y^{\frac{1}{2}}) = \frac{\sqrt{2}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \frac{1}{c(n)} \quad \Rightarrow \quad E(\hat{\sigma}) = \sigma.$$

let

$$\hat{\sigma} = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}\left(\sum_{i=1}^n (X_i - 1)^2\right)^{1/2}$$

$$E(\hat{\sigma}) = \sigma.$$

By L-S theorem, $\hat{\sigma}$ is the UMVUE of σ.

Problem 6

Denote$\mathbf{X} = (x_1, ..., x_m)$,$\mathbf{Y} = (y_1, ..., y_n)$ Note that

$$\begin{aligned} f_{\mu,\sigma}(\mathbf{X}, \mathbf{Y}) &= (\frac{1}{\sqrt{2\pi}\sigma})^m(\frac{1}{\sqrt{2\pi}\sigma})^n exp(-\frac{\sum_{i=1}^m (x_i-\mu)^2}{2\sigma^2} - \frac{\sum_{j=1}^n (y_i - 2\mu)^2}{2\sigma^2}) \\ &= (\frac{1}{\sqrt{2\pi}\sigma})^m(\frac{1}{\sqrt{2\pi}\sigma})^n exp(-\frac{1}{2\sigma^2}(\sum_{i=1}^m x_i^2 + \sum_{j=1}^n y_j^2) + \frac{\mu}{\sigma^2}(\sum_{i=1}^m x_i + 2\sum_{j=1}^n y_j) - \frac{(m+4n)\mu^2}{2\sigma^2}) \end{aligned}$$

from the properties in an exponential family,$T(\mathbf{X},\mathbf{Y}) = (T_1(\mathbf{X},\mathbf{Y}), T_2(\mathbf{X},\mathbf{Y})) = (\sum X_i + 2\sum Y_j, \sum X_i^2 + \sum Y_j^2)$ is sufficient and complete statistic.

$$
\begin{aligned}
ET_1 &= E[\sum_{i=1}^{m} X_i + 2\sum_{j=1}^{n} Y_j] = (m+2n)\mu \quad (T_1 \sim N((m+4n)\mu, (m+4n)\sigma^2)) \\
ET_1^2 &= (m+4n)^2\mu^2 + (m+4n)\sigma^2 \\
ET_2 &= E[\sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{n} Y_j^2] = (m+4n)\mu^2 + (m+n)\sigma^2
\end{aligned}
$$

then

$$E(\frac{T_1}{m+2n}) = \mu$$

$$E[\frac{T_2 - \frac{1}{m+4n}{T_1}^2}{(m+n-1)}] = \sigma^2$$

By L-S theorem,$(\frac{T_1}{m+2n}, \frac{T_2 - \frac{1}{m+4n}{T_1}^2}{(m+n-1)})$ are UMVUE of (μ, σ^2)

Problem 7

(1) MME is

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n x_i^2}{n}$$

MLE is the same one.

(2)

$$I(\theta) = E[\frac{\partial \log f(X;\theta)}{\partial\theta}]^2 = E[-\frac{\partial^2 \log f(X;\theta)}{\partial\theta^2}]$$

$$
\begin{aligned}
I(\sigma^2) &= E[-\frac{1}{2\sigma^2} + \frac{x^2}{2\sigma^4}]^2 \\
&= \frac{1}{4\sigma^4} + \frac{1}{4\sigma^8}E[x^4] - \frac{1}{2\sigma^6}E[x^2] \\
&= \frac{1}{4\sigma^4} + \frac{3\sigma^4}{4\sigma^8} - \frac{\sigma^2}{2\sigma^6} \\
&= \frac{1}{2\sigma^4}
\end{aligned}
$$

hence C-R lower bound is $\frac{1}{nI(\sigma)} = \frac{2\sigma^4}{n}$.

(3) $\sum_{i=1}^n X_i^2$ is sufficient and complete according to the property of exponential family.

$$E(\frac{\sum_{i=1}^n X_i^2}{n}) = \sigma^2$$

By L-S theorem,$\frac{\sum_{i=1}^n X_i^2}{n}$ are UMVUE of σ^2.

Problem 8

(1)The joint p.d.f of (X_i, Y_i) is

$$f(x_i, y_i) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}\sigma_1\sigma_2} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_i-\mu_1}{\sigma_1}\right)\left(\frac{y_i-\mu_2}{\sigma_2}\right) + \left(\frac{y_i-\mu_2}{\sigma_2}\right)^2\right]}$$

$$f(\mathbf{x}) = \frac{1}{2\pi|\mathbf{\Sigma}|^{1/2}} e^{-(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})/2}$$

The likelihoodo function is

$$\begin{aligned} L\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \boldsymbol{\mu}, \mathbf{\Sigma}\right) &= \prod_{i=1}^{n} f\left(\mathbf{x}_i, \boldsymbol{\mu}, \mathbf{\Sigma}\right) \\ &= \prod_{i=1}^{n} \frac{1}{2\pi|\mathbf{\Sigma}|^{1/2}} e^{-(\mathbf{x}_i-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})/2} \\ &= \frac{1}{(2\pi)^n|\mathbf{\Sigma}|^{n/2}} e^{-\sum_{i=1}^{n}(\mathbf{x}_i-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})/2} \end{aligned}$$

For the multivariate normal distribution, the maximum likelihood estimations of μ and $\mathbf{\Sigma}$ is

$$\hat{\mu} = \overline{\mathbf{x}}, \quad \hat{\mathbf{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i - \overline{\mathbf{x}}\right)\left(\mathbf{x}_i - \overline{\mathbf{x}}\right)'$$

So

$$\begin{cases} \hat{\mu_1} = \bar{x} \\ \hat{\mu_2} = \bar{y} \\ \hat{\sigma_1} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\ \hat{\sigma_2} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \\ \hat{\rho} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}} \end{cases}$$

(2)

$$\begin{cases} \hat{\mu_1} = 67.86305 \\ \hat{\mu_2} = 126.8677 \\ \hat{\sigma_1} = 3.739692 \\ \hat{\sigma_2} = 142.0589 \\ \hat{\rho} = 0.4483118 \end{cases}$$

Problem *1

The likelihood function is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{(2\sigma)^n}\exp\{-\frac{1}{\sigma}\sum_{i=1}^{n}|x_i - a|\}.$$

The log-likelihood function is

$$l(\theta|\mathbf{x}) = -n\log(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^{n}|x_i - a|.$$

To maximize $l(\theta|\mathbf{x})$,

$$\hat{a} = \arg\min_{a\in\mathbb{R}}\sum_{i=1}^{n}|x_i - a| = median(\mathbf{x}),$$

then

$$\frac{\partial l(\theta|\mathbf{x})}{\partial\sigma}|_{\sigma=\hat{\sigma},a=\hat{a}} = -\frac{n}{\hat{\sigma}} + \frac{\sum_{i=1}^{n}|x_i - \hat{a}|}{\hat{\sigma}^2} = 0 \quad \Rightarrow \quad \hat{\sigma} = \frac{\sum_{i=1}^{n}|x_i - \hat{a}|}{n}.$$

Therefore, the MLE of a and σ are

$$\hat{a} = median(\mathbf{X}),$$
$$\hat{\sigma} = \frac{\sum_{i=1}^{n}|X_i - median(\mathbf{X})|}{n}.$$

Problem *2

The likelihood function is

$$L(\mu,\sigma|\mathbf{x}) = \prod_{i=1}^{n} f(x_i;\mu,\sigma) = \alpha^n \beta^n (\prod_{i=1}^{n} x_i)^{\beta-1} e^{-n\alpha \sum_{i=1}^{n} x_i^{\beta}}$$

The log-likelihood function is

$$l(\mu,\sigma|\mathbf{x}) = n(log\alpha + \log\beta) + (\beta-1)\cdot\sum_{i=1}^{n}\log x_i - n\alpha\sum_{i=1}^{n} x_i^{\beta}$$

$$\frac{\partial\alpha}{\partial x} = \frac{n}{\alpha} - \sum x_i^{\beta}$$

since β is known, MLE for α is $\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} x_i^{\beta}}$