$$\mathrm{cov}(X, y) = E[(X-\mu_X)(y-\mu_y)^T] = E(Xy^T) - E(X)E(y)$$

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

$$\mathrm{cov}(X) = \mathrm{cov}([x_1 \quad x_2 \quad \cdots \quad x_n])$$

$$= \begin{bmatrix} \mathrm{cov}(x_1, x_1) & \mathrm{cov}(x_1, x_2) & \cdots & \mathrm{cov}(x_1, x_n) \\ \mathrm{cov}(x_2, x_1) & \mathrm{cov}(x_2, x_2) & \cdots & \mathrm{cov}(x_2, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(x_n, x_1) & \mathrm{cov}(x_n, x_2) & \cdots & \mathrm{cov}(x_n, x_n) \end{bmatrix}$$

代入协方差公式可得

$$\mathrm{cov}(X) = \begin{bmatrix} E[(x_1-\mu_{x_1})^T(x_1-\mu_{x_1})] & E[(x_1-\mu_{x_1})^T(x_2-\mu_{x_2})] & \cdots & E[(x_1-\mu_{x_1})^T(x_n-\mu_{x_n})] \\ E[(x_2-\mu_{x_2})^T(x_1-\mu_{x_1})] & E[(x_2-\mu_{x_2})^T(x_2-\mu_{x_2})] & \cdots & E[(x_2-\mu_{x_2})^T(x_n-\mu_{x_n})] \\ \vdots & \vdots & \ddots & \\ E[(x_n-\mu_{x_n})^T(x_1-\mu_{x_1})] & (x_n-\mu_{x_n})^T(x_2-\mu_{x_2}) & \cdots & (x_n-\mu_{x_n})^T(x_n-\mu_{x_n})] \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} (x_1-\mu_{x_1})^T(x_1-\mu_{x_1}) & (x_1-\mu_{x_1})^T(x_2-\mu_{x_2}) & \cdots & (x_1-\mu_{x_1})^T(x_n-\mu_{x_n}) \\ (x_2-\mu_{x_2})^T(x_1-\mu_{x_1}) & (x_2-\mu_{x_2})^T(x_2-\mu_{x_2}) & \cdots & (x_2-\mu_{x_2})^T(x_n-\mu_{x_n}) \\ \vdots & \vdots & \ddots & \\ (x_n-\mu_{x_n})^T(x_1-\mu_{x_1}) & (x_n-\mu_{x_n})^T(x_2-\mu_{x_2}) & \cdots & (x_n-\mu_{x_n})^T(x_n-\mu_{x_n}) \end{bmatrix}$$

样本自由度为n-1，设 $\bar{x}_i = x_i - \mu_{x_i}$，$\bar{X} = [\bar{x}_1 \quad \bar{x}_2 \quad \cdots \quad \bar{x}_n]$，

则 $\mathrm{cov}(X) = \frac{1}{m-1}\bar{X}^T\bar{X}$