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An Introduction to Barrier Options - Closed Form Solution and a Monte Carlo Approach

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1 Introduction

In recent years barrier options have become increasingly popular and frequently traded financial instruments, especially appearing in retail-products, so-called “certificates”, broadly offered in the German retail market. Barrier options are path-dependent exotic options that become activated or null if the underlying reaches certain levels. There are four main types of barrier options that can either have call or put feature: Down-and-In, Down-and-Out, Up-and-In and Up-and-Out. The “down” and “up” refer to the position of the barrier relative to the initial underlying price. The “in” and “out” specify the type of the barrier, referring to activating and nullifying when the barrier is breached respectively. Barrier options always come at a cheaper price than ordinary options with same features (Taleb 1997 [19]). A Down-and-Out call option for instance becomes nullified if the price of the underlying falls below the barrier and is therefore always cheaper than its counterpart with no barrier feature.

Despite being frequently traded nowadays, barrier options are still known as exotic options since they cannot be replicated by a finite combination of standard products, i.e. vanilla call and put options, future contracts etc. (Hausmann 2002 [6]). Already in 1973 Robert C. Merton described in his article (Merton 1973 [11]) a closed form solution for the price of a Down-and-Out call option. Since then the market for barrier options literally exploded.

In this paper we want to give an introduction to barrier options and its properties and investigate its pricing by discussing both analytic solution and numerical simulation procedures. We determine the analytic solution by risk-neutral valuation and the numerical price by Monte Carlo simulation. We furthermore introduce techniques to reduce both the statistical and the discretization error of our simulation. Our final goal guiding and motivating us through the whole paper is to price a very popular German retail product: The European bonus certificate. Investors see bonus certificates as alternatives to direct investments into the underlying. They protect the holder against losses of the underlying as long as the price does not breach a lower boundary. Those products are primarily purchased by investors who believe that the underlying will not fluctuate a lot. If this expectation turns out to be true the bonus certificate will yield a higher payoff than the underlying.

The first part of this paper introduces the theoretical framework and the mathematical tools that we need throughout this paper. Basic assumptions are given and definitions stated. We introduce the Black Scholes framework, take a look at the reflection principle and basic properties of bonus certificates and barrier options. The second part is devoted to pricing barrier options analytically. We price a single barrier European option by risk-neutral valuation and take a look at the sensitivities with respect to the price of the underlying and the volatility, namely delta (Δ) and vega (ν). In the third part we introduce numerical simulation and explore a variance reduction technique called Control Variates and a discretization error reducing technique exploiting the idea of Brownian bridges. We conclude with a discussion on the scope and limitations of the introduced techniques.

2 Theoretical Framework

2.1 Basic Assumptions and Definitions

Throughout this paper we assume a filtered probability space $(\Omega, \mathcal{M}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with respect to the filtration $\{\mathcal{F}_t\}_{t>0}$. Suppose $W = W_t$ to be a standard Brownian motion.

Definition 1. ***Brownian Motion***
A ***standard Brownian motion*** $W = W_t$ *living on a filtered probability space* $(\Omega, \mathcal{M}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *is defined by following properties:*

i. W *is adapted*

ii. $W_t - W_s$ *is independent of* $\mathcal{F}_s$, $0 \leq s \leq t$

iii. $W_t - W_s \sim N(0; t-s)$

For deterministic, but time-varying $\alpha(t)$ *and* $\sigma(t) > 0$ *we may define a* ***Brownian motion*** *with drift* $\alpha(t)$ *and diffusion coefficient* $\sigma^2(t)$ *through the SDE:*

$$dS_t = \alpha(t)dt + \sigma(t)dW_t \tag{2.1}$$

i.e. through

$$S_t = S_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW_s \tag{2.2}$$

with arbitrary constant S_0, *expectation* $\mathbb{E}[S_t - S_s] = \int_s^t \alpha(u)du$ *and variance* $Var[S_t - S_s] = Var[\int_s^t \sigma(u)dW_u] = \int_s^t \sigma^2(u)du$.

With constant α *and* σ *the stochastic process* S_t *is said to follow a* ***geometric Brownian motion*** *if it satisfies:*

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{2.3}$$

where W_t *is a Brownian motion.*

Figure 1 shows a typical path of a geometric Brownian motion with 1,000 steps, $T = 1$, $S_0 = 74.9225$ *and* $\alpha = 0.0138$

Furthermore we assume a world satisfying the Black Scholes conditions.

Definition 2. *We call a financial world to be **Black Scholes** if:*

i. Transaction costs are zero and the underlying Stock S does not pay any dividends[1]

ii. It is possible to borrow any fraction of the price of a security at the risk-free interest rate r which is constant over time

iii. Short selling is not prohibited

iv. The money market account follows the following model:

$$dB_t = rB_t dt \tag{2.4}$$

v. The underlying S follows the following geometric Brownian motion model:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^P, \tag{2.5}$$

with W_t^P *denoting a standard Brownian motion under the measure* P *and* σ *and* α *fixed.*

2.2 Important Theorems

For pricing options through risk-neutral valuation we introduce the important Girsanov's theorem in its one-dimensional version:

Theorem 1. ***Girsanov's Theorem (one-dimensional)***
Let $\{W_t^P\}_{0 \le t \le T}$ *be a Brownian motion on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$. *Let* $\{\theta_t\}_{t \ge 0}$ *be a process adapted to the filtration* $\{\mathcal{F}_t\}_{t \ge 0}$. *For* $0 \le t \le T$ *define*

$$W_t^Q := \int_0^t \theta_u du + W_t^P \quad \text{and} \quad Z_t := \exp\left(-\int_0^t \theta_u dW_u^P - \tfrac{1}{2}\int_0^t \theta_u^2 du\right) \tag{2.6}$$

and

$$Q(A) := \int_A Z_T d\mathbb{P}, \forall\, A \in \mathcal{F} \tag{2.7}$$

[1] Dividends are easy to implement if we assume a constant dividend growth rate (Bjoerk 2004 [1]). However in this paper we do not take dividends into account.

Then Q defines a risk-neutral probability measure and the process W_t^Q is a Brownian motion under Q.

*The **proof** is well presented in Jeanblanc 2009 [9] for both the one-dimensional and the general case.*

To find a analytical solution for barrier options we make use of a strong theorem in stochastic calculus - the **Reflection Principle**:

Theorem 2. ***The Reflection Principle***
Let S_t be a standard Brownian motion. Then for $H \geq 0$

$$S_t^* = \begin{cases} S_t, & \text{if } t < \inf\{t : S_t = H\} \\ 2H - S_t, & \text{if } t > \inf\{t : S_t = H\} \end{cases}$$

S_t^* is also a standard Brownian motion.

*The **proof** exploits the fact that Brownian motions satisfy the strong Markov property and can be found in Steele 2001 [18].*

Figure 2 shows a standard Brownian motion that is reflected at time T.

The crucial part in developing the analytic formula for barrier options includes the calculation of the joint distribution $Q\left(W_T^Q > k; \inf_{t\in[0;T]} W_t^Q < h\right)$. In the following Lemma we derive a representation dismissing the *"inf"* term.

Lemma 1. ***Joint Distribution***
Let Q be a probability measure like defined above and $\left\{W_t^Q\right\}$ a Brownian motion under Q. Suppose $h \leq k$, then:

$$Q\left(W_T^Q > k; \inf_{t\in[0;T]} W_t^Q < h\right) = Q\left(W_T^Q < 2h - k\right) \tag{2.8}$$

Proof: *Splitting up the term yields*

$$Q\left(W_T^Q > k; \inf\nolimits_{t\in[0;T]} W_t^Q < h\right) = Q(W_T^Q > k) - Q(W_T^Q > k; \inf\nolimits_{t\in[0;T]} W_t^Q \geq h)$$

Now we apply the aforementioned reflection principle and get:

$$\begin{aligned} &= Q(2h - W_T^Q > k) - Q(2h - W_T^Q > k; \inf\nolimits_{t\in[0;T]} W_t^Q \geq h) \\ &= Q(2h - k > W_T^Q) - Q(2h - k > W_T^Q; \inf\nolimits_{t\in[0;T]} W_t^Q \geq h) \end{aligned}$$

Since $h \leq k$ we know $2h - k \leq h$:

$$= Q(2h - k > W_T^Q) - Q(W_T^Q < 2h - k \leq h; \inf_{t \in [0;T]} W_t^Q \geq h)$$

We notice that the second term is zero since W_T^Q is strictly smaller than h:

$$= Q(W_T^Q < 2h - k)$$

□

2.3 Introduction to Bonus Certificates

As mentioned in the introduction we will not only stick to the theory of barrier option pricing, but want to elaborate our results on real-world problems. For this reason we introduce the European bonus certificate. Bonus certificates are popular products in the German retail market. They offer the holder the chance to earn more than holding the underlying as long as the underlying stays between a strike price K and a boundary H with $K > H$. The payoff is expressed as:

$$\text{Payoff} = \begin{cases} S_T, & \text{if } \exists\, t: \; S_t \leq H \text{ or } S_T > K \\ K, & \text{else} \end{cases} \tag{2.9}$$

The payoff of such an ordinary bonus certificate is shown in Figure 3.

Those products are primarily purchased by investors who believe that the underlying will not fluctuate a lot. If this expectation turns out to be correct the bonus certificate yields a higher payoff than the underlying.

A bonus certificate is a portfolio consisting of a zero-strike call and a long position in a Down-and-Out put option[2] (Reinmuth 2005 [13]):

$$p_{bc} = p_{dkop} + p_{zero\,call} \tag{2.10}$$

where p_{bc} is the price of a bonus certificate, p_{dkop} that of a Down-and-Out put option and $p_{zero\,call}$ that of a zero-strike call option.

The specific bonus certificate that we want to price is a Goldman Sachs certificate with ISIN DE000GS3DWL0 [14] and properties shown in Table 1 on May 8, 2011[3].

[2]Notation: The price of barrier options is denoted as p_{ABC} where $A = d$ or u stands for "down" or "up", $B = ko$ or ki for "knock-out" or "knock-in" and $C = p$ or c for "put" or "call".

[3]The actual price, the actual level of the barrier and the actual exercise price have been multiplied by 0.01.

The ask price of this European bonus certificate is **84.06** at day of pricing, so this price acts as our benchmark after taking into account limits of our model like the issuer risk and profit margins of Goldman Sachs.

2.4 Introduction to Barrier Options

In the following we regard a simple European Down-and-Out put option since we want to price the European bonus certificate above:

$$p_{bc} = p_{dkop} + p_{zero\,call}$$

Nonetheless we keep in mind that there also exist "non-simple" barrier options such as multi-barrier options or barrier options that require the asset price to not only cross a barrier, but spend a certain length of time across the barrier in order to knock in or knock out.

The analytical challenge in our case is to calculate p_{dkop}. Therefore we apply the technique of risk-neutral valuation to calculate the expectation of the payoff under the risk-neutral measure and discount it with the risk-free spot rate similar to pricing a vanilla put option (Steele 2001 [18]).

$$p_{dkop} = e^{-rT} E^Q \left[(K - S_T) \mathbf{1}_{\{K \geq S_T ; \inf_{t \in [0;T]} S_t \geq H\}} \right]$$

with time to maturity T, strike K, barrier $H \leq K$ and Q a risk-neutral measure with the money market account as numéraire.

In our numerical simulation part we will simulate our asset price according to geometric Brownian motion and implement the barrier as nullifying condition for each path.

3 Analytical Solution

3.1 Pricing Single Barrier European Options

From now on we assume a Black-Scholes world. We assume the underlying asset of our option to follow a geometric Brownian motion under a probability measure P.

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^P \tag{3.1}$$

Applying risk-neutral valuation we can eliminate the drift by switching to a risk-neutral measure Q through Girsanov's theorem. We define a new stochastic process:

$$W_t^Q = W_t^P + \frac{\alpha - r}{\sigma} t \tag{3.2}$$

Then Girsanov 's theorem states that there exists a measure Q, under which $\left\{W_t^Q\right\}$ is a Brownian motion. This yields the new SDE for the price process:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q \tag{3.3}$$

with solution (Hull 2007 [7]):

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^Q\right\} \tag{3.4}$$

With this evolution we can examine the price of our Down-and-Out barrier option:

$$p_{dkop} = e^{-rT} E^Q \left[(K - S_T)\mathbf{1}_{\{K \geq S_T; \inf_{t\in[0;T]} S_t \geq H\}}\right] \tag{3.5}$$

with time to maturity T, strike K and barrier $H \leq K$.

Given linearity of the expectation the price of our option can be written as:

$$\begin{aligned}
p_{dkop} &= e^{-rT} E^Q \left[(K - S_T)\mathbf{1}_{\{K \geq S_T; \inf_{t\in[0;T]} S_t \geq H\}}\right] \\
&= e^{-rT}\Bigg(E^Q \left[(K - S_T)\mathbf{1}_{\{S_T < K\}}\right] - E^Q \left[(K - S_T)\mathbf{1}_{\{S_T < H\}}\right] \\
&- E^Q \left[(K - S_T)\mathbf{1}_{\{S_T > H; \inf_{t\in[0;T]} S_t < H\}}\right] \\
&+ E^Q \left[(K - S_T)\mathbf{1}_{\{S_T > K; \inf_{t\in[0;T]} S_t < H\}}\right]\Bigg)
\end{aligned}$$

We immediately see that the first expectation is just the price of a plain-vanilla European put option with strike K. So we know (Black 1973 [2]):

$$\begin{aligned}
& e^{-rT} E^Q \left[(K - S_T)\mathbf{1}_{\{S_T < K\}}\right] = e^{-rT} K N\left(-\frac{ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& - S_0 N\left(-\sigma\sqrt{T} - \frac{ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& = K e^{-rT} N(-d_2) - S_0 N(-d_1)
\end{aligned}$$

with $d_1 = \sigma\sqrt{T} + \frac{ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Applying the same technique to the second expectation it takes us little effort to see that we only need to replace K by H in d_1 and d_2 to get the second expectation:

$$e^{-rT}E^Q\left[(K-S_T)\mathbf{1}_{\{S_T<H\}}\right]=e^{-rT}KN\left(-\frac{ln(\frac{S_0}{H})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{3.6}$$

$$-S_0N\left(-\sigma\sqrt{T}-\frac{ln(\frac{S_0}{H})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{3.7}$$

$$=Ke^{-rT}N\left(-d_{2(H)}\right)-S_0N\left(-d_{1(H)}\right)$$

with $d_{1(H)}=\sigma\sqrt{T}+\frac{ln(\frac{S_0}{H})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2=d_1-\sigma\sqrt{T}$.

Now we turn to the third and fourth term. Exemplarily we calculate the fourth expectation setting $m:=\frac{r-\frac{1}{2}\sigma^2}{\sigma}$, $h:=\frac{1}{\sigma}\ln\left(\frac{H}{S_0}\right)$ and $k:=\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)$ for sake of readability. We plug in (3.4) for S_T and S_t to get:

$$\begin{aligned}&e^{-rT}E^Q\left[(K-S_T)\mathbf{1}_{\{S_T>K;\inf_{t\in[0;T]}S_t<H\}}\right]\\&=e^{-rT}\left(KE^Q\left[\exp\left\{mW_T^Q-\frac{1}{2}m^2T\right\}\mathbf{1}_{\{W_T^Q>k;\inf_{t\in[0;T]}W_t^Q<h\}}\right]\right.\\&\left.-E^Q\left[\exp\left\{mW_T^Q-\frac{1}{2}m^2T\right\}S_0\exp\left\{\sigma W_T^Q\right\}\mathbf{1}_{\{W_T^Q>k;\inf_{t\in[0;T]}W_t^Q<h\}}\right]\right)\end{aligned} \tag{3.8}$$

Now we can apply Lemma 1 to the joint distribution $Q\left(W_T^Q>k;\inf_{t\in[0;T]}W_t^Q<h\right)$ since $h\leq k$. This yields:

$$\begin{aligned}&=e^{-rT}\left(Ke^{-\frac{1}{2}m^2T+2hm}E^Q\left[\exp\left\{-mW_T^Q\right\}\mathbf{1}_{\{W_T^Q<2h-k\}}\right]\right.\\&\left.-e^{-\frac{1}{2}m^2T+2h(m+\sigma)}E^Q\left[\exp\left\{(m+\sigma)(-W_T^Q)\right\}\mathbf{1}_{\{W_T^Q<2h-k\}}\right]\right)\\&=e^{-rT}\left(Ke^{2hm}N\left(\frac{(2h-k)+(m+\sigma)T}{\sigma\sqrt{T}}\right)\right.\\&\left.-S_0e^{2h(m+\sigma)+m\sigma T+\frac{1}{2}\sigma^2T}N\left(\frac{(2h-k)+(m+\sigma)T}{\sqrt{T}}\right)\right)\end{aligned} \tag{3.9}$$

Now plugging back m,h and k we get the desired expression:

$$\begin{aligned}&=e^{-rT}K\left(\frac{H}{S_0}\right)^{\frac{2r-\sigma^2}{\sigma^2}}N\left(\frac{\ln\left(\frac{H^2}{S_0K}\right)+rT\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)\\&-S_0\left(\frac{H}{S_0}\right)^{\frac{2r+\sigma^2}{\sigma^2}}N\left(\frac{\ln\left(\frac{H^2}{S_0K}\right)+rT+\frac{1}{2}\sigma^2T}{\sigma^2\sqrt{T}}\right)\end{aligned} \tag{3.10}$$

Finally we can put all four expectations together and get the price:

$$\begin{aligned}
\mathbf{p_{dkop}} &= Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0N(-x_1) \\
&- Ke^{-rT}N(-x_1+\sigma\sqrt{T}) - S_0\left(\frac{H}{S_0}\right)^{2\lambda}[N(y)-N(y_1)] \\
&+ Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\lambda-2}[N(y-\sigma\sqrt{T})-N(y_1-\sigma\sqrt{T})], \text{with} \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
\lambda :&= \frac{r+\frac{\sigma^2}{2}}{\sigma^2} \quad y := \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
x_1 :&= \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \quad y_1 := \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}
\end{aligned}$$

Now let us turn back to our bonus certificate. By letting K going to zero we can calculate the price of our zero-call on the DAX[4] using the Black-Scholes formula for European call options (Hull 2007 [7]) as:

$$p_{zero\ call} = S_0N(d_1) - Ke^{-rT}N(d_2) = 74.9225 \qquad (3.12)$$

Plugging the data of our bonus certificate into the above derived formula (3.11) for pricing European Down-and-Out put options we get:

$$\mathbf{p_{dkop}} = 9.4625 \qquad (3.13)$$

In summing up the two prices we get the final price of the European bonus certificate **84.39** which is close to the quoted price of **84.06**.

3.2 The Greeks: Delta and Vega

The **Delta** of a Down-and-Out put option describes the sensitivity of the price with respect to the underlying price. It must be zero when S is lower or equal to H since the option then has been knocked out. The more S approaches H the larger Delta should become in absolute terms. If S is far from touching H the option behaves similar to ordinary options.

[4]The price of a zero-call on the DAX should intuitively be equal to the underlying price itself since the holder does not receive any dividends and thus participates linearly on changes in the price.

The Delta is calculated as:

$$
\begin{aligned}
\Delta &= \frac{\partial p_{dkop}}{\partial S} \\
&= -\frac{Ke^{-rT}}{S\sigma\sqrt{T}}N'(-d_2) - N(-d_1) + \frac{1}{\sigma\sqrt{T}}N'(-d_1) \\
&+ N(-x_1) - \frac{1}{\sigma\sqrt{T}}N'(-x_1) + \frac{Ke^{-rT}}{\sigma\sqrt{T}S}N'(-x_1 + \sigma\sqrt{T}) \\
&+ 2\left(\frac{H}{S}\right)^{2\lambda}\frac{r}{\sigma^2}[N(y) - N(y_1)] + \left(\frac{H}{S}\right)^{2\lambda}\frac{1}{\sigma\sqrt{T}}[N'(y) - N'(y_1)] \\
&- Ke^{-rT}\frac{(\frac{H}{S})^{\frac{2r-\sigma^2}{\sigma^2}}(2r - \sigma^2)}{\sigma^2 S}[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})] \\
&- Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2}\frac{1}{\sigma\sqrt{T}S}[N'(y - \sigma\sqrt{T}) - N'(y_1 - \sigma\sqrt{T})]
\end{aligned}
$$

The **Vega** of a Down-and-Out put option describes the sensitivity of the price with respect to the underlying volatility. It is high when the underlying price is close to the barrier. As S moves away from the barrier the option behaves more like a plain vanilla option. Vega tends to zero for very high values of S.

The Vega is calculated as:

$$
\begin{aligned}
\nu &= \frac{\partial p_{dkop}}{\partial \sigma} \\
&= Ke^{-rT}\frac{d_1}{\sigma}N'(-d_2) - S(-\sqrt{T} + \frac{1}{\sigma}d_1)N'(-d_1) \\
&+ S(\frac{1}{\sigma}x_1 - \sqrt{T})N'(-x_1) - Ke^{-rT}\frac{1}{\sigma}x_1 N'(-x_1 + \sigma\sqrt{T}) \\
&- S\left(\frac{H}{S}\right)^{2\lambda}\left(\frac{2}{\sigma} - \frac{4(r + \frac{1}{2})}{\sigma^3}\right)\ln\left(\frac{H}{S}\right)[N(y) - N(y_1)] \\
&- S\left(\frac{H}{S}\right)^{2\lambda}\left[\left(-\frac{1}{\sigma}y + \sqrt{T}\right)N'(y) + \left(\frac{1}{\sigma}y_1 - \sqrt{T}\right)N'(y_1)\right] \\
&+ Ke^{-rT}(\frac{H}{S})^{2\lambda-2}(\frac{2}{\sigma} - \frac{4(r + \frac{1}{2}\sigma^2)}{\sigma^3})\ln(\frac{H}{S})[N(y - \sigma\sqrt{T} \\
&- N(y_1 - \sigma\sqrt{T})] - Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2}\frac{1}{\sigma}(yN'(y - \sigma\sqrt{T}) \\
&- y_1 N'(y_1 - \sigma\sqrt{T}))
\end{aligned}
$$

4 Numerical Simulation

Now that we have developed the closed-form solution of a Down-and-Out barrier put option, we focus on calculating the price through numerical simulation using Monte Carlo methods. The great advantage of Monte Carlo simulation lies in the fact that it is robust and can be easily extended to options depending on multiple assets when no analytical solutions exist (Moon 2008 [12]). We introduce the technique first and subsequently discuss two error reduction techniques.

First of all we need to generate the underlying asset prices $S_{t_1}, ..., S_T$ on a fixed set of points in time $0 < t_1 < ... < t_n = T$.

Recall that we assumed the asset price to follow a geometric Brownian motion satisfying the SDE in the risk-neutral world:

$$dS_t = rS_t dt + \sigma S_t dZ_t \tag{4.1}$$

where r and σ are constants and Z_t a standard Brownian motion.

The analytic solution for this SDE is:

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right)$$

which is a log-normally distributed random variable with expected value $\mathbb{E}[S_t] = e^{rt} S_0$ and variance Var $[S_t] = e^{2rt} S_0^2 (e^{\sigma^2 t} - 1)$.

Because our process S has independent normally distributed increments it is quite straightforward how to simulate the values $(S_{t_1}, ..., S_{t_n})$.

Let $Z_1, ..., Z_n$ be independent standard normally distributed random variables and $t_0 = 0$. Then subsequent values can be generated as:

$$S_{t_{i+1}} = S_{t_i} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) \sqrt{t_{i+1} - t_i} + \sigma \sqrt{t_{i+1} - t_i}\; Z_i\right\} \tag{4.2}$$

So our approach is to generate M paths with n time steps $\{(S_{t_1}^1, ..., S_{t_n}^1), ..., (S_{t_1}^M, ..., S_{t_n}^M)\}$ each and calculate the payoff of the option at maturity for each of the M paths by:

$$\text{Payoff}^k = \begin{cases} \max(K - S_{t_n}^k) & \text{, if } S_{t_i}^k > H \;\; \forall i = 1, ..., n \\ 0 & \text{, else} \end{cases} \tag{4.3}$$

Discounting all those payoffs to the present date and taking the average of them over the M paths we get a price for the option by numerical simulation:

$$p_{dkop} = \frac{1}{M} \sum_{k=1}^{M} \left(e^{-rT} \text{Payoff}^k\right) \tag{4.4}$$

This estimator is unbiased and converges with probability 1 as $n \to \infty$. The statistical error is of order $\mathcal{O}(1/\sqrt{M})$. For Barrier Options with continuous knock-out observation the discretization error - here the hitting time error (i.e. missing a barrier-breach that happens between two simulated time steps) - is of order $\mathcal{O}(1/\sqrt{n})$ (Gobet 2000 [5]).

The algorithm is as follows:

Algorithm 1. ***Standard Monte Carlo Method***

for $k = 1, ..., M$

  for $i = 1, ..., n$

    generate a $\mathcal{N}(0,1)$ *sample* Z_i

    set $S^k_{t_{i+1}} = S^k_{t_i} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1} - t_i} + \sigma\sqrt{t_{i+1} - t_i}\ Z_i\right\}$

  end

  if $\max_{1 \le i \le n} S_{t_i} > H$ *then* $Payoff^k = \max(K - S^k_{t_n})$

  else $Payoff^k = 0$

end

set $p_{dkop} = \frac{1}{M}\sum_{k=1}^{M}\left(e^{-rT}\mathit{Payoff}^k\right)$

In our case we choose a discretization of N = 1,000 equidistant time steps per year and simulate M = 20,000 paths in total for our Monte Carlo simulation in order to calculate the price of the Down-and-Out put option. The resulting price is **9.4976** compared to the theoretical value of **9.4625**. The difference in value can firstly be explained by the fact that we have a very high variance of **89.06** due to the knockout feature of the option. Secondly we have a discretization of only 1,000 time steps, so the option can only knockout on those nodes whereas our initial barrier option theoretically can knock out at any time.

In the following we will tackle the first problem by reducing the variance of our results. The second problem will be addressed thereafter through the concept of Brownian bridges, where we estimate the probability of the option knocking out between two time steps.

4.1 Control Variates

4.1.1 Theory of Control Variates

After having introduced the ordinary Monte Carlo simulation we now turn our focus to improving the efficiency of our simulation. In the following we introduce a variance reducing technique called Control Variates (Glasserman 2003 [4]).

The idea behind this technique is to exploit information about the errors in estimates of known variables that have a dependence to our variable (in our case Payoff^k) and thus reduce the variance of our variable. For sake of readability we define Payoff$^k = Y^k$.

First let us take another output X_k that is i.i.d. of which we know the expectation $E[X]$. Then we can regard for any fixed b:

$$Y^k(b) = Y^k - b(X_k - E[X]) \tag{4.5}$$

Then we can calculate the mean:

$$\overline{Y}(b) = \overline{Y} - b(\overline{X} - E[X]) = \frac{1}{M}\sum_{k=1}^{M}(Y^k - b(X_k - E[X])) \tag{4.6}$$

which we call our control variate estimator. Here $\overline{X} - E[X]$ serves as our control in estimation of $E[Y]$. We can easily show that this estimator is unbiased, because:

$$E[\overline{Y}(b)] = E[\overline{Y} - b(\overline{X} - E[X])] = E[\overline{Y}] = E[Y]$$

Furthermore we can show that it is also consistent:

$$\begin{aligned}
\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\overline{Y}(b) &= \lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}(Y^k - b(X_k - E[X])) \\
&= E[\overline{Y} - b(X - E[X])] \\
&= E[\overline{Y}]
\end{aligned} \tag{4.7}$$

Now we can calculate the variance of $Y^k(b)$ as:

$$\begin{aligned}
Var[Y^k(b)] &= Var[Y^k - b(X_k - E[X])] \\
&= \sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2 := \sigma^2(b)
\end{aligned} \tag{4.8}$$

with $\sigma_X^2 = Var[X]$, $\sigma_Y^2 = Var[Y]$ and ρ_{XY} representing the correlation between X and Y. So obviously the control variate estimator $\overline{Y}(b)$ has smaller variance than $\overline{Y}$ if $b^2\sigma_X < 2b\sigma_Y\rho_{XY}$.

We find that the variance-minimizing b^* in (4.8) is given by:

$$b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov[X,Y]}{Var[X]} \tag{4.9}$$

If we plug b^* back into (4.8) we find that the ratio of the variance of the optimally controlled estimator to the variance of the uncontrolled estimator - which represents a measure for the effectiveness of the control variate - is:

$$\frac{Var[\overline{Y}(b^*)]}{Var[\overline{Y}]} = 1 - \rho_{XY}^2 \tag{4.10}$$

From equation (4.10) we see that the effectiveness of a control variate is determined by the correlation between X and Y. We furthermore need $M/(1-\rho_{XY}^2)$ replications of Y^k to achieve the same variance as with M replications of $Y^k(b)$. This factor increases sharply as $|\rho_{XY}|$ approaches 1.

Thus indeed we could significantly reduce variance with a sufficiently well correlated X if we knew σ_Y and ρ_{XY}. Unfortunately this is in most cases not likely if we don't know $E[Y]$ which we - in fact - want to calculate. One way to deal with this problem is to use an estimate for b^*. In the following we use a least-square regression of (X_k, Y^k). The slope of this regression line then can be used as an estimate for b^*:

$$\hat{b}_M = \frac{\sum_{k=1}^{M}(X_k - \overline{X})(Y^k - \overline{Y})}{\sum_{k=1}^{M}(X_k - \overline{X})^2} \tag{4.11}$$

If we divide both numerator and denominator by M we can apply the strong law of large numbers and show that indeed $\hat{b}_M \to b^*$ with probability 1. However replacing b^* by $\hat{b}_M$ introduces some bias, but the bias vanishes quickly as M becomes sufficiently large (Glasserman 2003 [4]).

The algorithm is described in the following:

Algorithm 2. ***Control Variate Monte Carlo Method***

for $k = 1, ..., M$

  generate X_k

  for $i = 1, ..., n$

    generate a $\mathcal{N}(0,1)$ *sample* Z_i

    set $S_{t_{i+1}}^k = S_{t_i}^k \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1}-t_i} + \sigma\sqrt{t_{i+1}-t_i}\ Z_i\right\}$

  end

  if $\max_{1\le i\le n} S_{t_i} > H$ *then* $Y^k = \max\left(K - S_{t_n}^k\right)$

  else $Y^k = 0$

end

set $\hat{b}_M = \frac{\sum_{k=1}^{M}(X_k-\overline{X})(Y^k-\overline{Y})}{\sum_{k=1}(X_k-\overline{X})^2}$

for $k = 1, ..., M$

  set $Y^k = Y^k - \hat{b}_M(X_k - E[X])$

end

set $p_{dkop} = \frac{1}{M}\sum_{k=1}^{M}\left(e^{-rT}Y^k\right)$

4.1.2 Application of Control Variates to Barrier Options

Now after having introduced the theory of control variates we want to apply this technique to our simulation and discuss the impacts. We therefore start with a primitive control, then move forward to use the underlying asset as control variate and finally we examine a plain vanilla option as control variate.

4.1.2.1 Primitive Control What we call primitive control is nothing else than defining our X_k as independent standard normally distributed random variables. As our crude Monte Carlo paths are simulated from a sequence $Z_1, Z_2...$ of independent standard normal random variables it is straightforward that there exists some correlation between those random variables and our desired Y^k.

We know that $E[Z_k] = 0$ and $Var[Z_k] = 1$ so we can estimate the optimal b^* and thus calculate the controlled Y^k:

$$\hat{b}_M = \frac{\sum_{k=1}^{M}(Z_k - \overline{Z})(Y^k - \overline{Y})}{\sum_{k=1}^{M}(Z_k - \overline{Z})^2} = \frac{\sum_{k=1}^{M} Z_k(Y^k - \overline{Y})}{\sum_{k=1}^{M} Z_k^2} \tag{4.12}$$

$$Y^k(\hat{b}_M) = Y^k - \hat{b}_M(Z_k - E[Z]) = Y^k - \hat{b}_M Z_k \tag{4.13}$$

Implementing this primitive control in our initial model we find out that we get an option price of **9.4976** and the variance becomes **89.0632640717** compared to **89.0632748101** as before. The variance reduction is minimal as we have very small correlation. But those generic control variates are always available in a simulation and are mostly easy to implement.

4.1.2.2 Underlying Asset as Control Variate Now we want to use the underlying asset as control variate. As the payoff - thus the price of an option - depends on the evolution of the underlying asset it is also straightforward to use the asset prices $S_T^1, S_T^2, ...$ as control variate.

We know that in our risk-neutral world we have $E[S_T = e^{rT} S_0]$, so consequently:

$$\hat{b}_M = \frac{\sum_{k=1}^{M}(S_T^k - \overline{S_T})(Y^k - \overline{Y})}{\sum_{k=1}^{M}(S_T^k - \overline{S_T})^2} \tag{4.14}$$

and we have the controlled option price as:

$$Y^k(\hat{b}_M) = Y^k - \hat{b}_M(S_T^k - E[S_T]) = Y^k - \hat{b}_M(S_T^k - e^{rT} S_0) \tag{4.15}$$

This simulation yields a variance of **17.02** which is significantly better than the primitive control (89.06). The correlation and thus the effectiveness of the control variate depends on the strike and the barrier. When we regard our Down-and-Out put option we can imagine that the higher the strike is the greater is $|\rho_{SY}|$. Also the lower our barrier is the more we expect the correlation to be high speaking in absolute terms.

4.1.2.3 Plain Vanilla Option as Control Variate Since we know the expected value of a European put option through the Black-Scholes formula (Black 1973 [2]) we can also use it as a control variate. Therefore we simulate a series of European put option prices P_k. From Black-Scholes we know that $E[P] = Ke^{rT}N(-d_2) - S_0 N(-d1)$ with:

$$d_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T}}; \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Therefore we have:

$$\hat{b}_M = \frac{\sum_{k=1}^{M}(P_k - \overline{P_k})(Y^k - \overline{Y})}{\sum_{k=1}^{M}(P_k - \overline{P_k})^2} \tag{4.16}$$

and

$$Y^k(\hat{b}_M) = Y^k - \hat{b}_M(S_T^k - E[P]) = Y^k - \hat{b}_M\left(P_k - \left(Ke^{rT}N(-d_2) - S_0N(-d1)\right)\right) \tag{4.17}$$

We observe that with plain vanilla put options as control variate our model yields a variance of **2.9934e-24** and the resulting price does not differ from our theoretical value up to the **fifth decimal**. This kind of control is very effective if the barrier is low. In fact if the barrier is set to be zero the correlation is very close to 1.

4.1.3 Discussion

After having introduced three possible control variates we observe substantial improvements in variance especially with the underlying asset and the European put option as control variates. Taking the value yielded from the last control variate the resulting price of our bonus certificate is therefore **84.38504** which is identical to our theoretical value up to the fifth decimal. However due to the digital feature of the barrier option our variance becomes still very large when the barrier is set to be tight. With a barrier of 60 our model from (4.1.2.3) yields a variance of **34.94**.

4.2 Brownian Bridges

4.2.1 Theory of Brownian Bridges

Now that we have introduced a powerful technique reducing the statistical error of our simulation we turn our focus to reducing the discretization error, in our case the hitting-time error. Hitting-time error refers to the error that arises from not sufficiently fine discretization, i.e. when a continuously observed barrier option knocks out between two simulated time steps, but the underlying asset subsequently recovers at the latter time step, formally:

For simulated time steps t_i and t_{i+1} we have $S_{t_i} > H$ and $S_{t_{i+1}} > H$, but $S_t < H$ for at least one $t \in [t_i, t_{i+1}]$.

If we cannot simulate a large number of time steps due to limited computational time or resources the hitting-time error can become substantially large. Just imagine the case that we only simulate two time steps for our barrier option. If the option ends up in the money we would assume that the underlying asset has never touched the barrier during the whole life of the option. Furthermore assume that we have a very close barrier then it is quite unlikely that the underlying stayed above the barrier over its whole life.

Inspired by Mannella 1999 [10] Moon 2008 [12] proposed in his paper an efficient technique using Brownian bridges and the uniform distribution to reduce this hitting-time error.

The idea is to calculate an exit probability for each pair of time steps, i.e. the probability that the underlying breaches a lower boundary between two time steps. We then use a uniformly distributed random variable to decide whether this probability is sufficiently large to let our option knock out.

Formally we regard a domain $D = (H, \infty)$ and define the probability P_i that the process S exits D at $t \in [t_i, t_{i+1}]$ given that S_{t_i} and $S_{t_{i+1}}$ are in D. Then P_i is exactly the exit probability that we have described above. To calculate this exit probability we can use the law of Brownian bridges and get:

$$
\begin{aligned}
P_i &= P\left[\min_{t\in[t_i,t_{i+1}]} S_t \leq H \middle| S_{t_i} = s_1, S_{t_{i+1}} = s_2\right] && (4.18)\\
&= \exp\left(-2\frac{(H-s_1)(H-s_2)}{\sigma^2(t_{i+1}-t_i)}\right) && (4.19)
\end{aligned}
$$

with s_1 and s_2 in D.

In our algorithm we sample for each time step t_i a uniformly distributed random variable $u_i \sim \mathcal{U}(0,1)$. If in t_i the exit probability P_i exceeds u_i we dismiss the path and set the payoff as zero.

The algorithm with Brownian bridges becomes:

Algorithm 3. ***Brownian Bridges***

for $k = 1, ..., M$

for $i = 1, ..., n$

generate a $\mathcal{N}(0,1)$ *sample* Z_i

set $S^k_{t_{i+1}} = S^k_{t_i} \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\sqrt{t_{i+1}-t_i} + \sigma\sqrt{t_{i+1}-t_i}\ Z_i\right\}$

set $P_{t_{i+1}} = \exp\left\{-2\frac{(H-S_{t_i})(H-S_{t_{i+1}})}{\sigma^2(t_{i+1}-t_i)}\right\}$

end

generate a $\mathcal{U}(0,1)$ *sample* $u_i, i = 1, ..., n$

if $S_{t_i} > H$ *and* $P_{t_i} < u_i, \forall i \;\; 1 \leq i \leq n$ *then* $Y^k = \max\left(K - S^k_{t_n}\right)$

else $Y^k = 0$

end

set $p_{dkop} = \frac{1}{M} \sum_{k=1}^{M} \left(e^{-rT} Y^k\right)$

4.2.2 Application of Brownian Bridges to Barrier Options

As the barrier option that we have discussed until now has a very low barrier, thus our simulation with control variates yields a very good result, we want to illustrate the Brownian bridge approach in an example where the barrier is tight. We regard a Down-and-Out barrier put option on the DAX with the properties in Table 2 again priced on May 8, 2011.

Using our analytic formula (3.11) we get a theoretical price of **0.4313**. In this case the barrier is only about 7% below the actual price. As we can imagine the simulated price using simple Monte Carlo from Chapter 4 should be quite bad since it is very likely that the option undetectedly knocks out between two time steps.

Again simulating with 20,000 paths and 1,000 time steps we see a price of **0.4936** for our barrier option which is more than 14% above the theoretical price. In contrast, applying the Brownian bridge technique introduced in Subsection 4.2.1 we get a price of **0.4484** which represents a significant improvement against the simple Monte Carlo simulation.

4.2.3 Discussion

The Brownian bridge Monte Carlo method effectively reduces the discretization error of our simulation and represents a significant improvement to the simple Monte Carlo approach especially when the barrier is tight. When we increase the number of steps of our simulation it is furthermore numerically shown that our method converges much faster than that of the standard Monte Carlo simulation (Moon 2008 [12]).

5 Scope and Limitations

Summarizing our results in Table 3 we observe that our analytical price is different from the quoted price in the market. Several factors lead to possible discrepancy:

1. There exists an **issuer risk** for Goldman Sachs to default during the holding period which is in fact quite low (see credit rating of the issuer in Table 1), but possibly lowers the offered price.

2. We assumed the **implied volatility** of the underlying for the simple put option and the barrier option to be equal to that of a K-strike vanilla option as we could not get the actual implied volatilities. This may not be correct regarding problems for instance arising from the volatility skew of options (Hull 2007 [7]).

3. Goldman Sachs is a shareholder value serving company, thus wants to make **profit** with its products. For that reason the quoted price might be higher than the fair price.

4. To narrow the ask-bid spread issuers often **under- or overquote** both the ask and the bid price. This leads to a bias up to 5% away from the fair price in either direction.

5. Practitioners often price securities with digital features with a so-called **barrier shift**. This means that they try to approximate the digital payoff with a very steep but not exactly vertical payoff. This typically leads to higher prices as investors have to pay for the buffer (de Weert 2008 [20]).

6. We used the yield of the German government bond that matures in 1 year as a proxy for the 11-month **risk-free** rate which is not entirely correct. Despite highly improbable it is possible that Germany defaults.

Furthermore we noticed that applying variance reduction techniques we can indeed reduce variance of our simple Monte Carlo approach. Especially when the barrier is low we can significantly reduce the variance in our results using the underlying and a vanilla option as control. In case of our first option we even managed to implement a variance of 2.99e-24.

However we also see that the numerical results for barrier options with tight knock-out features are comparably poor resulting from the discretization error. Here our suggested Brownian bridge approach leads to a significant improvement to the simple Monte Carlo simulation and enables us to also price options with tight barriers using a relatively small number of time steps.

So for practically pricing barrier options we would carefully choose our simulation model depending on the tightness of our barrier and adjust the path and step numbers accordingly. With the appropriate model and adjusted path and step numbers we can make sure to generate a reasonably good approximation for the option price.

6 Concluding Remarks

Using the example of a European bonus certificate we examined in this paper basic properties and the pricing of barrier options both analytically and numerically. We practically introduced two techniques to reduce variance of the simple Monte Carlo simulation on the one hand and to reduce discretization error on the other hand. In the last part we discussed the scope and limitations of our models and gave a practical guideline how to handle different barrier options and assess their risks.

To conclude our results we note that the problems discussed in section 5 can be countered by adjusting for issuer risk, volatility skew, barrier shift and risk-free rate. Also given our foundation it is not a difficult task to extend our model to pricing more complex barrier options, such as multi-barrier, Asian or even Parisian barrier options. For loose barriers we can reduce variance significantly by using appropriate control variates and going further various other numerical techniques like for instance importance sampling. For relatively tight barriers we can reduce discretization error by using the Brownian bridge approach.

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A Figures

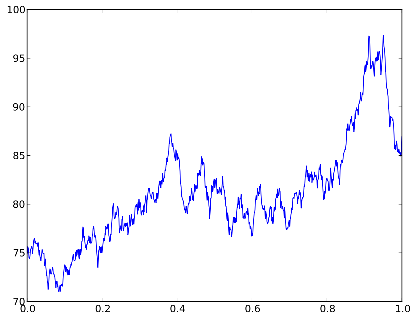

Figure 1: Geometric Brownian Motion
with 1,000 steps, $T = 1$ $S_0 = 74.9225$, $\alpha = 0.0138$

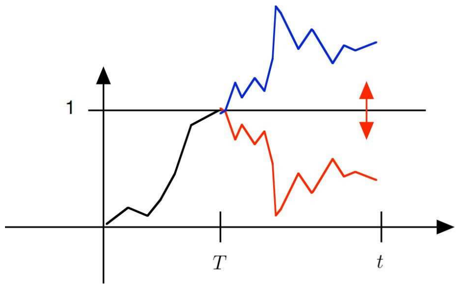

Figure 2: The Reflection Principle
Source: K. Igusa, Brandeis University [8]

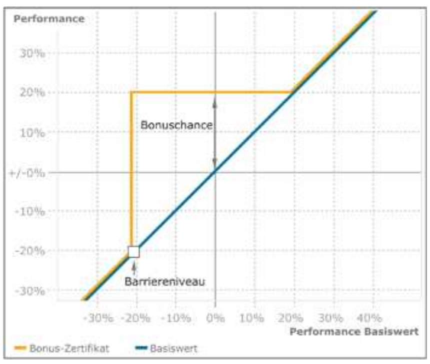

Figure 3: Payoff of a bonus certificate
Source: Goldman Sachs [16]

B Tables

Underlying	DAX Performance Index
Time to maturity	11 months
Exercise price (K)	82.5
Barrier (H)	27
Price of the underlying	74.9225
Credit Rating of the Issuer	A+/A1 (Fitch/Moody's) [17] CDS +71,08bp
Risk-free interest rate	1.38% (one year German government bond) [3]
Implied volatility	18.2071% (annualized implied volatility of the DAX) [15]

Table 1: Bonus Certificate

Underlying	DAX Performance Index
Time to maturity	1 Year
Exercise price (K)	82.5
Barrier (H)	70
Price of the underlying	74.9225
Risk-free interest rate	1.38% (one year German government bond) [3]
Implied volatility	18.2071% (annualized implied volatility of the DAX) [15]

Table 2: Second Barrier Option

First Barrier Option	**Option Price**	**Bonus Certificate**	**Variance**
Quoted	NA	84.06	-
Analytical	9.4625	84.39	-
Simple MC Simulation	9.4976	84.42	89.06
Primitive Control	9.4976	84.42	89.06
Underlying as Control	9.4906	84.41	17.02
Vanilla Option as Control	9.4625	84.39	2.99e-24
Second Barrier Option			
Analytical	0.4313	-	-
Simple MC Simulation	0.4808	-	3.02
Brownian Bridge Monte Carlo	0.4465	-	2.68

Table 3: Summary of Results

C Code

All codes are written in Python with following packages:

```
from math import * sqrt, *
from numpy.random import normal, lognormal, uniform
from numpy import ones, zeros, maximum, exp, array, dot, log
from scipy.stats import norm
```