A Simple FPTAS for Counting Edge Covers

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Overview

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Edge cover

Definition

For an undirected input graph $G = (V, E)$, an **edge cover** of G is a set of edges C covering all vertices.

Example

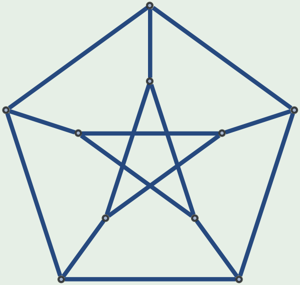

Figure : An edge cover for Petersen graph

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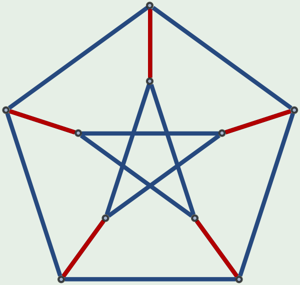

Figure : An edge cover for Petersen graph, with edges chosen being highlighted in red. Note that this is also a perfect matching.

Overview

1 Introduction
- Definition
- Counting Problems
- Edge Covers and Matching
- Edge Covers and Rtw-Mon-CNF Formulae

2 Main Result
- Previous Results and Our Result
- Counting via Marginal Probability
- Computation Tree Recursion
- Correlation Decay

Why counting?

Remark (Counting edge covers)

- *As the name suggests, the problem of counting edge covers simply asks for the number of edge covers in a graph.*
- *While both decision and optimization (minimum edge covers) version of edge cover is easy, the counting version is more challenging and interesting.*

Besides theoretical computer science, counting problems could also be related to many problems from other discipline such as:

- Partition function in statistical physics.
- Graph polynomials.
- Sampling, learning and inference in probabilistic graphical models.
- Pricing in combinatorial prediction markets.
- Query evaluations of probabilistic database.
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Approximate Counting

Many interesting problems in the exact counting regimes, including counting edge cover, is hard (#P-complete). We look for their approximate version.

Definition (FPTAS)

For given parameter $\varepsilon > 0$ and an instance of a particular problem class, if the algorithm outputs a number $\hat{N}$ such that

$$(1-\varepsilon)N \le \hat{N} \le (1+\varepsilon)N,$$

where N is the accurate answer of the problem instance, and the running time is bounded by $poly(n, 1/\varepsilon)$ with n being the size of instance, this is called the **FPTAS (fully polynomial time approximation scheme)**.

A randomized relaxation of FPTAS is known as **FPRAS (fully polynomial time randomized approximation scheme)**, which uses random bits and only outputs $\hat{N}$ to the desired precision with high probability.

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Edge cover

Besides its own interests, edge cover is closely related to many other problems such as:

- Matching problem.
- Rtw-Mon-CNF. (read twice monotone CNF)
- Holant problem.
- ....

Overview

Relation to Matching

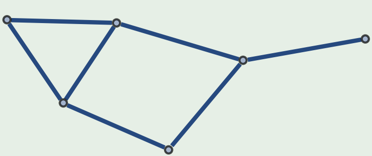
A perfect matching is always a minimum edge cover.

$$\#\text{perfect-matchings} \leq_{AP} \#\text{minimum-edge-covers}.$$

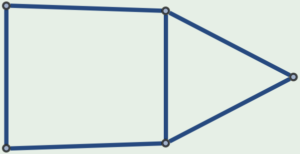
Besides,

Example (Minimum edge covers)

Find a minimum edge cover by maximal matching?



(a) G has a perfect matching.



(b) G doesn't have a perfect matching.

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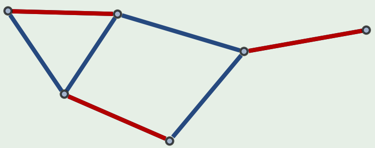

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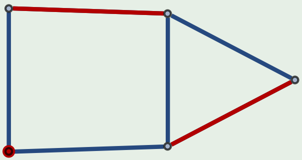

(b) G doesn't have a perfect matching.

Overview

Relation to Rtw-Mon-CNF

Definition

A formula is **read twice** if every variables appears at most twice.
A formula is **monotone** if every variables appears positively.

By treating edges as variables, and vertices as clauses,

$$\text{Edge-covers} \equiv \text{Rtw-Mon-CNF}.$$

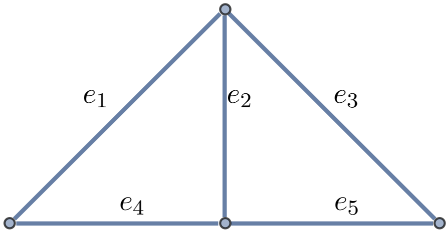

Figure : Graph representation for
$\phi = (e_1 \vee e_2 \vee e_3) \wedge (e_1 \vee e_4) \wedge (e_4 \vee e_5 \vee e_2) \wedge (e_3 \vee e_5)$.

Overview

Previous Results

- For counting edge covers, only an Markov Chain Monte Carlo (MCMC) based FPRAS is known for graphs with maximum degree 3 [I. Bezkov and W. Rummler 2009].
- For #DNF[Karp and Luby 1983] and counting matchings[Jerrum and Sinclair 1989], only FPRAS is known.
- For counting perfect matchings, it's still open whether or not it admits FPRAS (or FPTAS).
- For anti-ferromagnetic 2-spins systems (e.g. counting independent sets), an FPTAS is known, and it's correlation decay based, and goes beyond the best known MCMC based FPRAS and achieves the boundary of approximability (in terms of maximum degree).

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Our Result

- An FPTAS for counting edge covers in general graphs.
- Our algorithm is correlation decay based.
- This provides another example where the tractable range of correlation decay based FPTAS exceeds the best-known sampling based FPRAS.

Proof Sketch

Let $EC(G)$ be the set of edge covers. Here is an overall work-flow:

- Relate $|EC(G)|$ with a marginal probability $P(G, e)$.
- Derive a computation tree recursion for $P(G, e)$.
- $P(G, e, L)$: Truncate the tree at depth L for some notion of depth.
- Show exponential correlation decay with respect to that tree depth:

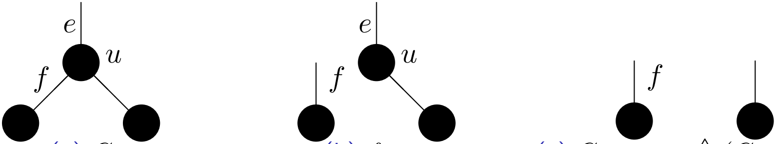
$$|P(G, e, L) - P(G, e)| \leq exp(-\Omega(L))$$

Definition (Dangling edge)

A **dangling edge** $e = (u, \_)$ of a graph is such a singleton edge with exactly one end-point vertex u, as shown in the Figure 4a.

$$G - e \triangleq (V, E \setminus e)$$

$$G - u \triangleq (V \setminus u, E - u)$$



(a) G (b) $f - u$ (c) $G - e - u \triangleq (G - e) - u$

Figure : Dangling edges examples.

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Counting v.s. Marginal Probability

Problem

Goal: estimate $|EC(G)|$.

Let X be an edge cover sampled uniformly from $EC(G)$, consider the following marginal probability:
for an edge e, we write $P(G, e) \triangleq \Pr(e \notin X)$.

Solution: estimate $P(G, e)$.

Why $P(G, e)$?

Recall that the set of all edges E is an edge cover. For a randomly sampled edge cover X, what is $\Pr(X = E)$?

Let $E = \{e_i\}$, and $e_i = (u_i, v_i)$.

$$\Pr(X = E) = \frac{1}{|EC(G)|}$$

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where $G_1 = G, G_{i+1} = G_i - e_i - u_i - v_i$.

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Therefore,

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Computation Tree Recursion

We focus on e is dangling.

$$P(G,e) = \frac{1 - \prod_{i=1}^{d} P(G_i, e_i)}{2 - \prod_{i=1}^{d} P(G_i, e_i)},$$

where $G_1 \triangleq G - e - u$, and $G_{i+1} \triangleq G_i - e_i$.

But the computation tree could be exponentially large, how do we get an estimate out of it?

Computationally Efficient Correlation Decay

A natural estimate: Truncate the tree!

But without a degree bound, the tree could still be exponentially large. So we truncate with a modified recursion depth:

$$P(G,e,L) = \begin{cases} \frac{1}{2}, & \text{if } L \leq 0; \\ \frac{1-\prod_{i=1}^{d} P(G_i,e_i,L-\lceil \log_6 (d+1) \rceil)}{2-\prod_{i=1}^{d} P(G_i,e_i,L-\lceil \log_6 (d+1) \rceil)}, & \text{otherwise.} \end{cases}$$

This is also known as computational efficient correlation decay.

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Key Lemma

Given graph G, edge e and depth L,

$$|P(G,e,L) - P(G,e)| \leq 3 \cdot (\frac{1}{2})^{L+1}$$

Now we just take $L = \log_2 \left(\frac{6m}{\varepsilon}\right)$, this gives the desired FPTAS.

Conclusions and Upcoming Results

As a conclusion, we have shown an FPTAS for counting edge covers (or Rtw-Mon-CNF) in general graphs.

A natural question: what about counting Read-k-Mon-CNF?

Upcoming: FPTAS for counting Read-5-Mon-CNF, while Read-6-Mon-CNF does not admit FPTAS unless $P = NP$.

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Q & A.

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