Time domain decomposition methods for parabolic optimal control problems

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Joint work with Martin J. Gander

Model problem

For $\hat{y} \in L^2(Q)$, $\gamma \geq 0$ and $\nu > 0$, minimize the cost functional

$$J(y,u) := \frac{1}{2}\|y-\hat{y}\|^2_{L^2(Q)} + \frac{\gamma}{2}\|y(T)-\hat{y}(T)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

subject to

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q := (0,T)\times\Omega, \\ y &= 0 && \text{on } \Sigma := (0,T)\times\partial\Omega, \\ y &= y_0 && \text{on } \Sigma_0 := \{0\}\times\Omega, \end{aligned}$$

with $\Omega \subset \mathbb{R}^n$.

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Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda\rangle.$$

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First-order optimality system:

$$\begin{aligned} \partial_t y - \Delta y &= u && \text{in } Q, & \qquad \partial_t \lambda + \Delta\lambda &= y-\hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \qquad \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \qquad \lambda &= -\gamma(y-\hat{y}) && \text{in } \Sigma_T := \{T\}\times\Omega, \end{aligned}$$

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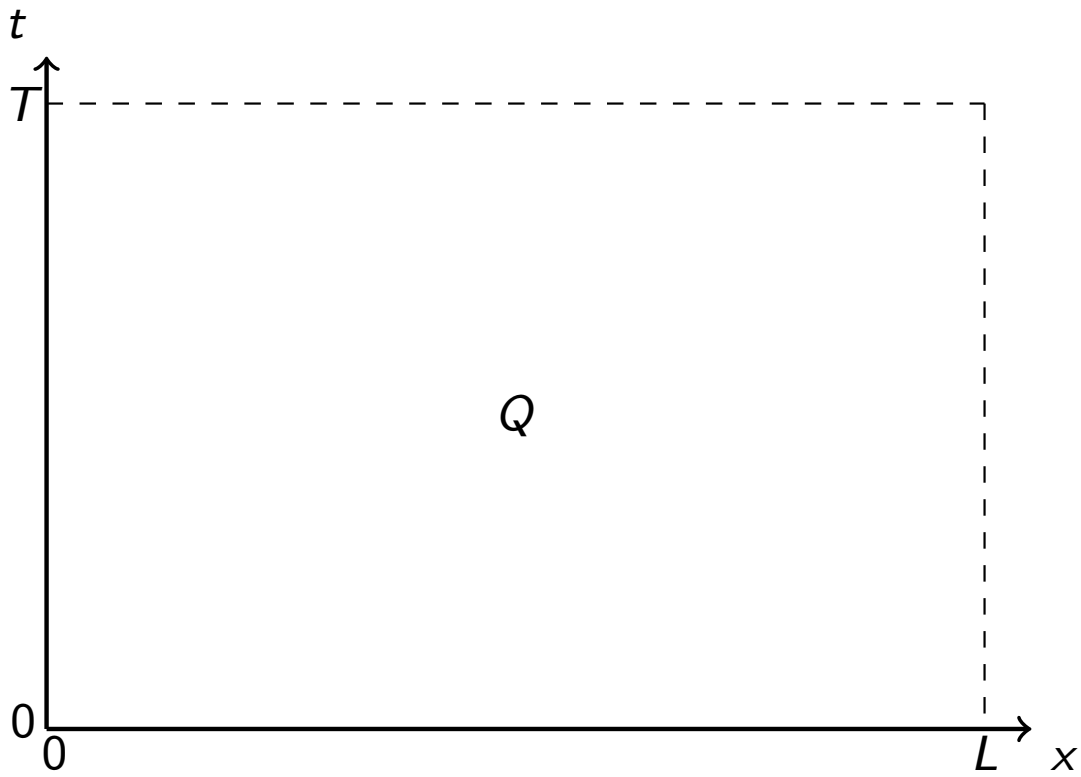
Lagrange multipliers method

$$\mathcal{L}(y,u,\lambda) = J(y,u) + \langle \partial_t y - \Delta y - u, \lambda\rangle.$$

Reduced optimality system (forward-backward):

$$\begin{aligned} \partial_t y - \Delta y &= \nu^{-1}\lambda && \text{in } Q, & \qquad \partial_t \lambda + \Delta\lambda &= y-\hat{y} && \text{in } Q, \\ y &= 0 && \text{in } \Sigma, & \qquad \lambda &= 0 && \text{in } \Sigma, \\ y &= y_0 && \text{in } \Sigma_0, & \qquad \lambda &= -\gamma(y-\hat{y}) && \text{in } \Sigma_T. \end{aligned}$$

Dirichlet-Neumann Waveform Relaxation



$Q = (0, L) \times (0, T)$,

$$\partial_t y - \Delta y = \nu^{-1}\lambda, \qquad \partial_t \lambda + \Delta \lambda = y - \hat{y},$$
$$y(0,t) = 0, \qquad \lambda(0,t) = 0,$$
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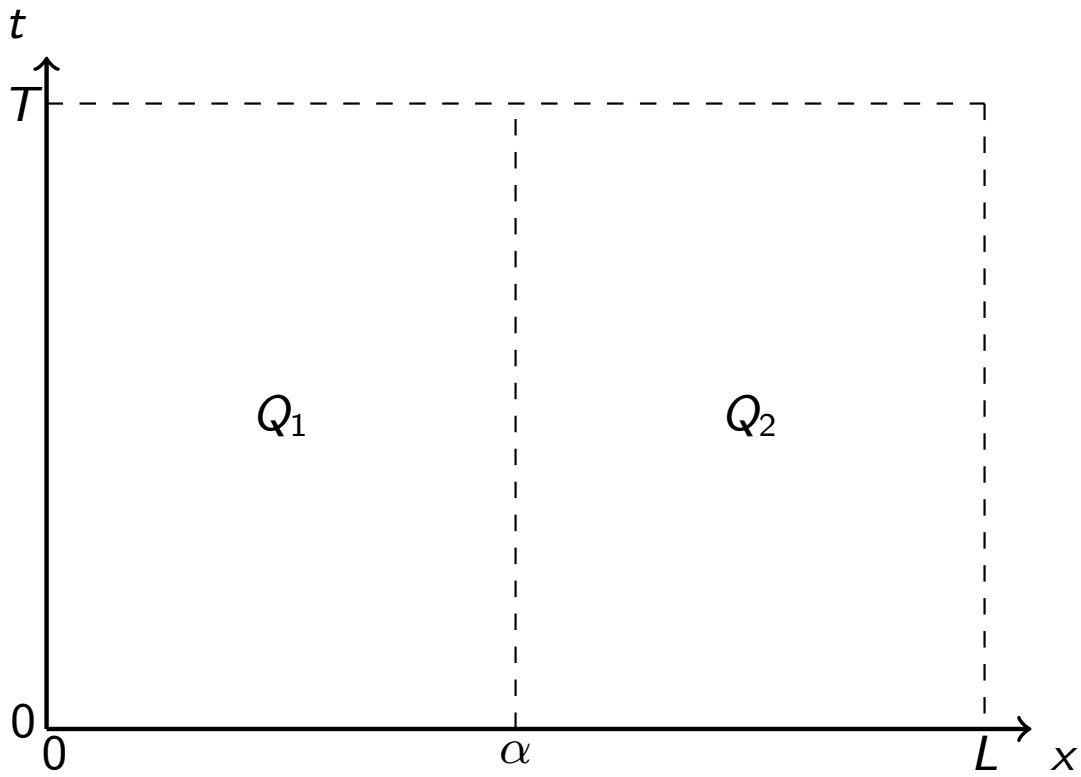

$Q_1 = (0, \alpha) \times (0, T)$ and $Q_2 = (\alpha, L) \times (0, T)$,

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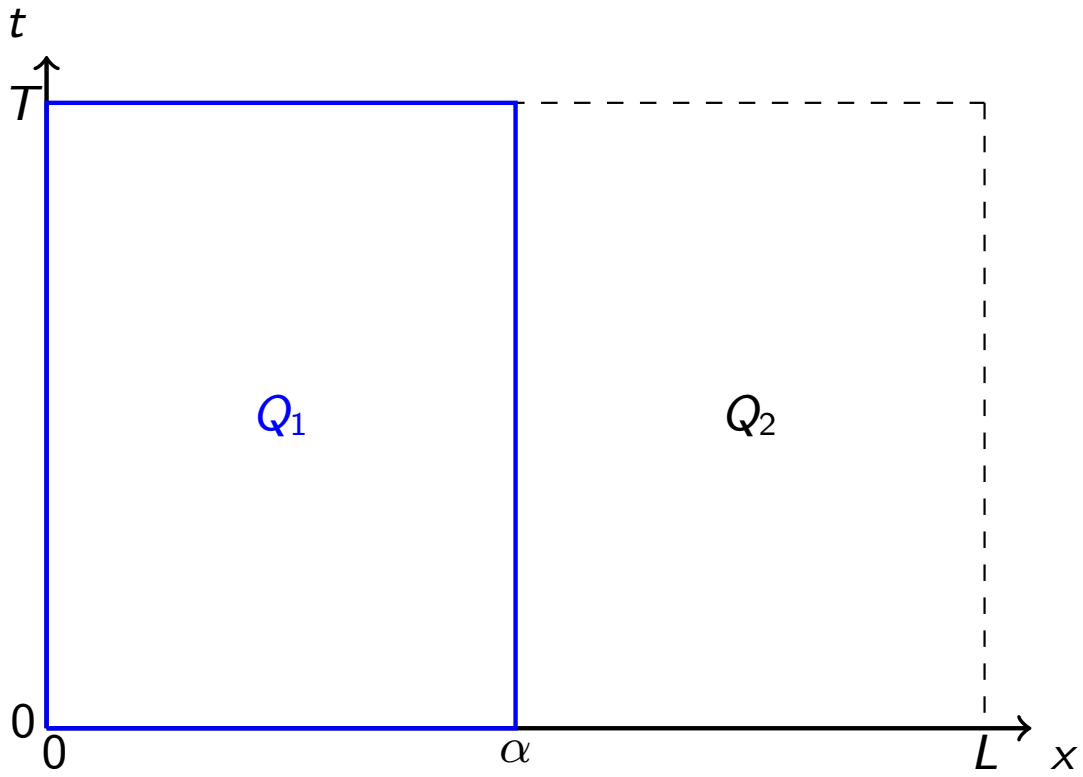

$Q_1 = (0, \alpha) \times (0, T)$,

$$\partial_t y_1^{k+1} - \Delta y_1^{k+1} = \nu^{-1} \lambda_1^{k+1}, \qquad \partial_t \lambda_1^{k+1} + \Delta \lambda_1^{k+1} = y_1^{k+1} - \hat{y}_1,$$

$$y_1^{k+1}(0, t) = 0, \qquad \lambda_1^{k+1}(0, t) = 0,$$

$$y_1^{k+1}(\alpha, t) = y_2^{k}(\alpha, t), \qquad \lambda_1^{k+1}(\alpha, t) = \lambda_2^{k}(\alpha, t),$$

$$y_1^{k+1}(x, 0) = y_{1,0}(x), \qquad \lambda_1^{k+1}(x, T) + \gamma y_1^{k+1}(x, T) = \gamma \hat{y}_1(x, T).$$

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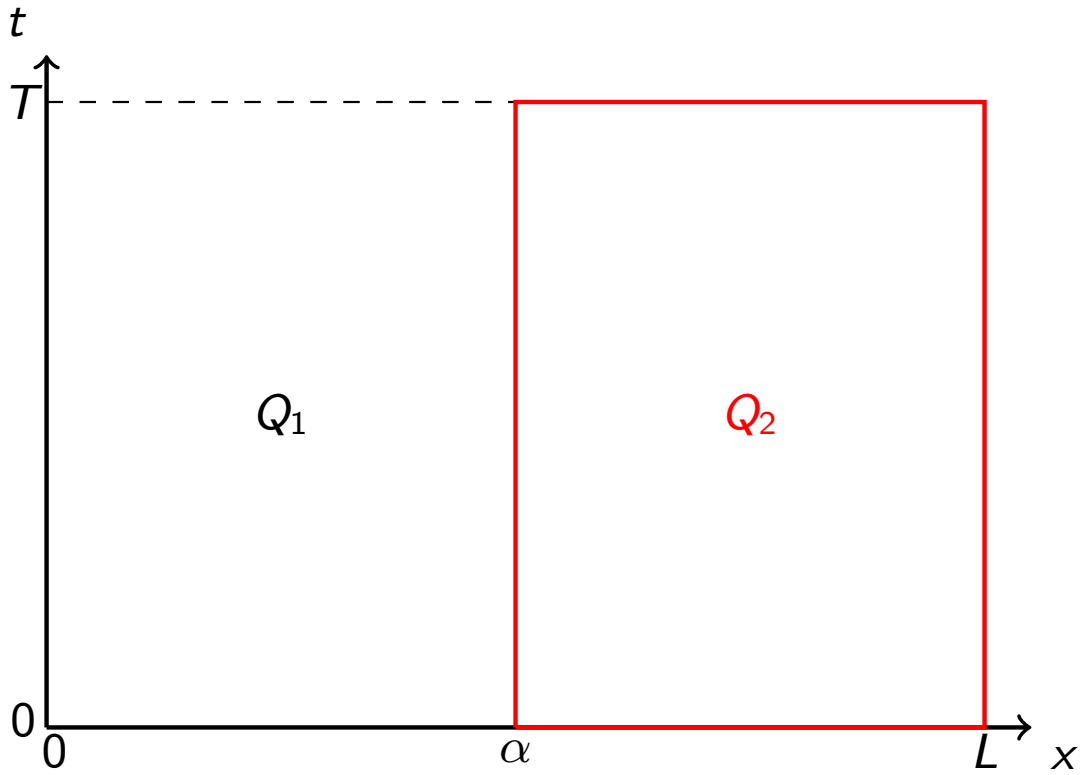

$Q_2 = (\alpha, 1) \times (0, T)$,

$$\partial_t y_2^{k+1} - \Delta y_2^{k+1} = \nu^{-1} \lambda_2^{k+1}, \qquad \partial_t \lambda_2^{k+1} + \Delta \lambda_2^{k+1} = y_2^{k+1} - \hat{y}_2,$$

$$\partial_x y_2^{k+1}(\alpha, t) = \partial_x y_1^{k+1}(\alpha, t), \qquad \partial_x \lambda_2^{k+1}(\alpha, t) = \partial_x \lambda_1^{k+1}(\alpha, t),$$

$$y_2^{k+1}(L, t) = 0, \qquad \lambda_2^{k+1}(L, t) = 0,$$

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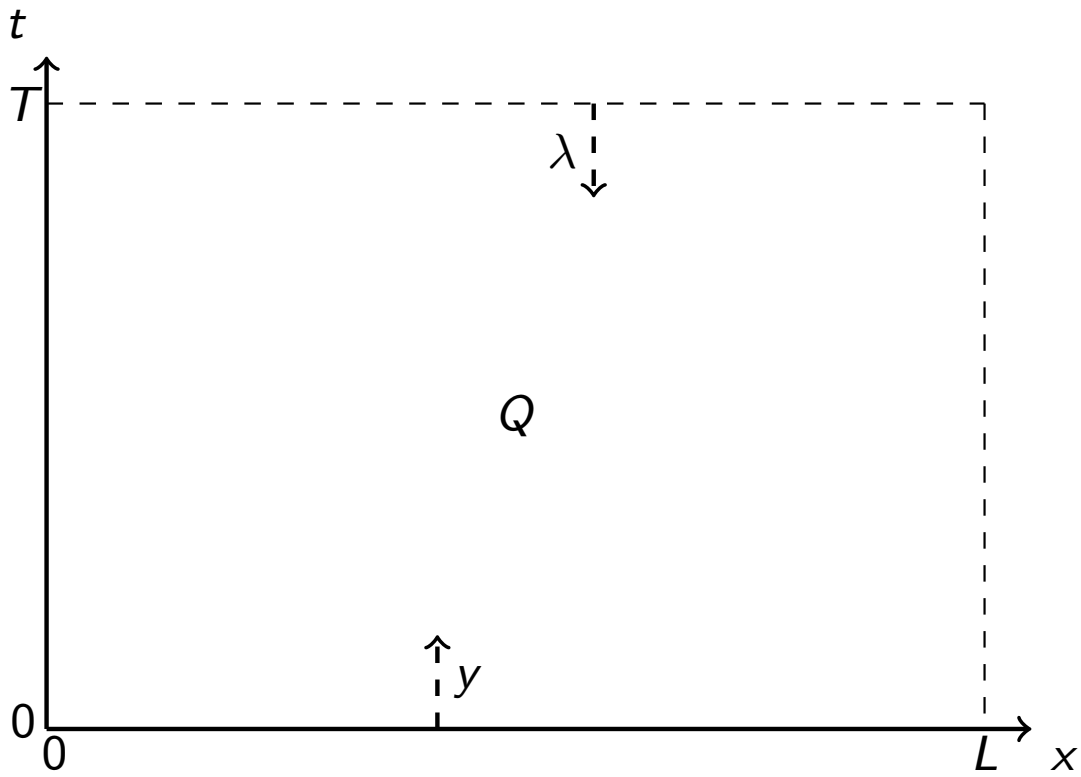
Time domain decomposition



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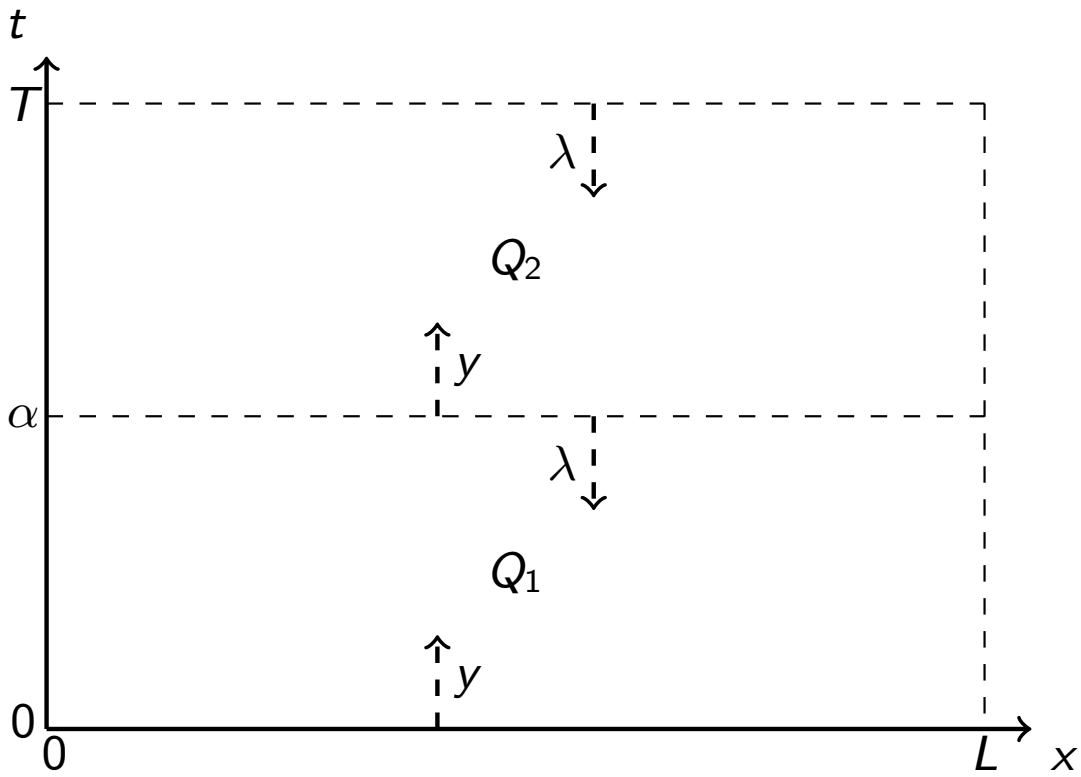

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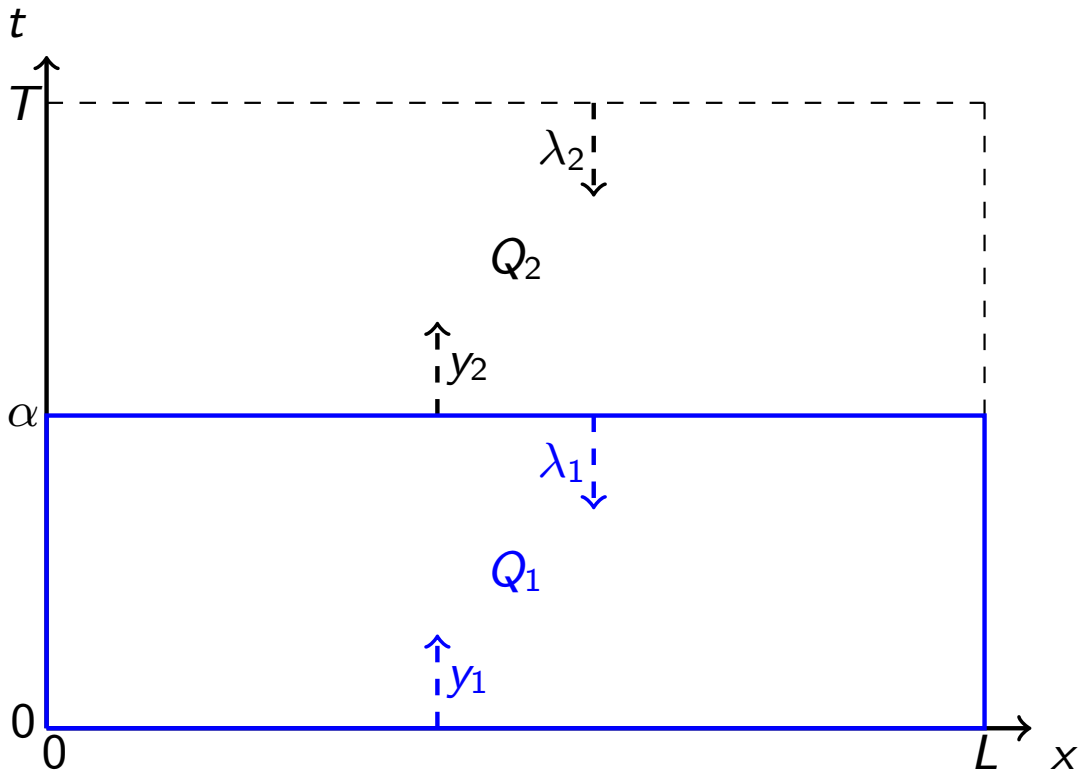

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$$y_1^{k+1}(0, t) = 0, \qquad \lambda_1^{k+1}(0, t) = 0,$$

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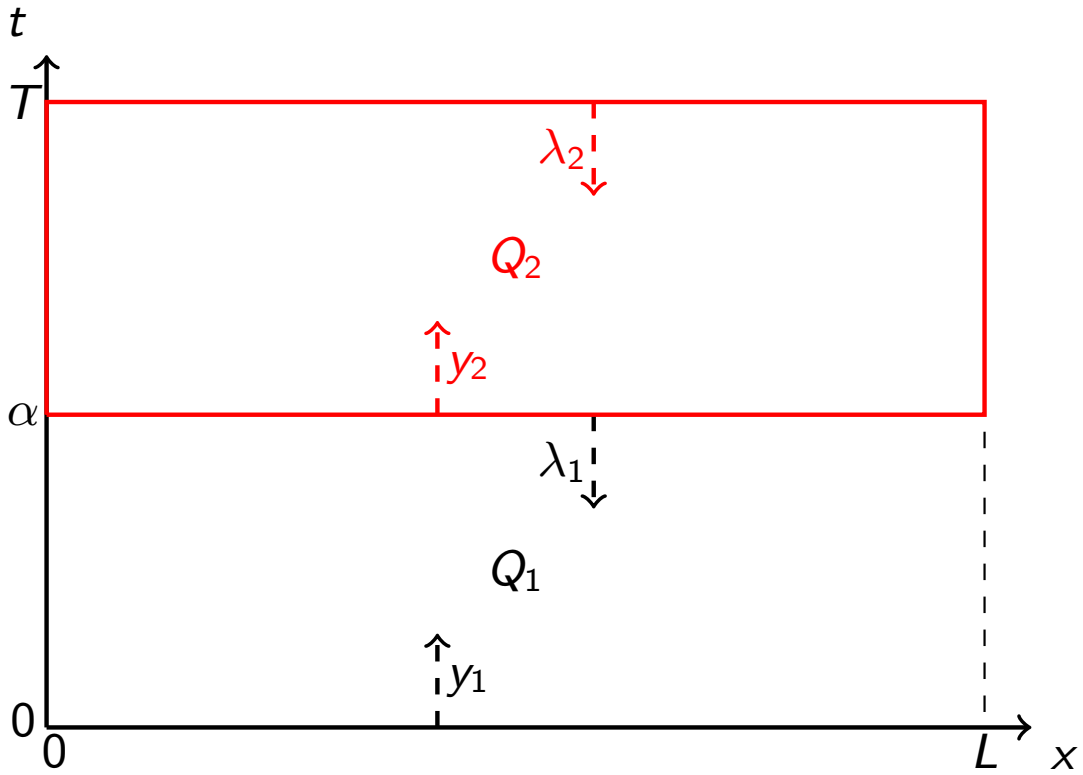

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$$\partial_t y_2^{k+1}(x, \alpha) = \partial_t y_1^{k+1}(x, \alpha), \qquad \lambda_2^{k+1}(x, T) + \gamma y_2^{k+1}(x, T) = \gamma \hat{y}_2(x, T).$$

Semi-discretization

Reduced optimality system:

$$\begin{cases} \partial_t \begin{pmatrix} y \\ \lambda \end{pmatrix} + \begin{pmatrix} -\Delta y - \nu^{-1}\lambda \\ -y + \Delta\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{y} \end{pmatrix}, \\ y(\cdot, 0) = y_0(\cdot), \\ \lambda(\cdot, T) + \gamma y(\cdot, T) = \gamma \hat{y}(\cdot, T). \end{cases}$$

Semi-discretization

Ex: finite difference $-\Delta \approx A$

$$\begin{cases} \begin{pmatrix} \dot{Y} \\ \dot{\Lambda} \end{pmatrix} + \begin{pmatrix} A & -\nu^{-1} I \\ -I & -A \end{pmatrix} \begin{pmatrix} Y \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{Y} \end{pmatrix} \text{ in } (0, T), \\ Y(0) = Y_0, \\ \Lambda(T) + \gamma Y(T) = \gamma \hat{Y}(T), \end{cases}$$

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$A = PDP^{-1}$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$,

$$\begin{cases} \begin{pmatrix} \dot{z}_i \\ \dot{\mu}_i \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_i \\ \mu_i \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{z}_i \end{pmatrix} \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \mu_i(T) + \gamma z_i(T) = \gamma \hat{z}_i(T), \end{cases}$$

with $z = P^{-1} Y$, $\hat{z} = P^{-1} \hat{Y}$ and $\mu = P^{-1} \Lambda$.

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n **independent** 2×2 **systems**.

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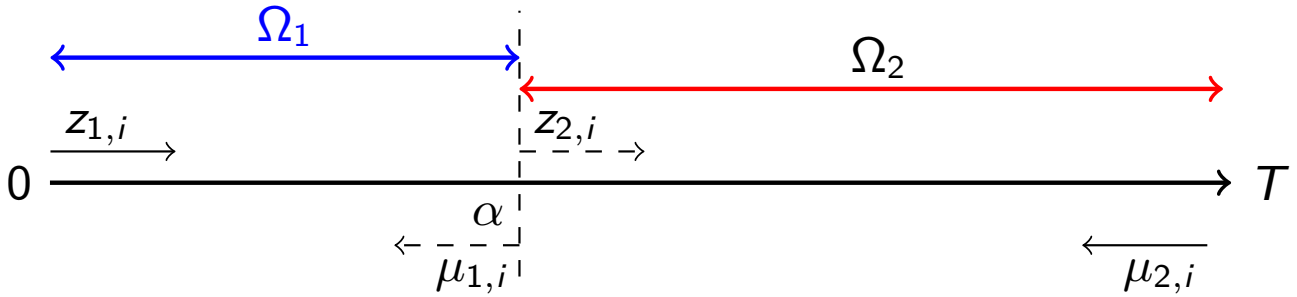
Second-order ODE:

$$\begin{cases} \ddot{z}_i - \sigma_i^2 z_i = -\nu^{-1} \hat{z}_i \text{ in } (0, T), \\ z_i(0) = z_{0,i}, \\ \dot{z}_i(T) + \omega_i z_i(T) = \nu^{-1} \gamma \hat{z}_i(T), \end{cases} \quad \text{or} \quad \begin{cases} \ddot{\mu}_i - \sigma_i^2 \mu_i = -(\dot{\hat{z}}_i + d_i \hat{z}_i) \text{ in } (0, T), \\ \mu_i(0) - d_i \mu_i(0) = z_{0,i} - \hat{z}_i(0), \\ \gamma \mu_i(T) + \beta_i \mu_i(T) = 0, \end{cases}$$

with $\sigma_i := \sqrt{d_i^2 + \nu^{-1}}$, $\omega_i := \nu^{-1} \gamma + d_i$ and $\beta_i := 1 - \gamma d_i$.

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

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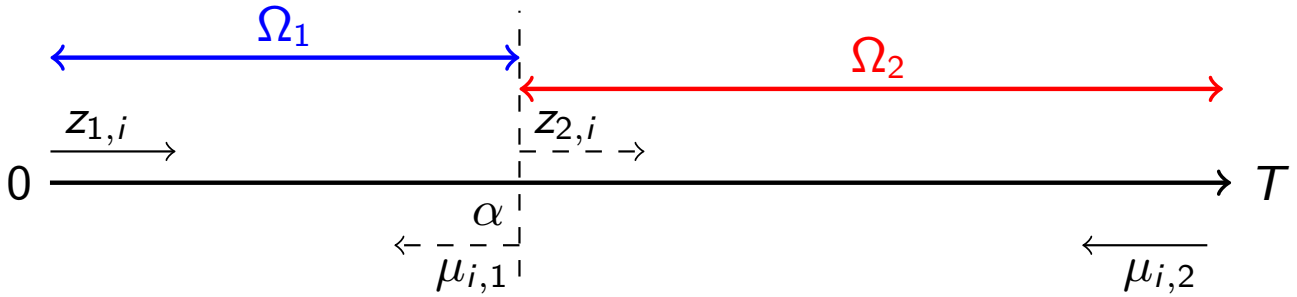

Dirichlet:

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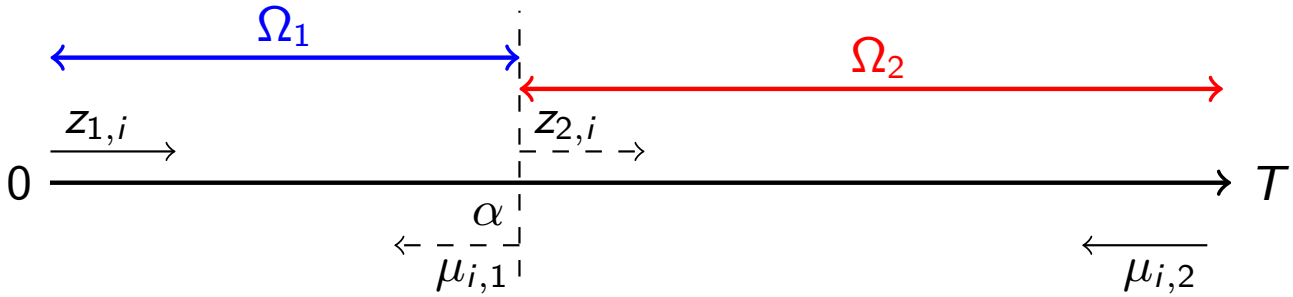
Update:

$$f^k_{\alpha,i} := (1-\theta) f^{k-1}_{\alpha,i} + \theta \mu^k_{2,i}(\alpha), \quad \theta \in (0,1).$$

Dirichlet-Neumann convergence analysis



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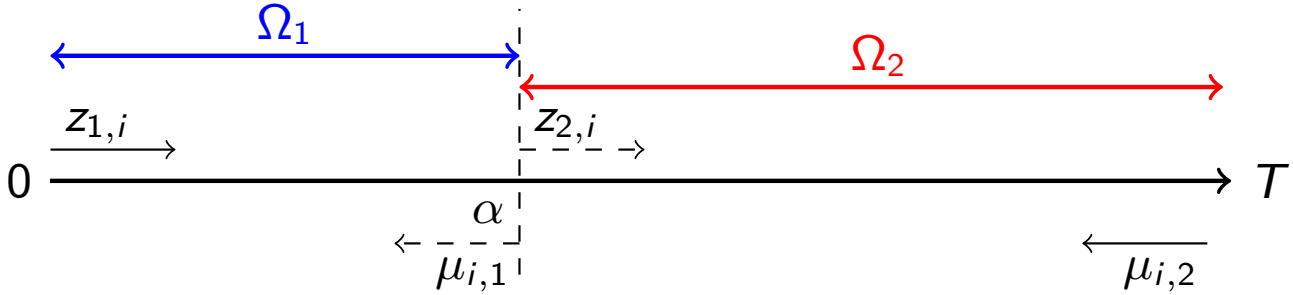
Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}^k_{2,i} \\ \dot{\mu}^k_{2,i} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z^k_{2,i} \\ \mu^k_{2,i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}^k_{2,i}(\alpha) = \dot{z}^k_{1,i}(\alpha), \\ \mu^k_{2,i}(T) + \gamma z^k_{2,i}(T) = 0, \end{cases}$$

Update:

$$f^k_{\alpha,i} := (1-\theta) f^{k-1}_{\alpha,i} + \theta \mu^k_{2,i}(\alpha), \quad \theta \in (0,1).$$

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \ddot{z}^k_{1,i} - \sigma_i^2 z^k_{1,i} = 0 \text{ in } \Omega_1, \\ z^k_{1,i}(0) = 0, \\ \dot{z}^k_{1,i}(\alpha) + d_i z^k_{1,i}(\alpha) = f^{k-1}_{\alpha,i}, \end{cases}$$

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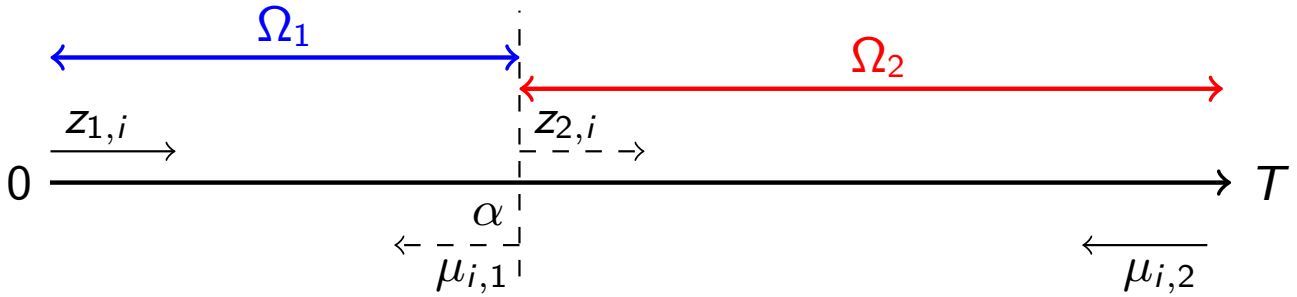
$$f^k_{\alpha,i} = (1-\theta) f^{k-1}_{\alpha,i} + \theta\left(\dot{z}^k_{2,i}(\alpha) + d_i z^k_{2,i}(\alpha)\right).$$

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \ddot{z}^k_{1,i} - \sigma_i^2 z^k_{1,i} = 0 \text{ in } \Omega_1, \\ z^k_{1,i}(0) = 0, \\ \dot{z}^k_{1,i}(\alpha) + d_i z^k_{1,i}(\alpha) = f^{k-1}_{\alpha,i}, \end{cases}$$

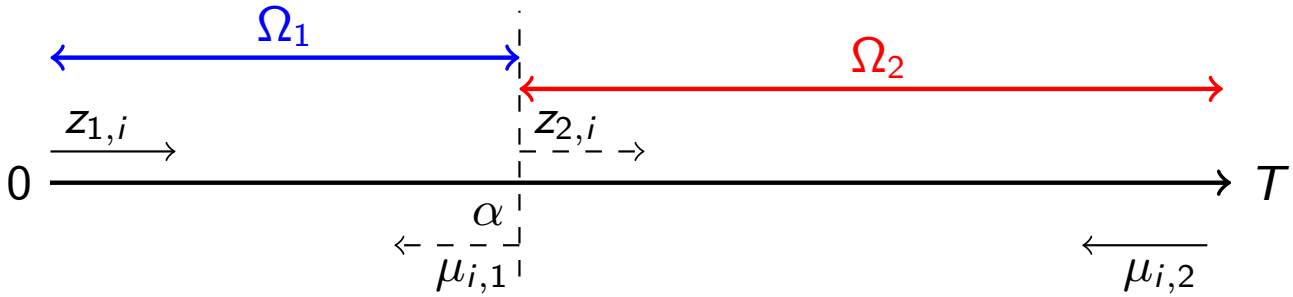
Neumann:

$$\begin{cases} \ddot{z}^k_{2,i} - \sigma_i^2 z^k_{2,i} = 0 \text{ in } \Omega_2, \\ \dot{z}^k_{2,i}(\alpha) = \dot{z}^k_{1,i}(\alpha), \\ \dot{z}^k_{2,i}(T) + \omega_i z^k_{2,i}(T) = 0, \end{cases}$$

Update:

$$f^k_{\alpha,i} = (1-\theta) f^{k-1}_{\alpha,i} + \theta\left(\dot{z}^k_{2,i}(\alpha) + d_i z^k_{2,i}(\alpha)\right).$$

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \begin{pmatrix} \dot{z}^k_{1,i} \\ \dot{\mu}^k_{1,i} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z^k_{1,i} \\ \mu^k_{1,i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z^k_{1,i}(0) = 0, \\ \mu^k_{1,i}(\alpha) = f^{k-1}_{\alpha,i}, \end{cases}$$

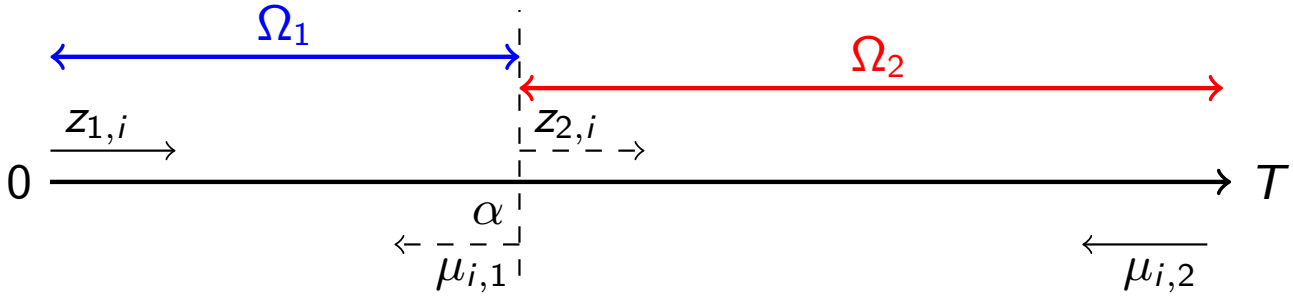
Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}^k_{2,i} \\ \dot{\mu}^k_{2,i} \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z^k_{2,i} \\ \mu^k_{2,i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}^k_{2,i}(\alpha) = \dot{z}^k_{1,i}(\alpha), \\ \mu^k_{2,i}(T) + \gamma z^k_{2,i}(T) = 0, \end{cases}$$

Update:

$$f^k_{\alpha,i} := (1-\theta) f^{k-1}_{\alpha,i} + \theta \mu^k_{2,i}(\alpha), \quad \theta \in (0,1).$$

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \ddot{\mu}_{1,i}^k - \sigma_i^2 \mu_{1,i}^k = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

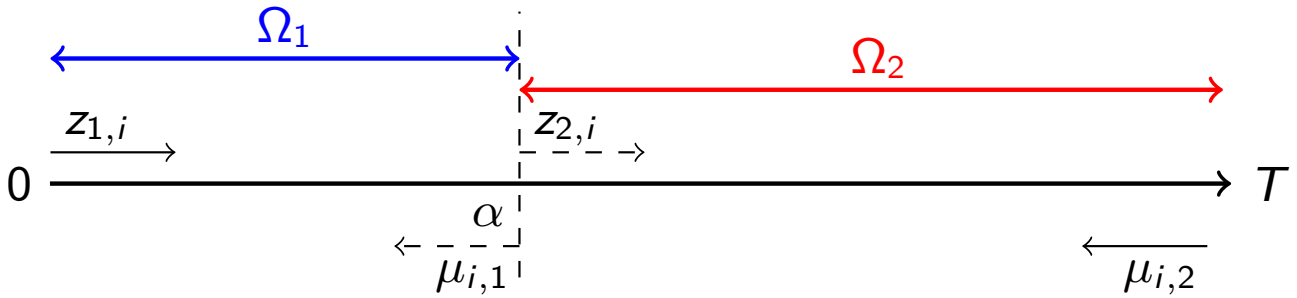
Neumann:

$$\begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^k = (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha).$$

Dirichlet-Neumann convergence analysis



Dirichlet:

$$\begin{cases} \ddot{\mu}_{1,i}^k - \sigma_i^2 \mu_{1,i}^k = 0 \text{ in } \Omega_1, \\ \dot{\mu}_i(0) - d_i \mu_i(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

Neumann:

$$\begin{cases} \ddot{\mu}_{2,i}^k - \sigma_i^2 \mu_{2,i}^k = 0 \text{ in } \Omega_2, \\ \ddot{\mu}_{2,i}^k(\alpha) - d_i \dot{\mu}_{2,i}^k(\alpha) = \ddot{\mu}_{1,i}^k(\alpha) - d_i \dot{\mu}_{1,i}^k(\alpha), \\ \gamma \dot{\mu}_i^k(T) + \beta_i \mu_i^k(T) = 0, \end{cases}$$

Update:

$$f_{\alpha,i}^k = (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha).$$

Dirichlet-Neumann convergence analysis

$$\begin{cases} \ddot{z}_{1,i}^k - \sigma_i^2 z_{1,i}^k = 0 \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \dot{z}_{1,i}^k(\alpha) + d_i z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases}$$

$$\begin{cases} \ddot{z}_{2,i}^k - \sigma_i^2 z_{2,i}^k = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \dot{z}_{2,i}^k(T) + \omega_i z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k = (1-\theta) f_{\alpha,i}^{k-1} + \theta\left(\dot{z}_{2,i}^k(\alpha) + d_i z_{2,i}^k(\alpha)\right).$$

Observations:

- ◇ three systems are equivalent,
- ◇ same convergence using z or μ,
- ◇ not anymore a "DN" algorithm,
- ◇ forward-backward structure less important.

Variants of DN and ND algorithms[1]

Category	Ω_1	Ω_2	type
(z_i, μ_i)	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
z_i	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
μ_i	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)
	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

[1] Gander and L. 2023

Variants of DN and ND algorithms[1]

Category	Ω_1	Ω_2	type
(z_i, μ_i)	μ_i	$\dot{z}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\dot{z}_i$	(RN)
	$\dot{\mu}_i$	z_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	z_i	(RD)
z_i	z_i	$\dot{z}_i$	(DN)
	z_i	$\dot{z}_i$	(DN)
	$\dot{z}_i$	z_i	(ND)
	$\dot{z}_i$	z_i	(ND)
μ_i	μ_i	$\dot{\mu}_i$	(DN)
	$\dot{z}_i + d_i z_i$	$\ddot{z}_i + d_i \dot{z}_i$	(RR)
	$\dot{\mu}_i$	μ_i	(ND)
	$\ddot{z}_i + d_i \dot{z}_i$	$\dot{z}_i + d_i z_i$	(RR)

[1] Gander and L. 2023

Dirichlet-Neumann convergence analysis

Natural Dirichlet-Neumann (DN_1):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta \mu_{2,i}^k(\alpha).$$

Dirichlet-Neumann on both level (DN_2):

$$\begin{cases} \begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta z_{2,i}^k(\alpha).$$

Forward-backward can always be recovered !

$$z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1} \Rightarrow \dot{\mu}_{1,i}^k(\alpha) - d_i \mu_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}.$$

Dirichlet-Neumann convergence analysis

Natural Dirichlet-Neumann (DN_1):

$$\begin{cases} \ddot{z}_{1,i}^k - \sigma_i^2 z_{1,i}^k = 0 \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \dot{z}_{1,i}^k(\alpha) + d_i z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \ddot{z}_{2,i}^k - \sigma_i^2 z_{2,i}^k = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \dot{z}_{2,i}^k(T) + \omega_i z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta\left(\dot{z}_{2,i}^k(\alpha) + d_i z_{2,i}^k(\alpha)\right).$$

Dirichlet-Neumann on both level (DN_2):

$$\begin{cases} \ddot{z}_{1,i}^k - \sigma_i^2 z_{1,i}^k = 0 \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ z_{1,i}^k(\alpha) = f_{\alpha,i}^{k-1}, \end{cases} \quad \begin{cases} \ddot{z}_{2,i}^k - \sigma_i^2 z_{2,i}^k = 0 \text{ in } \Omega_2, \\ \dot{z}_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha), \\ \dot{z}_{2,i}^k(T) + \omega_i z_{2,i}^k(T) = 0, \end{cases}$$

$$f_{\alpha,i}^k := (1-\theta) f_{\alpha,i}^{k-1} + \theta z_{2,i}^k(\alpha).$$

Dirichlet-Neumann convergence analysis

Solve the problem and find

$$f_{\alpha,i}^{k} := \rho\, f_{\alpha,i}^{k-1}.$$

Dirichlet-Neumann convergence analysis

Solve the problem and find

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Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\left(\sigma_i + d_i \tanh(a_i)\right)\left(\omega_i + \sigma_i \tanh(b_i)\right)} \right) \right|,$$

$$\rho_{\mathrm{ND}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \coth(b_i)}{\left(\sigma_i + d_i \coth(a_i)\right)\left(\omega_i + \sigma_i \coth(b_i)\right)} \right) \right|.$$

Solve the problem and find

$$f_{\alpha,i}^{k} := \rho\, f_{\alpha,i}^{k-1}.$$

Convergence factor with analytical form

$$\rho_{\mathrm{DN}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \tanh(b_i)}{\left(\sigma_i + d_i \tanh(a_i)\right)\left(\omega_i + \sigma_i \tanh(b_i)\right)} \right) \right|,$$

$$\rho_{\mathrm{ND}_1} := \max_{d_i \in \lambda(A)} \left| 1 - \theta \left(1 - \nu^{-1} \frac{\sigma_i \gamma + \beta_i \coth(b_i)}{\left(\sigma_i + d_i \coth(a_i)\right)\left(\omega_i + \sigma_i \coth(b_i)\right)} \right) \right|.$$

"Optimal" relaxation parameter with equioscillation

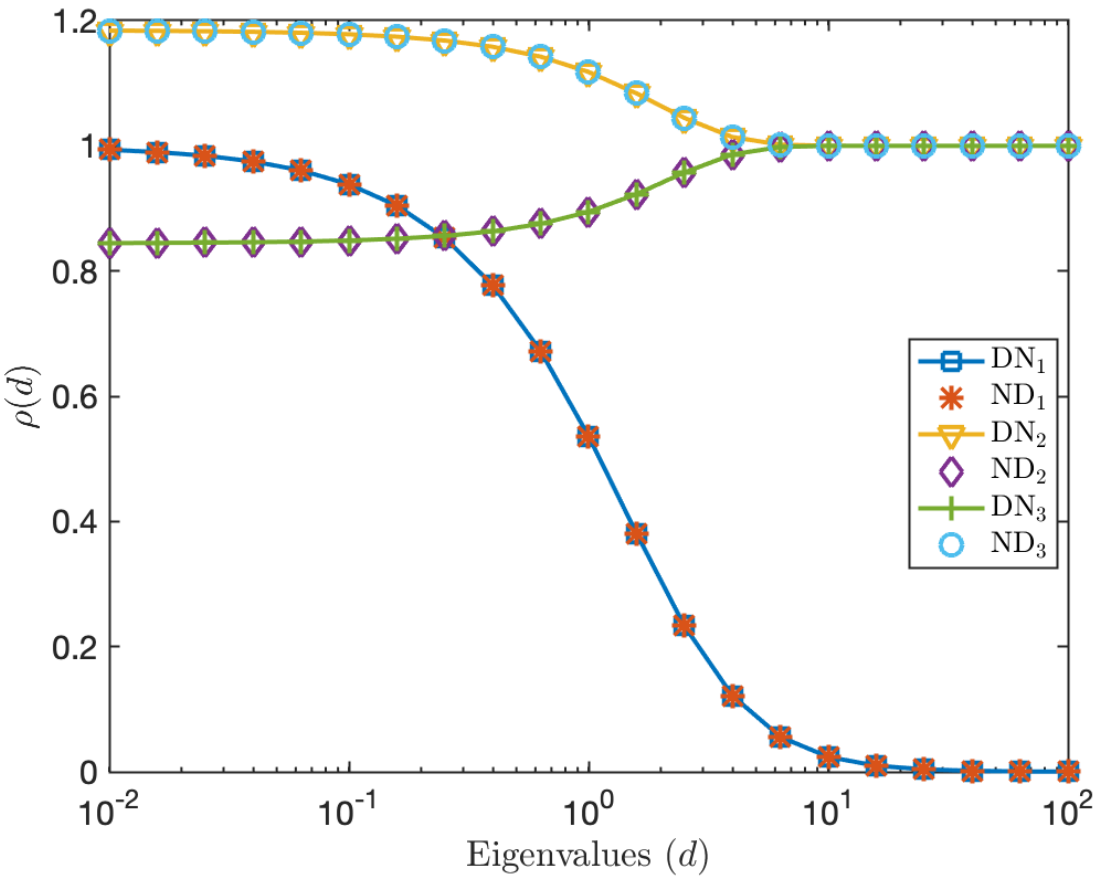
$$\theta_{\mathrm{DN}_2}^{*} = \frac{2}{3 + \coth(\sqrt{\nu^{-1}}\alpha) \frac{\coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}} \coth\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$$\theta_{\mathrm{ND}_2}^{*} = \frac{2}{3 + \tanh(\sqrt{\nu^{-1}}\alpha) \frac{\tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right) + \gamma\sqrt{\nu^{-1}}}{1 + \gamma\sqrt{\nu^{-1}} \tanh\left(\sqrt{\nu^{-1}}(T-\alpha)\right)}},$$

$\theta_{\mathrm{DN}_2}^{*} = \theta_{\mathrm{ND}_3}^{*}$ and $\theta_{\mathrm{ND}_2}^{*} = \theta_{\mathrm{DN}_3}^{*}$.

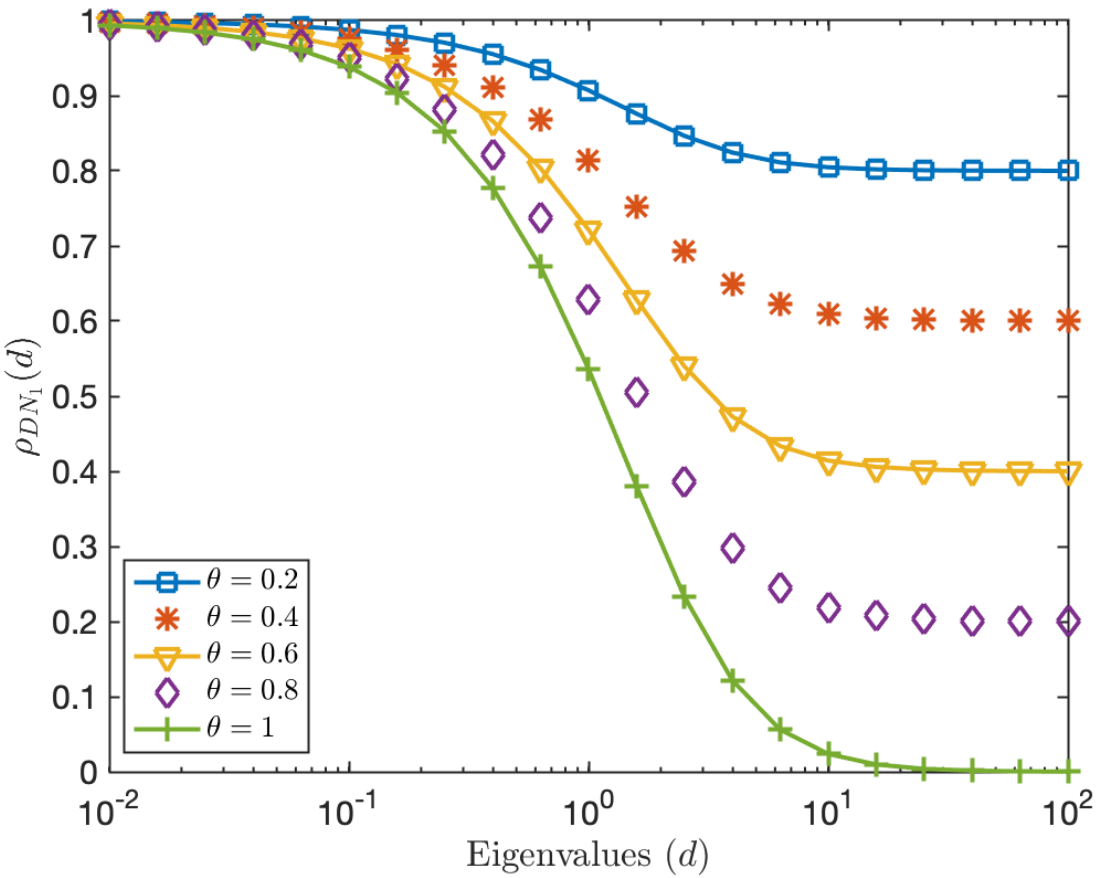
Numerical experiments

$\nu = 0.1$, $\gamma = 0$, $\alpha = \frac{T}{2}$ and $\theta = 1$.



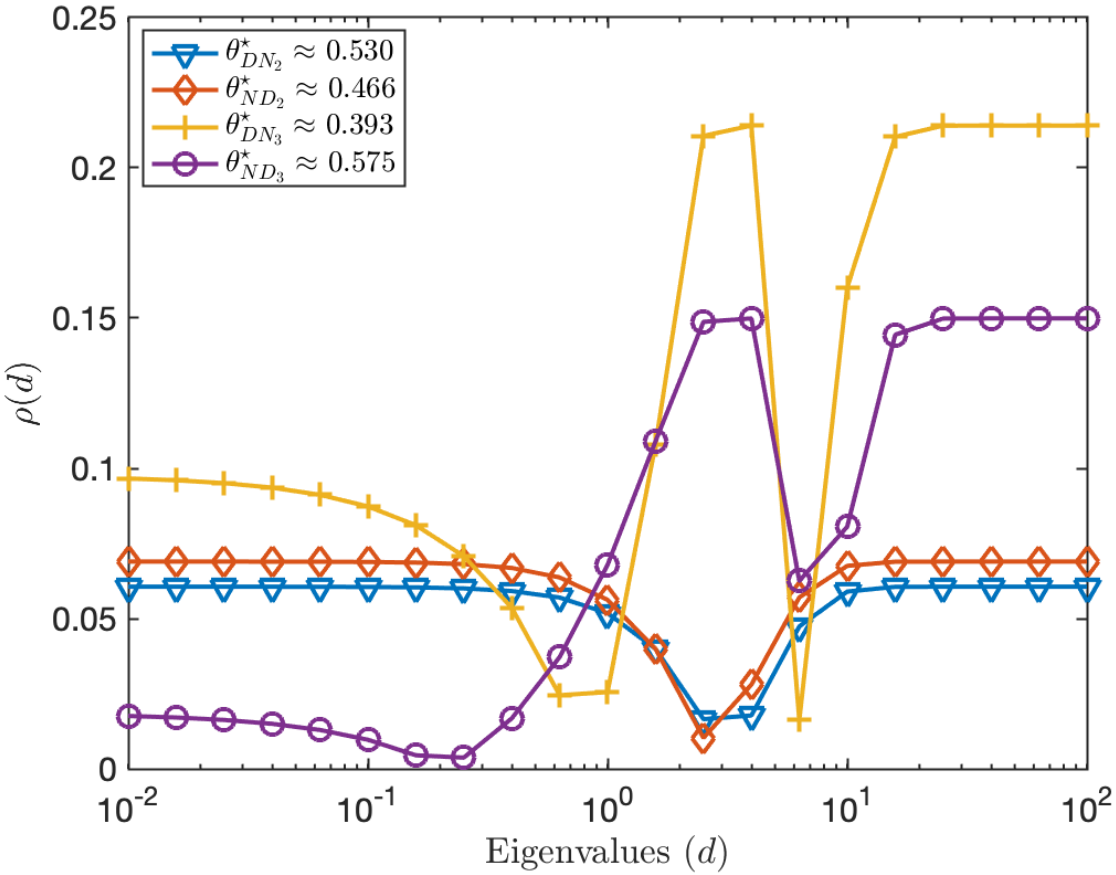
Numerical experiments

$\nu = 0.1$, $\gamma = 0$ and $\alpha = \frac{T}{2}$.



Numerical experiments

$\nu = 0.1$, $\gamma = 10$ and $\alpha = \frac{7}{10}T$.



$\theta^\star_{\mathrm{DN}_2} = \theta^*_{\mathrm{DN}_2}$ and $\theta^\star_{\mathrm{ND}_2} = \theta^*_{\mathrm{ND}_2}$.

Non-overlap

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\
z_{1,i}^k(0) = 0, \\
\mu_{1,i}^k(\alpha) = \mu_{2,i}^{k-1}(\alpha), \\
\begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\
z_{2,i}^k(\alpha) = z_{1,i}^k(\alpha), \\
\mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0,
\end{cases}
$$

Non-overlap

$$
\begin{cases}
\ddot{z}_{1,i}^{k} - \sigma_i^2 z_{1,i}^{k} = 0 \text{ in } \Omega_1, \\
z_{1,i}^{k}(0) = 0, \\
\dot{z}_{1,i}^{k}(\alpha) + d_i z_{1,i}^{k}(\alpha) = \dot{z}_{1,i}^{k-1}(\alpha) + d_i z_{1,i}^{k-1}(\alpha),
\end{cases}
$$

$$
\begin{cases}
\ddot{z}_{2,i}^{k} - \sigma_i^2 z_{2,i}^{k} = 0 \text{ in } \Omega_2, \\
z_{2,i}^{k}(\alpha) = z_{1,i}^{k}(\alpha), \\
\dot{z}_{2,i}^{k}(T) + \omega_i z_{2,i}^{k}(T) = 0,
\end{cases}
$$

Non-overlap

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$$\begin{cases} \ddot{z}_{2,i}^k - \sigma_i^2 z_{2,i}^k = 0 \text{ in } \Omega_2, \\ z_{2,i}^k(\alpha) = z_{1,i}^k(\alpha), \\ \dot{z}_{2,i}^k(T) + \omega_i z_{2,i}^k(T) = 0, \end{cases}$$

RD instead of DD !

Non-overlap

$$\begin{cases} \ddot{z}_{1,i}^k - \sigma_i^2 z_{1,i}^k = 0 \text{ in } \Omega_1, \\ z_{1,i}^k(0) = 0, \\ \dot{z}_{1,i}^k(\alpha) + d_i z_{1,i}^k(\alpha) = \dot{z}_{1,i}^{k-1}(\alpha) + d_i z_{1,i}^{k-1}(\alpha), \end{cases}$$

$$\begin{cases} \ddot{z}_{2,i}^k - \sigma_i^2 z_{2,i}^k = 0 \text{ in } \Omega_2, \\ z_{2,i}^k(\alpha) = z_{1,i}^k(\alpha), \\ \dot{z}_{2,i}^k(T) + \omega_i z_{2,i}^k(T) = 0, \end{cases}$$

RD instead of DD !

Same classical Schwarz problem

Ω_1	Ω_2	type
z_i	z_i	(DD)
μ_i	μ_i	(DD)

A little bit further

Parallel Schwarz $\mu + pz, z + q\mu$ [Gander and Kwok 2016]

A little bit further

Parallel Schwarz $\mu + pz, z + q\mu$ [Gander and Kwok 2016]
"Natrual" Optimized Schwarz $(\partial_n + p)(\cdot), (\partial_n + q)(\cdot)$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\
z_{1,i}^k(0) = 0, \\
\dot{\mu}_{1,i}^k(\alpha) + p\mu_{1,i}^k(\alpha) = \dot{\mu}_{2,i}^{k-1}(\alpha) + p\mu_{2,i}^{k-1}(\alpha),
\end{cases}
$$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\
\dot{z}_{2,i}^k(\alpha) + qz_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) + qz_{1,i}^k(\alpha), \\
\mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0.
\end{cases}
$$

A little bit further

Parallel Schwarz $\mu + pz, z + q\mu$ [Gander and Kwok 2016]
"Natrual" Optimized Schwarz $(\partial_n + p)(\cdot), (\partial_n + q)(\cdot)$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\
z_{1,i}^k(0) = 0, \\
\dot{\mu}_{1,i}^k(\alpha) + p\mu_{1,i}^k(\alpha) = \dot{\mu}_{2,i}^{k-1}(\alpha) + p\mu_{2,i}^{k-1}(\alpha),
\end{cases}
$$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\
\dot{z}_{2,i}^k(\alpha) + qz_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) + qz_{1,i}^k(\alpha), \\
\mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0.
\end{cases}
$$

36 Variants of OS:

Ω_1	Ω_2	tran. cond. type
$z + p\mu$	$z + q\mu$	(DD)
	$\dot{z} + q\mu$	(DR)
	$z + q\dot{\mu}$	(DR)
	$\dot{z} + q\dot{\mu}$	(DN)
	$\dot{z} + qz$	(DR)
	$\dot{\mu} + q\mu$	(DR)

A little bit further

Parallel Schwarz $\mu + pz, z + q\mu$ [Gander and Kwok 2016]
"Natrual" Optimized Schwarz $(\partial_n + p)(\cdot), (\partial_n + q)(\cdot)$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{1,i}^k \\ \dot{\mu}_{1,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{1,i}^k \\ \mu_{1,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_1, \\
z_{1,i}^k(0) = 0, \\
\dot{\mu}_{1,i}^k(\alpha) + p\mu_{1,i}^k(\alpha) = \dot{\mu}_{2,i}^{k-1}(\alpha) + p\mu_{2,i}^{k-1}(\alpha),
\end{cases}
$$

$$
\begin{cases}
\begin{pmatrix} \dot{z}_{2,i}^k \\ \dot{\mu}_{2,i}^k \end{pmatrix} + \begin{pmatrix} d_i & -\nu^{-1} \\ -1 & -d_i \end{pmatrix} \begin{pmatrix} z_{2,i}^k \\ \mu_{2,i}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega_2, \\
\dot{z}_{2,i}^k(\alpha) + qz_{2,i}^k(\alpha) = \dot{z}_{1,i}^k(\alpha) + qz_{1,i}^k(\alpha), \\
\mu_{2,i}^k(T) + \gamma z_{2,i}^k(T) = 0.
\end{cases}
$$

36 Variants of OS:

Ω_1	Ω_2	tran. cond. type
$z + p\mu$	$z + q\mu$	(DD)
	$\dot{z} + q\mu$	(DR)
	$z + q\dot{\mu}$	(DR)
	$\dot{z} + q\dot{\mu}$	(DN)
	$\dot{z} + qz$	(DR)
	$\dot{\mu} + q\mu$	(DR)

Thanks for your attention !