~~Crite~~

Good:

1. $|\Delta_S| > |\Delta_R|$

and

$|\theta_S^*| < |\theta_R^*|$

2. ~~$|\Delta_S| < |\Delta_R|$~~

$|\Delta_S| - |\Delta_R| > |\theta_S^*| - |\theta_R^*|$

Setup 1:

~~$\mu_A = \mu_{I\phi}$~~

~~$\mu_B = \mu_{I\psi}$~~

~~$\Delta_{S1} = \mu_{I\psi} - \mu_{I\phi}$~~

~~$\theta_{S1}^* = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times$~~

$\mu_A = \mu_{I\phi}$ $\quad \sigma_A^2 = \sigma_{I\phi}^2$ $\quad n_A = n_3/2$

$\mu_B = \mu_{I\psi}$ $\quad \sigma_B^2 = \sigma_{I\psi}^2$ $\quad n_B = \cancel{n_3}\, n_3/2$

$\Delta_{S1} = \mu_{I\psi} - \mu_{I\phi}$

$$\theta_{S1}^* = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\left(\frac{\sigma_{I\phi}^2}{n_3/2} + \frac{\cancel{\sigma_{I\phi}^2}\ \sigma_{I\psi}^2}{\cancel{n_3}\, n_3/2} \right)}$$

$$= (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\frac{2(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)}{n_3}}$$

Setup 2:

~~μ_{A2}~~ ~~μ_{A20}~~

$$n_B = n_A = \frac{n_0 + n_1 + n_2 + n_3}{2}$$

$$\mu_A = \frac{\frac{n_0}{2}\mu_{C0} + \frac{n_1}{2}\mu_{I1} + \frac{n_2}{2}\mu_{C2} + \frac{n_3}{2}\mu_{I\phi}}{n_A} = \frac{n_0\mu_{C0} + n_1\mu_{I1} + n_2\mu_{C2} + n_3\mu_{I\phi}}{n_0 + n_1 + n_2 + n_3}$$

$$\mu_B = \frac{\frac{n_0}{2}\mu_{C0} + \frac{n_1}{2}\mu_{C1} + \frac{n_2}{2}\mu_{I2} + \frac{n_3}{2}\mu_{I\psi}}{n_B} = \frac{n_0\mu_{C0} + n_1\mu_{C1} + n_2\mu_{I2} + n_3\mu_{I\psi}}{n_0 + n_1 + n_2 + n_3}$$

$$\sigma_A^2 = \frac{\frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{I1}^2 + \frac{n_2}{2}\sigma_{C2}^2 + \frac{n_3}{2}\sigma_{I\phi}^2}{n_A}$$

$$\sigma_B^2 = \frac{\frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{C1}^2 + \frac{n_2}{2}\sigma_{I2}^2 + \frac{n_3}{2}\sigma_{I\psi}^2}{n_B}$$

$$\Delta_{S2} = \frac{n_0\mu_{C0} + n_1\mu_{C1} + n_2\mu_{I2} + n_3\mu_{I\psi} - n_0\mu_{C0} - n_1\mu_{I1} - n_2\mu_{C2} - n_3\mu_{I\phi}}{n_0 + n_1 + n_2 + n_3}$$

~~$n_0\mu_{C0} + n_1\mu_{I1} + n_2\mu_{C2} + n_3\mu_{I\phi}$~~

$$= \frac{n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I\phi})}{n_0 + n_1 + n_2 + n_3}$$

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{4}{2} = 2$$

$$\theta^*_{S2} = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times$$

$$\sqrt{\left(\frac{\frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{I1}^2 + \frac{n_2}{2}\sigma_{C2}^2 + \frac{n_3}{2}\sigma_{I\phi}^2}{n_A^2} + \frac{\frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{C1}^2 + \frac{n_2}{2}\sigma_{I2}^2 + \frac{n_3}{2}\sigma_{I\psi}^2}{n_B^2}\right)}$$

$$= (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times \sqrt{\frac{\frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{I1}^2 + \frac{n_2}{2}\sigma_{C2}^2 + \frac{n_3}{2}\sigma_{I\phi}^2 + \frac{n_0}{2}\sigma_{C0}^2 + \frac{n_1}{2}\sigma_{C1}^2 + \frac{n_2}{2}\sigma_{I2}^2 + \frac{n_3}{2}\sigma_{I\psi}^2}{\left(\frac{n_0+n_1+n_2+n_3}{2}\right)^2}}$$

$$= (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times \sqrt{\frac{2\left(n_0(2\sigma_{C0}^2) + n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)\right)}{(n_0+n_1+n_2+n_3)^2}}$$

Set up 3:

$$n_B = n_A = \frac{n_1+n_2+n_3}{2}$$

$$\mu_A = \frac{\frac{n_1}{2}\mu_{I1} + \frac{n_2}{2}\mu_{C2} + \frac{n_3}{2}\mu_{I\phi}}{n_A} = \frac{n_1\mu_{I1} + n_2\mu_{C2} + n_3\mu_{I\phi}}{n_1+n_2+n_3}$$

$$\mu_B = \frac{\frac{n_1}{2}\mu_{C1} + \frac{n_2}{2}\mu_{I2} + \frac{n_3}{2}\mu_{I\psi}}{n_B} = \frac{n_1\mu_{C1} + n_2\mu_{I2} + n_3\mu_{I\psi}}{n_1+n_2+n_3}$$

$$\sigma_A^2 = \frac{\frac{n_1}{2}\sigma_{I1}^2 + \frac{n_2}{2}\sigma_{C2}^2 + \frac{n_3}{2}\sigma_{I\phi}^2}{n_A}$$

$$\sigma_B^2 = \frac{\frac{n_1}{2}\sigma_{C1}^2 + \frac{n_2}{2}\sigma_{I2}^2 + \frac{n_3}{2}\sigma_{I\psi}^2}{n_B}$$

$$= (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times \sqrt{\frac{2(n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2))}{(n_1+n_2+n_3)^2}}$$

$$\Delta_{S3} = \frac{n_1(\mu_{C1}-\mu_{I1}) + n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I\psi}-\mu_{I\phi})}{n_1+n_2+n_3}$$

$$\theta^*_{S3} = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \times$$

$$\sqrt{\left(\frac{\frac{n_1}{2}\sigma_{I1}^2 + \frac{n_2}{2}\sigma_{C2}^2 + \frac{n_3}{2}\sigma_{I\phi}^2}{n_A^2} + \frac{\frac{n_1}{2}\sigma_{C1}^2 + \frac{n_2}{2}\sigma_{I2}^2 + \frac{n_3}{2}\sigma_{I\psi}^2}{n_B^2}\right)}$$

Setup 4:

$$n_{A1} = n_{A2} = \frac{n_1 + n_3}{4} \quad ; \quad n_{B1} = n_{B2} = \frac{n_2 + n_3}{4}$$

$$\mu_{A1} = \frac{\frac{n_1}{4}\mu_{C1} + \frac{n_3}{4}\mu_{C3}}{n_{A1}} = \frac{n_1\mu_{C1} + n_3\mu_{C3}}{n_1 + n_3} \qquad \mu_{A2} = \frac{\frac{n_1}{4}\mu_{I1} + \frac{n_3}{4}\mu_{I\phi}}{n_{A2}} = \frac{n_1\mu_{I1} + n_3\mu_{I\phi}}{n_1 + n_3}$$

$$\mu_{B1} = \frac{\frac{n_2}{4}\mu_{C2} + \frac{n_3}{4}\mu_{C3}}{n_{B1}} = \frac{n_2\mu_{C2} + n_3\mu_{C3}}{n_2 + n_3} \qquad \mu_{B2} = \frac{\frac{n_2}{4}\mu_{I2} + \frac{n_3}{4}\mu_{I4}}{n_{B2}} = \frac{n_2\mu_{I2} + n_3\mu_{I4}}{n_2 + n_3}$$

$$\Delta_{S4} = (\mu_{B2} - \mu_{B1}) - (\mu_{A2} - \mu_{A1})$$

$$= \frac{n_2\mu_{I2} + n_3\mu_{I4}}{n_2 + n_3} - \frac{n_2\mu_{C2} + n_3\mu_{C3}}{n_2 + n_3} - \left(\frac{n_1\mu_{I1} + n_3\mu_{I\phi}}{n_1 + n_3} - \frac{n_1\mu_{C1} + n_3\mu_{C3}}{n_1 + n_3}\right)$$

$$= \frac{n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I4} - \mu_{C3})}{n_2 + n_3} - \frac{n_1(\mu_{I1} - \mu_{C1}) + n_3(\mu_{I\phi} - \mu_{C3})}{n_1 + n_3}$$

$$\sigma_{A1}^2 = \frac{\frac{n_1}{4}\sigma_{C1}^2 + \frac{n_3}{4}\sigma_{C3}^2}{n_{A1}} = \frac{n_1\sigma_{C1}^2 + n_3\sigma_{C3}^2}{n_1 + n_3}, \qquad \sigma_{A2}^2 = \frac{\frac{n_1}{4}\sigma_{I1}^2 + \frac{n_3}{4}\sigma_{I\phi}^2}{n_{A2}} = \frac{n_1\sigma_{I1}^2 + n_3\sigma_{I\phi}^2}{n_1 + n_3}$$

$$\sigma_{B1}^2 = \frac{\frac{n_1}{4}\sigma_{C2}^2 + \frac{n_3}{4}\sigma_{C3}^2}{n_{B1}} = \frac{n_2\sigma_{C2}^2 + n_3\sigma_{C3}^2}{n_2 + n_3} \qquad \sigma_{B2}^2 = \frac{\frac{n_2}{4}\sigma_{I2}^2 + \frac{n_3}{4}\sigma_{I4}^2}{n_{B2}} = \frac{n_2\sigma_{I2}^2 + n_3\sigma_{I4}^2}{n_2 + n_3}$$

$$\theta^*_{S4} = (z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{\sigma_{A1}^2}{n_{A1}} + \frac{\sigma_{A2}^2}{n_{A2}} + \frac{\sigma_{B1}^2}{n_{B1}} + \frac{\sigma_{B2}^2}{n_{B2}}}$$

$$= (z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{4(n_1\sigma_{C1}^2 + n_3\sigma_{C3}^2)}{(n_1+n_3)^2} + \frac{4(n_1\sigma_{I1}^2 + n_3\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{4(n_2\sigma_{C2}^2 + n_3\sigma_{C3}^2)}{(n_2+n_3)^2} + \frac{4(n_2\sigma_{I2}^2 + n_3\sigma_{I\phi}^2)}{(n_2+n_3)^2}}$$

Comparing setup 2 and 3:

$$\Delta_{S2} = \frac{n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I\phi})}{n_0 + n_1 + n_2 + n_3}$$

$n_0 > 0$
$\Rightarrow |\Delta_{S3}| > |\Delta_{S2}|$

$$\Delta_{S3} = \frac{n_1(\mu_{C1} - \mu_{I1}) + n_2(\mu_{I2} - \mu_{C2}) + n_3(\mu_{I\psi} - \mu_{I\phi})}{n_1 + n_2 + n_3}$$

$$\theta^*_{S2} = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\frac{2(n_0(2\sigma_{C0}^2) + n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2))}{(n_0 + n_1 + n_2 + n_3)^2}}$$

$$\theta^*_{S3} = (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\frac{2(n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2))}{(n_1 + n_2 + n_3)^2}}$$

$$|\theta^*_{S3}| < |\theta^*_{S2}|$$

$$\Leftrightarrow \theta^*_{S3} < \theta^*_{S2}$$

(This is true assuming $z_{1-\alpha/2} > z_{1-\pi_{min}}$, which is almost always true for experiment design purposes, and noting $\sqrt{\cdot}$ gives the positive root.

$$\Leftrightarrow (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\frac{2(n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2))}{(n_1 + n_2 + n_3)^2}}$$

$$< (z_{1-\alpha/2} - z_{1-\pi_{min}}) \sqrt{\frac{2(n_0(2\sigma_{C0}^2) + n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2))}{(n_0 + n_1 + n_2 + n_3)^2}}$$

Cancelling the z-terms, squaring both sides of the ~~ineq~~ inequality, and cancelling the factor 2 on both sides yield:

$$\Leftrightarrow \frac{n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)}{(n_1 + n_2 + n_3)^2} < \frac{n_0(2\sigma_{C0}^2) + n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)}{(n_0 + n_1 + n_2 + n_3)^2}$$

Separating the first term on the RHS

$$\Leftrightarrow \frac{n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)}{(n_1 + n_2 + n_3)^2} < \frac{2n_0\sigma_{C0}^2}{(n_0 + n_1 + n_2 + n_3)^2} + \frac{n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)}{(n_0 + n_1 + n_2 + n_3)^2}$$

$$\Leftrightarrow \left(n_1(\sigma_{I1}^2 + \sigma_{C1}^2) + n_2(\sigma_{C2}^2 + \sigma_{I2}^2) + n_3(\sigma_{I\phi}^2 + \sigma_{I\psi}^2)\right)\left(\frac{1}{(n_1 + n_2 + n_3)^2} - \frac{1}{(n_0 + n_1 + n_2 + n_3)^2}\right) < \frac{2n_0\sigma_{C0}^2}{(n_0 + n_1 + n_2 + n_3)^2}$$

Noting $\frac{1}{(n_1 + n_2 + n_3)^2} - \frac{1}{(n_0 + n_1 + n_2 + n_3)^2}$

$$= \frac{(n_0 + n_1 + n_2 + n_3)^2 - (n_1 + n_2 + n_3)^2}{(n_1 + n_2 + n_3)^2 (n_0 + n_1 + n_2 + n_3)^2}$$

$$= \frac{(n_0 + 2n_1 + 2n_2 + 2n_3)n_0}{(n_1 + n_2 + n_3)^2 (n_0 + n_1 + n_2 + n_3)^2}$$

we have

$$\Leftrightarrow \left(n_1(\sigma_{I1}^2+\sigma_{c1}^2)+n_2(\sigma_{c2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\varphi}^2+\sigma_{I4}^2)\right)\left(\frac{(n_0+2n_1+2n_2+2n_3)\cancel{n_0}}{(n_1+n_2+n_3)^2\cancel{(n_0+n_1+n_2+n_3)}}\right) < \frac{2\cancel{n_0}\sigma_{c0}^2}{\cancel{(n_0+n_1+n_2+n_3)^2}}$$

$$\Leftrightarrow \frac{1}{2}\frac{\left(n_1(\sigma_{I1}^2+\sigma_{c1}^2)+n_2(\sigma_{c2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\varphi}^2+\sigma_{I4}^2)\right)(n_0+2n_1+2n_2+2n_3)}{(n_1+n_2+n_3)^2} < \sigma_{c0}^2$$

i.e. if σ_{c0}^2 is very large then setup 3 is superior under the first criteria.

As a rule of thumb, if $\sigma_{I1}^2, \sigma_{c1}^2, \ldots, \sigma_{I4}^2$ are comparable (say $\approx \sigma_S^2$), we require

$$\frac{1}{2}\frac{\left(n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)\right)(n_0+2n_1+2n_2+2n_3)}{(n_1+n_2+n_3)^2} < \sigma_{c0}^2$$

$$\Leftrightarrow \sigma_S^2\frac{(n_1+n_2+n_3)(n_0+2(n_1+n_2+n_3))}{(n_1+n_2+n_3)^2} < \sigma_{c0}^2$$

$$\Leftrightarrow \sigma_S^2\left(\frac{(n_1+n_2+n_3)n_0}{(n_1+n_2+n_3)^2}+\frac{(n_1+n_2+n_3)2(n_1+n_2+n_3)}{(n_1+n_2+n_3)^2}\right) < \sigma_{c0}^2$$

$$\Leftrightarrow \sigma_S^2\left(\frac{n_0}{(n_1+n_2+n_3)}+2\right) < \sigma_{c0}^2$$

~~i.e. $\sigma_{c0}^2/18$~~

~~Realisticly~~

We consider the compromise condition: $|\Delta_{S3}| - |\Delta_{S2}| > |\Theta^*_{S3}| - |\Theta^*_{S2}|$,

~~assuming~~ given $|\Delta_{S3}| > |\Delta_{S2}|$ and assuming $|\Theta^*_{S3}| > |\Theta^*_{S2}|$

This is equivalent to:

$$\left|\frac{n_1(\mu_{C1}-\mu_{I1})+n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{I\phi})}{n_1+n_2+n_3}\right| - \left|\frac{n_1(\mu_{C1}-\mu_{I1})+n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{I\phi})}{n_0+n_1+n_2+n_3}\right|$$

$$> (z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{2(n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2))}{(n_1+n_2+n_3)^2}}$$

$$- (z_{(1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{2(n_0\sigma_{C0}^2)+n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\phi}^2))}{(n_0+n_1+n_2+n_3)^2}}$$

Noting $\left|\frac{x}{n_1+n_2+n_3}\right| - \left|\frac{x}{n_0+n_1+n_2+n_3}\right| = |x|\left(\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3}\right)$ for any x as $n_0, n_1, n_2, n_3 > 0$

we pull out the common terms to obtain

$$\eta\left(\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3}\right) > \sqrt{2}\, z\left(\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma_{C0}^2+\xi}}{n_0+n_1+n_2+n_3}\right),$$

where $\eta = |n_1(\mu_{C1}-\mu_{I1})+n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{I\phi})|$,

$\xi = n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)$, and

$z = (z_{1-\alpha/2} - z_{1-\pi_{min}})$.

~~It can b~~

we note

$$\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma_{C0}^2+\xi}}{n_0+n_1+n_2+n_3} = \frac{(n_0+n_1+n_2+n_3)\sqrt{\xi} - (n_1+n_2+n_3)\sqrt{2n_0\sigma_{C0}^2+\xi}}{n_0}\left(\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3}\right),$$

and the difference in fractions ~~cancel~~ cancels on both sides giving:

$$\eta > \sqrt{2}\, z\left(\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma_{C0}^2+\xi}\right)$$

Moving $\sqrt{2}z$ to the LHS and rearranging the terms yields

$$\frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma_{C0}^2+\xi} > \frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}\,z} \quad (\dagger)$$

Given

$$\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3} = \frac{(n_0+n_1+n_2+n_3)-(n_1+n_2+n_3)}{(n_1+n_2+n_3)(n_0+n_1+n_2+n_3)} = \frac{n_0}{(n_1+n_2+n_3)(n_0+n_1+n_2+n_3)},$$

we have

$$\frac{\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\sqrt{2n_0\sigma_{C0}^2+\xi}}{n_0+n_1+n_2+n_3} = \frac{(n_0+n_1+n_2+n_3)\sqrt{\xi} - (n_1+n_2+n_3)\sqrt{2n_0\sigma_{C0}^2+\xi}}{(n_1+n_2+n_3)(n_0+n_1+n_2+n_3)}$$

$$= \frac{(n_0+n_1+n_2+n_3)\sqrt{\xi} - (n_1+n_2+n_3)\sqrt{2n_0\sigma_{C0}^2+\xi}}{\cancel{(n_1+n_2+n_3)(n_0)}\; n_0} \times \frac{n_0}{(n_1+n_2+n_3)(n_0+n_1+n_2+n_3)}$$

$$= \frac{(n_0+n_1+n_2+n_3)\sqrt{\xi} - (n_1+n_2+n_3)\sqrt{2n_0\sigma_{C0}^2+\xi}}{n_0}\left(\frac{1}{n_1+n_2+n_3} - \frac{1}{n_0+n_1+n_2+n_3}\right)$$

Since the LHS of (†) is always +ve, the inequality can easily be satisfied if the RHS is non-positive:

$$\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}z} \le 0$$

This trivial case can be converted ~~as~~ to a useful rule of thumb by recalling $|\Delta_{S3}| = \frac{\eta}{n_1+n_2+n_3}$ and $|\theta^*_{S3}| = \frac{\sqrt{2}z\sqrt{\xi}}{n_1+n_2+n_3}$, and rearranging the ineq above as:

$$\frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} \le \frac{\eta}{\sqrt{2}z}$$

$$\Leftrightarrow \frac{n_0+n_1+n_2+n_3}{n_0}\,\frac{\sqrt{2}z\sqrt{\xi}}{n_1+n_2+n_3} \le \frac{\eta}{n_1+n_2+n_3}$$

$$\Leftrightarrow \frac{n_0+n_1+n_2+n_3}{n_0}\left|\theta^*_{S3}\right| \le \left|\Delta_{S3}\right|, \quad (‡)$$

~~A any dilution~~ At this point, any dilution will drop the actual effect quicker than the loss of the MDE

i.e. ~~∃ a large~~ when the effect size is sufficiently large ~~such that~~, ~~adding~~ many ~~samples who not qualify for treatment~~ ~~any diluto~~ it ~~continue to stand a high chance to be detected even when heavily diluted~~. ~~One~~ Note the rule of thumb does not require knowledge on how samples in group 0 react (i.e. μ_{C0} and σ^2_{C0}), but only the number of samples one planned to ~~put~~ include in analysis.

Consider an example where we are evaluating on a binomial metric, where the variance of each group maxes out at $1/4$. In this case

$$\theta^*_{S3} = \frac{\sqrt{2}z\sqrt{n_1(\frac{1}{4}+\frac{1}{4})+n_2(\frac{1}{4}+\frac{1}{4})+n_3(\frac{1}{4}+\frac{1}{4})}}{n_1+n_2+n_3} = \frac{\sqrt{2}\frac{1}{\sqrt{2}}z\sqrt{n_1+n_2+n_3}}{n_1+n_2+n_3} = \frac{z}{\sqrt{n_1+n_2+n_3}},$$

and (‡) becomes

$$\frac{n_0+n_1+n_2+n_3}{n_0}\,\frac{z}{\sqrt{n_1+n_2+n_3}} \le |\Delta_{S3}|$$

$$\Leftrightarrow \frac{z(n_0+n_1+n_2+n_3)}{n_0\sqrt{n_1+n_2+n_3}} \le |\Delta_{S3}|$$

TO BE DEVELOPED FURTHER

Say $\sigma_{I_1}^2 = \sigma_{C_1}^2 = \dots = \sigma_s^2$

$$\frac{1}{2}\left(n_1(2\sigma_s^2)+n_2(2\sigma_s^2)+n_3(2\sigma_s^2)\right)\left(\frac{n_0+2n_1+2n_2+2n_3}{(n_1+n_2+n_3)^2}\right) < \sigma_{c0}^2$$

$$\Leftrightarrow \sigma_s^2(n_1+n_2+n_3)\left(\frac{n_0+2n_1+2n_2+2n_3}{(n_1+n_2+n_3)^2}\right) < \sigma_{c0}^2$$

$$\cancel{< 2\sigma_s^2\left(\tfrac{1}{2}n_0(n_1+n_2+n_3)+(n_1+\dots\right.}$$

$$\Leftrightarrow \sigma_s^2\,\frac{n_0(n_1+n_2+n_3)+(2n_1+2n_2+2n_3)(n_1+n_2+n_3)}{(n_1+n_2+n_3)^2} < \sigma_{c0}^2$$

$$\Leftrightarrow \sigma_s^2\left(\frac{n_0}{(n_1+n_2+n_3)}+2\right) < \sigma_{c0}^2$$

Note

$|\Delta_{S3}| = \frac{\eta}{n_1+n_2+n_3}$

$|\theta^*_{S3}| = \frac{\sqrt{2}\,z\sqrt{\xi}}{n_1+n_2+n_3}$

$\Leftrightarrow (n_1+n_2+n_3)|\theta^*_{S3}| = \sqrt{2}\,z\sqrt{\xi}$

$\Leftrightarrow \frac{n_1+n_2+n_3}{\sqrt{2}\,z}|\theta^*_{S3}| = \sqrt{\xi}$

$\Leftrightarrow \left(\frac{n_1+n_2+n_3}{\sqrt{2}\,z}\right)^2|\theta^*_{S3}|^2 = \xi$

$$\frac{n_1+n_2+n_3}{n_0}\sqrt{2n_0\sigma_{c0}^2+\xi} > \frac{n_0+n_1+n_2+n_3}{n_0}\sqrt{\xi} - \frac{\eta}{\sqrt{2}\,z}$$

$$\Leftrightarrow \sqrt{2n_0\sigma_{c0}^2+\xi} > \frac{n_0+n_1+n_2+n_3}{n_1+n_2+n_3}\sqrt{\xi} - \frac{\eta}{\sqrt{2}\,z}\frac{n_0}{n_1+n_2+n_3}$$

$$\Leftrightarrow \sqrt{2}\,z\sqrt{2n_0\sigma_{c0}^2+\xi} > (n_0+n_1+n_2+n_3)\frac{\sqrt{2}\,z\sqrt{\xi}}{n_1+n_2+n_3} - \frac{\eta}{n_1+n_2+n_3}(n_0)$$

$$\Leftrightarrow \sqrt{2}\,z\sqrt{2n_0\sigma_{c0}^2+\xi} > (n_0+n_1+n_2+n_3)|\theta^*_{S3}| - n_0|\Delta_{S3}|$$

$$\Leftrightarrow 2z^2(2n_0\sigma_{c0}^2+\xi) > \left((n_0+n_1+n_2+n_3)|\theta^*_{S3}|-n_0|\Delta_{S3}|\right)^2$$

$$\Leftrightarrow 2n_0\sigma_{c0}^2+\xi > \frac{1}{2z^2}\left((n_0+n_1+n_2+n_3|\theta^*_{S3}|-n_0|\Delta_{S3}|\right)^2$$

$$\Leftrightarrow 2n_0\sigma_{c0}^2 > \frac{1}{2z^2}\left[\left((n_0+n_1+n_2+n_3)|\theta^*_{S3}|-n_0|\Delta_{S3}|\right)^2 - \left((n_1+n_2+n_3)|\theta^*_{S3}|\right)^2\right]$$

$$\Leftrightarrow 2n_0\sigma_{c0}^2 > \frac{1}{2z^2}\left[\left(n_0(|\theta^*_{S3}|-|\Delta_{S3}|)+(n_1+n_2+n_3)|\theta^*_{S3}|\right)^2 - \left((n_1+n_2+n_3)|\theta^*_{S3}|\right)^2\right]$$

$$\Leftrightarrow \frac{2\sigma_{c0}^2}{n_0} > \frac{1}{\cancel{2}z^2}\left[\left(|\theta^*_{S3}|-|\Delta_{S3}|+\frac{n_1+n_2+n_3}{n_0}|\theta^*_{S3}|\right)^2 - \left(\frac{n_1+n_2+n_3}{n_0}|\theta^*_{S3}|\right)^2\right]$$

Automatically true if RHS is -ve

$\Leftarrow |\theta^*_{S3}| \le |\Delta_{S3}|$

as variance is > 0, always

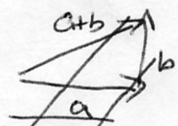
For $|\theta^*_{S3}| > |\Delta_{S3}|$

[Interpretation required.]

Assumed

$$\frac{n_0+n_1+n_2+n_3}{n_0}|\theta^*_{S3}| > |\Delta_{S3}|$$

$$\Leftrightarrow |\theta^*_{S3}| > \frac{n_0}{n_0+n_1+n_2+n_3}|\Delta_{S3}|$$



$\cancel{\sqrt{a^2+b^2} = a+b}$

$\cancel{\|a\|^2+\|b\|^2 = \|a+b\|^2}$

Reverse condition also hold?

$$\Delta_{S3} = \frac{n_1(\mu_{C1}-\mu_{I1}) + n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I\psi}-\mu_{I\phi})}{n_1+n_2+n_3}$$

$$\theta^*_{S3} = (z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{2(n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2))}{(n_1+n_2+n_3)^2}}$$

$$\Delta_{S4} = \frac{n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I\psi}-\mu_{C3})}{n_2+n_3} - \frac{n_1(\mu_{I1}-\mu_{C1}) + n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}$$

$$\theta^*_{S4} = 2(z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{n_1(\sigma_{C1}^2+\sigma_{I1}^2) + n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)}{(n_2+n_3)^2}}$$

When is $\Delta_{S3} > \Delta_{S4}$?

$|\theta^*_{S4}| > |\theta^*_{S3}|$ always: (and hence cannot be superior under criterion 1)

$$\Leftrightarrow \sqrt{2}\,\cancel{2}(\cancel{z_{1-\alpha/2} - z_{1-\pi_{min}}})\sqrt{\frac{n_1(\sigma_{C1}^2+\sigma_{I1}^2) + n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)}{(n_2+n_3)^2}}$$

$$\cancel{\sqrt{2}}(\cancel{z_{1-\alpha/2} - z_{1-\pi_{min}}})\sqrt{\frac{n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}{(n_1+n_2+n_3)^2}}$$

Cancelling out the terms and squaring both side of the inequality we have

$$2\left(\frac{n_1(\sigma_{C1}^2+\sigma_{I1}^2) + n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)}{(n_2+n_3)^2}\right)$$

$$> \cancel{\frac{n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)}{(n}}$$

$$> \frac{n_1(\sigma_{I1}^2+\sigma_{C1}^2) + n_2(\sigma_{C2}^2+\sigma_{I2}^2) + n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}{(n_1+n_2+n_3)^2}$$

~~We group the terms on both side~~ Rearranging the inequality by σ^2-terms we have

$$\cancel{2\left(\frac{n_1}{(n_1+n_3)^2}(\sigma_{C1}^2+\sigma_{I1}^2) + \frac{n_2}{(n_2+n_3)^2}(\sigma_{C2}^2+\sigma_{I2}^2)\right) + \frac{n_3}{(n_1+n_3)^2}(\sigma}$$

$$2\left(\frac{n_1}{(n_1+n_3)^2}(\sigma_{C1}^2+\sigma_{I1}^2) + \frac{n_2}{(n_2+n_3)^2}(\sigma_{C2}^2+\sigma_{I2}^2) + \frac{n_3}{(n_1+n_3)^2}\sigma_{I\phi}^2 + \frac{n_3}{(n_2+n_3)^2}\sigma_{I\psi}^2 + \left(\frac{n_3}{(n_1+n_3)^2} + \frac{n_3}{(n_2+n_3)^2}\right)\sigma_{C3}^2\right)$$

$$> \frac{n_1}{(n_1+n_2+n_3)^2}(\sigma_{I1}^2+\sigma_{C1}^2) + \frac{n_2}{(n_1+n_2+n_3)^2}(\sigma_{C2}^2+\sigma_{I2}^2) + \frac{n_3}{(n_1+n_2+n_3)^2}\sigma_{I\phi}^2 + \frac{n_3}{(n_1+n_2+n_3)^2}\sigma_{I\psi}^2$$

~~Inspecting the inequality above~~

As all the terms featured in the inequality are non-negative. The inequality holds in all cases: all matching terms in the LHS are greater than the RHS ($\frac{x}{(n_1+n_3)^2} > \frac{x}{(n_1+n_2+n_3)^2}$ and $\frac{x}{(n_2+n_3)^2} > \frac{x}{(n_1+n_2+n_3)^2}$ for all $x, n_1, n_2, n_3 > 0$), there is an extra, unmatched term σ_{C3}^2 on the LHS, and the LHS is multiplied by a factor of 2.

In this case we fall back to the other condition and explore when

$\cancel{\Delta_{33}}$ $|\Delta_{S4}| - |\Delta_{S3}| > |\theta^*_{S4}| - |\theta^*_{S3}|$

$\boxed{\text{we need the actual effect size to be even greater for S4 to be considered worthwhile}}$

$$\left(\frac{|n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I4}-\mu_{C3})|}{n_2+n_3} - \frac{|n_1(\mu_{I1}-\mu_{C1}) + n_3(\mu_{I\phi}-\mu_{C3})|}{n_1+n_3}\right)$$

$$- \frac{|n_1(\mu_{C1}-\mu_{I1}) + n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I4}-\mu_{I\phi})|}{n_1+n_2+n_3} \quad (\dagger)$$

$$> 2(z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_3)^2} + \frac{n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{C3}^2+\sigma_{I4}^2)}{(n_2+n_3)^2}}$$

$$-\sqrt{2}(z_{1-\alpha/2} - z_{1-\pi_{min}})\sqrt{\frac{n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)}{(n_1+n_2+n_3)^2}} \quad (\ddagger)$$

LHS: we first change the denominator of the fractions to $n_1+n_2+n_3$:

$$(\dagger) = \frac{\frac{n_1+n_2+n_3}{n_2+n_3} n_2(\mu_{I2}-\mu_{C2}) + \frac{n_1+n_2+n_3}{n_2+n_3} n_3(\mu_{I4}-\mu_{C3})}{n_1+n_2+n_3}$$

$$- \frac{\frac{n_1+n_2+n_3}{n_1+n_3} n_1(\mu_{I1}-\mu_{C1}) + \frac{n_1+n_2+n_3}{n_1+n_3} n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_2+n_3}$$

$$- \frac{n_1(\mu_{C1}-\mu_{I1}) + n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I4}-\mu_{I\phi})}{n_1+n_2+n_3}$$

$$\frac{x}{n_2+n_3} = \frac{n_1+n_2+n_3}{n_1+n_2+n_3}\,\frac{x}{n_2+n_3} = \frac{n_1+n_2+n_3}{n_2+n_3}\,\frac{x}{n_1+n_2+n_3}$$

we then group ~~the~~ the expression by ~~[illegible]~~ terms by the μ ~~[illegible]~~ -terms

$$= \frac{1}{n_1+n_2+n_3}\Big[\cancel{\tfrac{n_1+n_2+n_3}{n_1+n_3} n_1(\mu_{C1}-\mu_{I1})} \quad \cancel{n_1(\mu_{I1}-\mu_{C1})}$$

$$(\mu_{I1}-\mu_{C1})\left[-n_1\frac{n_1+n_2+n_3}{n_1+n_3} + n_1\right] +$$

$$(\mu_{I2}-\mu_{C2})\left[n_2\frac{n_1+n_2+n_3}{n_2+n_3} - n_2\right] +$$

$$\mu_{I4}\left[n_3\frac{n_1+n_2+n_3}{n_2+n_3} - n_3\right] +$$

$$\mu_{I\phi}\left[-n_3\frac{n_1+n_2+n_3}{n_1+n_3} + n_3\right] +$$

$$\mu_{C3}\left[-n_3\frac{n_1+n_2+n_3}{n_2+n_3} + n_3\frac{n_1+n_2+n_3}{n_1+n_3}\right]\Big]$$

$$= \frac{1}{n_1+n_2+n_3}\Big[(\mu_{C1}-\mu_{I1})\, n_1\left(\frac{n_1+n_2+n_3}{n_1+n_3} - 1\right) +$$

$$(\mu_{I2}-\mu_{C2})\, n_2\left(\frac{n_1+n_2+n_3}{n_1+n_3} - 1\right) +$$

$$(\mu_{I4}-\mu_{I\phi})\, n_3\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right.$$

$$(+) = \frac{1}{n_1+n_2+n_3}\Big[(\mu_{C1}-\mu_{I1})\, n_1\left(\frac{n_1+n_2+n_3}{n_1+n_3}-1\right)+$$
$$(\mu_{I2}-\mu_{C2})\, n_2\left(\frac{n_1+n_2+n_3}{n_2+n_3}-1\right)+$$
$$\mu_{I\psi}\, n_3\left(\frac{n_1+n_2+n_3}{n_2+n_3}-1\right)+$$
$$\mu_{I\phi}\, n_3\left(1-\frac{n_1+n_2+n_3}{n_1+n_3}\right)+$$
$$\mu_{C3}\, n_3\left(\frac{n_1+n_2+n_3}{n_1+n_3}-\frac{n_1+n_2+n_3}{n_2+n_3}\right)\Big]$$

$$= \frac{1}{n_1+n_2+n_3}\Big[(\mu_{C1}-\mu_{I1})\, n_1\left(\frac{n_2}{n_1+n_3}\right)+(\mu_{I2}-\mu_{C2})\, n_2\left(\frac{n_1}{n_2+n_3}\right)+$$
$$\mu_{I\psi}\, n_3\left(\frac{n_1}{n_2+n_3}\right)+\mu_{I\phi}\, n_3\left(-\frac{n_2}{n_1+n_3}\right)+$$
$$\mu_{C3}\, n_3\left(\frac{n_2}{n_1+n_3}-\frac{n_1}{n_2+n_3}\right)\Big]$$

Notice the emergence of common terms $\frac{n_1}{n_2+n_3}$ and $\frac{n_2}{n_1+n_3}$, we further group by them to obtain

$$\frac{1}{n_1+n_2+n_3}\Big[\cancel{\left(\frac{n_1}{n_2+n_3}\right)}\left(\frac{n_1}{n_2+n_3}\right)\left[n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})\right]+$$
$$\left(\frac{n_2}{n_1+n_3}\right)\left[n_1(\mu_{C1}-\mu_{I1})+n_3(\mu_{C3}-\mu_{I\phi})\right]\Big]$$

$$= \frac{1}{n_1+n_2+n_3}\Big[\left(\frac{n_1}{n_2+n_3}\right)\left[n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\psi}-\mu_{C3})\right]-$$
$$\left(\frac{n_2}{n_1+n_3}\right)\left[n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})\right]\Big]$$

RHS: We first group the z-terms and change the denominator in the square roots to $\frac{1}{(n_1+n_2+n_3)^2}$, so that it can cancell with that on the LHS later.

$$(\neq)\ (z_{1-\alpha/2}-z_{1-\pi_{min}})\Bigg[2\sqrt{\frac{\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2 n_1(\sigma_{C1}^2+\sigma_{I1}^2)+\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)}{(n_1+n_2+n_3)^2}+\frac{\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2 n_2(\sigma_{C2}^2+\sigma_{I2}^2)+\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I\psi}^2)}{(n_1+n_2+n_3)^2}}$$

$$-\sqrt{2}\sqrt{\frac{n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}{(n_1+n_2+n_3)^2}}\Bigg]$$

$$= \frac{(z_{1-\alpha/2} - z_{1-\pi_{min}})}{n_1+n_2+n_3}\left[2\sqrt{\begin{aligned}&\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2 n_1(\sigma_{C1}^2+\sigma_{I1}^2) + \left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I\phi}^2) + \\ &\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2 n_2(\sigma_{C2}^2+\sigma_{I2}^2) + \left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I4}^2)\end{aligned}} - \sqrt{2}\sqrt{n_1(\sigma_{I1}^2+\sigma_{C1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)}\right]$$

$$= \frac{(z_{1-\alpha/2} - z_{1-\pi_{min}})}{n_1+n_2+n_3}\left[2\sqrt{\begin{aligned}&\left(1+\frac{n_2}{n_1+n_3}\right)^2 n_1(\sigma_{C1}^2+\sigma_{I1}^2) + \left(1+\frac{n_2}{n_1+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I\phi}^2) + \\ &\left(\frac{n_1}{n_2+n_3}+1\right)^2 n_2(\sigma_{C2}^2+\sigma_{I2}^2) + \left(\frac{n_1}{n_2+n_3}+1\right)^2 n_3(\sigma_{C3}^2+\sigma_{I4}^2)\end{aligned}} - \sqrt{2}\sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)}\right]$$

Combining both sides of the ineq.

LHS > RHS

$$\frac{1}{n_1+n_2+n_3}\left[\left(\frac{n_1}{n_2+n_3}\right)\left[n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{C3})\right] - \left(\frac{n_2}{n_1+n_3}\right)\left[n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})\right]\right]$$

$$> \frac{(z_{1-\alpha/2} - z_{1-\pi_{min}})}{n_1+n_2+n_3}\left[2\sqrt{\begin{aligned}&\left(1+\frac{n_2}{n_1+n_3}\right)^2 n_1(\sigma_{C1}^2+\sigma_{I1}^2) + \left(1+\frac{n_2}{n_1+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I\phi}^2) + \\ &\left(1+\frac{n_1}{n_2+n_3}\right)^2 n_2(\sigma_{C2}^2+\sigma_{I2}^2) + \left(1+\frac{n_1}{n_2+n_3}\right)^2 n_3(\sigma_{C3}^2+\sigma_{I4}^2)\end{aligned}} - \sqrt{2}\sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)}\right]$$

Cancelling the $1/(n_1+n_2+n_3)$ term on both sides

$$\Leftrightarrow \frac{n_1}{n_2+n_3}\left[n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{C3})\right] - \left(\frac{n_2}{n_1+n_3}\right)\left[n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})\right] >$$

$$\sqrt{2}(z_{1-\alpha/2} - z_{1-\pi_{min}})\left[\sqrt{2}\sqrt{\begin{aligned}&\left(1+\frac{n_2}{n_1+n_3}\right)^2 n_1(\sigma_{C1}^2+\sigma_{I1}^2) + \left(1+\frac{n_1}{n_2+n_3}\right)^2 n_2(\sigma_{C2}^2+\sigma_{I2}^2) + \\ &\left(1+\frac{n_2}{n_1+n_3}\right)^2 n_3\sigma_{I\phi}^2 + \left(1+\frac{n_1}{n_2+n_3}\right)^2 n_3\sigma_{I4}^2 + \\ &\left[\left(1+\frac{n_2}{n_1+n_3}\right)^2+\left(1+\frac{n_1}{n_2+n_3}\right)^2\right] n_3\sigma_{C3}^2\end{aligned}} - \sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I4}^2)}\right]$$

~~LHS~~ If we size up the samples by n_x, LHS grows at $O(n_x)$ while RHS grows at $O(\sqrt{n_x})$

$$2\left(1+\frac{n_2}{n_1+n_3}\right)^2 - 1$$
$$= 2 + 4\frac{n_2}{n_1+n_3} + 2\left(\frac{n_2}{n_1+n_3}\right)^2 - 1$$
$$= 1 + 4\frac{n_2}{n_1+n_3} + 2\left(\frac{n_2}{n_1+n_3}\right)^2$$

$$\cancel{\left(1+\frac{n_2}{n_1+n_3}\right)^2 + \left(1+\frac{n_1}{n_2+n_3}\right)^2 = 1 + 2\frac{n_2}{n_1+n_3} + \left(\frac{n_2}{n_1+n_3}\right)^2 + 1 + 2\frac{n_1}{n_2+n_3} + \left(\frac{n_1}{n_2+n_3}\right)^2}$$

$(1+b)^2$

Ineq:

$$\Delta_{S4} - \Delta_{S3} > \theta^*_{S4} - \theta^*_{S3}$$

$$\Leftrightarrow \frac{n_1}{n_2+n_3}\left[n_2(\mu_{I2}-\mu_{C2}) + n_3(\mu_{I4}-\mu_{C3})\right] - \frac{n_2}{n_1+n_3}\left[n_1(\mu_{I1}-\mu_{C1}) + n_3(\mu_{I4}-\mu_{C3})\right] >$$

$$\sqrt{2}\, z \left[\sqrt{2\left(1+\frac{n_2}{n_1+n_3}\right)^2\left[n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_3(\sigma_{C3}^2+\sigma_{I4}^2)\right] + 2\left(1+\frac{n_1}{n_2+n_3}\right)^2\left[n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{C3}^2+\sigma_{I4}^2)\right]} - \sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I4}^2+\sigma_{I4}^2)}\right]$$

Case 1: $n_3 = 0$

Ineq becomes

$$\frac{n_1}{n_2}\left[n_2(\mu_{I2}-\mu_{C2})\right] - \frac{n_2}{n_1}\left[n_1(\mu_{I1}-\mu_{C1})\right] >$$

$$\sqrt{2}\, z \left[\sqrt{2\left(1+\frac{n_2}{n_1}\right)^2\left[n_1(\sigma_{C1}^2+\sigma_{I1}^2)\right] + 2\left(1+\frac{n_1}{n_2}\right)^2\left[n_2(\sigma_{C2}^2+\sigma_{I2}^2)\right]} - \sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)}\right]$$

$$\Leftrightarrow n_1(\mu_{I2}-\mu_{C2}) - n_2(\mu_{I1}-\mu_{C1}) >$$

$$\sqrt{2}\, z \left[\sqrt{2\left(1+\frac{n_2}{n_1}\right)^2\left[n_1(\sigma_{C1}^2+\sigma_{I1}^2)\right] + 2\left(1+\frac{n_1}{n_2}\right)^2\left[n_2(\sigma_{C2}^2+\sigma_{I2}^2)\right]} - \sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)}\right]$$

Important special case : $n_1 = n_2 = n$ (Liu & Chamberlain (2018))

Ineq becomes $n\left[(\mu_{I2}-\mu_{C2}) - (\mu_{I1}-\mu_{C1})\right] >$

$$\sqrt{2}\, z \left[\sqrt{8n(\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2)} - \sqrt{n(\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2)}\right]$$

$$\Leftrightarrow \sqrt{n}\left[(\mu_{I2}-\mu_{C2}) - (\mu_{I1}-\mu_{C1})\right] > z(4-\sqrt{2})\sqrt{\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2}$$

$$\Leftrightarrow n > \left(\frac{(4-\sqrt{2})\, z \sqrt{\sigma_{C1}^2+\sigma_{I1}^2+\sigma_{C2}^2+\sigma_{I2}^2}}{(\mu_{I2}-\mu_{C2}) - (\mu_{I1}-\mu_{C1})}\right)^2$$

$$\left((4-\sqrt{2})z\right)^2 \simeq \left((4-\sqrt{2})\times(1.96+0.84)\right)^2 = 52.42$$

for $\alpha = 0.05$

$\pi_{min} = 0.8$

Back to the ineq,
if we divide both sides by the second square root of the RHS we have

$$\cancel{\frac{n_1}{n_2+n_3}[n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I4}-\mu_{C3})]} - \cancel{\frac{n_2}{n_1+n_3}[n_1}$$

$$\frac{n_1\dfrac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\phi}-\mu_{C3})}{n_2+n_3} - n_2\dfrac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\phi}^2)}}$$

$$> \sqrt{2}\,z\left[\sqrt{2}\sqrt{\frac{\left(1+\frac{n_2}{n_1+n_3}\right)^2\left[n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)\right]+\left(1+\frac{n_1}{n_2+n_3}\right)^2\left[n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{C3}^2+\sigma_{I\phi}^2)\right]}{n_1(\sigma_{C1}^2+\sigma_{I1}^2)+n_2(\sigma_{C2}^2+\sigma_{I2}^2)+n_3(\sigma_{I\phi}^2+\sigma_{I\phi}^2)}}-1\right] \quad (*)$$

Assume $\sigma_{C1}^2 \simeq \sigma_{I1}^2 \simeq \cdots \simeq \sigma_S^2$ i.e the variance between groups are similar we have

$$\frac{n_1\dfrac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\phi}-\mu_{C3})}{n_2+n_3} - n_2\dfrac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)}}$$

$$\sqrt{n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)} = \sqrt{2\sigma_S^2(n_1+n_2+n_3)}$$

$$> \sqrt{2}\,z\left[\sqrt{2}\sqrt{\frac{\left(1+\frac{n_2}{n_1+n_3}\right)^2\left[n_1(2\sigma_S^2)+n_3(2\sigma_S^2)\right]+\left(1+\frac{n_1}{n_2+n_3}\right)^2\left[n_2(2\sigma_S^2)+n_3(2\sigma_S^2)\right]}{n_1(2\sigma_S^2)+n_2(2\sigma_S^2)+n_3(2\sigma_S^2)}}-1\right]$$

Cancelling the $2\sigma_S^2$ on in the fraction

$$= \sqrt{2}\,z\left[\sqrt{2}\sqrt{\frac{\left(1+\frac{n_2}{n_1+n_3}\right)^2(n_1+n_3)+\left(1+\frac{n_1}{n_2+n_3}\right)^2(n_2+n_3)}{(n_1+n_2+n_3)}}-1\right]$$

$$= \sqrt{2}\,z\left[\sqrt{2}\sqrt{\left(1+\frac{n_2}{n_1+n_3}\right)^2\frac{(n_1+n_3)}{(n_1+n_2+n_3)}+\left(1+\frac{n_1}{n_2+n_3}\right)^2\frac{n_2+n_3}{(n_1+n_2+n_3)}}-1\right]$$

$$= \sqrt{2}\,z\left[\sqrt{2}\sqrt{\left(\frac{n_1+n_2+n_3}{n_1+n_3}\right)^2\left(\frac{n_1+n_3}{n_1+n_2+n_3}\right)+\left(\frac{n_1+n_2+n_3}{n_2+n_3}\right)^2\left(\frac{n_2+n_3}{n_1+n_2+n_3}\right)}-1\right]$$

$$= \sqrt{2}\,z\left[\sqrt{2}\sqrt{\frac{n_1+n_2+n_3}{n_1+n_3}+\frac{n_1+n_2+n_3}{n_2+n_3}}-1\right]$$

(*) Assume $n_1=n_2=n_3=n$:

(*) becomes
$$\frac{n\,\frac{n(\mu_{I2}-\mu_{c2})+n(\mu_{I\psi}-\mu_{c3})}{n+n}-n\,\frac{n(\mu_{I1}-\mu_{c1})+n(\mu_{I\phi}-\mu_{c3})}{n+n}}{\sqrt{n(\sigma_{c1}^2+\sigma_{I1}^2)+n(\sigma_{c2}^2+\sigma_{I2}^2)+n(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}}$$

$$>\sqrt{2}\,z\left[\sqrt{2\,\frac{(1+\frac{n}{n+n})^2[n(\sigma_{c1}^2+\sigma_{I1}^2)+n(\sigma_{c3}^2+\sigma_{I\phi}^2)]+(1+\frac{n}{n+n})^2[n(\sigma_{c2}^2+\sigma_{I2}^2)+n(\sigma_{c3}^2+\sigma_{I\psi}^2)]}{n(\sigma_{c1}^2+\sigma_{I1}^2)+n(\sigma_{c2}^2+\sigma_{I2}^2)+n(\sigma_{I\phi}^2+\sigma_{I\psi}^2)}}-1\right]$$

$$\Leftrightarrow \frac{n\left[\frac{n}{2n}[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-\frac{n}{2n}[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]\right]}{\sqrt{n\,[\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2]}}$$

$$>\sqrt{2}\,z\left[\sqrt{2\,\frac{(1+\frac{1}{2})^2 n(\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c3}^2+\sigma_{I\phi}^2)+(1+\frac{1}{2})^2 n(\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{c3}^2+\sigma_{I\psi}^2)}{n(\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2)}}-1\right]$$

$$\Leftrightarrow \sqrt{n}\,\frac{\frac{1}{2}\left[[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]\right]}{\sqrt{\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}}$$

$$>\sqrt{2}\,z\left[\sqrt{2\,\frac{(1+\frac{1}{2})^2(\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2+2\sigma_{c3}^2)}{\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}}-1\right]$$

The top can further simplify

$$\Leftrightarrow \frac{\sqrt{n}\left[[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]\right]}{\sqrt{\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}}$$

$$>2\sqrt{2}\,z\left[\sqrt{2\left(\tfrac{3}{2}\right)^2\left(1+\frac{2\sigma_{c3}^2}{\sigma_{c1}^2+\sigma_{I1}^2+\sigma_{c2}^2+\sigma_{I2}^2+\sigma_{I\phi}^2+\sigma_{I\psi}^2}\right)}-1\right]$$

$$(2\sqrt{12}[\sqrt{6}-1]z)^2$$
$$\approx 790.65604\ldots$$

If $\sigma_{c1}^2=\sigma_{I1}^2=\sigma_{c2}^2=\sigma_{I2}^2=\sigma_{I\phi}^2=\sigma_{I\psi}^2=\sigma_{c3}^2=\sigma_s^2$,

$$\frac{\sqrt{n}\left[[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]\right]}{\sqrt{6\sigma_s^2}}$$

$$>2\sqrt{2}\,z\left[\sqrt{2\,\frac{9}{4}\left(1+\frac{2\sigma_s^2}{6\sigma_s^2}\right)}-1\right]$$

$$\Leftrightarrow \frac{\sqrt{n}\left[[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]\right]}{\sqrt{6\sigma_s^2}}>2\sqrt{2}\,z\left[\sqrt{\cancel{2}\,\frac{\cancel{9}^{3}}{\cancel{4}}\,\frac{\cancel{8}^{2}}{\cancel{6}\,\cancel{2}}}-1\right]$$

$$\Leftrightarrow \sqrt{n}>\frac{2\sqrt{2}\,z[\sqrt{6}-1]\sqrt{6}\sqrt{\sigma_s^2}}{[(\mu_{I2}-\mu_{c2})+(\mu_{I\psi}-\mu_{c3})]-[(\mu_{I1}-\mu_{c1})+(\mu_{I\phi}-\mu_{c3})]}=\frac{2\sqrt{12}[\sqrt{6}-1]\,z\,\sqrt{\sigma_s^2}}{\Delta}$$

$$\frac{n_1 \dfrac{n_2(\mu_{I2}-\mu_{C2})+n_3(\mu_{I\phi}-\mu_{C3})}{n_2+n_3} - n_2 \dfrac{n_1(\mu_{I1}-\mu_{C1})+n_3(\mu_{I\phi}-\mu_{C3})}{n_1+n_3}}{\sqrt{2\,\sigma_s^2\,(n_1+n_2+n_3)}}$$

$$> \sqrt{2}\, z \left[\sqrt{2\left(\frac{n_1+n_2+n_3}{n_1+n_3} + \frac{n_1+n_2+n_3}{n_2+n_3}\right)} - 1\right]$$

$n_1 = n_2 = n, \quad n_3 = 0$

$$\frac{n\,\dfrac{n(\mu_{I2}-\mu_{C2})}{n} - n\,\dfrac{n(\mu_{I1}-\mu_{C1})}{n}}{\sqrt{2\,\sigma_s^2\,(n+n)}} > \sqrt{2}\, z \left[\sqrt{2\left(\frac{n+n}{n} + \frac{n+n}{n}\right)} - 1\right]$$

$$\Leftrightarrow \quad \frac{n\left[(\mu_{I2}-\mu_{C2})-(\mu_{I1}-\mu_{C1})\right]}{\underbrace{\sqrt{4n\,\sigma_s^2}}_{2\sqrt{n\,\sigma_s^2}}} > \sqrt{2}\, z \left[\sqrt{2(2+2)} - 1\right]$$

$$\Leftrightarrow \quad \sqrt{n} > \frac{(8-2\sqrt{2})\, z\, \cancel{\sqrt{\sigma_s^2}}\, \sigma_s}{(\mu_{I2}-\mu_{C2})-(\mu_{I1}-\mu_{C1})}$$

$$\Leftrightarrow \quad n > \underbrace{\left[(8-2\sqrt{2})\, z\right]^2}_{209.6821} \left[\frac{\sigma_s^2}{\left[(\mu_{I2}-\mu_{C2})-(\mu_{I1}-\mu_{C1})\right]^2}\right]$$