The 3-Dimensional Matching Problem(3-DM)

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- It is a partitioning problem.
  - Partitioning problems: We are searching over ways of dividing a collection of objects into subsets.
- It is considered as a harder version of the Bipartite Matching Problem.
  - Things get more complicated as we move from ordered pairs to ordered triples.
  - No polynomial algorithm is known for 3-DM.
- It also forms a special case of Set Cover Problem.

The 3-Dimensional Matching Problem (3-DM)

INSTANCE: Disjoint sets B, G, P, each of size a,
and a set of ordered triples $T \subseteq B \times G \times P$.

QUESTION (decision version): Does there exist a subset $M \subseteq T$ with $|M| = a$ such that
for each pair $(b, g, p), (b', g', p') \in M$ it holds that
$b \neq b', g \neq g', p \neq p'$?

We are searching for a **perfect** three-dimensional matching.

3-DM Is NP-Complete

- ✓ 3-DM is NP
  - Given a collection of triples $M \subseteq T$, we could verify that each element in $B \cup G \cup P$ belongs to exactly one of the triples in M, in polynomial time.
- ✓ 3-DM is NP-Complete
  - We prove the NP-completeness of 3-DM by a polynomial time reduction from 3-SAT to 3-DM.
  - Strategy of the reduction:
    We must design sets of triples (“gadgets”) that model Boolean variables and clauses.

3-SAT $\leq_p$ 3-DM

Instance Φ of 3-SAT

- n variables $v_1,\ldots,v_n$
- m clauses $c_1,\ldots,c_m$

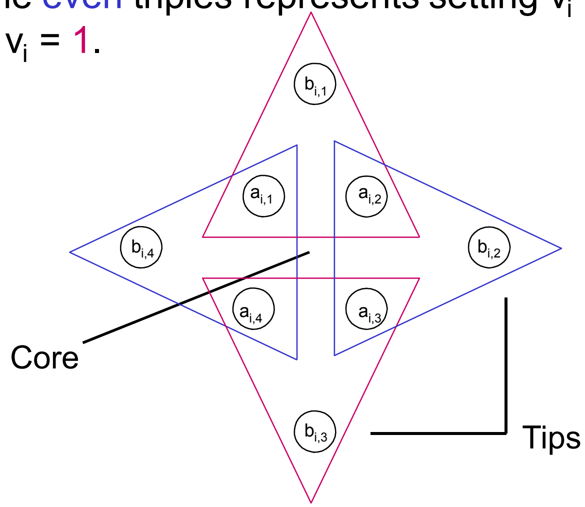
Instance I of 3-DM

- n variable gadgets
- m clause gadgets
- $(n-1)m$ cleanup gadgets

Note: Initially, we will describe all the elements in I without trying to specify for each one whether it comes from B, G, or P.

3-SAT $\leq_p$ 3-DM

- Gadget associated with variable v_i
  - We define $A_i = \{a_{i,1},\ldots,a_{i,2m}\}$: core elements of gadget i.
  - We define $B_i = \{b_{i,1},\ldots,b_{i,2m}\}$: elements at the tips of gadget i.
  - We define $t_{i,j} = (a_{i,j}\ a_{i,(j+1) \bmod 2m}, b_{i,j})$, for each j = 1,…,2m.
  - We call a $t_{i,j}$ even if j is even and odd if j is odd.
  - Elements in A_i are involved **only** in $\{t_{i,j}\}$.
  - **Perfect matching** in gadget i: We must use either all the even triples and leave the odd tips **free**, or all the odd triples and leave the even tips **free** in gadget i.
  - Encoding: Using the even triples represents setting $v_i = 0$, and using the odd triples represents setting $v_i = 1$.



Variable v_i

3-SAT $\leq_p$ 3-DM

- Gadget associated with clause c_j
  - We define $P_j = \{p_{j,1}, p_{j,2}\}$: core elements of gadget j.
  - We involve them in three triples, one for each literal in the clause.
  - The $b_{i,j}$ elements in these triples reflect the three ways whereby the clause can be satisfied.

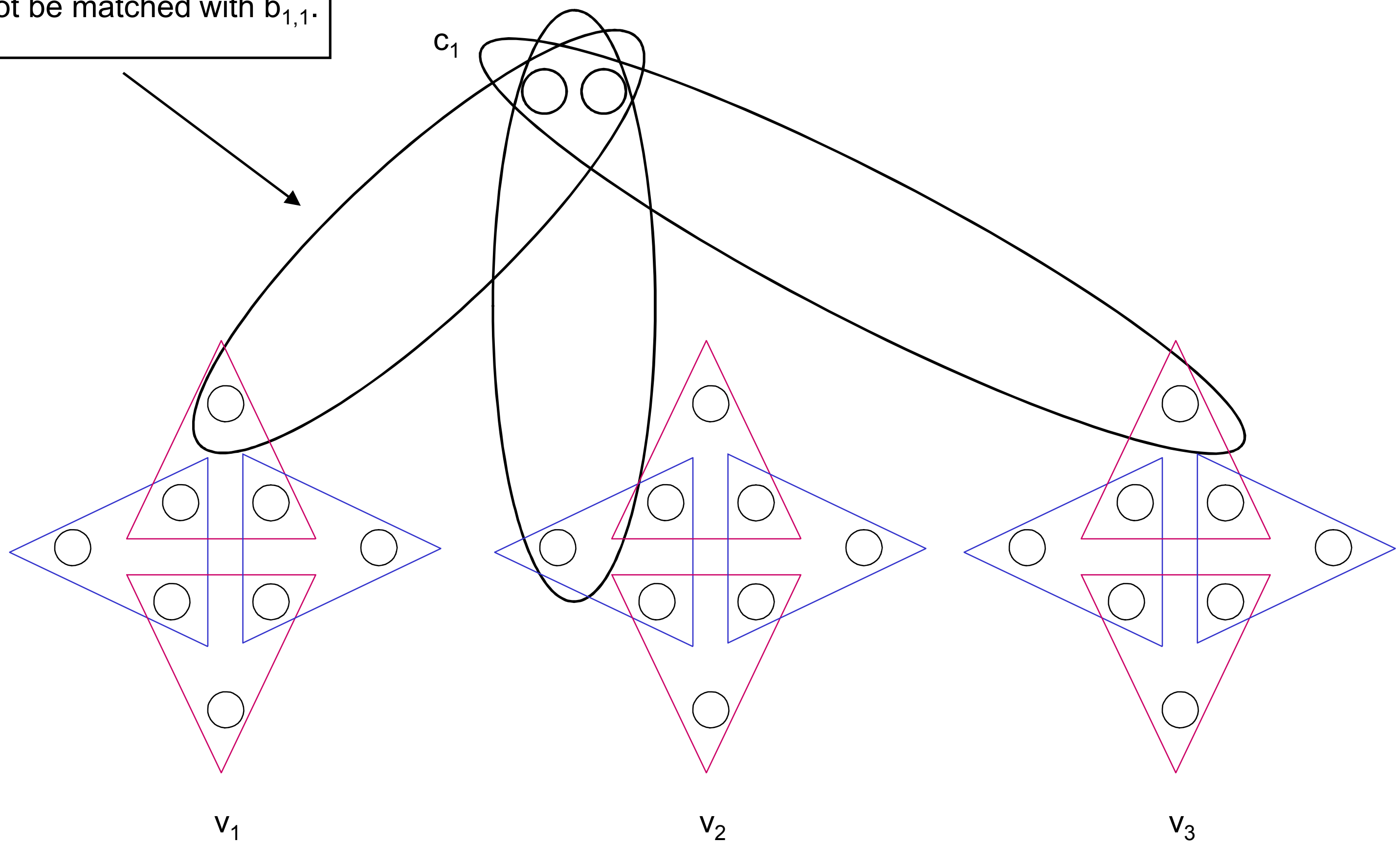
    Suppose l is a literal of c_j.

    If $l = v_i$, we define a triple $(p_{j,1}, p_{j,2}, b_{i,2j})$.
    ($v_i = 1$: We cover the odd tips and leave the even tips **free**. Hence, we select an even tip.)

    If $l = \neg v_i$, we define a triple $(p_{j,1}, p_{j,2}, b_{i,2j-1})$.
    ($v_i = 0$: We cover the even tips and leave the odd tips **free**. Hence, we select an odd tip.)
  - These are the **only** three triples that cover P_j. Thus, one of them must be used.
  - Elements in P_j can only be matched if some variable gadget leaves the corresponding tip **free**.

3-SAT $\leq_p$ 3-DM

If v_1 = 1 the core elements can not be matched with $b_{1,1}$.



$c_1 = \neg v_1 \vee v_2 \vee \neg v_3$

3-SAT $\leq_p$ 3-DM

- Gadget associated with unmatched tip b_i
  - In the matching defined **(n − 1)m tips** are left **unmatched**.
    $n \cdot 2m$ (initially) − nm (covered by $\{t_{i,\,j}\}$) − m (covered by clause gadgets) = $(n - 1)m$
  - We define $Q_i = \{q_{i,1}, q_{i,2}\}$: core elements of gadget i.
  - There is a triple $(q_{i,1}, q_{i,2}, b)$ for every tip b in every variable gadget.

- Decomposition of elements of 3-DM instance into three disjoint sets B, G, P
  - $B = \{a_{i,\,j} \mid j \text{ is odd}\} \cup \{p_{j,1}\} \cup \{q_{i,1}\}$
  - $G = \{a_{i,\,j} \mid j \text{ is even}\} \cup \{p_{j,2}\} \cup \{q_{i,2}\}$
  - $P = \{b_{i,\,j}\}$

3-SAT $\leq_p$ 3-DM

Given an instance Φ of 3-SAT, we construct an instance I of 3-DM that has a perfect matching iff Φ is satisfiable.

- If Φ has a satisfying truth assignment then I has a perfect matching.
  - We make the corresponding choices of even/odd $\{t_{i,\,j}\}$ for each variable gadget i.
  - We match P_j with $b_{i,\,j}$ that corresponds to one of its satisfying literals for each clause gadget j.
  - We use the cleanup gadgets to cover the unmatched tips.
  - Thus, I has a perfect matching.

- If I has a perfect matching then Φ has a satisfying truth assignment.
  - For each variable v_i:
    We set $v_i = 0$, if the even $\{t_{i,\,j}\}$ have been chosen in the corresponding variable gadget.
    We set $v_i = 1$, if the odd $\{t_{i,\,j}\}$ have been chosen in the corresponding variable gadget.
  - Any clause c_j is satisfiable.
    The core elements in P_j have been covered.
    At least one of the three variable gadgets corresponding to a literal in c_j chose the correct matching.
    There is a variable assignment that satisfies c_j.
  - Thus, Φ has a satisfying truth assignment.