# A Simple FPTAS for Counting Edge Covers

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# Overview

# Edge cover

## Definition

For an undirected input graph $G = (V, E)$, an **edge cover** of $G$ is a set of edges $C$ covering all vertices.

## Example

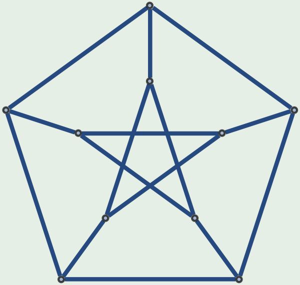

Figure : An edge cover for Petersen graph

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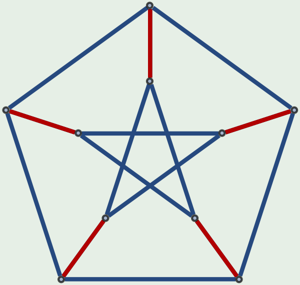


Figure : An edge cover for Petersen graph, with edges chosen being highlighted in red. Note that this is also a perfect matching.

# Edge cover

Edge cover is related to many other problems such as:

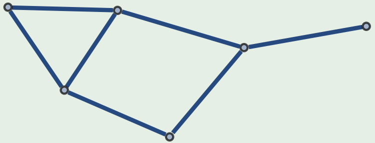
- Matching problem.
- Rtw-Mon-CNF. (read twice monotone CNF)
- Holant problem.
- . . . .

# Relation to Matching

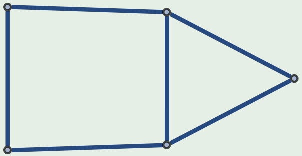
The minimal edge cover could be found via a greedy algorithm based on a maximum matching.

## Example

Find edge covers by maximal matching?



(a) $G$ has a perfect matching.



(b) $G$ doesn't have a perfect matching.

## Remark

*For a graph with a perfect matching, enumerating (sampling) perfect matchings is equivalent to enumerating (sampling) minimum edge covers.*

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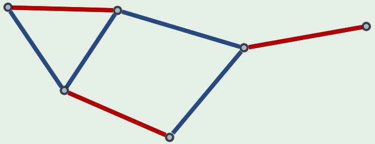

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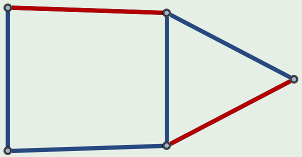

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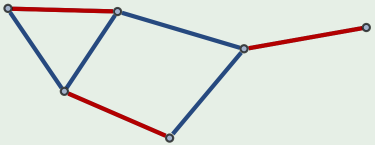

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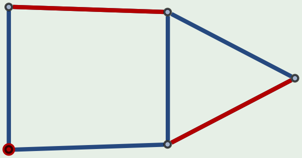

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Consider the following graph, its edge covers are exactly satisfying assignments to the following CNF formula, if we treat edges as variables, and vertices as clauses.

$$\phi = (e_1 \vee e_2 \vee e_3) \wedge (e_1 \vee e_4) \wedge (e_4 \vee e_5 \vee e_2) \wedge (e_3 \vee e_5).$$

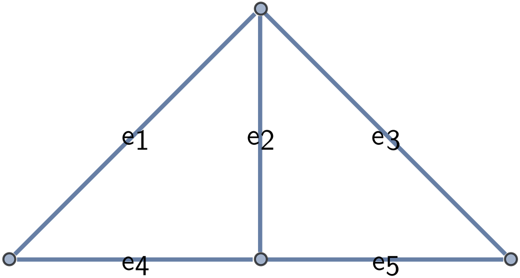


Figure : Graph representation for $\phi$.

# Counting Problems

A list of problems in their search, optimization, and counting versions.

**Search problems:**

- SAT.
- Find a (perfect) matching.
- Find an edge cover.
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**Optimizations:**

- MAX-SAT.
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Besides theoretical computer science, counting problems are also related to many problems from other discipline such as:

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# Approximate Counting

Many interesting problems in the exact counting regimes, including counting edge cover, is hard (#P-complete). Instead we look for these two types of polynomial time approximation scheme:

### Definition (FPTAS)

For given parameter $\varepsilon > 0$ and an instance of a particular problem class, if the algorithm outputs a number $\hat{N}$ such that $(1-\varepsilon)N \leq \hat{N} \leq (1+\varepsilon)N$, where $N$ is the accurate answer of the problem instance, and the running time is bounded by $poly(n, 1/\varepsilon)$ with $n$ being the size of instance, this is called the **FPTAS (fully polynomial time approximation scheme)**.

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A randomized relaxation of FPTAS is known as **FPRAS (fully polynomial time randomized approximation scheme)**, which uses random bits and only outputs $\hat{N}$ to the desired precision with high probability.

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# Main Result

Previous work:

- Only an MCMC-based FPRAS is known for counting edge covers in graphs with maximum degree $3$.
- The correlation decay based FPTAS for anti-ferromagnetic 2-spins systems (e.g. counting independent sets) goes beyond the best known MCMC based FPRAS and achieves the boundary of approximability.

Our main result is a correlation based FPTAS for counting edge covers for general graphs, which provides another example where the tractable range for correlation decay based FPTAS exceeds the sampling based FPRAS.

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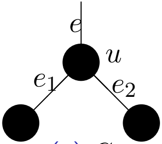
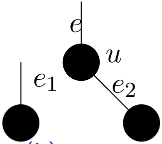
# Dangling instance

**Definition**

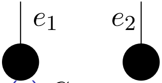
A **dangling edge** $e = (u, \_)$ of a graph is such singleton edge with exactly one end-point vertex $u$, as shown in the Figure 4a.




(a) $G$




(b) $e_1 - u$




(c) $G - e - u$

Figure : Dangling edges examples.

# Counting v.s. Marginal Probability

Recall our counting problem:

**Problem**

Let $EC(G)$ be the set of edge covers.
Goal: estimate $|EC(G)|$.

Let $X$ be an edge cover sampled uniformly from $EC(G)$, consider the following marginal probability:
for an edge $e$, we write $P(G, e) \triangleq \Pr(e \notin X)$.

Solution: estimate $P(G, e)$.

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# Counting from marginal probability

Recall that the set of all edges $E$ is an edge cover. For a randomly sampled edge cover $X$, what is $\Pr(X = E)$?

$$\Pr(X = E) = \frac{1}{|EC(G)|}$$

where $G_1 = G, G_{i+1} = G_i - e_i - u_i - v_i$.
Therefore,

$$\frac{1}{|EC(G)|} = \prod_i (1 - P(G_i, e_i)) .$$

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# Estimating Marginal Probability

We focus on $e$ is dangling.

$$P(G, e) = \frac{1 - \prod_{i=1}^{d} P(G_i, e_i)}{2 - \prod_{i=1}^{d} P(G_i, e_i)},$$

where $G_1 \triangleq G - e - u$, and $G_{i+1} \triangleq G_i - e_i$.
Truncate with modified recursion depth:

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## Proposition

*Given graph $G$, edge $e$ and depth $L$,*

$$|P(G, e, L) - P(G, e)| \leq 3 \cdot (\frac{1}{2})^{L+1}$$

THANK YOU!