# Non-overlapping domain decomposition methods for time parallel solution of PDE-constrained optimization problems

Liu-Di LU

Section of Mathematics
University of Geneva

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Séminaire de Mathématiques et de leurs Applications

The **alternating Schwarz method** is the earliest domain decomposition method invented by **H.A. Schwarz** in 1870 (About a border crossing through an alternating procedure).

Ueber einen Grenzübergang durch alternirendes Verfahren.

Von

**H. A. Schwarz.**

(Aus einem am 30. Mai gehaltenen Vortrage.)

Die unter dem Namen Dirichlet'sches Princip bekannte Schlussweise, welche in gewissem Sinne als das Fundament des von Riemann entwickelten Zweiges der Theorie der analytischen Funktionen angesehen werden muss, unterliegt, wie jetzt wohl allgemein zugestanden wird, hinsichtlich der Strenge sehr begründeten Einwendungen, deren vollständige Entfernung, soviel ich weiss, den Anstrengungen der Mathematiker bisher nicht gelungen ist.

Problem: Show existence of harmonic functions

$$\Delta y = 0 \text{ in } \Omega, \quad y = g \text{ on } \partial\Omega.$$

Available tools: **Fourier (1807)** for rectangular domain $\Omega$ and **Poisson (1815)** for circular domain $\Omega$.

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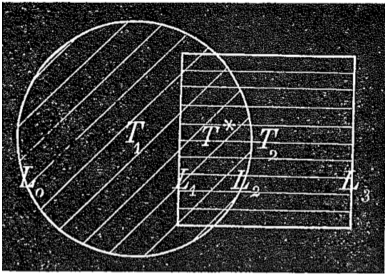
A *door handle* type domain $\Omega$




Prove convergence of this iterative method with maximum principle.

# Interpretation and illustration

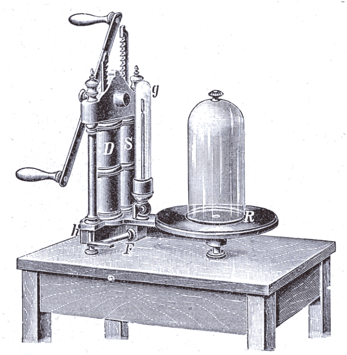
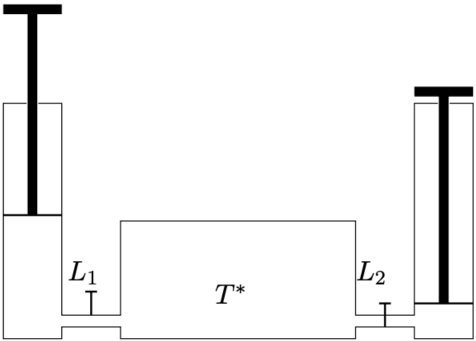
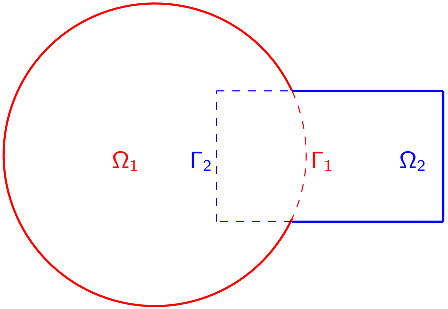



Fig. 103. Zweistiefelige Hahnluftpumpe



**Domain**: $\Omega = \Omega_1 \cup \Omega_2$



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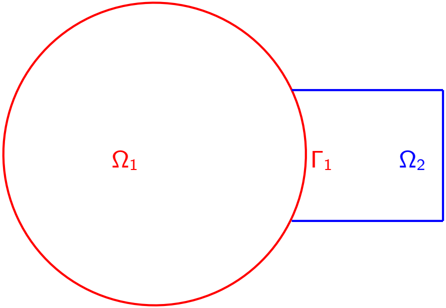
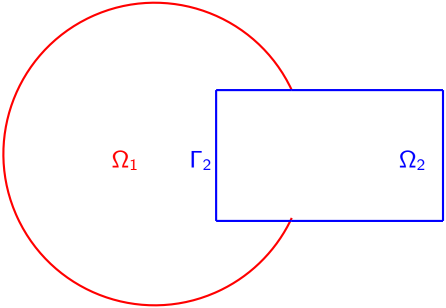



$$
\begin{aligned}
\Delta y_1^1 &= 0 && \text{in } \Omega_1, \\
y_1^1 &= g && \text{on } \partial\Omega \cap \bar{\Omega}_1, \\
y_1^1 &= y_2^0 && \text{on } \Gamma_1
\end{aligned}
$$

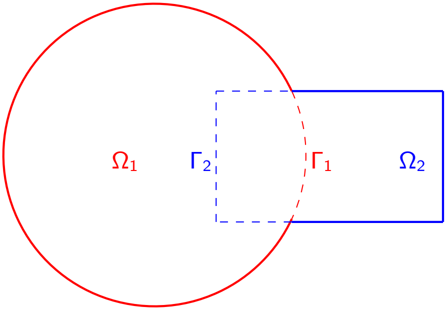
**Domain**: $\Omega = \Omega_1 \cup \Omega_2$




$$\begin{aligned} \Delta y_2^1 &= 0 && \text{in } \Omega_2, \\ y_2^1 &= g && \text{on } \partial\Omega \cap \bar{\Omega}_2, \\ y_2^1 &= y_1^1 && \text{on } \Gamma_2 \end{aligned}$$

# Interpretation and illustration

**Domain**: $\Omega = \Omega_1 \cup \Omega_2$




$$\begin{aligned} \Delta y_1^\ell &= 0 && \text{in } \Omega_1, \\ y_1^\ell &= g && \text{on } \partial\Omega \cap \bar{\Omega}_1, \\ y_1^\ell &= y_2^{\ell-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta y_2^\ell &= 0 && \text{in } \Omega_2, \\ y_2^\ell &= g && \text{on } \partial\Omega \cap \bar{\Omega}_2, \\ y_2^\ell &= y_1^\ell && \text{on } \Gamma_2 \end{aligned}$$

# Towards a computational tool

- **Miller (1965)**: Numerical analogs to the Schwarz alternating procedure.
- **Dryja and Widlund (1987)**: An additive variant of the Schwarz alternating method for the case of many subregions.
- **Lions (1988,1989,1990)**: On the Schwarz alternating method I, II, III.
- **Quarteroni and Valli (1999)**: Domain decomposition methods for partial differential equations.
- **Smith, Bjorstad and Gropp (2004)**: Domain decomposition: parallel multilevel methods for elliptic partial differential equations.
- **Toselli and Widlund (2006)**: Domain decomposition methods-algorithms and theory.
- **Dolean, Jolivet and Nataf**: An Introduction to domain decomposition methods: algorithms, theory, and parallel Implementation.

Some domain decomposition methods: **alternating Schwarz**, **parallel Schwarz**, optimized Schwarz, **Dirichlet–Neumann**, Neumann–Neumann (BDD), FETI (Dirichlet–Dirichlet), ...

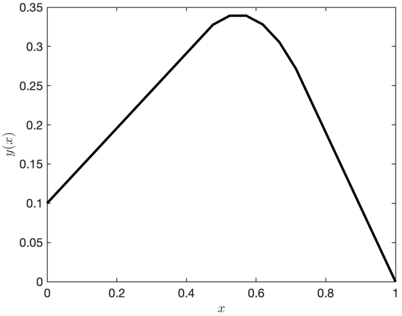
# Alternating Schwarz method

**Problem**:

$$-\partial_{xx} y = f \quad \text{in } \Omega = (0,1),$$
$$y(0) = 0.1, \quad y(1) = 0,$$

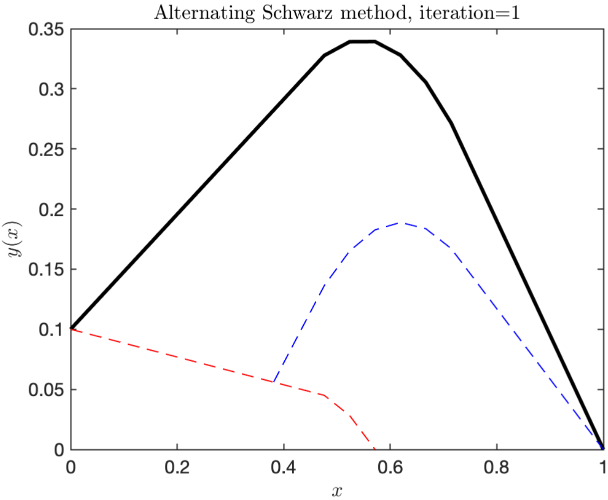
with

$$f = \begin{cases} 5 & \text{if } 0.4 < x < 0.7, \\ 0 & \text{else.} \end{cases}$$



# Alternating Schwarz method

**Subdomains**: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **an initial guess**: $y_2^0(0.57) = 0$.



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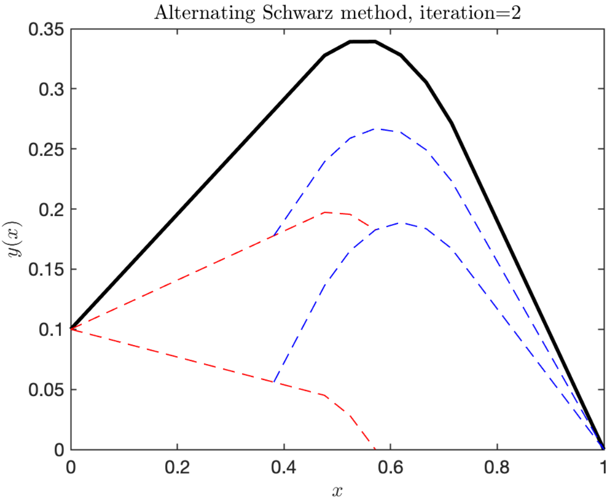

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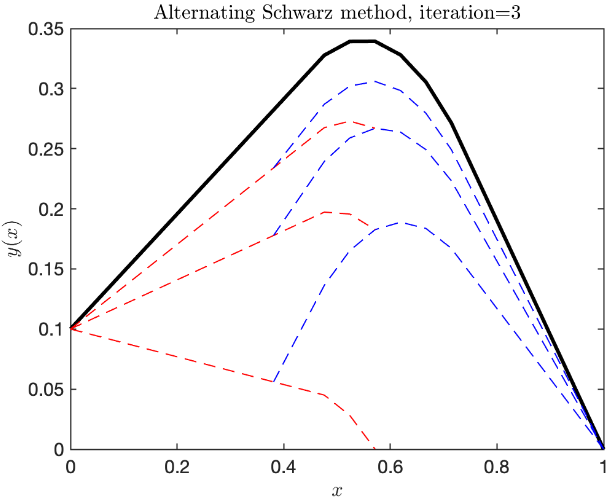

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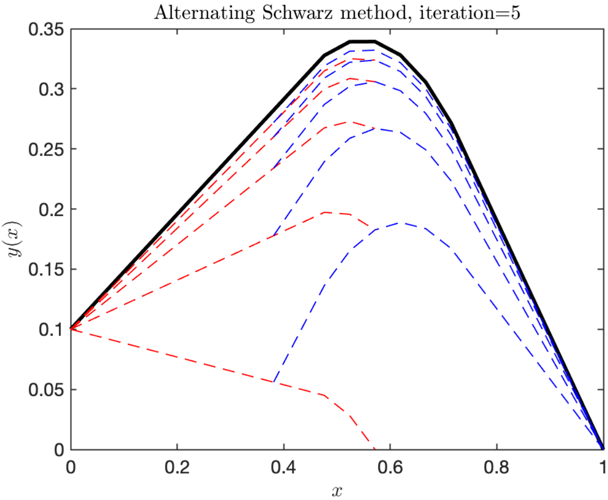

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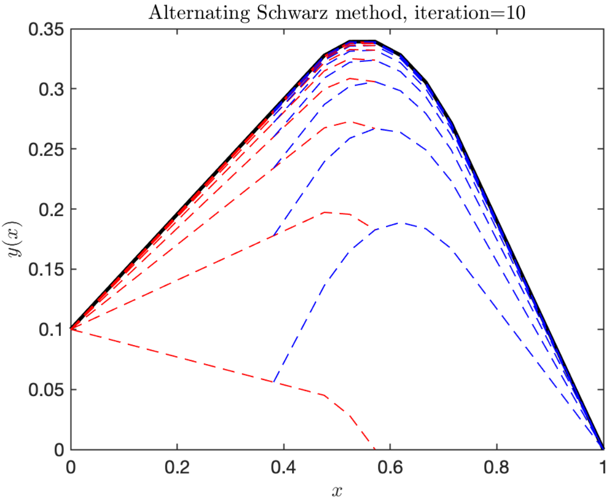

**Problem**:

$$-\partial_{xx} y = f \quad \text{in } \Omega = (0,1),$$
$$y(0) = g_l, \quad y(1) = g_r.$$

**Subdomains**: $\Omega_1 = (0, b)$ and $\Omega_2 = (a, 1)$ with $b > a$, and **an initial guess**: $y_2^0$, we solve

$$\begin{aligned} -\partial_{xx} y_1^\ell &= f \quad \text{in } \Omega_1, & -\partial_{xx} y_2^\ell &= f \quad \text{in } \Omega_2, \\ y_1^\ell(0) &= g_l, & y_2^\ell(1) &= g_r, \\ y_1^\ell(b) &= y_2^{\ell-1}(b), & y_2^\ell(a) &= y_1^\ell(a). \end{aligned}$$

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Denote the error $e_j^\ell := y - y_j^\ell$ that satisfies

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Analytical solutions: $e_1^\ell(x) = C_1^\ell x$ and $e_2^\ell(x) = C_2^\ell(x-1)$.

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Analytical solutions: $e_1^\ell(x) = C_1^\ell x$ and $e_2^\ell(x) = C_2^\ell(x-1)$.
Evaluation of the coefficients $C_1^\ell$ and $C_2^\ell$:

$$
C_1^\ell = \frac{e_2^{\ell-1}(b)}{b}, \qquad C_2^\ell = \frac{e_1^\ell(a)}{a-1} = e_2^{\ell-1}(b)\frac{1}{a-1}\frac{a}{b}.
$$

The convergence factor

$$
e_2^\ell(b) = \rho(a,b)e_2^{\ell-1}(b), \quad \rho(a,b) := \frac{1-b}{1-a}\frac{a}{b}.
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# Convergence analysis

Denote the error $e_j^\ell := y - y_j^\ell$ that satisfies

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e_1^\ell(0) &= 0, & e_2^\ell(1) &= 0,\\
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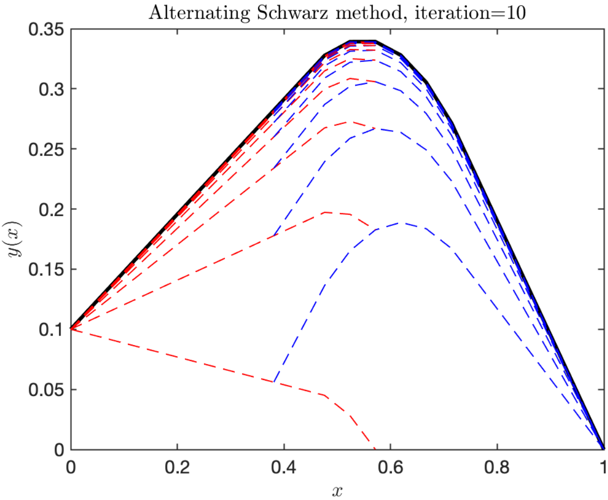
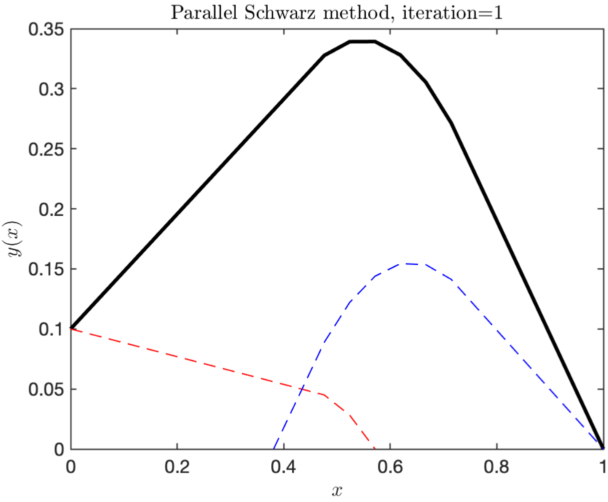
(i) The alternating Schwarz method always converges when $b > a$.
(ii) The larger the overlap size $b-a$, the better the convergence.
(iii) The alternating Schwarz method **does not converge** when $a = b$ !

**Subdomains**: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **an initial guess**: $y_2^0(0.57) = 0$.



# Parallel Schwarz method

**Subdomains**: $\Omega_1 = (0, 0.57)$ and $\Omega_2 = (0.38, 1)$ with **two initial guesses**: $y_2^0(0.57) = 0$ and $y_1^0(0.38) = 0$.



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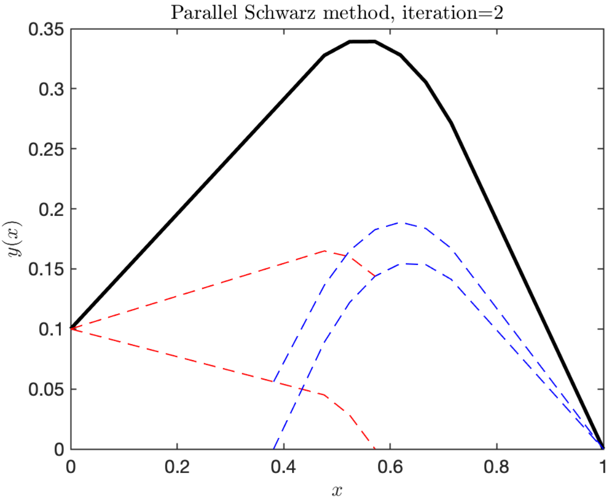

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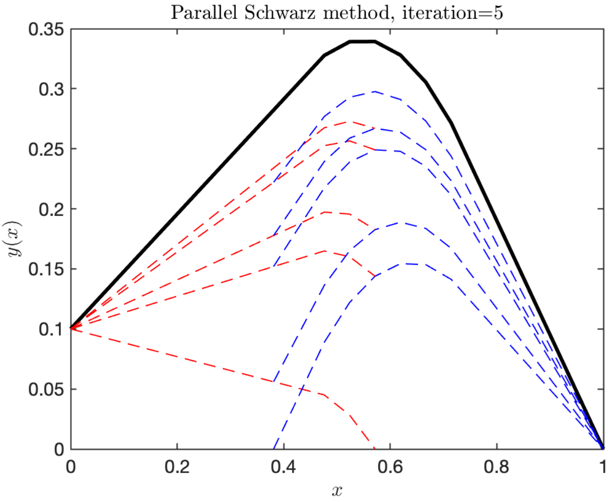

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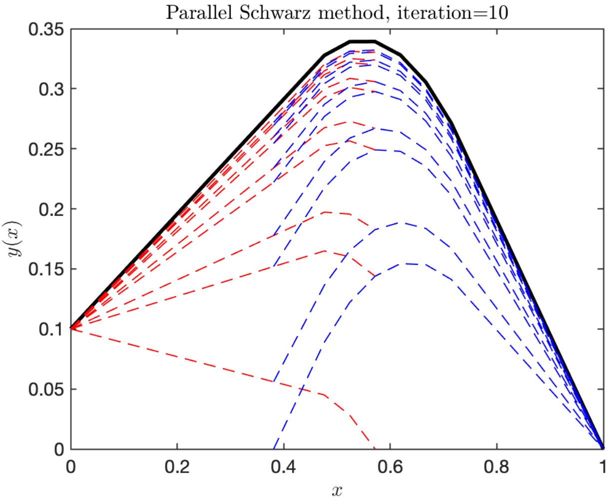

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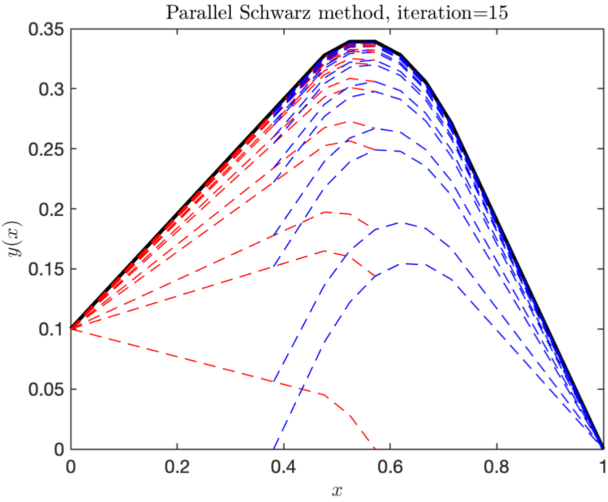

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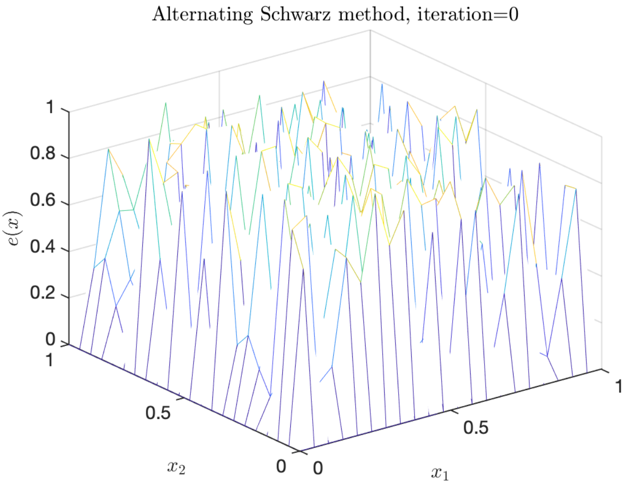
The convergence factor is now

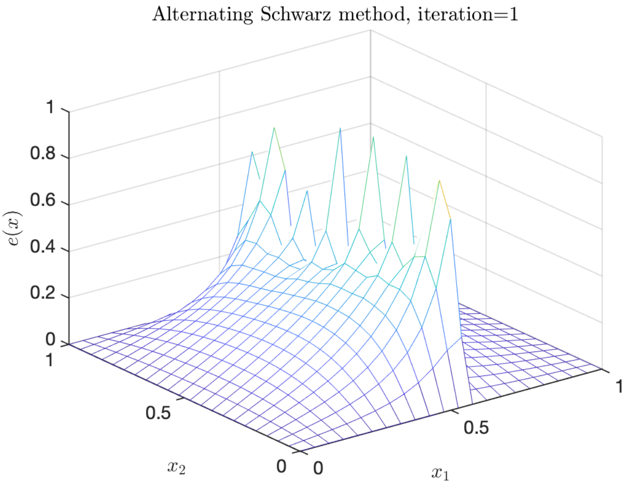
$$e_2^{\ell+1}(b) = \frac{1-b}{1-a}\frac{a}{b}\, e_2^{\ell-1}(b).$$

# Smoother



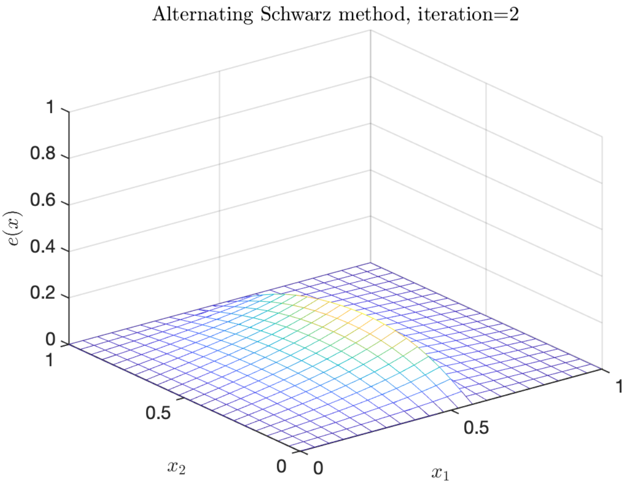
Alternating Schwarz method, iteration=0
1
0.8
0.6
0.4
0.2
0
$e(x)$
1
0.5
0
$x_2$
0
0.5
1
$x_1$

# Smoother



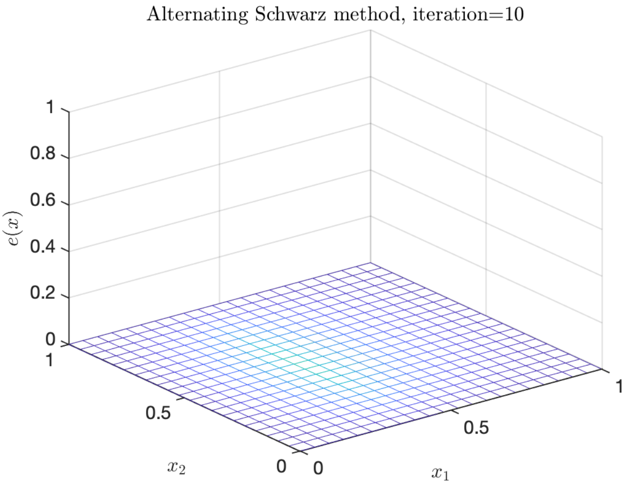
Alternating Schwarz method, iteration=1
$e(x)$
1
0.8
0.6
0.4
0.2
0
1
0.5
$x_2$
0
0
0.5
$x_1$
1

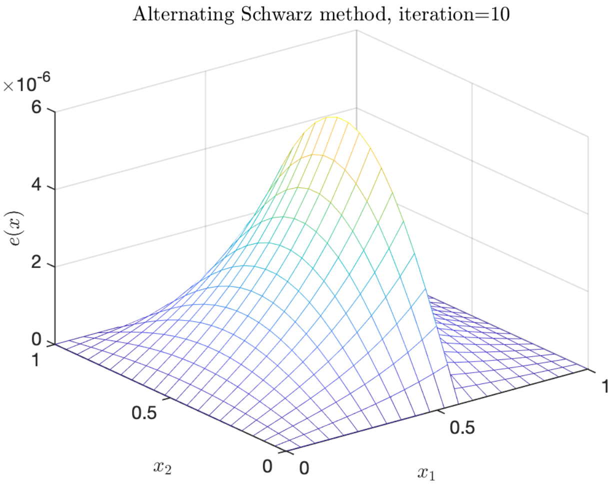
# Smoother



Alternating Schwarz method, iteration=2
$e(x)$
1
0.8
0.6
0.4
0.2
0
1
0.5
0
$x_2$
0
0.5
1
$x_1$

# Smoother


Alternating Schwarz method, iteration=10
1
0.8
0.6
0.4
0.2
0
$e(x)$
1
0.5
0
$x_2$
0
0.5
1
$x_1$


Alternating Schwarz method, iteration=10
×10^-6
6
4
2
0
e(x)
1
0.5
0
x_2
0
0.5
1
x_1

# Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u \qquad \text{in } Q := (0,T) \times \Omega, \\ y &= 0 \qquad \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 \qquad \text{on } \Sigma_0 := \{0\} \times \Omega, \end{aligned}$$

with $\Omega \subset \mathbb{R}^n$.

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Lagrange multipliers method

$$\mathcal{L}(y, u, \lambda) = J(y, u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

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$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u \quad \text{in } Q, & \partial_t \lambda + \Delta \lambda &= y - \hat{y} \quad \text{in } Q, \\ y &= 0 \quad \text{in } \Sigma, & \lambda &= 0 \quad \text{in } \Sigma, \\ y &= y_0 \quad \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) \quad \text{in } \Sigma_T := \{T\} \times \Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

# Model problem

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$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q, & \partial_t \lambda + \Delta \lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T := \{T\} \times \Omega, \end{aligned}$$

$$-\lambda + \nu u = 0 \quad \text{in } Q.$$

# Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y - \hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T) - \hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u \qquad \text{in } Q := (0,T) \times \Omega, \\ y &= 0 \qquad \text{on } \Sigma := (0,T) \times \partial\Omega, \\ y &= y_0 \qquad \text{on } \Sigma_0 := \{0\} \times \Omega, \end{aligned}$$

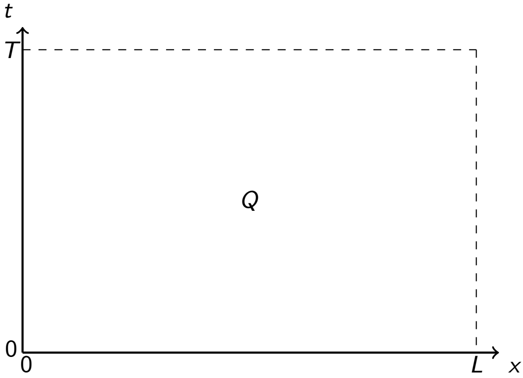
with $\Omega \subset \mathbb{R}^n$.

Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda \rangle.$$

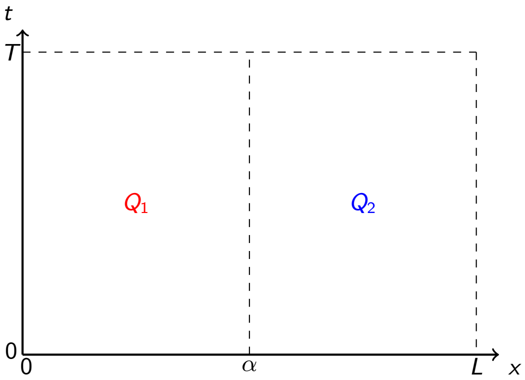
Reduced optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda && \text{in } Q, & \partial_t \lambda + \Delta \lambda &= y - \hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \lambda &= -\gamma(y - \hat{y}) && \text{in } \Sigma_T. \end{aligned}$$



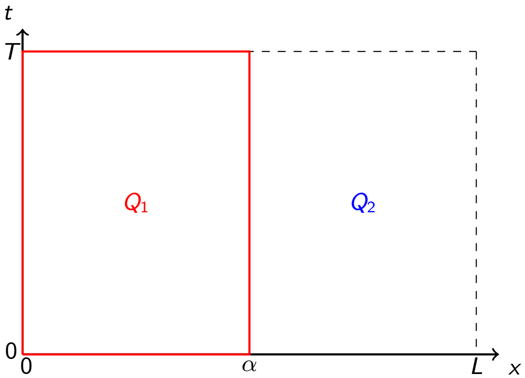

Space-time domain: $Q = (0, L) \times (0, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0,t) &= 0, & \lambda(0,t) &= 0, \\ y(L,t) &= 0, & \lambda(L,t) &= 0, \\ y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T), \end{aligned}$$



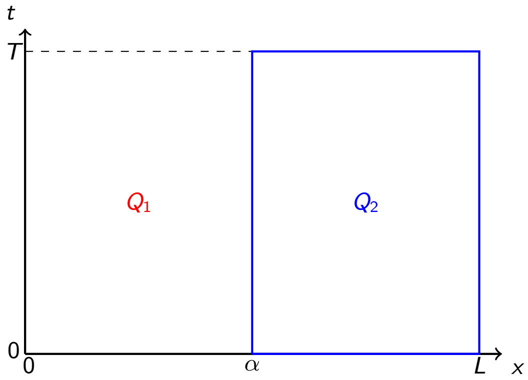

**Subdomains**: $Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$,

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1} \lambda, & \partial_t \lambda + \partial_{xx} \lambda &= y - \hat{y}, \\
y(0, t) &= 0, & \lambda(0, t) &= 0, \\
y(L, t) &= 0, & \lambda(L, t) &= 0, \\
y(x, 0) &= y_0(x), & \lambda(x, T) + \gamma y(x, T) &= \gamma \hat{y}(x, T),
\end{aligned}
$$




**Subdomain**: $Q_1 = (0, \alpha) \times (0, T)$,

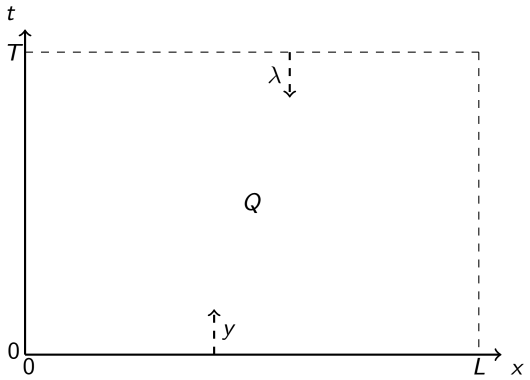
$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1}\lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx}\lambda_1^\ell &= y_1^\ell - \hat{y}_1,\\
y_1^\ell(0,t) &= 0, & \lambda_1^\ell(0,t) &= 0,\\
y_1^\ell(\alpha,t) &= y_2^{\ell-1}(\alpha,t), & \lambda_1^\ell(\alpha,t) &= \lambda_2^{\ell-1}(\alpha,t),\\
y_1^\ell(x,0) &= y_{1,0}(x), & \lambda_1^\ell(x,T) + \gamma y_1^\ell(x,T) &= \gamma\hat{y}_1(x,T).
\end{aligned}
$$




**Subdomains**: $Q_2 = (\alpha, 1) \times (0, T)$,

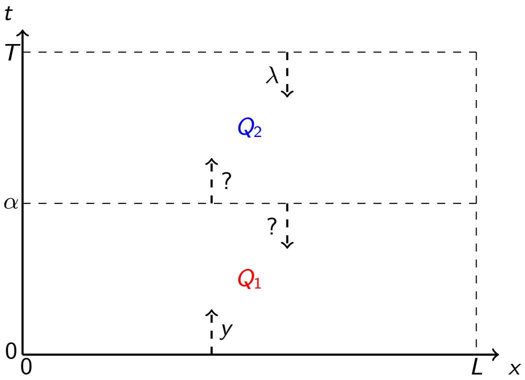
$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} \lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2, \\ \partial_x y_2^\ell(\alpha, t) &= \partial_x y_1^\ell(\alpha, t), & \partial_x \lambda_2^\ell(\alpha, t) &= \partial_x \lambda_1^\ell(\alpha, t), \\ y_2^\ell(L, t) &= 0, & \lambda_2^\ell(L, t) &= 0, \\ y_2^\ell(x, 0) &= y_{2,0}(x), & \lambda_2^\ell(x, T) + \gamma y_2^\ell(x, T) &= \gamma \hat{y}_2(x, T). \end{aligned}$$

# Time domain decomposition



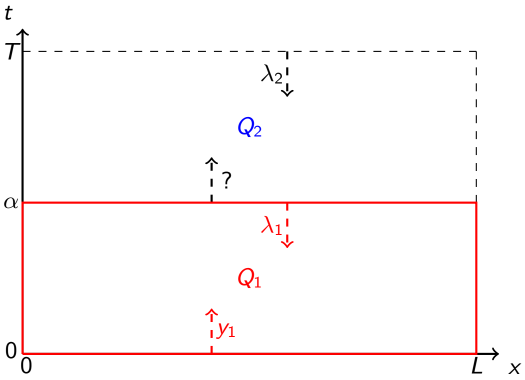

Space-time domain: $Q = (0, L) \times (0, T)$,

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y},\\
y(0,t) &= 0, & \lambda(0,t) &= 0,\\
y(L,t) &= 0, & \lambda(L,t) &= 0,\\
y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T).
\end{aligned}
$$



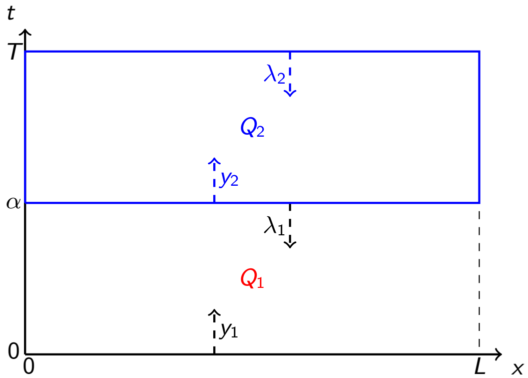

**Subdomains**: $Q_1 = (0, L) \times (0, \alpha)$ and $Q_2 = (0, L) \times (\alpha, T)$,

$$\begin{aligned} \partial_t y - \partial_{xx} y &= \nu^{-1}\lambda, & \partial_t \lambda + \partial_{xx}\lambda &= y - \hat{y}, \\ y(0, t) &= 0, & \lambda(0, t) &= 0, \\ y(L, t) &= 0, & \lambda(L, t) &= 0, \\ y(x, 0) &= y_0(x), & \lambda(x, T) + \gamma y(x, T) &= \gamma \hat{y}(x, T). \end{aligned}$$




**Subdomain**: $Q_1 = (0, L) \times (0, \alpha)$,

$$
\begin{aligned}
\partial_t y_1^\ell - \partial_{xx} y_1^\ell &= \nu^{-1} \lambda_1^\ell, & \partial_t \lambda_1^\ell + \partial_{xx} \lambda_1^\ell &= y_1^\ell - \hat{y}_1, \\
y_1^\ell(0, t) &= 0, & \lambda_1^\ell(0, t) &= 0, \\
y_1^\ell(L, t) &= 0, & \lambda_1^\ell(L, t) &= 0, \\
y_1^\ell(x, 0) &= y_0(x), & \lambda_1^\ell(x, \alpha) &= \lambda_2^{\ell-1}(x, \alpha),
\end{aligned}
$$




**Subdomain**: $Q_2 = (0, L) \times (\alpha, T)$,

$$\begin{aligned} \partial_t y_2^\ell - \partial_{xx} y_2^\ell &= \nu^{-1} \lambda_2^\ell, & \partial_t \lambda_2^\ell + \partial_{xx} \lambda_2^\ell &= y_2^\ell - \hat{y}_2, \\ y_2^\ell(0, t) &= 0, & \lambda_2^\ell(0, t) &= 0, \\ y_2^\ell(L, t) &= 0, & \lambda_2^\ell(L, t) &= 0, \\ \partial_t y_2^\ell(x, \alpha) &= \partial_t y_1^\ell(x, \alpha), & \lambda_2^\ell(x, T) + \gamma y_2^\ell(x, T) &= \gamma \hat{y}_2(x, T). \end{aligned}$$

Reduced optimality system:

$$\begin{cases} \partial_t \begin{pmatrix} y \\ \lambda \end{pmatrix} + \begin{pmatrix} -\partial_{xx} y - \nu^{-1}\lambda \\ -y + \partial_{xx}\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ y(\cdot, 0) = y_0(\cdot), \\ \lambda(\cdot, T) + \gamma y(\cdot, T) = \gamma \hat{y}(\cdot, T), \end{cases} \quad \text{in } (0, L) \times (0, T)$$

# Semi-discretization

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0, T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

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Diagonalization: $A = PDP^{-1}$ and $D = \operatorname{diag}(d_1, \dots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So $n$ **independent** $2 \times 2$ **systems**.

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Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

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with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So $n$ **independent** $2 \times 2$ **systems**.

Second-order ODE:

$$\begin{cases} \ddot{z}_i - \sigma_i^2 z_i = -\nu^{-1} \hat{z}_i \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \dot{z}_i(T) + \omega_i z_i(T) = \nu^{-1} \gamma \hat{z}_i(T), \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Finite difference discretization: $-\partial_{xx} \approx A \in \mathbb{R}^{n\times n}$,

$$\begin{cases} \begin{pmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{\lambda}} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\boldsymbol{y}} \end{pmatrix} \text{ in } (0,T), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \boldsymbol{\lambda}(T) + \gamma \boldsymbol{y}(T) = \gamma \hat{\boldsymbol{y}}(T), \end{cases}$$

Diagonalization: $A = PDP^{-1}$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$,

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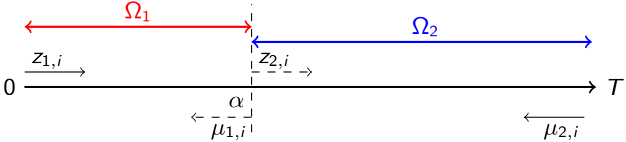
with $\boldsymbol{z} = P^{-1}\boldsymbol{y}$, $\hat{\boldsymbol{z}} = P^{-1}\hat{\boldsymbol{y}}$ and $\boldsymbol{\mu} = P^{-1}\boldsymbol{\lambda}$. So $n$ **independent** $2\times 2$ **systems**.
Second-order ODE:

$$\begin{cases} \ddot{\mu}_i - \sigma_i^2 \mu_i = -(\dot{\hat{z}}_i + d_i \hat{z}_i) \text{ in } (0,T), \\ \mu_i(0) - d_i \mu_i(0) = z_{0,i} - \hat{z}_i(0), \\ \gamma \dot{\mu}_i(T) + \beta_i \mu_i(T) = 0, \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1}\gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.



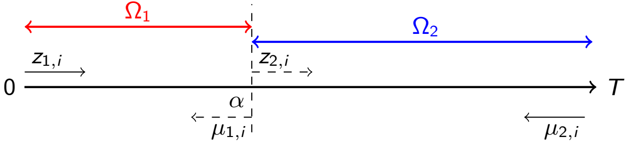

Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

# Transformation




Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

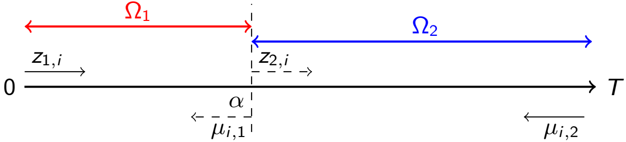
Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).$$

# Transformation




Dirichlet:

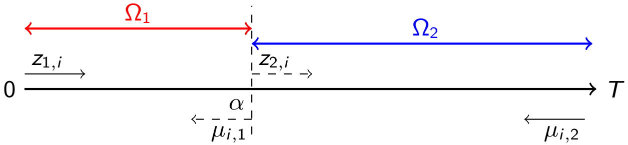
$$\begin{cases} \ddot{z}_{1,i}^{\ell} - \sigma_i^2 z_{1,i}^{\ell} = 0 \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \dot{z}_{1,i}^{\ell}(\alpha) + d_i z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{z}_{2,i}^{\ell} - \sigma_i^2 z_{2,i}^{\ell} = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \dot{z}_{2,i}^{\ell}(T) + \omega_i z_{2,i}^{\ell}(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^{\ell} = (1-\theta) f_{\alpha,i}^{\ell-1} + \theta\big(\dot{z}_{2,i}^{\ell}(\alpha) + d_i z_{2,i}^{\ell}(\alpha)\big).$$




Dirichlet:

$$\begin{cases} \ddot{\mu}^{\ell}_{1,i} - \sigma_i^2 \mu^{\ell}_{1,i} = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu^{\ell}_{1,i}(\alpha) = f^{\ell-1}_{\alpha,i}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{\mu}^{\ell}_{2,i} - \sigma_i^2 \mu^{\ell}_{2,i} = 0 \text{ in } \Omega_2, \\ \ddot{\mu}^{\ell}_{2,i}(\alpha) - d_i \dot{\mu}^{\ell}_{2,i}(\alpha) = \ddot{\mu}^{\ell}_{1,i}(\alpha) - d_i \dot{\mu}^{\ell}_{1,i}(\alpha), \\ \gamma \dot{\mu}^{\ell}_i(T) + \beta_i \mu^{\ell}_i(T) = 0, \end{cases}$$

Update:

$$f^{\ell}_{\alpha,i} = (1-\theta) f^{\ell-1}_{\alpha,i} + \theta \mu^{\ell}_{2,i}(\alpha), \quad \theta \in (0,1).$$

# Transformation

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\
z_{1,i}^{\ell}(0) = 0, \\
\mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1},
\end{cases}
$$

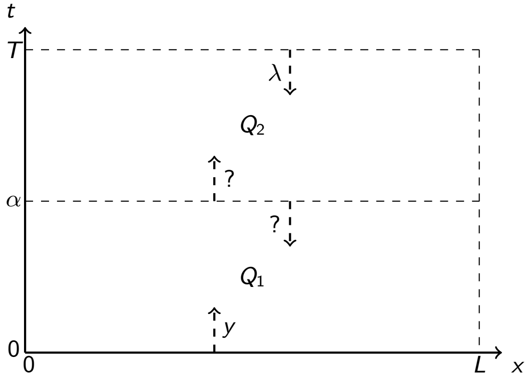
$$
\begin{cases}
\begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\
\dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\
\mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0,
\end{cases}
$$

$$
f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha), \quad \theta \in (0,1).
$$

**Two observations**:

(1) Three systems are equivalent, so same convergence using $z$ or $\mu$;

(2) Not anymore a "DN" algorithm, and forward-backward structure is less important.




Space-time domain: $Q = (0, L) \times (0, T)$,

$$
\begin{aligned}
\partial_t y - \Delta y &= \nu^{-1}\lambda, & \partial_t \lambda + \Delta \lambda &= y - \hat{y},\\
y(0,t) &= 0, & \lambda(0,t) &= 0,\\
y(L,t) &= 0, & \lambda(L,t) &= 0,\\
y(x,0) &= y_0(x), & \lambda(x,T) + \gamma y(x,T) &= \gamma \hat{y}(x,T).
\end{aligned}
$$

# Variants of DN and ND algorithms

| Category | $\Omega_1$ | $\Omega_2$ | type |
|---|---|---|---|
| $(z_i, \mu_i)$ | $\mu_i$ | $\dot{z}_i$ | (DN) |
| | $\dot{z}_i + d_i z_i$ | $\dot{z}_i$ | (RN) |
| | $\dot{\mu}_i$ | $z_i$ | (ND) |
| | $\ddot{z}_i + d_i \dot{z}_i$ | $z_i$ | (RD) |
| $z_i$ | $z_i$ | $\dot{z}_i$ | (DN) |
| | $z_i$ | $\dot{z}_i$ | (DN) |
| | $\dot{z}_i$ | $z_i$ | (ND) |
| | $\dot{z}_i$ | $z_i$ | (ND) |
| $\mu_i$ | $\mu_i$ | $\dot{\mu}_i$ | (DN) |
| | $\dot{z}_i + d_i z_i$ | $\ddot{z}_i + d_i \dot{z}_i$ | (RR) |
| | $\dot{\mu}_i$ | $\mu_i$ | (ND) |
| | $\ddot{z}_i + d_i \dot{z}_i$ | $\dot{z}_i + d_i z_i$ | (RR) |

# Comparaison of two DN variants

Natural Dirichlet–Neumann ($DN_1$):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^{\ell} \\ \dot{\mu}_{1,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^{\ell} \\ \mu_{1,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^{\ell}(0) = 0, \\ \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^{\ell} \\ \dot{\mu}_{2,i}^{\ell} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^{\ell} \\ \mu_{2,i}^{\ell} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^{\ell}(\alpha) = \dot{z}_{1,i}^{\ell}(\alpha), \\ \mu_{2,i}^{\ell}(T) + \gamma z_{2,i}^{\ell}(T) = 0, \end{cases}$$

$$f_{\alpha,i}^{\ell} := (1-\theta) f_{\alpha,i}^{\ell-1} + \theta \mu_{2,i}^{\ell}(\alpha).$$

Dirichlet–Neumann at two levels ($DN_2$):

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**Forward–backward structure can always be recovered !**

$$z_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1} \Rightarrow \dot{\mu}_{1,i}^{\ell}(\alpha) - d_i \mu_{1,i}^{\ell}(\alpha) = f_{\alpha,i}^{\ell-1}.$$

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Dirichlet–Neumann at two levels ($\text{DN}_2$):

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Solve the problem and find

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Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left( 1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\left(\sigma_i + d_i \tanh(a_i)\right)\left(\omega_i + \sigma_i \tanh(b_i)\right)} \right) \right|,$$

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"Optimal" relaxation parameter with equioscillation

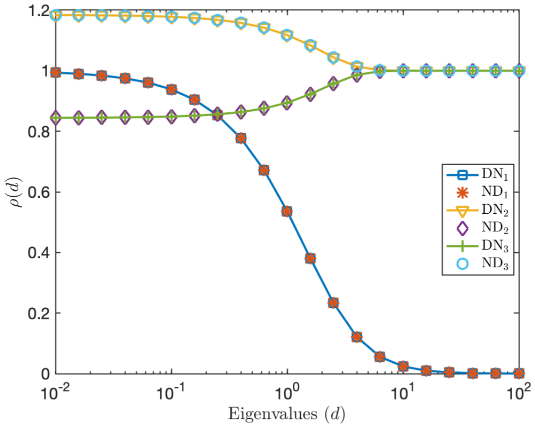
$$\theta^*_{\mathrm{DN}_2} = \frac{2}{3 + \coth(\sqrt{\nu^{-1}}\alpha)\frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$$\theta^*_{\mathrm{ND}_2} = \frac{2}{3 + \tanh(\sqrt{\nu^{-1}}\alpha)\frac{\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}}\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

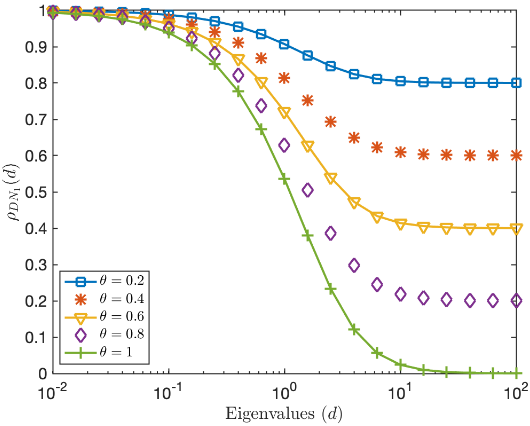
$\theta^*_{\mathrm{DN}_2} = \theta^*_{\mathrm{ND}_3}$ and $\theta^*_{\mathrm{ND}_2} = \theta^*_{\mathrm{DN}_3}$.

# Numerical experiments

Convergence factor of different DN and ND variants, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$, and relaxation parameter: $\theta = 1$.

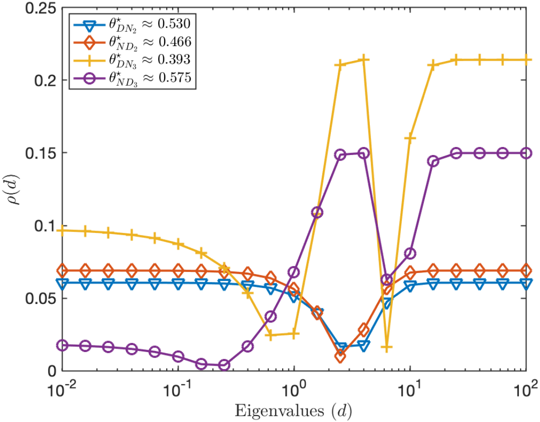


Convergence factor of different $\mathrm{DN}_1$, with penalization parameters: $\nu = 0.1$, $\gamma = 0$, interface: $\alpha = \frac{T}{2}$.



# Numerical experiments

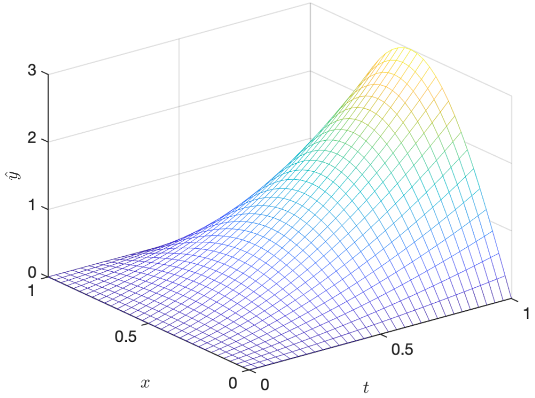
Optimal convergence factors with penalization parameters: $\nu = 0.1$, $\gamma = 10$ and interface: $\alpha = \frac{7}{10}T$.



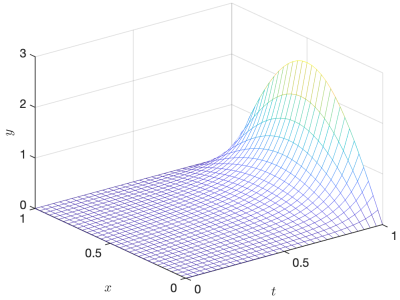

$\theta^{\star}_{\mathrm{DN}_2} = \theta^{*}_{\mathrm{DN}_2}$ and $\theta^{\star}_{\mathrm{ND}_2} = \theta^{*}_{\mathrm{ND}_2}$.

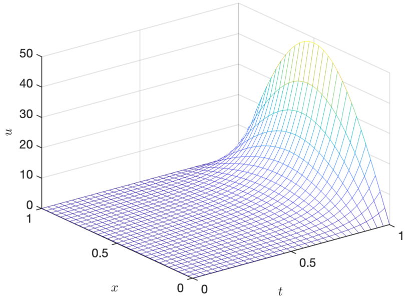
# Numerical experiments

Numerical tests with penalization parameters: $\nu = 0.1$, $\gamma = 10$, final time: $T = 1$, and a target function $\hat{y}(t, x) = \sin(\pi x)(2t^2 + t)$.



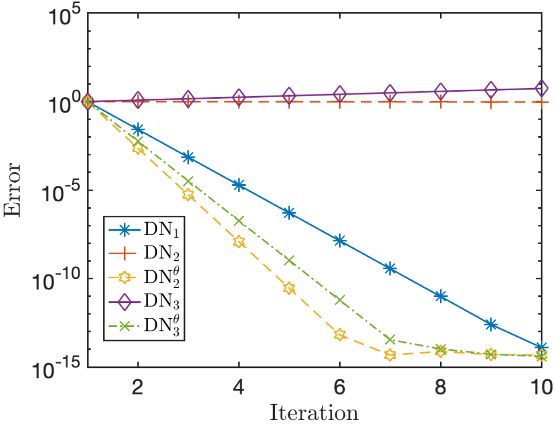
Numerical scheme: Crank-Nicolson, and mesh size: $h_t = h_x = \frac{1}{32}$.



# Numerical experiments

With an interface $\alpha = \frac{7}{10}T$.



# Conclusion

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- Like always, there are a lot of interesting things to be further discovered ;)

**Thanks for your attention !**