Approximate Counting Edge Covers via Correlation Decay

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Abstract

An edge cover for an undirected graph is the set of edges covering every vertices in the graph. We demonstrate the correlation decay property of sampling edge covers in general graphs, and present a PTAS for the approximate counting edge covers in general graphs.

1 Introduction

Since we are only dealing with undirected graph, we will simply write graph to refer to undirected graph. And for notational convenience we only consider graphs with at least 6 vertices.

TBA.

2 Preliminaries

2.1 Definitions

Definition 1. *A hanging edge $e = (u, \_)$ of a graph is such singleton edge with only one end-point vertex u, as shown in the Figure 1a.*

A completely hanging edge $e = (\_, \_)$ of a graph is such edge with no end-point vertex. Note that a completely hanging edge is not a hanging edge.

A graph with at least one hanging edge is called a non-trivial hanging graph.

Here's the list of configurations of trivial hanging graphs:(FIXME DRAW FIGURES)

- $E = \emptyset$.
- $V = \emptyset$.
- *(FIXME MAYBE NOT NEEDED [1]) Graphs containing only a vertex with only hanging edges.*
- *(FIXME MAYBE NOT NEEDED [2]) Graphs containing only completely hanging edges.*

Other definitions TBA.

2.2 Notations

For a graph $G = (V, E)$ and edge $e = (u, v) \in E$, let $G - e \triangleq (V, E - e)$, and $G - e - u - v \triangleq (V - \{u, v\}, E - e)$.

For example, given a degree-3 vertex u with hanging edge e, the result of $G - e - u$ is shown in Figure 1b. FIXME: also define $G - e - u$.

(a) G (b) $G - e - u$

Figure 1: Hanging graphs examples.

We use 0 to denote scalar value 0, and $\mathbf{0}$ to denote vector value 0, and $\{e_i\}_{i=1}^{d}$ denote the d-dimensional vector with i-th coordinate being e_i, so $\{e_i\} = \mathbf{0}$ means $\forall i, e_i = 0$.

[1] check if each G_i is still hanging graphs, either trivial or not

[2] check if each G_i is still hanging graphs, either trivial or not

3 The Recursion

Given a graph $G = (V, E)$ with edge e, now define $P(G, e)$ to be the probability over all edge coverings of G that edge e is not chosen, namely

$$P(G, e) \triangleq \mathbb{P}(\text{edge } e \text{ is not chosen}) \tag{1}$$

Now we're ready for writing out the recursion.

3.1 Nontrivial Hanging Graphs

Proposition 2. *For nontrivial hanging graph $G = (V, E)$ with a hanging edge $e = (u, \_)$, denote the d edges incident with u except e as $e_1, e_2, \ldots, e_d$, let $G_i = G - e - u - \sum_{k=1}^{i-1} e_k$ (specifically, $G_1 = G - e - u$), so we have*

$$P(G, e) = \frac{1 - \prod_{i=1}^{d} P(G_i, e_i)}{2 - \prod_{i=1}^{d} P(G_i, e_i)} = \frac{1}{2} - \frac{0.5 \prod_{i=1}^{d} P(G_i, e_i)}{2 - \prod_{i=1}^{d} P(G_i, e_i)} \tag{2}$$

Proof. For $\alpha \in \{0, 1\}^d$, let E_α be the set of edge coverings in $G - e - u$ such that its restriction onto $\{e_i\}_{i=1}^d$ is consistent with α, denote $Z_\alpha = \|E_\alpha\|$, and $Z = \sum_{\alpha \in \{0,1\}^d} Z_\alpha$.

Also note that counting edge coverings with restriction α is the same in either G, $G - e$, or $G - e - u$, so it's enough to work with $G - e - u$.

$$\begin{aligned} P(G, e) =& \frac{\text{number of solutions with } e \text{ not chosen}}{\text{total number of solutions}} \\ =& \frac{\sum_{\alpha \in \{0,1\}^d, \alpha \neq \mathbf{0}} Z_\alpha}{Z_{\mathbf{0}} + 2 \sum_{\alpha \in \{0,1\}^d, \alpha \neq \mathbf{0}} Z_\alpha} \\ =& \frac{1 - \frac{Z_{\mathbf{0}}}{Z}}{2 - \frac{Z_{\mathbf{0}}}{Z}}. \end{aligned}$$

Now consider the term $\frac{Z_{\mathbf{0}}}{Z}$, it says the probability over all solutions in $G - e - u$ that none of $\{e_i\}_{i=1}^d$ is picked, so

$$\frac{Z_{\mathbf{0}}}{Z} = \mathbb{P}\left(\{e_i\} = \mathbf{0}\right) = \mathbb{P}(e_1 = 0) \prod_{i=2}^{d} \mathbb{P}\left(e_i = 0 \mid \{e_j\}_{j=1}^{i-1} = \mathbf{0}\right) = \prod_{i=1}^{d} P(G_i, e_i).$$

Hence concludes the proof. □

Corollary 3. *For nontrivial hanging graphs,*

$$P(G, e) \leq \frac{1}{2}$$

As a side note, note that $\forall i, G_i$ is a hanging graph (maybe trivial hanging graphs though), although e_i can be a completely hanging edge.

TODO: sum of partial derivatives < 1, hence the correlation decay property on hanging graphs.

3.2 Trivial Hanging Graphs

TBA.

3.3 General Graphs

Here we focus on graphs with no hanging edges and no completely hanging edges, as trivial hanging graphs is just trivial base cases, and nontrivial hanging graphs has been handled in the previous section.

Here's a typical example of converting a general graph to hanging graphs. Say we picked $e = (u, v)$ out of any general graph as in figure 2a, again we want to write the recursion of $P(G, e)$ for G. By definition we have

$$P(G, e) = \frac{(\text{number of solutions in } G - e)}{(\text{number of solutions in } G - e) + (\text{number of solutions in } G - e - u - v)}. \tag{3}$$

For $\alpha \in \{0, 1\}^{d_1}, \beta \in \{0, 1\}^{d_2}$, let $E^G_{\alpha,\beta}$ be the set of edge coverings in G such that its restriction to $\{e_i\}_{i=1}^{d_1}$ is consistent with α, and restriction to $\{f_i\}_{i=1}^{d_2}$ is consistent with β, where $\{e_i\}$ is the set of edges incident with vertex u except e, and $\{f_i\}$ is the set of edges incident with vertex v except e, and $d_1 = \|\{e_i\}\|, d_2 = \|\{f_i\}\|$.

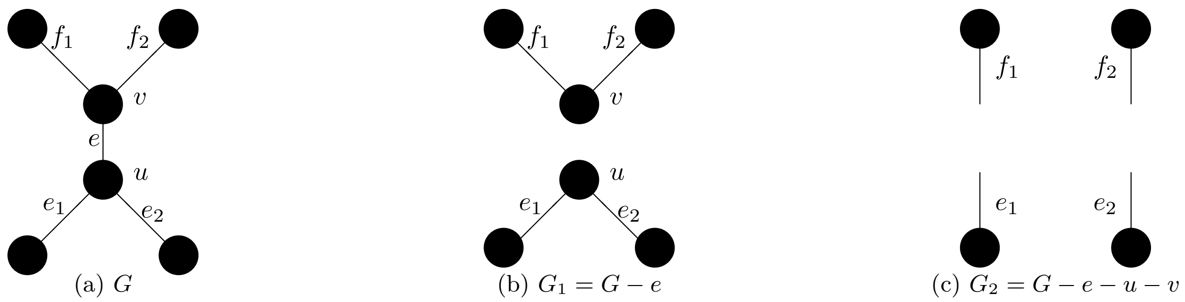
Denote $Z^G_{\alpha,\beta} \triangleq \left\| E^G_{\alpha,\beta} \right\|$, $G_1 \triangleq G - e, G_2 \triangleq G - e - u - v$, C_1 be the number of solutions in G_1, C_2 be the number of solutions in G_2, now we have

Proposition 4.

$$C_1 = \sum_{\alpha \neq \mathbf{0}, \beta \neq \mathbf{0}} Z^{G_1}_{\alpha,\beta} = \sum_{\alpha \neq \mathbf{0}, \beta \neq \mathbf{0}} Z^{G_2}_{\alpha,\beta}$$
$$C_2 = \sum_{\alpha,\beta} Z^{G_2}_{\alpha,\beta}$$

And denote $Z = \sum_{\alpha,\beta} Z^{G_2}_{\alpha,\beta}$*, we also have,*

$$P(G,e) = \frac{\sum_{\alpha \neq \mathbf{0}, \beta \neq \mathbf{0}} Z^{G_2}_{\alpha,\beta}}{Z + \sum_{\alpha \neq \mathbf{0}, \beta \neq \mathbf{0}} Z^{G_2}_{\alpha,\beta}} = 1 - \frac{1}{2 + \mathbb{P}(\alpha = 0, \beta = 0) - \mathbb{P}(\alpha = 0) - \mathbb{P}(\beta = 0)}$$



(a) G (b) $G_1 = G - e$ (c) $G_2 = G - e - u - v$

Figure 2: General graphs examples.

First consider the term $\mathbb{P}(\alpha = 0, \beta = 0) = \frac{\sum_{\alpha = \mathbf{0}, \beta = \mathbf{0}} Z^{G_2}_{\alpha,\beta}}{Z}$, it says the probability in edge coverings of G_2 that none of the edges $\{e_i\}$ and none of $\{f_i\}$ is chosen.

Denote $G_i^1 \triangleq G - e - u - v - \sum_{k=1}^{i-1} e_k$, and $G_i^2 \triangleq G - e - u - v - \sum_{k=1}^{d_1} e_k - \sum_{k=1}^{i-1} f_k$, so we have

Proposition 5.

$$\mathbb{P}(\alpha = 0, \beta = 0) = \prod_{i=1}^{d_1} P(G_i^1, e_i) \cdot \prod_{i=1}^{d_2} P(G_i^2, f_i)$$

Proof.

$$\begin{aligned}
\mathbb{P}(\alpha = 0, \beta = 0) =& \mathbb{P}(\alpha = 0) \cdot \mathbb{P}(\beta = 0 \mid \alpha = 0) \\
=& \mathbb{P}(\{e_i\} = \mathbf{0}, \{f_i\} = \mathbf{0}) \\
=& \prod_{i=1}^{d_1} \mathbb{P}\left(e_i = 0 \mid \{e_j\}_{j=1}^{i-1} = \mathbf{0}\right) \cdot \prod_{i=1}^{d_2} \mathbb{P}\left(f_i = 0 \mid \{e_j\}_{j=1}^{d_1} = \mathbf{0}, \{f_j\}_{j=1}^{i-1} = \mathbf{0}\right) \\
=& \prod_{i=1}^{d_1} P(G_i^1, e_i) \cdot \prod_{i=1}^{d_2} P(G_i^2, f_i)
\end{aligned}$$

□

Note that $\forall i$, both G_i^1, G_i^2 must be hanging graphs.(FIXME [3])

TODO: show that $\mathbb{P}(\alpha = 0, \beta = 0), \mathbb{P}(\alpha = 0), \mathbb{P}(\beta = 0)$ has correlation decay property, hence the correlation decay property on $P(G,e)$ on general graphs.

[3] Can we say nontrivial hanging graphs if we restrict graphs to have at least 6 vertices?