Financial Statistics
Homework 6

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1 8.2

What is the consumption based CAPM? What is the expected value of the stochastic discount factor?

Solution:

Consumption-based CAPM is

$$\boldsymbol{S}_t = E_t\left[\frac{p_t}{p_{t+1}}\delta\frac{U'\left(C_{t+1}\right)}{U'\left(C_t\right)}\boldsymbol{S}_{t+1}\right]$$

Sopposed that there exists a risk-free asset, the expected value of the stochastic discount factor is

$$E_t M_{t+1} = \left(1 + r_{\mathrm{f},t+1}\right)^{-1}$$

2 8.4

What are the two key assumptions in deriving the Hansen-Singleton formula from the consumption based CAPM? According to the formula, what are the risk-free interest and the expected excess return?

Solution:

Key assumptions:

1. Power utility: $U(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$
2. Normal returns: $Y_{i,t+1}$ is normally distributed

Risk-free interest:

$$r_{\mathrm{f},t+1} = -\log\delta + \gamma E_t \Delta C_{t+1} - \frac{\gamma^2}{2}\operatorname{var}_t\left(\Delta C_{t+1}\right)$$

Expected excess return:

$$E_t\left(R_{i,t+1} - r_{\mathrm{f},t+1}\right) = -\frac{1}{2}\operatorname{var}_t\left(R_{i,t+1}\right) + \gamma \operatorname{cov}_t\left(R_{i,t+1}, \Delta C_{t+1}\right)$$

3 9.3

Assume that the mean-adjusted return follows the AR(1) model:

$$r^*_{t+1} = \gamma r^*_t + \eta_{t+1},$$

where η_{t+1} is a white noise series with mean zero and variance σ^2. Let $P^*_{Dt} = \sum_{j=0}^{\infty} \rho^j r^*_{t+1+j}$

(a) Show that $P^*_{Dt} = (1-\gamma\rho)^{-1}\left(\gamma r^*_t + \sum_{j=0}^{\infty} \rho^j \eta_{j+1+j}\right)$.

(b) Deduce from (a) that $E_t P^*_{Dt} = \gamma r^*_t/(1-\gamma\rho)$.

(c) Deduce from (a) that $\operatorname{var}_t\left(P^*_{Dt}\right) = \frac{\sigma^2}{(1-\gamma\rho)^2(1-\rho^2)}$.

Solutions:

(a)

Iterative expansion gives

$$r^*_{t+1+k} = \gamma^{1+k} r^*_t + \sum_{j=0}^{k} \gamma^{k-j}\eta_{t+1+j}$$

Therefore,

$$\begin{aligned}
P^*_{Dt} &= \sum_{k=0}^{\infty} \rho^k r^*_{t+1+k} = \sum_{k=0}^{\infty} \rho^k \left(\gamma^{1+k} r^*_t + \sum_{j=0}^{k} \gamma^{k-j}\eta_{t+1+j}\right) \\
&= \sum_{k=0}^{\infty} \rho^k \gamma^{1+k} r^*_t + \sum_{k=0}^{\infty} \rho^k \sum_{j=0}^{k} \gamma^{k-j}\eta_{t+1+j} = \gamma r^*_t \sum_{k=0}^{\infty} (\gamma\rho)^k + \sum_{j=0}^{\infty} \rho^j \eta_{t+1+j} \sum_{k=j}^{\infty} (\gamma\rho)^{k-j} \\
&= \gamma r^*_t \sum_{k=0}^{\infty} (\gamma\rho)^k + \sum_{j=0}^{\infty} \rho^j \eta_{t+1+j} \sum_{l=0}^{\infty} (\gamma\rho)^l \\
&= \left(\sum_{k=0}^{\infty} (\gamma\rho)^k\right)\left(\gamma r^*_t + \sum_{j=0}^{\infty} \rho^j \eta_{t+1+j}\right) \\
&= (1-\gamma\rho)^{-1}\left(\gamma r^*_t + \sum_{j=0}^{\infty} \rho^j \eta_{t+1+j}\right)
\end{aligned}$$

(b)

Since $\{\eta_t\}$ is white noise with mean zero, then,

$$\begin{aligned}
E_t P^*_{Dt} &= (1-\gamma\rho)^{-1}\left(\gamma E_t r^*_t + \sum_{j=0}^{\infty}\rho^j E_t \eta_{t+1+j}\right) \\
&= (1-\gamma\rho)^{-1}\left(\gamma r^*_t + \sum_{j=0}^{\infty}\rho^j 0\right) \\
&= \frac{\gamma r^*_t}{1-\gamma\rho}
\end{aligned}$$

(c)

Since $\{\eta_t\}$ is white noise with variance σ^2, then,

$$\begin{aligned}
\operatorname{var}_t\left(P^*_{Dt}\right) &= (1-\gamma\rho)^{-2}\left(\gamma^2 \operatorname{var}_t\left(r^*_t\right) + \sum_{j=0}^{\infty}\rho^{2j}\operatorname{var}_t\left(\eta_{t+1+j}\right)\right) \\
&= (1-\gamma\rho)^{-2}\left(\gamma^2 0 + \sum_{j=0}^{\infty}\rho^{2j}\sigma^2\right) \\
&= \frac{\sigma^2}{(1-\gamma\rho)^2\left(1-\rho^2\right)}
\end{aligned}$$

4 9.4

Suppose that the log-dividend growth $\Delta d_{t+1} = d_{t+1} - d_t$ follows the $AR(1)$ model:

$$\Delta d_{t+1} = (1-\theta)d + \theta\Delta d_t + \varepsilon_{t+1}$$

where ε_t is a white noise series with mean zero. Assume further that the return follows the AR(1) model:

$$r_{t+1} = (1-\gamma)r + \gamma r_t + \eta_{t+1}$$

where η_{t+1} is a white noise series with mean zero.

(a) What is $E_t d_{t+j}$?

(b) What is the expected discounted log-dividend $P_{Dt} = \sum_{j=0}^{\infty}\rho^j E_t d_{t+1+j}$? Hint: An easier way is to show that

$$P_{Dt} = \sum_{j=0}^{\infty}\rho^j E_t \Delta d_{t+j+1} + d_t + \rho P_{Dt}$$

with and then compute $E_t\Delta d_{t+j+1}$ or use Exercise 9.3.

(c) What is the present value of the stock?

Solutions:

(a)

Note that

$$E_t \Delta d_{t+k} = (1-\theta) d \sum_{j=0}^{k-1} \theta^j + \theta^k \Delta d_t = \left(1-\theta^k\right) d + \theta^k \Delta d_t$$

Therefore,

$$\begin{aligned} E_t d_{t+j} &= E_t d_t + \sum_{k=1}^{j} E_t \Delta d_{t+k} \\ &= d_t + \sum_{k=1}^{j} \left(\left(1-\theta^k\right) d + \theta^k \Delta d_t \right) \\ &= d_t + jd + \frac{\theta\left(1-\theta^j\right)}{1-\theta} \left(\Delta d_t - d\right) \end{aligned}$$

(b)

Note that

$$\begin{aligned} P_{dt} &= \sum_{j=0}^{\infty} \rho^j E_t d_{t+1+j} = \sum_{j=0}^{\infty} \rho^j E_t \left(d_{t+j} + \Delta d_{t+1+j}\right) \\ &= d_t + \sum_{j=1}^{\infty} \rho^j E_t d_{t+j} + \sum_{j=0}^{\infty} \rho^j E_t \Delta d_{t+1+j} = d_t + \rho \sum_{j=0}^{\infty} \rho^j E_t d_{t+1+j} + \sum_{j=0}^{\infty} \rho^j E_t \Delta d_{t+1+j} \\ &= d_t + \rho P_{Dt} + \sum_{j=0}^{\infty} \rho^j E_t \Delta d_{t+1+j} \end{aligned}$$

Thus,

$$\begin{aligned} P_{dt} &= \frac{1}{1-\rho} \left(d_t + \sum_{j=0}^{\infty} \rho^j E_t \Delta d_{t+1+j} \right) \\ &= \frac{1}{1-\rho} \left(d_t + \sum_{j=0}^{\infty} \rho^j \left(\left(1-\theta^{1+j}\right) d + \theta^{1+j} \Delta d_t \right) \right) \\ &= \frac{1}{1-\rho} \left(d_t + d \sum_{j=0}^{\infty} \rho^j + \theta \left(\Delta d_t - d\right) \sum_{j=0}^{\infty} \rho^j \theta^j \right) \\ &= \frac{1}{1-\rho} \left(d_t + \frac{d}{1-\rho} + \frac{\theta\left(\Delta d_t - d\right)}{1-\rho\theta} \right) \end{aligned}$$

(c)

Let $r_t^* = r_t - r$. Then, $\{r_t^*\}$ is still an AR(1) process

$$r_{t+1}^* = \gamma r_t^* + \eta_{t+1}$$

Using the result in $9.3(b)$,

$$\begin{aligned} P_{rt} &= \sum_{j=0}^{\infty} \rho^j E_t r_{t+1+j} = \sum_{j=0}^{\infty} \rho^j r + \sum_{j=0}^{\infty} \rho^j E_t r_{t+1+j}^* \\ &= \frac{r}{1-\rho} + \frac{\gamma r_t^*}{1-\rho\gamma} = \frac{\gamma r_t}{1-\rho\gamma} + \frac{(1-\gamma) r}{(1-\rho)(1-\rho\gamma)} \end{aligned}$$

Thus, the present value of the stock is

$$\begin{aligned}
s_t &= \frac{\kappa}{1-\rho} + (1-\rho)P_{dt} - P_{rt} \\
&= -\log(1-\rho) + d_t + \frac{\theta \Delta d_t}{1-\rho\theta} + \frac{(1-\theta)d}{(1-\rho)(1-\rho\theta)} - \frac{\gamma r_t}{1-\rho\gamma} - \frac{(1-\gamma)r}{(1-\rho)(1-\rho\gamma)}
\end{aligned}$$