The Ins and Outs of Barrier Options: Part 2

EMANUEL DERMAN AND IRAJ KANI

Barrier options are extensions of standard calls and puts on stocks. Standard calls and puts have payoffs that depend on one market level: the strike. Barrier options have payoffs that depend on two market levels: the strike and the barrier. Investors can use them to gain exposure to (or enhance returns from) future market scenarios more complex than the simple bullish or bearish expectations embodied in standard options. In addition, their premiums are usually lower than those of standard options with the same strike and experience.

In Part 1 of this article (see Derman and Kani [1996]), we explored the basics of barrier options, and compared them to standard puts and calls. In this article, we explore binary options and other related barrier options in a similar manner.

The article closes with an appendix outlining the mathematics of barrier options. For some end-users, this will be useful as an extension of the analytics underlying standard options. Others may simply want to keep the equations handy as a back-up reference that completes the analysis of the barrier options begun in Part 1.

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LESS COMMON BARRIER OPTIONS

There are four less common barrier options that we describe and analyze next.

- A *binary up-and-in call* has no strike and a single barrier. It pays the holder a one-time fixed amount, say $1, the first time the stock price crosses the barrier from below during the option's lifetime. If the stock price has not crossed the barrier by expiration, the option expires worthless.

In the previous section on barrier options (see Part 1), we assumed the rebate was always zero. For barrier options with non-zero rebates, you can value the contribution to the barrier option value of a $1 non-deferred cash rebate associated with an up barrier by noting that it is equal to the value of a binary up-and-in call.

- A *binary down-and-in put* has no strike and a single barrier. It pays the holder a one-time fixed amount, say $1, the first time the stock price crosses the barrier from above during the option's lifetime. If the stock price has not crossed the barrier by expiration, the option expires worthless.

  Similarly, you can value the contribution to the barrier option value of a $1 non-deferred cash rebate associated with a down barrier by noting that it is equal to the value of a binary down-and-in put.
- A *capped European-style call* is characterized by both a strike and a *cap barrier* above the strike. It has the same payoff as a standard call with the same strike and expiration, except that, if the stock price ever crosses the barrier from below, the option immediately terminates and the holder receives a cash payment equal in value to the difference between the barrier and the strike. If this cash is received immediately, the capped call is termed *non-deferred*. If it is received on the date the option would have expired, it is termed *deferred*.
- A *floored European-style put* is characterized by both a strike and a *floor barrier* below the strike. It has the same payoff as a standard put with the same strike and expiration, except that if the stock price ever crosses the barrier from above, the option immediately terminates and the holder receives a cash payment equal in value to the difference between the strike and the barrier. The cash value can similarly be deferred or non-deferred.

We analyze some examples of these options next using the same format as in Part 1. For each option, the first chart describes the value of the option, and the second and third capture the delta and gamma, respectively. The assumptions behind these evaluations are the same as in Part 1, and summarized below:

Strike:	100
Barrier:	80 if down, 120 if up
Rebate:	0
Dividend Yield:	5.0% (annually compounded)
Volatility:	20% per year
Risk-Free Rate	10.0% (annually compounded)

In Panel A of Exhibit 1, as the stock rises toward the 120 barrier, the up-and-in binary call value increases toward its constant payoff of $1. Because its payoff is somewhat like the delta of a standard call at expiration (one in the money, zero otherwise), its value varies with stock price somewhat like the delta of the standard call option.

In Panel B, delta increases from zero far below the barrier, and then drops back to zero above the barrier. The increase is more rapid for shorter expirations.

The call owner is long volatility (positive gamma) below the barrier, and short volatility near the barrier as delta drops to zero (Panel C). Gamma is infinite at the barrier.

As Panel A of Exhibit 2 shows, as the stock falls toward the 80 barrier, the down-and-in binary put value increases toward its constant payoff of $1. Its value varies with stock price somewhat like that of the delta of the standard put option.

Delta decreases from zero far above the barrier, and then rises back to zero below the barrier (Panel B). The increase is more rapid for shorter expirations.

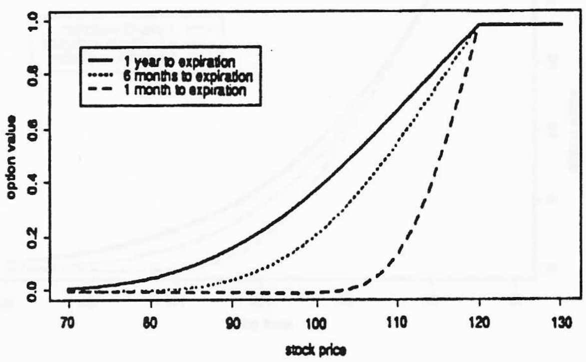
The down-and-in binary put owner is long volatility (positive gamma) above the barrier (Panel C). For some binary puts, though not this one, the put holder can also be short volatility nearer the barrier. Gamma is infinite at the barrier.

Panel A of Exhibit 3 shows that, for stock prices far below the barrier, it is unlikely that the barrier will be crossed, and the value of the

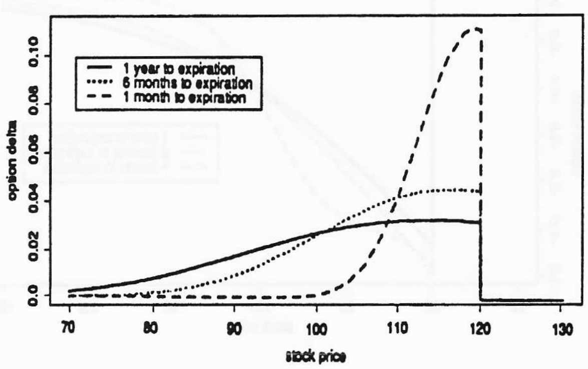
Exhibit 1

Binary Up-and-In Call: Barrier 120

Panel A. Value



Panel B. Delta



Panel C. Gamma

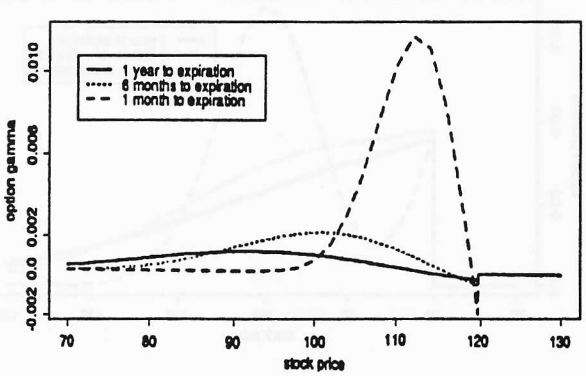
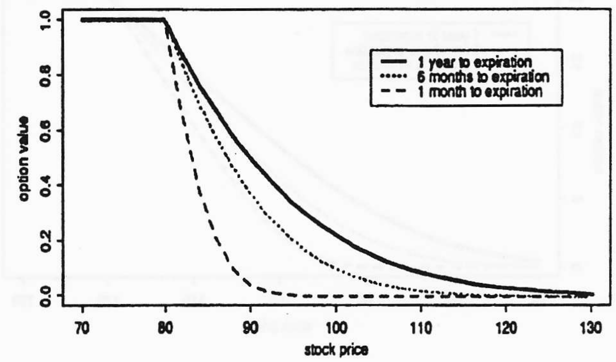


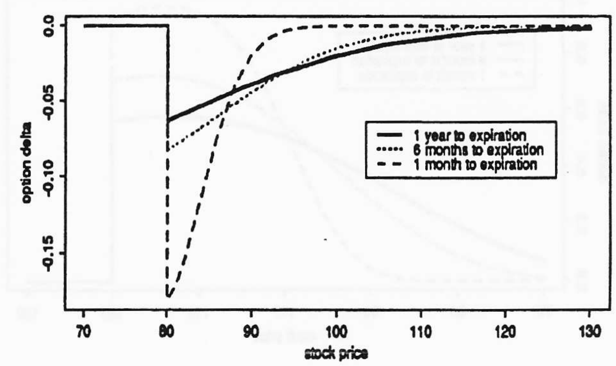
Exhibit 2

Binary Down-and-In Put: Barrier 80

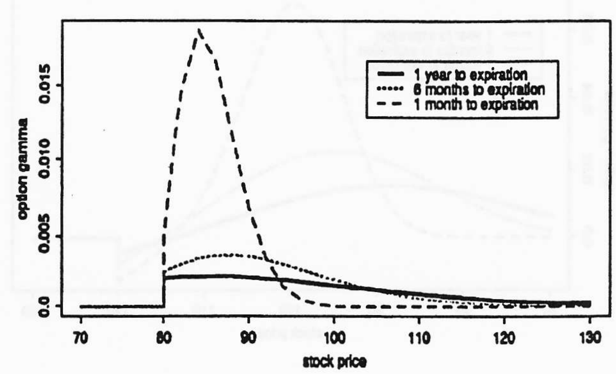
Panel A. Value



Panel B. Delta



Panel C. Gamma



capped call is close to that of a standard call. Closer to 120, the call rises steadily in value toward its maximum payoff of 20.

Delta increases at low stock prices (Panel B). Closer to the 120 barrier, delta actually decreases because of the limited upside.

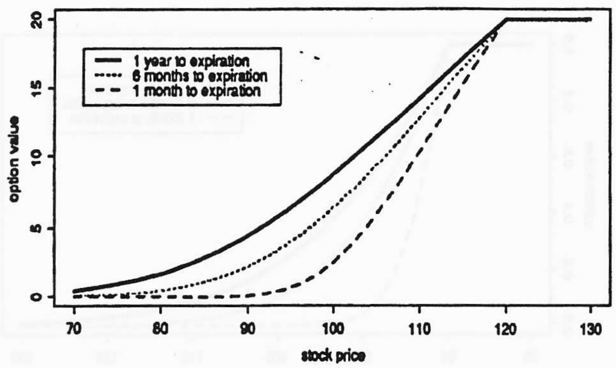
The call owner is long volatility below the barrier and short volatility nearer the barrier (Panel C).

As Panel A of Exhibit 4 shows, for stock prices far above the barrier, it is unlikely that the barrier will be crossed, and the value of the

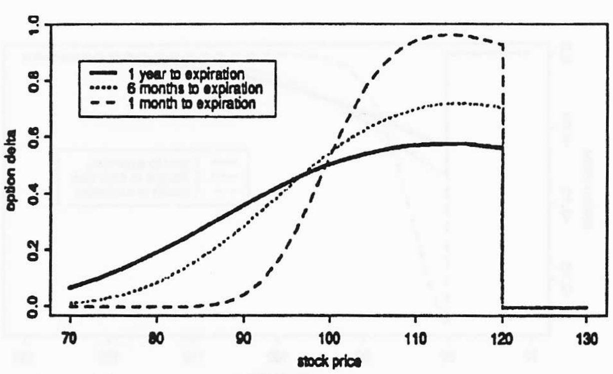
Exhibit 3

Capped Call: Strike 100, Barrier 120, Non-Deferred

Panel A. Value



Panel B. Delta



Panel C. Gamma

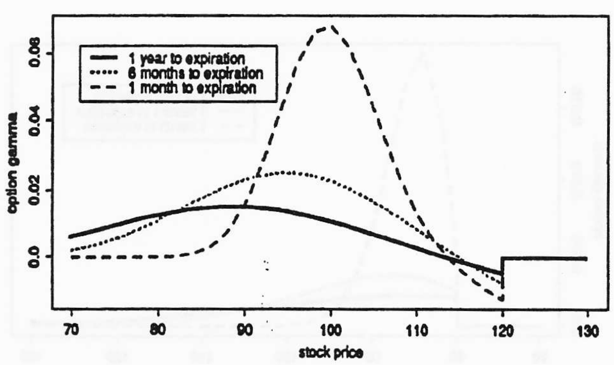
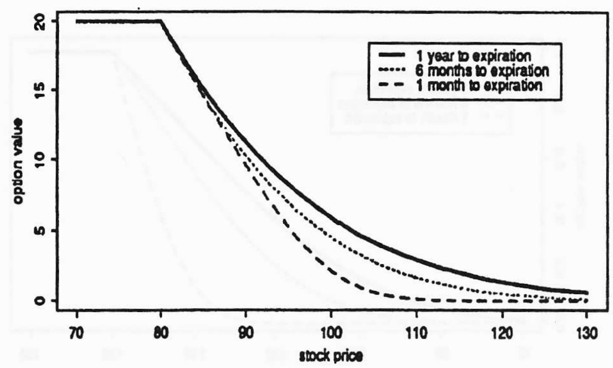


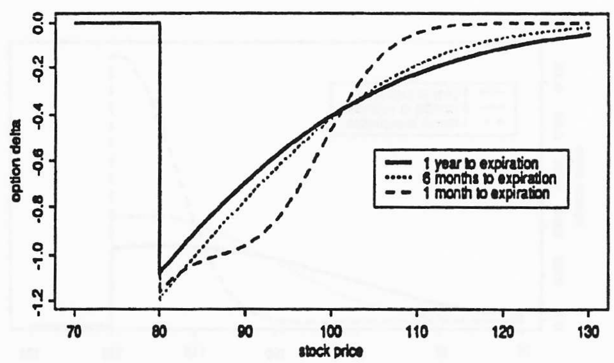
Exhibit 4

Floored Put: Strike 100, Barrier 80, Non-Deferred

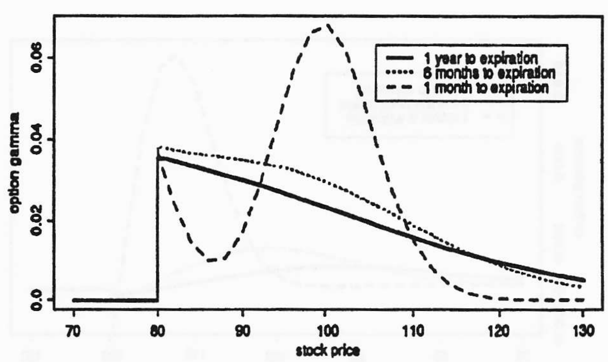
Panel A. Value



Panel B. Delta



Panel C. Gamma



floored put is close to that of a standard put. Closer to 80, the put rises steadily in value toward its maximum payoff of 20.

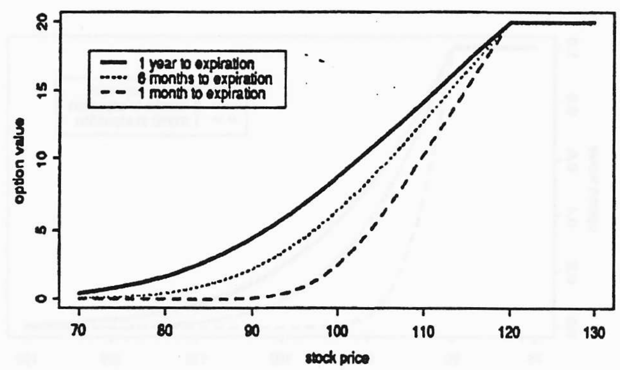
Delta decreases from zero at high stock prices to larger negative values at lower stock prices (Panel B). Close to the 80 barrier, delta has magnitude greater than one, the maximum allowed for a standard put.

The put owner is long volatility above the barrier for this particular put, though some floored puts can have negative gamma close to the barrier (Panel C).

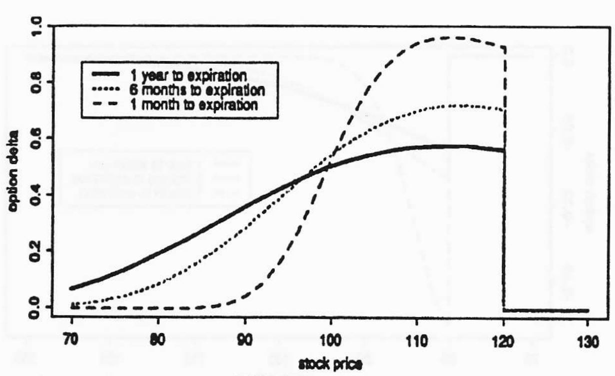
Exhibit 3

Capped Call: Strike 100, Barrier 120, Non-Deferred

Panel A. Value



Panel B. Delta



Panel C. Gamma

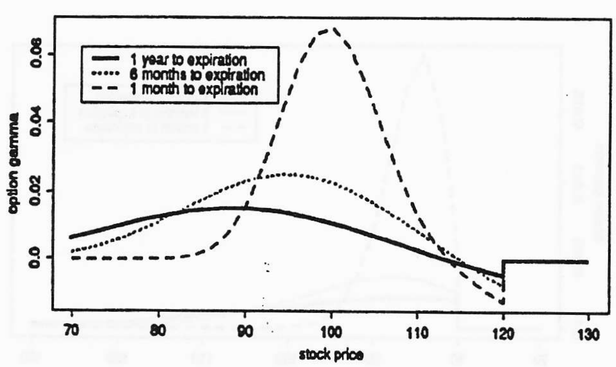
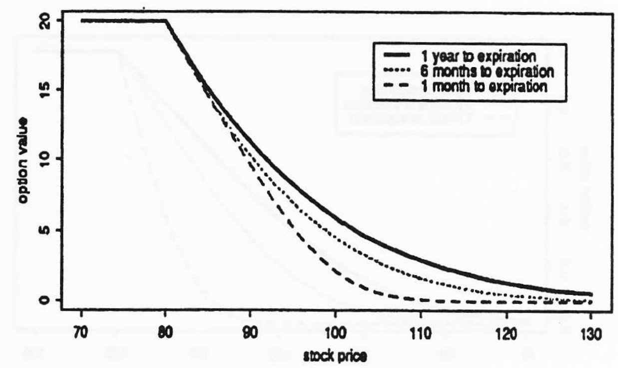


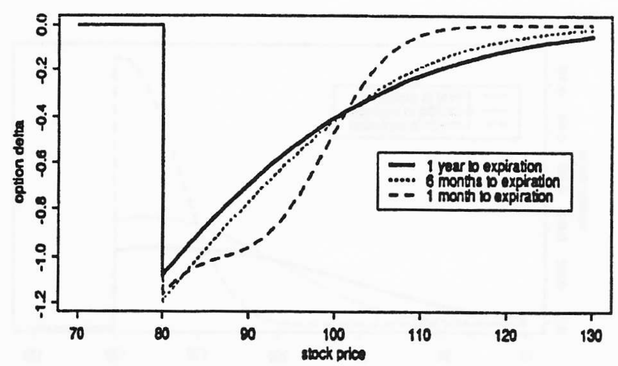
Exhibit 4

Floored Put: Strike 100, Barrier 80, Non-Deferred

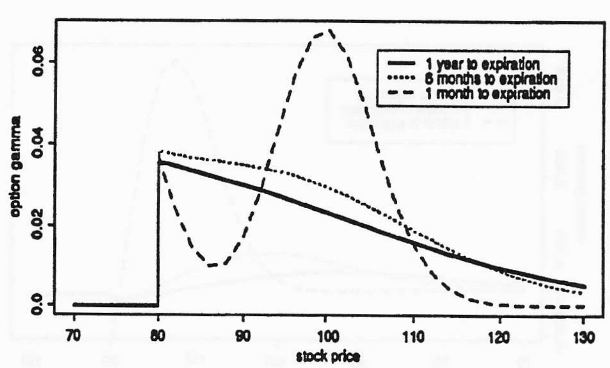
Panel A. Value



Panel B. Delta



Panel C. Gamma



floored put is close to that of a standard put. Closer to 80, the put rises steadily in value toward its maximum payoff of 20.

Delta decreases from zero at high stock prices to larger negative values at lower stock prices (Panel B). Close to the 80 barrier, delta has magnitude greater than one, the maximum allowed for a standard put.

The put owner is long volatility above the barrier for this particular put, though some floored puts can have negative gamma close to the barrier (Panel C).

THE MATHEMATICS OF BARRIER OPTIONS

In this section we present a more detailed mathematical description of barrier options and derive some of their analytical properties. We assume no fee for borrowing stock.

DOWN-AND-OUT CALL OPTIONS

We denote the spot stock price by S, the strike price by K, and the barrier by b. Assume that $S > b$ (otherwise the option has already terminated and has zero value). For the moment also assume that $K > b$, and that the option rebate R is zero. The price of the down-and-out call option is given by

$$C_{DO}(S, K, \sigma, \tau; b) =$$

$$C_{BS}(S, K, \sigma, \tau) - \left(\frac{b}{S}\right)^{2(\hat{r}/\sigma^2)-1} C_{BS}(b^2/S, K, \sigma, \tau) \quad (1)$$

where $\hat{r}$ is the continuous cost of carry rate for the stock given by the difference between the risk-free rate and the continuous dividend yield.

In Equation (1), $C_{BS}(S, K, \sigma, \tau)$ is the Black-Scholes (B-S) expression for the European-style call option with strike K, volatility σ, and time to expiration τ. The second term is the "barrier" term, which is necessary to guarantee that the value of the option given by Equation (1) is zero at the knockout barrier. That is, when $(S = b)$,

$$C_{DO}(S = b, K, \sigma, \tau; b) = 0 \quad (2)$$

Equation (1) resembles the formula for a call spread. One can make this resemblance more complete by transforming it to give:*

$$C_{DO}(S, K, \sigma, \tau; b) =$$

$$C_{BS}(S, K, \sigma, \tau) - \left(\frac{b}{S}\right)^{2(\hat{r}/\sigma^2)+1} \times$$

$$C_{BS}(S, KS^2/b^2, \sigma, \tau) \quad (3)$$

For stock prices much larger than the barrier $(S >> b)$, the contribution of the second term in Equation (1) is negligible, and so we have:

$$C_{DO}(S, K, \sigma, \tau; b) \sim C_{BS}(S, K, \sigma, \tau)$$

$$\text{for } S >> b \quad (4)$$

The value of the call is given by the B-S formula when the stock price is much greater than the barrier level. This is completely expected, as the stock price has very small probability of ever crossing the barrier level. Therefore, the option is unlikely to terminate early by knockout, and will pay its terminal payoff to the investor at expiration.

It is interesting to examine the behavior of a down-and-out call option when the stock price is close to the barrier level. In this region, it is not too difficult to show that one has the following approximation for the option price:

$$C_{DO}(S, K, \sigma, \tau; b) =$$

$$\varepsilon(2Se^{-\delta\tau}N(X) + \left(\frac{2\hat{r}}{\sigma^2} - 1\right)C_{BS}(S, K, \sigma, \tau)) \quad (5)$$

Here $S = (1 + \varepsilon)b$, where $\varepsilon > 0$ is assumed to be small so that S is close to b, δ is the dividend yield on the stock, and

$$X = \frac{\left[\log\left(\frac{S}{K}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]}{\sigma\sqrt{\tau}} \quad (6)$$

It is clear that the option price vanishes as the stock price approaches the barrier level $(\varepsilon \to 0)$. Equation (5) has an interesting form in the limit where volatility or time to expiration is very large. For $S \sim b$ and σ large,

$$C_{DO}(S, K, \sigma, \tau; b) \sim \varepsilon Se^{-\delta\tau} \quad (7)$$

If the holder of the down-and-out option receives a rebate R on knockout, this adds the following value C_R to the premium:

$$C_R = R\left[\left(\frac{b}{S}\right)^{h_1} N(Z_1) + \left(\frac{b}{S}\right)^{h_2} N(Z_2)\right] \quad (8)$$

where

$$Z_1 = \left[\log\left(\frac{b}{S}\right) - (2r\sigma^2 + (\hat{r} - \frac{1}{2}\sigma^2)^2)^{1/2}\tau\right]/v \quad (9)$$

$$Z_2 = \left[\log\left(\frac{b}{S}\right) + (2r\sigma^2 + (\hat{r} - \frac{1}{2}\sigma^2)^2)^{1/2}\tau\right] / \nu \tag{10}$$

$$h_1 = \left[(\hat{r} - \frac{1}{2}\sigma^2) - (2r\sigma^2 + (\hat{r} - \frac{1}{2}\sigma^2)^2)^{1/2}\right] / \sigma^2 \tag{11}$$

$$h_2 = \left[(\hat{r} - \frac{1}{2}\sigma^2) + (2r\sigma^2 + (\hat{r} - \frac{1}{2}\sigma^2)^2)^{1/2}\right] / \sigma^2 \tag{12}$$

and where we have set $\nu = \sigma\sqrt{\tau}$.

In deriving Equation (8), we assume that the rebate is *non-deferred* — that is, it is received by the optionholder as soon as the stock price crosses the barrier level. Occasionally, however, this payoff is *deferred* until the designated expiration of the contract; in that case, the optionholder immediately receives the rebate value discounted from the designated option expiration to the crossing time. In this *deferred* payment situation Equations (9)-(12) are modified to read:

$$Z_1 = \left[\log\left(\frac{b}{S}\right) - (\hat{r} - \frac{1}{2}\sigma^2)\tau\right] / \nu \tag{13}$$

$$Z_2 = \left[\log\left(\frac{b}{S}\right) + (\hat{r} - \frac{1}{2}\sigma^2)\tau\right] / \nu \tag{14}$$

$$h_1 = 0 \tag{15}$$

$$h_2 = (2(\hat{r} - \frac{1}{2}\sigma^2)) / \sigma^2 \tag{16}$$

The justification for Equation (1) begins with the knockout barrier distribution function for the final stock price S_τ at time τ conditional upon the initial value S of the stock price, a barrier at b, and a stock volatility σ. This distribution function $u(S_\tau, \tau, \sigma; b \mid S)$ is given by the following expression for all $S > b$:

$$u(S_\tau, \tau, \sigma; b \mid S) = U(S_\tau, \tau, \sigma \mid S) - \left(\frac{b}{S}\right)^{2(\hat{r}/\sigma^2)-1} \times$$

$$U\left(S_\tau, \tau, \sigma \,\middle|\, \frac{b^2}{S}\right) \text{ for } S > b$$

$$= 0 \text{ for } S \le b \tag{17}$$

Here $U(S_\tau, \tau, \sigma)$ denotes the Black-Scholes lognormal distribution function for the final stock price at time τ corresponding to the volatility parameter σ and having the following well-known expression:

$$U(S_\tau, \tau, \sigma) =$$

$$\frac{1}{S_\tau^\sigma \sqrt{2\pi\tau}} \exp \times$$

$$\left[(\log(S_\tau / S) - (\hat{r} - \frac{1}{2}\sigma^2)\tau)^2 / (2\sigma^2\tau)\right] \tag{18}$$

The knockout distribution function of Equation (17) vanishes at the knockout barrier. First assume that the rebate R = 0. The value of the knockout call is then given by:

$$C(S, K, \tau, \sigma; b) =$$

$$\int_{MAX(K,b)}^{\infty} u(S_\tau, \tau, \sigma; b \mid S) MAX(S - K, 0) dS_\tau \tag{19}$$

It is a simple matter to perform the integration in Equation (19) and so derive Equation (1).

The addition of the non-zero rebate values to the computation proceeds with the construction of the probability distribution function for first-time absorption in the time interval τ to $\tau + \Delta\tau$. This is the first passage time distribution function. This density function is readily constructed as the time derivative of the total absorption probability function and is given by:

$$G(\tau, \sigma; b \mid S) = -\frac{d}{d\tau} \int_b^{\infty} u(S_\tau, \tau, \sigma; b \mid S)\, dS_\tau \tag{20}$$

If non-deferred, the contribution of the rebate to the value of the down-and-out call option C_R is then computed simply as follows:

$$C_R = R \int_0^{\tau} G(t, \sigma; b \mid S)\, e^{-rt} dt \tag{21}$$

For deferred payoffs, the rebate contribution to the value of the down-and-out call option is given by:

$$C_R = Re^{-r\tau} \int_0^{\tau} G(t, \sigma; b \mid S)\, dt \tag{22}$$

The integrations in Equations (21) and (22) can be readily performed using Equations (17) and (20).

So far, we have assumed that the strike is above the barrier, that is, K > b. In general, the strike of a down-and-out option may be below the barrier level and we must account for this condition by using the correct limits of integration as in Equation (19). The correct solution for any strike is given by

$$C_{DO}(S, K, \sigma, \tau; b) =$$

$$Se^{-\delta\tau}N(X) - Ke^{-r\tau}N(X - v) -$$

$$(Se^{-\delta\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2+1} N(Y) - Ke^{-r\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2-1} \times$$

$$N(Y - v)) + C_R \tag{23}$$

where

$$X = \left[\log\left(\frac{S}{MAX(b, K)}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{24}$$

$$Y = \left[\log\left(\frac{b^2}{SMAX(b, K)}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{25}$$

Down-and-In Call Options

A standard call option is clearly equivalent to a long position in a down-and-out call option and a long position in a down-and-in call option, each with the same strike and expiration as the standard call. Their payoffs are the same under all stock price scenarios. Consequently,

$$C_{DI}(S, K, \sigma, \tau; b) =$$

$$C(S, K, \sigma, \tau) - C_{DO}(S, K, \sigma, \tau; b) \tag{26}$$

In contrast to the holder of knockout options, the holder of knock-in options will always benefit from increases in volatility (and is thus long volatility) for all stock prices. This is intuitively clear, as any increase in volatility will increase both the knock-in probability and the expected payoff if the option is knocked in.

Up-and-Out Call Options

Using the same notation as in previous cases, the value of an up-and-out call option under the assumption that b > S, K is given by:

$$C_{UO}(S, K, \sigma, \tau; b) =$$

$$Se^{-\delta\tau}(N(X_1) - N(X_2)) -$$

$$Ke^{-r\tau}(N(X_1 - v) - N(X_2 - v)) -$$

$$Se^{-\delta\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2+1} (N(Y_1) - N(Y_2)) +$$

$$Ke^{-r\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2-1} (N(Y_1 - v) - N(Y_2 - v))) + C'_R \tag{27}$$

where

$$X_1 = \left[\log\left(\frac{S}{K}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{28}$$

$$X_2 = \left[\log\left(\frac{S}{b}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{29}$$

$$Y_1 = \left[\log\left(\frac{b^2}{SK}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{30}$$

$$Y_2 = \left[\log\left(\frac{b}{S}\right) + \left(\hat{r} + \frac{1}{2}\sigma^2\right)\tau\right]/v \tag{31}$$

The rebate-term C'_R in this case is a slightly modified form of C_R for down-and-out options:

$$C'_R = R\left[\left(\frac{b}{S}\right)^{h_1} N(-Z_1) + \left(\frac{b}{S}\right)^{h_2} N(-Z_2)\right] \tag{32}$$

Up-and-In Call Options

A standard call option is also identically equivalent to an up-and-in call combined with an up-and-out call, for zero rebate. Thus the following relationship holds:

$$C_{UI}(S, K, \sigma, \tau; b) =$$

$$C(S, K, \sigma, \tau) - C_{UO}(S, K, \sigma, \tau; b) \tag{33}$$

The holder of an up-and-in option will always benefit from increases in the market volatility. Because up-and-out call options may possess negative kappa values, Equation (33) shows that the sensitivity to volatility of up-and-in call options may be considerably larger than that of a standard call option.

Up-and-Out Put Options

Up-and-out puts are similar to down-and-out calls. Therefore, using our earlier notation, we simply give the final result for an up-and-out put option with barrier at b, assuming that the condition b > S is satisfied:

$$P_{UO}(S, K, \sigma, \tau; b) =$$

$$Ke^{-r\tau}N(-X + v) - Se^{-\delta\tau}N(-X) -$$

$$\left[Ke^{-r\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2-1} N(-Y + v) - \right.$$

$$\left. Se^{-\delta\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2+1} N(-Y) \right] + C'_R \quad (34)$$

The X and Y in this equation are obtained from Equations (24) and (25) by replacing MAX with MIN on the right-hand side. The rebate contribution C'_R is the same as in Equation (32). The proof of this formula is similar to the case of down-and-out call options. For large barrier values this formula approaches the value of a standard put.

Furthermore, for small volatilities the value of this option is given by $Ke^{-r\tau} - Se^{-\delta\tau}$. Also, for barrier values very close to the spot price (at fixed volatility), the only contribution to the put value is due to the rebate term and is simply equal to R (or $Re^{-r\tau}$ if the rebate is deferred).

Up-and-In Put Options

A zero rebate up-and-in put combined with a zero rebate up-and-out put is equivalent to a standard put. Thus:

$$P_{UI}(S, K, \sigma, \tau; b) =$$

$$P(S, K, \sigma, \tau) - P_{UO}(S, K, \sigma, \tau; b) \quad (35)$$

Down-and-Out Put Options

It is relatively simple to make the proper modification in the up-and-out call formulation to derive the formula for a down-and-out put:

$$P_{DO}(S, K, \sigma, \tau; b) =$$

$$Ke^{-r\tau}(N(-X_1 + v) - N(-X_2 + v)) -$$

$$Se^{-\delta\tau}(N(-X_1) - N(-X_2)) -$$

$$Ke^{-r\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2-1}(N(-Y_1 + v) - N(-Y_2 + v)) +$$

$$Se^{-\delta\tau}\left(\frac{b}{S}\right)^{2\hat{r}/\sigma^2+1}(N(-Y_1) - N(-Y_2)) + C_R$$

$$(36)$$

with variables X_1, X_2, Y_1, Y_2 as defined by Equations (28)-(31) and with the rebate term as in Equation (8).

Down-and-In Put Options

A zero rebate down-and-in put combined with a zero rebate down-and-out put is equivalent to a standard put. Thus the following relationship holds:

$$P_{DI}(S, K, \sigma, \tau; b) =$$

$$P(S, K, \sigma, \tau) - P_{DO}(S, K, \sigma, \tau; b) \quad (37)$$

ENDNOTES

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*Use the scaling property $C_{BS}(\lambda S, \lambda K, \sigma, \tau) = \lambda C_{BS}(S, K, \sigma, \tau)$.

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