A Simple FPTAS for Counting Edge Covers

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Overview

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Edge cover

Definition

For an undirected input graph $G = (V, E)$, an **edge cover** of G is a set of edges C covering all vertices.

Example

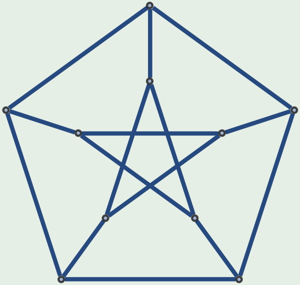

Figure: An edge cover for Petersen graph

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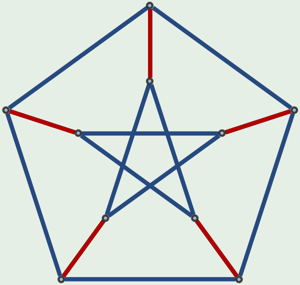

Figure: An edge cover for Petersen graph, with edges chosen being highlighted in red. Note that this is also a perfect matching.

Edge cover

Edge cover is related to many other problems such as:

- Matching problem.
- Rtw-Mon-CNF. (read twice monotone CNF)
- Holant problem.
- ....

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   - Definition
   - Edge Covers and Matching
   - Edge Covers and Rtw-Mon-CNF Formulae
   - Counting Problems

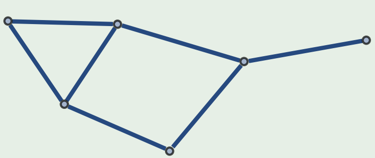
2. Main Result
   - Previous Results and Our Result
   - Counting via Marginal Probability
   - Computation Tree Recursion
   - Correlation Decay

Relation to Matching

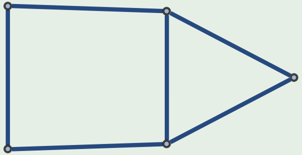
For a graph with a perfect matching, enumerating (sampling) perfect matchings is equivalent to enumerating (sampling) minimum edge covers.

Example (Minimum edge covers)

Find a minimum edge cover by maximal matching?



(a) G has a perfect matching.



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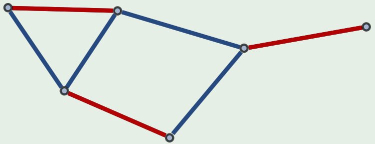
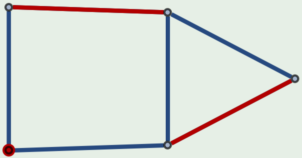


(c) G has a perfect matching.



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Relation to Rtw-Mon-CNF

Definition

A formula is **read twice** if every variables appears at most twice.
A formula is **monotone** if every variables appears positively.

Consider the following Rtw-Mon-CNF formula, its satisfying assignments are exactly edge covers of its graph representation, where we write edges as variables, and vertices as clauses.

$$\phi = (e_1 \vee e_2 \vee e_3) \wedge (e_1 \vee e_4) \wedge (e_4 \vee e_5 \vee e_2) \wedge (e_3 \vee e_5).$$

Figure: Graph representation for ϕ.

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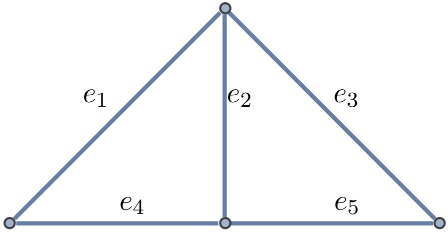

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Counting Problems

A list of problems in their search, optimization, and counting versions.

Search problems:

- SAT.
- Find a (perfect) matching.
- Find an edge cover.
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Optimizations:

- MAX-SAT.
- Find a maximum matching.
- Find a minimum edge cover.
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Why counting?

Besides theoretical computer science, counting problems are also related to many problems from other discipline such as:

- Partition function in statistical physics.
- Graph polynomials.
- Sampling, learning and inference in probabilistic graphical models.
- Pricing in combinatorial prediction markets.
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Approximate Counting

Many interesting problems in the exact counting regimes, including counting edge cover, is hard (#P-complete). Instead we look for these two types of polynomial time approximation scheme:

Definition (FPTAS)

For given parameter $\varepsilon > 0$ and an instance of a particular problem class, if the algorithm outputs a number $\hat{N}$ such that

$$(1-\varepsilon)N \leq \hat{N} \leq (1+\varepsilon)N,$$

where N is the accurate answer of the problem instance, and the running time is bounded by $poly(n, 1/\varepsilon)$ with n being the size of instance, this is called the **FPTAS (fully polynomial time approximation scheme)**.

A randomized relaxation of FPTAS is known as **FPRAS (fully polynomial time randomized approximation scheme)**, which uses random bits and only outputs $\hat{N}$ to the desired precision with high probability.

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- For counting edge covers, only an MCMC-based FPRAS is known graphs with maximum degree 3.
- For #DNF and counting matchings, only FPRAS is known.
- For counting perfect matchings, it's still open whether or not it admits FPRAS (or FPTAS).
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Our Result

- An FPTAS for counting edge covers in *general* graphs.
- Our algorithm is correlation decay based.
- This provides another example where the tractable range of correlation decay based FPTAS exceeds the sampling based FPRAS.

Proof Sketch

Let $EC(G)$ be the set of edge covers. Here is an overall work-flow:

- Relate $|EC(G)|$ with a marginal probability $P(G, e)$.
- Derive a computation tree recursion for $P(G, e)$.
- $P(G, e, L)$: Truncate the tree at depth L for some notion of depth.
- Show exponential correlation decay with respect to that tree depth:

$$|P(G, e, L) - P(G, e)| \leq exp(-\Omega(L))$$

Devising sub-problems

Definition (Dangling edge)

A **dangling edge** $e = (u, \_)$ of a graph is such singleton edge with exactly one end-point vertex u, as shown in the Figure 3a.

$$G - e \triangleq (V, E \setminus e)$$

$$G - u \triangleq (V \setminus u, E - u)$$

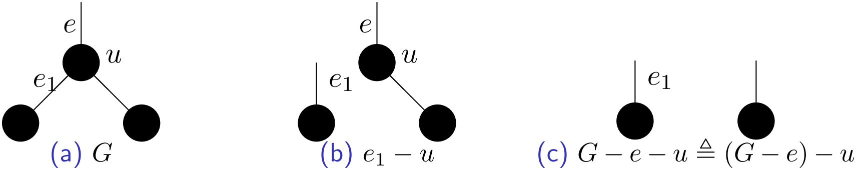

Figure: Dangling edges examples.

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Counting v.s. Marginal Probability

Problem

Goal: estimate $|EC(G)|$.

Let X be an edge cover sampled uniformly from $EC(G)$, consider the following marginal probability:
for an edge e, we write $P(G, e) \triangleq \Pr(e \notin X)$.

Solution: estimate $P(G, e)$.

Why $P(G, e)$?

Recall that the set of all edges E is an edge cover. For a randomly sampled edge cover X, what is $\Pr(X = E)$?

Let $E = \{e_i\}$, and $e_i = (u_i, v_i)$.

$$\Pr(X = E) = \frac{1}{|EC(G)|}$$

Therefore,

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Computation Tree Recursion

We focus on e is dangling.

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Truncate with a modified recursion depth:

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Correlation Decay

Proposition

Given graph G, edge e and depth L,

$$|P(G, e, L) - P(G, e)| \leq 3 \cdot (\frac{1}{2})^{L+1}$$

Now we just take $L = \log_2 \left(\frac{6m}{\varepsilon}\right)$, this gives the desired FPTAS.

Conclusions and Upcoming Results

As a conclusion, we have shown an FPTAS for counting edge covers in general graphs.

We later generalized our techniques for establishing correlation decay based FPTAS and obtained the following:

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- FPTAS for counting Read-5-Mon-CNF, while Read-6-Mon-CNF does not admit FPTAS unless $P = NP$.
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Q & A.