SUPPLEMENTARY DOCUMENT

In this supplementary document, we revisit Section 3 of the paper "An Evaluation Framework for Personalization Strategy Experiment Design" by Liu and McCoy (2020), and expand on the intermediate algebraic work when deriving:

(i) The actual & minimum detectable effect sizes, and
(ii) The conditions that lead to a setup being superior.

We will use the equation numbers featured in the original paper, and append letters for intermediate steps.

A EFFECT SIZE OF EXPERIMENT SETUPS

We begin with the actual effect size and the MDE of the four experiment setups that are featured in Section 3.1 of the paper. For each setup we first compute the sample size, mean response, and response variance in each analysis group (denoted n_g, μ_g, and σ_g^2 respectively for each analysis group g). These quantities arise as a mixture of user groups described in Section 2.1 of the paper. We then substitute the quantities computed into the definitions of Δ and θ^*:

$$\Delta \triangleq \mu_B - \mu_A, \qquad \text{(see (2))}$$

$$\theta^* \triangleq (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sigma_{\bar{D}}, \quad \text{where } \sigma_{\bar{D}} = \sigma_A^2/n_A + \sigma_B^2/n_B \qquad \text{(see (5))}$$

and z_q represents the q^{th} quantile of a standard normal, to obtain the setup-specific actual effect size and MDE for a setup with two analysis groups. For setups with more than two analysis groups, we will specify the actual and minimum detectable effect when we discuss specifics for each of the setups. We assume all random splits are done 50/50 in these setups to maximize the test power.

Setup 1 (Users in the intersection only). We recall the setup, which considers only users who qualify for both personalization strategies (i.e. the intersection), randomly splits user group 3 into two analysis groups, A and B, each with the following number of samples:

$$n_A = n_B = \frac{n_3}{2}.$$

Users in analysis group A are provided treatment under strategy 1, and users in analysis group B are provided treatment under strategy 2. This leads to the groups' responses having the following metric mean and variance:

$$\mu_A = \mu_{I\phi},\ \mu_B = \mu_{I\psi}, \quad \sigma_A^2 = \sigma_{I\phi}^2,\ \sigma_B^2 = \sigma_{I\psi}^2.$$

The actual effect size and MDE for Setup 1 are hence:

$$\Delta_{S1} = \mu_{I\psi} - \mu_{I\phi}, \tag{7}$$

$$\theta_{S1}^* = (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\frac{\sigma_{I\phi}^2}{n_3/2} + \frac{\sigma_{I\psi}^2}{n_3/2}}. \tag{8}$$

Setup 2 (All samples). The setup, which considers all users regardless of whether they qualify for any strategy or not, also contains two analysis groups, A and B, each taking half of the population:

$$n_A = n_B = \frac{n_0 + n_1 + n_2 + n_3}{2}.$$

The mean response and response variance for groups A and B are the weighted mean response and response variance of the constituent user groups respectively, weighted by the constituent groups' size. As we only provide treatment to those who qualify for strategy 1 in group A, and those who qualify for strategy 2 in group B, this leads to different responses in different constituent user groups:

$$\mu_A = \frac{n_0\mu_{C0} + n_1\mu_{I1} + n_2\mu_{C2} + n_3\mu_{I\phi}}{n_0 + n_1 + n_2 + n_3}, \qquad \mu_B = \frac{n_0\mu_{C0} + n_1\mu_{C1} + n_2\mu_{I2} + n_3\mu_{I\psi}}{n_0 + n_1 + n_2 + n_3}; \tag{9}$$

$$\sigma_A^2 = \frac{n_0\sigma_{C0}^2 + n_1\sigma_{I1}^2 + n_2\sigma_{C2}^2 + n_3\sigma_{I\phi}^2}{n_0 + n_1 + n_2 + n_3}, \qquad \sigma_B^2 = \frac{n_0\sigma_{C0}^2 + n_1\sigma_{C1}^2 + n_2\sigma_{I2}^2 + n_3\sigma_{I\psi}^2}{n_0 + n_1 + n_2 + n_3}. \tag{10}$$

Substituting the above into the definitions of actual effect size and MDE and simplifying the resultant expressions we have:

$$\Delta_{S2} = \frac{n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I_\phi})}{n_0 + n_1 + n_2 + n_3}, \tag{11}$$

$$\theta_{S2}^* = (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\frac{2\big(n_0(2\sigma_{C0}^2) + n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)\big)}{(n_0 + n_1 + n_2 + n_3)^2}}. \tag{12}$$

Setup 3 (Qualified users only). The setup is very similar to Setup 2, with members from user group 0 excluded:

$$n_A = n_B = \frac{n_1 + n_2 + n_3}{2}.$$

The absence of members from user group 0 means they are not featured in the weighted mean response and response variance of the two analysis groups:

$$\mu_A = \frac{n_1\mu_{I1} + n_2\mu_{C2} + n_3\mu_{I\phi}}{n_1 + n_2 + n_3},\ \mu_B = \frac{n_1\mu_{C1} + n_2\mu_{I2} + n_3\mu_{I\psi}}{n_1 + n_2 + n_3};$$

$$\sigma_A^2 = \frac{n_1\sigma_{I1}^2 + n_2\sigma_{C2}^2 + n_3\sigma_{I\phi}^2}{n_1 + n_2 + n_3},\ \sigma_B^2 = \frac{n_1\sigma_{C1}^2 + n_2\sigma_{I2}^2 + n_3\sigma_{I\psi}^2}{n_1 + n_2 + n_3}. \tag{13}$$

This lead to the following actual effect size and MDE for Setup 3:

$$\Delta_{S3} = \frac{n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I_\phi})}{n_1 + n_2 + n_3}, \tag{14}$$

$$\theta_{S3}^* = (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\frac{2\Big(n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)\Big)}{(n_1 + n_2 + n_3)^2}}. \tag{15}$$

Setup 4 (Dual control). Setup 4 is unique amongst the experiment setups introduced as it has four analysis groups. Two of the analysis groups are drawn from those who qualified into strategy 1 and are allocated into the first half, and the other two are drawn from those who are qualified into strategy 2 and are allocated into the second half:

$$n_{A1} = n_{A2} = \frac{n_1 + n_3}{4},\ n_{B1} = n_{B2} = \frac{n_2 + n_3}{4}. \tag{16}$$

The mean response and response variance of each analysis group are the weighted metric mean response and response variance of the user groups involved respectively:

$$\mu_{A1} = \frac{n_1\mu_{C1} + n_3\mu_{C3}}{n_1 + n_3},\ \mu_{A2} = \frac{n_1\mu_{I1} + n_3\mu_{I\phi}}{n_1 + n_3},$$
$$\mu_{B1} = \frac{n_1\mu_{C2} + n_3\mu_{C3}}{n_2 + n_3},\ \mu_{B2} = \frac{n_1\mu_{I2} + n_3\mu_{I\psi}}{n_2 + n_3};$$
$$\sigma^2_{A1} = \frac{n_1\sigma^2_{C1} + n_3\sigma^2_{C3}}{n_1 + n_3},\ \sigma^2_{A2} = \frac{n_1\sigma^2_{I1} + n_3\sigma^2_{I\phi}}{n_1 + n_3},$$
$$\sigma^2_{B1} = \frac{n_1\sigma^2_{C2} + n_3\sigma^2_{C3}}{n_2 + n_3},\ \sigma^2_{B2} = \frac{n_1\sigma^2_{I2} + n_3\sigma^2_{I\psi}}{n_2 + n_3}; \tag{17}$$

As the setup takes the difference of differences (i.e. the difference of groups $B2$ and $B1$, and the difference of groups $A2$ and $A1$), the actual effect size are specified, post-simplification, as follows:

$$\begin{aligned}\Delta_{S4} &= (\mu_{B2} - \mu_{B1}) - (\mu_{A2} - \mu_{A1}) \\ &= \frac{n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{C3})}{n_2 + n_3} - \frac{n_2(\mu_{I1} - \mu_{C1}) + n_3(\mu_{I\phi} - \mu_{C3})}{n_1 + n_3}.\end{aligned} \tag{18}$$

The MDE for Setup 4 is similar to that specified in the RHS of Inequality (5), albeit with more groups:

$$\begin{aligned}\theta^*_{S4} &= (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\sigma^2_{A1}/n_{A1} + \sigma^2_{A2}/n_{A2} + \sigma^2_{B1}/n_{B1} + \sigma^2_{B2}/n_{B2}} \\ &= 2 \cdot (z_{1-\alpha/2} - z_{1-\pi_{\min}}) \times \\ &\sqrt{\frac{n_1(\sigma^2_{C1} + \sigma^2_{I1}) + n_3(\sigma^2_{C3} + \sigma^2_{I\phi})}{(n_1 + n_3)^2} + \frac{n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{C3} + \sigma^2_{I\psi})}{(n_2 + n_3)^2}}.\end{aligned} \tag{19}$$

B METRIC DILUTION

We then expand the calculations in Section 3.2 of the paper, where we discuss the conditions where an experiment setup with metric dilution (Setup 2) will emerge superior to one without metric dilution (Setup 3), and vice versa.

B.1 The first criterion

We first show $\theta^*_{S3} < \theta^*_{S2}$, the condition which will lead to Setup 3 being superior to Setup 2 under the first criterion, is equivalent to

$$\frac{\big(n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})\big) \cdot (n_0 + 2n_1 + 2n_2 + 2n_3)}{2(n_1 + n_2 + n_3)^2} < \sigma^2_{C0}. \tag{20}$$

We start by substituting the expressions for θ^*_{S2} (Equation (12)) and θ^*_{S3} (Equation (15)) into the inequality $\theta^*_{S3} < \theta^*_{S2}$ to obtain

$$\begin{aligned}&(z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\frac{2\big(n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})\big)}{(n_1 + n_2 + n_3)^2}} \\ &< (z_{1-\alpha/2} - z_{1-\pi_{\min}})\sqrt{\frac{2\big(n_0(2\sigma^2_{C0}) + n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})\big)}{(n_0 + n_1 + n_2 + n_3)^2}}.\end{aligned} \tag{20a}$$

Canceling the $z_{1-\alpha/2} - z_{1-\pi_{\min}}$ and $\sqrt{2}$ terms on both sides, and then squaring them yields

$$\begin{aligned}&\frac{n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})}{(n_1 + n_2 + n_3)^2} \\ &< \frac{n_0(2\sigma^2_{C0}) + n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})}{(n_0 + n_1 + n_2 + n_3)^2}.\end{aligned} \tag{20b}$$

We then write $\xi = n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})$ and move the ξ terms on the RHS to the LHS:

$$\xi\left(\frac{1}{(n_1 + n_2 + n_3)^2} - \frac{1}{(n_0 + n_1 + n_2 + n_3)^2}\right) < \frac{n_0(2\sigma^2_{C0})}{(n_0 + n_1 + n_2 + n_3)^2}. \tag{20c}$$

As the partial fractions can be consolidated as

$$\begin{aligned}&\frac{1}{(n_1 + n_2 + n_3)^2} - \frac{1}{(n_0 + n_1 + n_2 + n_3)^2} \\ &= \frac{(n_0 + n_1 + n_2 + n_3)^2 - (n_1 + n_2 + n_3)^2}{(n_1 + n_2 + n_3)^2(n_0 + n_1 + n_2 + n_3)^2} \\ &= \frac{(n_0 + 2n_1 + 2n_2 + 2n_3)n_0}{(n_1 + n_2 + n_3)^2(n_0 + n_1 + n_2 + n_3)^2},\end{aligned} \tag{20d}$$

where the second step utilizes the identity $a^2 - b^2 = (a + b)(a - b)$, Inequality (20c) can be written as

$$\xi\left(\frac{(n_0 + 2n_1 + 2n_2 + 2n_3)n_0}{(n_1 + n_2 + n_3)^2(n_0 + n_1 + n_2 + n_3)^2}\right) < \frac{n_0(2\sigma^2_{C0})}{(n_0 + n_1 + n_2 + n_3)^2}. \tag{20e}$$

We finally cancel the n_0 and $(n_0 + n_1 + n_2 + n_3)^2$ terms on both sides of Inequality (20e), move the factor of two to the LHS, and write ξ in its full form to arrive at Inequality (20):

$$\frac{\big(n_1(\sigma^2_{I1} + \sigma^2_{C1}) + n_2(\sigma^2_{C2} + \sigma^2_{I2}) + n_3(\sigma^2_{I\phi} + \sigma^2_{I\psi})\big) \cdot (n_0 + 2n_1 + 2n_2 + 2n_3)}{2(n_1 + n_2 + n_3)^2} < \sigma^2_{C0}.$$

B.2 The second criterion

In the case where Inequality (20) does not hold, we consider when Setup 3 will emerge superior to Setup 2 under the second criterion:

$$\Delta_{S3} - \Delta_{S2} > \theta^*_{S3} - \theta^*_{S2}.$$

If this inequality does not hold either (and both sides are not equal), we consider Setup 2 as superior to Setup 3 under the same criterion as the following holds:

$$\Delta_{S3} - \Delta_{S2} < \theta^*_{S3} - \theta^*_{S2} \iff \Delta_{S2} - \Delta_{S3} > \theta^*_{S2} - \theta^*_{S3}.$$

The master inequality. We first show that the inequality $\Delta_{S3} - \Delta_{S2} > \theta^*_{S3} - \theta^*_{S2}$ is equivalent to

$$\frac{n_1 + n_2 + n_3}{n_0}\sqrt{2n_0\sigma^2_{C0} + \xi} > \frac{n_0 + n_1 + n_2 + n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}z}, \tag{22}$$

where

$$\begin{aligned}\eta &= n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I\phi}), \\ \xi &= n_1(\sigma^2_{C1} + \sigma^2_{I1}) + n_2(\sigma^2_{I2} + \sigma^2_{C2}) + n_3(\sigma^2_{I\psi} + \sigma^2_{I\phi}), \text{ and} \\ z &= z_{1-\alpha/2} - z_{1-\pi_{\min}}.\end{aligned}$$

Writing η, ξ, and z as shown above, we substitute in the expressions for Δ_{S2}, θ^*_{S2}, Δ_{S3}, and θ^*_{S3} (Equations (11), (12), (14), and (15) respectively) into the initial inequality to obtain

$$\frac{\eta}{n_1+n_2+n_3} - \frac{\eta}{n_0+n_1+n_2+n_3} > z\sqrt{\frac{2\xi}{(n_1+n_2+n_3)^2}} - z\sqrt{\frac{2\left(n_0(2\sigma^2_{C0})+\xi\right)}{(n_0+n_1+n_2+n_3)^2}}. \tag{22a}$$

Pulling out the common factors on each side we have

$$\eta\left(\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3}\right) > \sqrt{2}z\left(\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma^2_{C0}+\xi}}{n_0+n_1+n_2+n_3}\right). \tag{22b}$$

Writing the partial fraction on the LHS of Inequality (22b) as a composite fraction we have

$$\eta\left(\frac{n_0}{(n_1+n_2+n_3)(n_0+n_1+n_2+n_3)}\right) > \sqrt{2}z\left(\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma^2_{C0}+\xi}}{n_0+n_1+n_2+n_3}\right). \tag{22c}$$

We then move the composite fraction to the RHS and the $\sqrt{2}z$ term to the LHS:

$$\frac{\eta}{\sqrt{2}z} > \frac{(n_1+n_2+n_3)\cdot(n_0+n_1+n_2+n_3)}{n_0}\left(\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma^2_{C0}+\xi}}{n_0+n_1+n_2+n_3}\right), \tag{22d}$$

and expand the brackets, canceling terms that appear on both sides of the fractions in the RHS:

$$\frac{\eta}{\sqrt{2}z} > \frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma^2_{C0}+\xi}\,. \tag{22e}$$

Finally, we swap the position of the leftmost term with that of the rightmost term in Inequality (22e) to arrive at Inequality (22):

$$\frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma^2_{C0}+\xi} > \frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}z}.$$

The trivial case: RHS ≤ 0*.* We first observe that the LHS of Inequality (22) is always positive, and hence the inequality trivially holds if the RHS is non-positive. Here we show RHS ≤ 0 is equivalent to

$$\frac{n_0+n_1+n_2+n_3}{n_0}\theta^*_{S3} \leq \Delta_{S3}. \tag{23}$$

The can be done by writing RHS ≤ 0 in full:

$$\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}z} \leq 0, \tag{23a}$$

and moving the second term on the LHS to the RHS:

$$\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} \leq \frac{\eta}{\sqrt{2}z}. \tag{23b}$$

We then add a factor of $\sqrt{2}z/(n_1+n_2+n_3)$ on both sides to get

$$\frac{n_0+n_1+n_2+n_3}{n_0}\frac{\sqrt{\xi}\sqrt{2}z}{n_1+n_2+n_3} \leq \frac{\eta}{n_1+n_2+n_3}. \tag{23c}$$

Noting from Equations (14) and (15) that

$$\Delta_{S3} = \frac{\eta}{n_1+n_2+n_3} \text{ and } \theta^*_{S3} = \frac{\sqrt{2}\cdot z\cdot\sqrt{\xi}}{n_1+n_2+n_3},$$

we finally replace the terms in Inequality (23c) with Δ_{S3} and θ^*_{S3} to arrive at Inequality (23):

$$\frac{n_0+n_1+n_2+n_3}{n_0}\theta^*_{S3} \leq \Delta_{S3}.$$

The non-trivial case: RHS > 0*.* We then tackle the case where the RHS of the master inequality (Inequality (22)) is greater than zero. We show in this non-trivial case, Inequality (22) is equivalent to

$$\frac{2\sigma^2_{C0}}{n_0} > \frac{\left(\theta^*_{S3} - \Delta_{S3} + \frac{n_1+n_2+n_3}{n_0}\theta^*_{S3}\right)^2 - \left(\frac{n_1+n_2+n_3}{n_0}\theta^*_{S3}\right)^2}{2z^2}. \tag{24}$$

We first multiply both sides of Inequality (22) with the fraction $n_0\sqrt{2}z/(n_1+n_2+n_3)$ to get

$$\frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma^2_{C0}+\xi}\frac{n_0\sqrt{2}z}{n_1+n_2+n_3} > \left(\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}z}\right)\frac{n_0\sqrt{2}z}{n_1+n_2+n_3}\,. \tag{24a}$$

Canceling terms on both sides of the fractions we have

$$\sqrt{2n_0\sigma^2_{C0}+\xi}\sqrt{2}z > (n_0+n_1+n_2+n_3)\frac{\sqrt{\xi}\sqrt{2}z}{n_1+n_2+n_3} - n_0\frac{\eta}{n_1+n_2+n_3}\,. \tag{24b}$$

Again noting the identities for Δ_{S3} and θ^*_{S3} stated above, we can replace the fractions on the RHS and obtain

$$\sqrt{2n_0\sigma^2_{C0}+\xi}\sqrt{2}z > (n_0+n_1+n_2+n_3)\theta^*_{S3} - n_0\Delta_{S3}. \tag{24c}$$

We then square both sides of Inequality (24c) and move the $2z^2$ term to the RHS:

$$2n_0\sigma^2_{C0}+\xi > \frac{\left((n_0+n_1+n_2+n_3)\theta^*_{S3} - n_0\Delta_{S3}\right)^2}{2z^2}. \tag{24d}$$

Note the squaring still allows the implication to go both ways as both sides of Inequality (24c) are positive. Based on the identity for θ^*_{S3}, we observe ξ can also be written as

$$\xi = \frac{(n_1+n_2+n_3)^2(\theta^*_{S3})^2}{2z^2}. \tag{24e}$$

Thus, we can group all terms with a $2z^2$ denominator by moving ξ in Inequality (24d) to the RHS and substituting Equation (24e) into the resultant inequality:

$$2n_0\sigma^2_{C0} > \frac{\left((n_0+n_1+n_2+n_3)\theta^*_{S3} - n_0\Delta_{S3}\right)^2 - \left((n_1+n_2+n_3)\theta^*_{S3}\right)^2}{2z^2}. \tag{24f}$$

We finally normalize the inequality to one with unit Δ_{S3} and θ^*_{S3} to enable effective comparison. We divide both sides of Inequality (24f) by ${n_0}^2$:

$$\frac{2\sigma^2_{C0}}{n_0} > \frac{\left(\frac{n_0+n_1+n_2+n_3}{n_0}\theta^*_{S3} - \Delta_{S3}\right)^2 - \left(\frac{n_1+n_2+n_3}{n_0}\theta^*_{S3}\right)^2}{2z^2}, \tag{24g}$$

and split the coefficient of θ^*_{S3} in the first squared term into an integer (1) and a fractional ($(n_1+n_2+n_3)/n_0$) part to arrive at Inequality (24):

$$\frac{2\sigma^2_{C0}}{n_0} > \frac{\left(\theta^*_{S3} - \Delta_{S3} + \frac{n_1+n_2+n_3}{n_0}\theta^*_{S3}\right)^2 - \left(\frac{n_1+n_2+n_3}{n_0}\theta^*_{S3}\right)^2}{2z^2}.$$

C DUAL CONTROL

We finally clarify the calculations in Section 3.3 of the paper, where we determine the sample size required for Setup 4 (aka a dual control setup) to emerge superior to Setup 3 (a simpler A/B test setup). In the paper we showed that $\theta^*_{S4} > \theta^*_{S3}$ always holds, and hence Setup 4 will never be superior to Setup 3 under the first evaluation criterion. We thus focus on the second evaluation criterion $\Delta_{S4} - \Delta_{S3} > \theta^*_{S4} - \theta^*_{S3}$, and first show that the criterion is equivalent to Inequality (27). Assuming the ratio of user group sizes $n_1 : n_2 : n_3$ remains unchanged, we then show how

(i) The RHS of the Inequality (27) remains a constant and
(ii) The LHS of the Inequality (27) scales along $O(\sqrt{n})$, where n is the number of users.

The results mean Setup 4 could emerge superior to Setup 3 if we have sufficiently large number of users. We also show from the inequality that

(iii) The number of users required for a dual control setup to emerge superior to simpler setups is accessible only to the largest organizations and their top affiliates.

The master inequality. We first show the criterion $\Delta_{S4} - \Delta_{S3} > \theta^*_{S4} - \theta^*_{S3}$ is equivalent to

$$\frac{n_1\frac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - n_2\frac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{I\phi}+\sigma^2_{I\psi})}} > \tag{27}$$
$$\sqrt{2}z\left[\sqrt{2\cdot\frac{\begin{array}{c}(1+\frac{n_2}{n_1+n_3})^2[n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_3(\sigma^2_{C3}+\sigma^2_{I\phi})]+\\(1+\frac{n_1}{n_2+n_3})^2[n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{C3}+\sigma^2_{I\psi})]\end{array}}{n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{I\phi}+\sigma^2_{I\psi})}}-1\right],$$

where $z = z_{1-\alpha/2} - z_{1-\pi_{\min}}$. The number of terms involved is large, and hence we first simplify the LHS and RHS independently, and combine them in the final step.

For the LHS (i.e. $\Delta_{S4} - \Delta_{S3}$), we substitute in Equations (14) and (18) to obtain

$$\frac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - \frac{n_2(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3} - \frac{n_1(\mu_{C1}-\mu_{I1})+n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{I_\phi})}{n_1+n_2+n_3}. \tag{27a}$$

The expression can be rewritten in terms of multiplicative products between the n-terms and the (difference between) μ-terms:

$$\begin{aligned}&n_1(\mu_{I1}-\mu_{C1})\left[-\tfrac{1}{n_1+n_3}+\tfrac{1}{n_1+n_2+n_3}\right]+\\&n_2(\mu_{I2}-\mu_{C2})\left[\tfrac{1}{n_2+n_3}-\tfrac{1}{n_1+n_2+n_3}\right]+\\&n_3\mu_{I\psi}\left[\tfrac{1}{n_2+n_3}-\tfrac{1}{n_1+n_2+n_3}\right]+n_3\mu_{I\phi}\left[-\tfrac{1}{n_1+n_3}+\tfrac{1}{n_1+n_2+n_3}\right]+\\&n_3\mu_{C3}\left[-\tfrac{1}{n_2+n_3}+\tfrac{1}{n_1+n_3}\right].\end{aligned} \tag{27b}$$

We then extract a $1/(n_1+n_2+n_3)$ term from Expression (27b):

$$\begin{aligned}(n_1+n_2+n_3)^{-1}\Big[&n_1(\mu_{I1}-\mu_{C1})\left(-\tfrac{n_1+n_2+n_3}{n_1+n_3}+1\right)+\\&n_2(\mu_{I2}-\mu_{C2})\left(\tfrac{n_1+n_2+n_3}{n_2+n_3}-1\right)+\\&n_3\mu_{I\psi}\left(\tfrac{n_1+n_2+n_3}{n_2+n_3}-1\right)+n_3\mu_{I\phi}\left(-\tfrac{n_1+n_2+n_3}{n_1+n_3}+1\right)+\\&n_3\mu_{C3}\left(-\tfrac{n_1+n_2+n_3}{n_2+n_3}+\tfrac{n_1+n_2+n_3}{n_1+n_3}\right)\Big].\end{aligned} \tag{27c}$$

This allows us to perform some cancellation with the RHS, which also has a $1/(n_1+n_2+n_3)$ term, in the final step. Noting

$$\frac{n_1+n_2+n_3}{n_1+n_3} = 1+\frac{n_2}{n_1+n_3} \quad\text{and}\quad \frac{n_1+n_2+n_3}{n_2+n_3} = 1+\frac{n_1}{n_2+n_3},$$

we can write the inner square brackets as

$$\begin{aligned}(n_1+n_2+n_3)^{-1}\Big[&n_1(\mu_{I1}-\mu_{C1})\left(-\tfrac{n_2}{n_1+n_3}\right)+n_2(\mu_{I2}-\mu_{C2})\left(\tfrac{n_1}{n_2+n_3}\right)+\\&n_3\mu_{I\psi}\left(\tfrac{n_1}{n_2+n_3}\right)+n_3\mu_{I\phi}\left(-\tfrac{n_2}{n_1+n_3}\right)+\\&n_3\mu_{C3}\left(-1-\tfrac{n_1}{n_2+n_3}+1+\tfrac{n_2}{n_1+n_3}\right)\Big],\end{aligned} \tag{27d}$$

and group the $n_1/(n_2+n_3)$ and $n_2/(n_1+n_3)$ terms to arrive at

$$\frac{1}{n_1+n_2+n_3}\left[\frac{n_1}{n_2+n_3}\Big(n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})\Big) - \frac{n_2}{n_1+n_3}\Big(n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})\Big)\right]. \tag{27e}$$

For the RHS (i.e. $\theta^*_{S4} - \theta^*_{S3}$), we substitute in Equations (15) and (19) to obtain

$$2z\sqrt{\frac{n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_3(\sigma^2_{C3}+\sigma^2_{I\phi})}{(n_1+n_3)^2}+\frac{n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{C3}+\sigma^2_{I\psi})}{(n_2+n_3)^2}} - \sqrt{2}z\sqrt{\frac{n_1(\sigma^2_{I1}+\sigma^2_{C1})+n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{I\phi}+\sigma^2_{I\psi})}{(n_1+n_2+n_3)^2}}, \tag{27f}$$

where $z = z_{1-\alpha/2} - z_{1-\pi_{\min}}$. We then extract a $\sqrt{2}z/(n_1+n_2+n_3)$ term from Expression (27f) to arrive at

$$\frac{\sqrt{2}z}{n_1+n_2+n_3}\left[\sqrt{2}\sqrt{\begin{array}{c}\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2[n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_3(\sigma^2_{C3}+\sigma^2_{I\phi})]+\\\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2[n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{C3}+\sigma^2_{I\psi})]\end{array}} - \sqrt{n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{I\phi}+\sigma^2_{I\psi})}\right], \tag{27g}$$

where $(n_1+n_2+n_3)/(n_2+n_3)$ and $(n_1+n_2+n_3)/(n_1+n_3)$ can also be written as $1+n_1/(n_2+n_3)$ and $1+n_2/(n_1+n_3)$ respectively.

We finally combine both sides of the inequality by taking Expressions (27e) and (27g):

$$\frac{1}{n_1+n_2+n_3}\left[\frac{n_1}{n_2+n_3}\Big(n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})\Big) - \frac{n_2}{n_1+n_3}\Big(n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})\Big)\right] >$$
$$\frac{\sqrt{2}z}{n_1+n_2+n_3}\left[\sqrt{2}\sqrt{\begin{array}{c}(1+\frac{n_2}{n_1+n_3})^2[n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_3(\sigma^2_{C3}+\sigma^2_{I\phi})]+\\(1+\frac{n_1}{n_2+n_3})^2[n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{C3}+\sigma^2_{I\psi})]\end{array}} - \sqrt{n_1(\sigma^2_{C1}+\sigma^2_{I1})+n_2(\sigma^2_{C2}+\sigma^2_{I2})+n_3(\sigma^2_{I\phi}+\sigma^2_{I\psi})}\right]. \tag{27h}$$

Canceling the common $1/(n_1+n_2+n_3)$ terms on both sides, and dividing both sides by $\sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}$ leads us to Inequality (27):

$$\frac{n_1\frac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - n_2\frac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}} >$$

$$\sqrt{2}z\left[\sqrt{2\cdot\frac{\begin{array}{c}(1+\frac{n_2}{n_1+n_3})^2[n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)]+\\(1+\frac{n_1}{n_2+n_3})^2[n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)]\end{array}}{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}}-1\right].$$

RHS remains a constant. We then simplify the σ^2-terms in Inequality (27) by assuming that they are similar in magnitude, i.e.

$$\sigma_{C1}^2 \approx \sigma_{I1}^2 \approx \cdots \approx \sigma_{I\psi}^2 \approx \sigma_S^2,$$

and show the RHS of the inequality is equal to

$$\sqrt{2}z\left[\sqrt{\frac{n_1+n_2+n_3}{n_1+n_3}+\frac{n_1+n_2+n_3}{n_2+n_3}}-1\right]. \tag{28}$$

As long as the group size ratio $n_1 : n_2 : n_3$ remains unchanged, Expression (28) will remain a constant. It is safe to apply the simplifying assumption as we know from the evaluation framework specification that there are three classes of parameters in the inequality: the user group sizes (n), the mean responses (μ), and the response variances (σ^2). Among these three classes of parameters, only the user group sizes have the potential to scale in any practical settings, and thus we can effectively treat the means and variances as constants below.

We begin by substituting σ_S^2 into Inequality (27):

$$\frac{n_1\frac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - n_2\frac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)}} >$$

$$\sqrt{2}z\left[\sqrt{2\cdot\frac{\begin{array}{c}(1+\frac{n_2}{n_1+n_3})^2[n_1(2\sigma_S^2)+n_3(2\sigma_S^2)]+\\(1+\frac{n_1}{n_2+n_3})^2[n_2(2\sigma_S^2)+n_3(2\sigma_S^2)]\end{array}}{n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)}}-1\right]. \tag{28a}$$

Moving the common $2\sigma_S^2$ terms out and canceling the common terms in the RHS fraction we have

$$\frac{n_1\frac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - n_2\frac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{2\sigma_S^2(n_1+n_2+n_3)}} >$$

$$\sqrt{2}z\left[\sqrt{2\cdot\frac{(1+\frac{n_2}{n_1+n_3})^2(n_1+n_3)+(1+\frac{n_1}{n_2+n_3})^2(n_2+n_3)}{(n_1+n_2+n_3)}}-1\right]. \tag{28b}$$

We can already see the LHS of Inequality (28b) scales along $O(\sqrt{n})$—we will demonstrate this result in greater detail below.

Focusing on the RHS of the inequality, we express the squared terms as rational fractions and divide each term in the numerator by the denominator to obtain

$$\sqrt{2}z\left[\sqrt{2\left[\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2\frac{n_1+n_3}{n_1+n_2+n_3}+\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2\frac{n_2+n_3}{n_1+n_2+n_3}\right]}-1\right]. \tag{28c}$$

Canceling the common $n_1+n_2+n_3$ terms leads to that presented in Expression (28):

$$\sqrt{2}z\left[\sqrt{\frac{n_1+n_2+n_3}{n_1+n_3}+\frac{n_1+n_2+n_3}{n_2+n_3}}-1\right].$$

LHS scales along $O(\sqrt{n})$. We demonstrate the scaling relation between the LHS of Inequality (27) and the number of users in each group by simplifying the n-terms (but not the σ^2-terms as above), assuming $n_1 \approx n_2 \approx n_3 \approx n$, and showing the LHS of the inequality is equal to

$$\frac{\sqrt{n}\big((\mu_{I2}-\mu_{C2})-(\mu_{I1}-\mu_{C1})+\mu_{I\psi}-\mu_{I\phi}\big)}{2\sqrt{\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}}. \tag{29}$$

While the relationship (that the LHS of the inequality scales along $O(\sqrt{n})$) is evident by inspecting the LHS of Inequality (28b) or even Inequality (27) itself, we believe the simplification allows us to show the relationship more clearly.

We begin by substituting n into Inequality (27) to obtain

$$\frac{n\frac{n(\mu_{I2}-\mu_{C2})+n(\mu_{I\psi}-\mu_{C3})}{n+n} - n\frac{n(\mu_{I1}-\mu_{C1})+n(\mu_{I\phi}-\mu_{C3})}{n+n}}{\sqrt{n(\sigma_{C1}^2+\sigma_{I1}^2)+n(\sigma_{C2}^2+\sigma_{I2}^2)+n(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}} > \tag{29a}$$

$$\sqrt{2}z\left[\sqrt{2\cdot\frac{\begin{array}{c}(1+\frac{n}{n+n})^2[n(\sigma_{C1}^2+\sigma_{I1}^2)+n(\sigma_{C3}^2+\sigma_{I\phi}^2)]+\\(1+\frac{n}{n+n})^2[n(\sigma_{C2}^2+\sigma_{I2}^2)+n(\sigma_{C3}^2+\sigma_{I\psi}^2)]\end{array}}{n(\sigma_{C1}^2+\sigma_{I1}^2)+n(\sigma_{C2}^2+\sigma_{I2}^2)+n(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}}-1\right].$$

Moving the common n-terms out and canceling them in the fractions where appropriate lead to

$$\frac{\sqrt{n}\,\frac{1}{2}\left[((\mu_{I2}-\mu_{C2})+(\mu_{I\psi}-\mu_{C3}))-((\mu_{I1}-\mu_{C1})+(\mu_{I\phi}-\mu_{C3}))\right]}{\sqrt{\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}} >$$

$$\sqrt{2}z\left[\sqrt{2\cdot\frac{\begin{array}{c}(1+\frac{1}{2})^2(\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C3}^2+\sigma_{I\phi}^2)+\\(1+\frac{1}{2})^2(\sigma_{C2}^2+\sigma_{I2}^2+\sigma_{C3}^2+\sigma_{I\psi}^2)\end{array}}{\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}}-1\right], \tag{29b}$$

where the LHS is equal to Expression (29) as claimed above.

It is clear that there are no n-terms left on the RHS of Inequality (29), and hence the RHS remains a constant as shown previously. Setting up the inequality to demonstrate the third result—that the number of users required for a dual control setup to emerge superior is large—we further simplify the RHS of the inequality by rearranging the terms in the square root:

$$\frac{\sqrt{n}\left[\left((\mu_{I2}-\mu_{C2})+(\mu_{I\psi}-\mu_{C3})\right)-\left((\mu_{I1}-\mu_{C1})+(\mu_{I\phi}-\mu_{C3})\right)\right]}{2\sqrt{\sigma^2_{C1}+\sigma^2_{I1}+\sigma^2_{C2}+\sigma^2_{I2}+\sigma^2_{I\phi}+\sigma^2_{I\psi}}} > \sqrt{2}z\left[\sqrt{2\left(\frac{3}{2}\right)^2\left(1+\frac{2\sigma^2_{C3}}{\sigma^2_{C1}+\sigma^2_{I1}+\sigma^2_{C2}+\sigma^2_{I2}+\sigma^2_{I\phi}+\sigma^2_{I\psi}}\right)}-1\right]. \tag{29c}$$

Required number of users is large. We finally show that while Setup 4 could emerge superior to Setup 3 as the number of users increase, the number of users required is high. We do so by assuming both the σ^2- and n-terms are similar in magnitude, i.e. $\sigma^2_{C1} \approx \sigma^2_{I1} \approx \cdots \approx \sigma^2_{I\psi} \approx \sigma^2_S$ and $n_1 \approx n_2 \approx n_3 \approx n$, and show that Inequality (27) is equivalent to

$$n > \left(2\sqrt{12}\left(\sqrt{6}-1\right)z\right)^2 \frac{\sigma^2_S}{\Delta^2}, \tag{30}$$

where $\Delta = (\mu_{I2}-\mu_{C2})-(\mu_{I1}-\mu_{C1})+\mu_{I\psi}-\mu_{I\phi}$ is the actual effect size difference between Setups 4 and 3. Note we are determining when Setup 4 is superior to Setup 3 under the second evaluation criterion—that *the gain in actual effect* is greater than the loss in sensitivity—and thus assume Δ is positive.

The equivalence can be shown by substituting σ^2_S into Inequality (29c), which already assumes the n-terms are similar in magnitude:[8]

$$\frac{\sqrt{n}\left[\left((\mu_{I2}-\mu_{C2})+(\mu_{I\psi}-\mu_{C3})\right)-\left((\mu_{I1}-\mu_{C1})+(\mu_{I\phi}-\mu_{C3})\right)\right]}{2\sqrt{6\sigma^2_S}} > \sqrt{2}z\left[\sqrt{2\left(\frac{3}{2}\right)^2\left(1+\frac{2\sigma^2_S}{6\sigma^2_S}\right)}-1\right]. \tag{30a}$$

Noting the expression within the LHS square bracket is equal to Δ, we simplify the expression within the RHS square root, and move every non-n term to the RHS of the inequality to obtain

$$\sqrt{n} > \sqrt{2}z\left[\sqrt{6}-1\right]\frac{2\sqrt{6\sigma^2_S}}{\Delta}. \tag{30b}$$

As all quantities in the inequality are positive, we can square both sides and consolidate the coefficients on the RHS to arrive at Inequality (30):

$$n > \left(2\sqrt{12}\left(\sqrt{6}-1\right)z\right)^2 \frac{\sigma^2_S}{\Delta^2}.$$

[8] Alternatively we can substitute n into Inequality (28b), which already assumes the σ^2-terms are similar in magnitude. Simplifying the resultant inequality would yield the same end result.