Défi Turing
Unofficial Offline Edition

English Version

Revision **v2025.1**

1 Sum of multiples of 5 or 7

If we list all the natural numbers less than 20 that are multiples of 5 or 7, we get 5, 7, 10, 14 and 15. The sum of these numbers is 51.

Find the sum of all multiples of 5 or 7 less than 2013.

(Inspired by problem 1 of Project Euler)

2 Sum of odd-valued terms in Fibonacci sequence

Each new term in the Fibonacci sequence is generated by adding the two preceding terms. Starting with 1 and 1, the first 10 terms are as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

By considering the terms in the Fibonacci sequence whose values do not exceed 4 million, find the sum of the odd-valued terms.

(Inspired by problem 2 of Project Euler)

3 Largest prime factor of composite number

The prime factors of the number 2013 are 3, 11 and 61, because $3 \times 11 \times 61 = 2013$.

What is the largest prime factor of the number 20130101?

(Inspired by problem 3 of Project Euler)

4 Largest palindrome of multiplying two numbers

A palindromic number reads the same from left to right and from right to left.
The largest palindrome that can be obtained by multiplying two 2-digit numbers is $9009 = 99 \times 91$.

What is the largest palindrome that can be obtained by multiplying a 4-digit number by a 3-digit number?

(Inspired by problem 4 of Project Euler)

5 Sum of digits of 2^{2222}

$2^{15} = 32768$ and the sum of its digits is $3 + 2 + 7 + 6 + 8 = 26$.

What is the sum of the digits that make up the number 2^{2222}?

(Inspired by problem 16 of Project Euler)

6 Sum of digits in number $2013!$

$n!$ means $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.
For example, $10! = 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 3628800$.
The sum of the digits in the number $10!$ is $3+6+2+8+8+0+0 = 27$.

Find the sum of the digits in the number 2013!.

(Inspired by problem 20 of Project Euler)

7 | What is the 23456^{th} prime number?

By listing the first six prime numbers: 2, 3, 5, 7, 11 and 13, we see that the 6^{th} prime number is 13.

What is the 23456^{th} prime number?

(Inspired by problem 7 of Project Euler)

8 | Rectangle-6400

In a rectangle with a length of 4 and a width of 3, we can draw 12 squares with a side length of 1, 6 squares with a side length of 2, and 2 squares with a side length of 3. In total, we can draw 20 squares. We will call this a rectangle-20.

What is the area of the rectangle-6400 whose shape is closest to a square?

Note: The dimensions are whole numbers for both the rectangle and the squares.

(Inspired by the 2012 CMP problem from the mathematics and physics circle of the Société jurassienne d'émulation)

9 | Pythagorean triplet

A triplet of non-zero natural numbers (a, b, c) is Pythagorean if $a^2 + b^2 = c^2$. For example, $(3, 4, 5)$ is a Pythagorean triplet.

Among the Pythagorean triples (a, b, c) such that $a + b + c = 3600$, give the largest product $a \times b \times c$.

(Inspired by problem 9 of Project Euler)

10 Sum of prime numbers

The sum of the prime numbers between 1 and 10 is $2 + 3 + 5 + 7 = 17$.

Find the sum of the prime numbers between 1 and 10 000 000.

(Inspired by problem 10 of Project Euler)

11 Mirror of number

The "mirror image of a number n" is the number n written from right to left. For example, $\mathrm{mirror}(7423) = 3247$.

What is the largest number less than 10 million that has the property: $\mathrm{mirror}(n) = 4 \times n$?

(Proposed by Le Coyote)

12 Triangular numbers with many divisors

The sequence of triangular numbers is generated by adding the natural numbers. Thus, the 7th triangular number is $1+2+3+4+5+6+7=28$. The first ten triangular numbers are as follows:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

Let us list the divisors of the first seven triangular numbers:

$$\begin{aligned} 1 &: 1 \\ 3 &: 1, 3 \\ 6 &: 1, 2, 3, 6 \\ 10 &: 1, 2, 5, 10 \\ 15 &: 1, 3, 5, 15 \\ 21 &: 1, 3, 7, 21 \\ 28 &: 1, 2, 4, 7, 14, 28 \end{aligned}$$

We can see that 28 is the first triangular number with more than five divisors.

What is the value of the first triangular number with more than one thousand divisors?

(Inspired by problem 12 of Project Euler)

13 | Palindrome square

The smallest palindromic square with an even number of digits is $698896 = 836^2$.

What is the next palindromic square?

(Source: The Mathematical Gazette)

14 Longest Syracuse sequence

In mathematics, a "Syracuse sequence" is a sequence of natural numbers defined as follows:

We start with an integer greater than zero; if it is even, we divide it by 2; if it is odd, we multiply it by 3 and add 1.
By repeating the operation, we obtain a sequence of positive integers, each of which depends only on its predecessor.

There is a conjecture that says that the Syracuse sequence of any strictly positive integer reaches 1.

For example, starting from 14, we construct the sequence of numbers: 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. This is called the Syracuse sequence of the number 14. Here, it has a length of 18.

For what starting number less than 1 500 000 do we obtain the longest Syracuse sequence? There are two solutions, give the smallest one.

(Inspired by problem 14 of Project Euler)

15 | Numbers of Mage Hic

Fill in a 3×3 grid with all the numbers from 1 to 9 (green squares). Then calculate the products of the three numbers in each row and each column (grey squares). Finally, add these six products together to obtain the blue number (450 in the example given).

2	9	4	72
7	5	3	105
6	1	8	48
84	45	96	450

Of the 362 880 possible grids, what is the minimum blue number and the maximum blue number? Give the product of these two numbers as the result.

(Inspired by problem 12 from the 2013 FFJM individual quarterfinals: "Le nombre du Mage Hic")

16 | THREE x NINE = TROIS x NEUF

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found. Each different letter represents a different number, and each number is represented by the same letter.

Solve the cryptarithm below (give the product obtained as the answer):

```
THREE x NINE = TROIS x NEUF
```

Constraints:
`THREE` and `NEUF` are multiples of 9;
`TROIS` and `NINE` are multiples of 3.

(Problem 18 from the quarterfinals of the 2003 Mathematical and Logical Games)

17 | Amicable numbers

Let $d(n)$ be the sum of the proper divisors of n (divisors strictly smaller than n). If $d(a) = b$ and $d(b) = a$, with a different from b, then a and b are said to be amicable.
For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore $d(220) = 284$. The proper divisors of 284 are 1, 2, 4, 71 and 142; therefore $d(284) = 220$. 220 and 284 are two amicable numbers.
The sum of the amicable numbers between 1 and 1000 is $220 + 284 = 504$.

What is the sum of the amicable numbers between 1 and 100 000?

(Inspired by problem 21 of Project Euler)

18 Sum of non-abundant numbers

A perfect number is a number whose sum of proper divisors is exactly equal to the number itself. For example, the sum of the proper divisors of 28 would be $1 + 2 + 4 + 7 + 14 = 28$, which means that 28 is a perfect number.

A number n is called deficient if the sum of its proper divisors is less than n, and it is called abundant if this sum is greater than n.

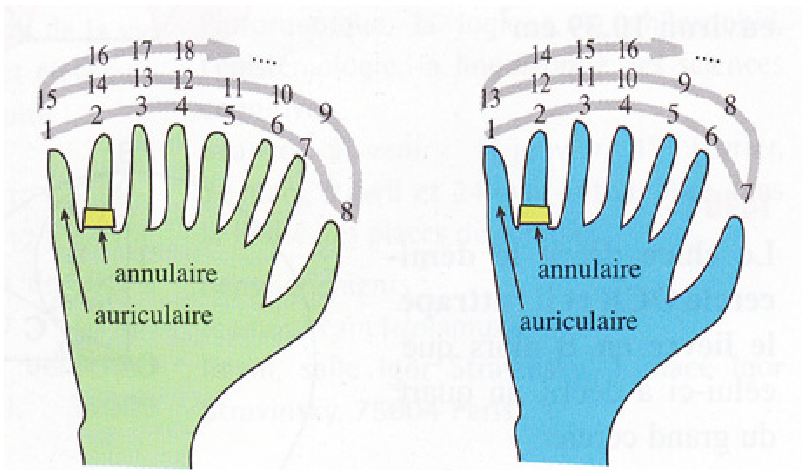
Since 12 is the smallest abundant number ($1 + 2 + 3 + 4 + 6 = 16$), the smallest number that can be written as the sum of two abundant numbers is 24.

Find the sum of all positive integers less than or equal to 2013 that <u>cannot</u> be written as the sum of two abundant numbers.

(Inspired by problem 23 of Project Euler)

19 Close encounters of the fourth kind

Little green men meet little blue men. To their great surprise, they notice that their hands do not have the same number of fingers: 7 for the blue ones and 8 for the green ones. But the scientists of both peoples notice that if you count on your fingers as shown in the figure, moving back and forth from the little finger to the thumb, then from the thumb to the little finger, certain numbers can be counted on both the ring finger of the blue hands and the ring finger of the green hands (2 and 14, for example).
These numbers have been termed "annular" by scholars.



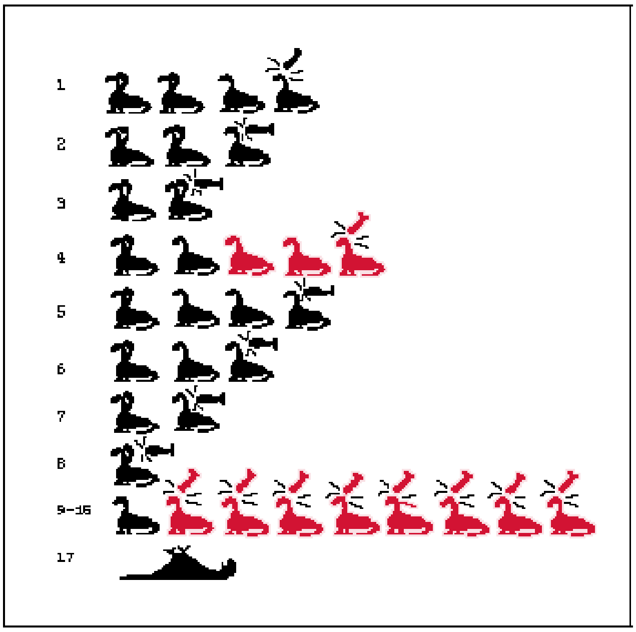
Calculate the sum of the ring numbers between 1 and 2013.

(Inspired by a problem from the Mathematical and Logical Games Championship)

20 Hercules and the hydra

A battle between Hercules and one (or more) hydra(s) can be described by the following steps:

- On the first blow, the Hydra loses a head, and a copy of the wounded Hydra (that is, one with one less head) appears. On the second blow, a Hydra loses a head, and two copies of the wounded Hydra appear. On the tenth blow, a Hydra loses a head, so ten copies appear, and so on.
- When a one-headed Hydra is struck, it disappears.
- To complicate matters, the treacherous Hera forces Hercules to cut off the heads in the worst possible order, so that it takes him as long as possible to destroy the Hydras.



Example: Hercules fights two two-headed hydras and two one-headed hydras.

Assuming Hercules is capable of cutting off a head every second, how many seconds will it take him to defeat a four-headed hydra?

(Jean-Bernard Roux's problem)

21 | Happy new year 2013!

2013 has an interesting feature: it is the first year since 1987 to be composed of all different digits. A period of 26 years separates these two dates.

Between year 1 and 2013 (inclusive):

a) How many years have there been with all different digits? (Years 1 through 9 will be counted in this number).
b) What was the length (in years) of the longest period between two dates with all different digits?

Give the product of the results of a) and b).

(Proposed by Le Coyote)

22 | Eightfold anagrams

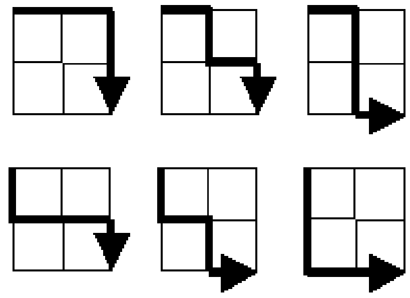
Mathilde found two surprising six-digit numbers. When multiplied by 8, they produce a six-digit number written with the same digits in a different order. Even better: these two numbers are also anagrams!

What are Mathilde's numbers? Give the sum of the two numbers.

(Inspired by problem 14 from the quarterfinals of the 2010 Mathematical and Logical Games)

23 Paths in a grid

Starting from the top left corner of a 2×2 grid, there are 6 paths to the bottom right corner, moving only to the right or down.



How many such paths are there in a 30×20 grid?

(Inspired by problem 15 of Project Euler)

24 | Lexicographic permutations

A permutation is an ordered arrangement of objects. For example, 3124 is a possible permutation of the digits 1, 2, 3 and 4.
The list of permutations in lexicographical order of 0, 1 and 2 is as follows:

$$012, 021, 102, 120, 201, 210$$

In lexicographical order, what is the two-millionth permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

(Inspired by problem 24 of Project Euler)

25 | First Fibonacci number with 2013 digits

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{where } F_1 = 1 \text{ and } F_2 = 1.$$

Thus, the first 12 terms are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …
The 12th term, F_{12}, is the first term that contains 3 digits.

What is the index of the first term in the Fibonacci sequence that contains 2013 digits?

(Inspired by problem 25 of Project Euler)

26 | Number divisible by pieces

Find an integer $c_1c_2c_3 \cdots c_9$ composed of all digits from 1 to 9, such that $c_1 \cdots c_k$ is divisible by k, for $k = 1, \ldots, 9$.

Let's take a 3-digit example: 321.

3 is divisible by 1.
32 is divisible by 2.
321 is divisible by 3.

(Source unknown)

27 | Euler's polynomial for prime numbers

Euler published the remarkable quadratic polynomial:

$$n^2 + n + 41$$

It turns out that this formula provides 40 prime numbers for successive values of n ranging from 0 to 39. However, when $n = 40$, $40^2 + 40 + 41 = 40(40 + 1) + 41$ is divisible by 41.

Using a computer, the incredible formula $n^2 - 79n + 1601$ was discovered. It provides 80 prime numbers for successive values of n ranging from 0 to 79 (each prime number appears twice):

1601, 1523, 1447, 1373, 1301, 1231, 1163, 1097, 1033, 971, 911, 853, 797, 743, 691, 641, 593, 547, 503, 461, 421, 383, 347, 313, 281, 251, 223, 197, 173, 151, 131, 113, 97, 83, 71, 61, 53, 47, 43, 41, 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601.

The product of the coefficients, -79 and 1601, gives -126479.

Consider quadratic polynomials of the form:
$n^2 + an + b$, where $|a| < 1500$ and $|b| < 1500$
where $|n|$ is the absolute value of n (for example, $|11| = 11$ and $|-4| = 4$).

Give the product of the coefficients a and b of the quadratic polynomial that generates the longest sequence of prime numbers for successive values of n, starting from $n = 0$.

(Inspired by problem 27 of Project Euler)

28 Spiral numbers

Starting with the number 1, and turning clockwise, a 5×5 spiral is constructed as follows:

21	22	23	24	**25**
20	**7**	8	**9**	10
19	6	**1**	2	11
18	**5**	4	**3**	12
17	16	15	14	**13**

We can verify that the sum of the numbers on the diagonals is 101.

What is the sum of the numbers on the diagonals of a 2013×2013 spiral constructed in the same way?

(Inspired by problem 28 of Project Euler)

29 Distinct powers

Consider a^b for $2 \leq a \leq 5$ and $2 \leq b \leq 5$:

$$2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$$
$$3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$$
$$4^2 = 16, 4^3 = 64, 4^4 = 256, 4^5 = 1024$$
$$5^2 = 25, 5^3 = 125, 5^4 = 625, 5^5 = 3125$$

If we sort these numbers in ascending order, removing any duplicates, we obtain a sequence of 15 distinct terms:

$$4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125$$

How many distinct terms are there in the sequence obtained as above for $2 \leq a \leq 1000$ and $2 \leq b \leq 1000$?

(Inspired by problem 29 of Project Euler)

30 | Sum of fifth powers

Surprisingly, there are only three numbers that can be written as the sum of the fourth powers of their digits:

$$1634 = 1^4 + 6^4 + 3^4 + 4^4$$
$$8208 = 8^4 + 2^4 + 0^4 + 8^4$$
$$9474 = 9^4 + 4^4 + 7^4 + 4^4$$

We do not count $1 = 1^4$ because it is not a sum.

The sum of these numbers is $1634 + 8208 + 9474 = 19316$.

Find the sum of all numbers that can be written as the sum of the fifth powers of their digits.

(Inspired by problem 30 of Project Euler)

31 | Swiss coins

In Switzerland, there are seven coins: 5 cents, 10 cents, 20 cents, 50 cents, 1 franc, 2 francs, and 5 francs.

You can make 10 francs like this: 1×5 francs $+ 2 \times 2$ francs $+ 10 \times 10$ cents.

How many ways can you make 10 francs using any number of Swiss coins?

(Inspired by problem 31 of Project Euler)

32 | Pandigital identities

The product 7254 is interesting because the identity $39 \times 186 = 7254$, composed in the order of multiplicand, multiplier, and product, uses exactly once all the digits from 1 to 9.

Find the sum of all products having this property.

Warning! The same product can be obtained in several ways. It should only be counted once in the total.

(Inspired by problem 32 of Project Euler)

33 Easter in April

During the years 2001 to 9999 (inclusive), how many times will the date of Easter fall in April, in the Gregorian calendar?

(Proposed by Le Coyote)

34 | $1! + 4! + 5! = 145$

145 is a curious number. Indeed, $1! + 4! + 5! = 1 + 24 + 120 = 145$.

Find the product of all numbers that are equal to the sum of the factorial of their digits.

Notes:
1! = 1 and 2! = 2 are not sums and will not be included in the product.
Remember that 0! = 1.

(Inspired by problem 34 of Project Euler)

35 Circular prime numbers

The prime number 197 is called "circular" because all circular permutations of its digits: 197, 971 and 719 are also prime numbers.

There are 13 circular prime numbers less than 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular prime numbers are there less than 100 000?

(Inspired by problem 35 of Project Euler)

36 Palindromic numbers in two bases

The decimal number 585 (1001001001 in binary) is a palindrome in both bases.

Find the sum of all numbers less than 10 million that are palindromes in base 10 and base 2.

Note: Whether in base 10 or base 2, the numbers must not start with a 0.

(Inspired by problem 36 of Project Euler)

37 | Truncable prime numbers

The number 3797 has an interesting property. It is prime, and it remains prime when we eliminate its digits one by one from left to right: 3797, 797, 97 and 7 are all prime numbers. The same is true when eliminating the digits from right to left: 3797, 379, 37, and 3 are also all prime numbers.

Find the sum of the eleven prime numbers that can be truncated from left to right and from right to left.

Note: 2, 3, 5 and 7 are not considered truncable prime numbers.

(Inspired by problem 37 of Project Euler)

38 | Conway sequence

The Conway sequence is a mathematical sequence invented in 1986 by mathematician John Horton Conway, initially under the name “audioactive sequence”. In this sequence, a term is determined by announcing the digits that form the previous term.

$$
\begin{aligned}
T_1 &= 1 \\
T_2 &= 11 \\
T_3 &= 21 \\
T_4 &= 1211 \\
T_5 &= 111221 \\
T_6 &= 312211 \\
T_7 &= 13112221 \\
T_8 &= 1113213211 \\
T_9 &= 31131211131221 \\
&\dots
\end{aligned}
$$

If $T_1 = 2$, how many 1s will there be in T_{50}?

(Proposed by Le Coyote)

39 | Right triangles with given perimeter

If p is the perimeter of a right triangle with integer side lengths, there are exactly three solutions for $p = 120$: $\{20, 48, 52\}$, $\{24, 45, 51\}$, $\{30, 40, 50\}$.

For what value of $p < 10\,000$ is the number of solutions maximum?

(Inspired by problem 39 of Project Euler)

40 | Champernowne's constant

The Champernowne constant is an irrational real number created by concatenating positive integers:

$$0.12345678910\mathbf{1}112131415161718192021\cdots$$

We can see that the 12th digit of the fractional part is 1.

If d_n represents the nth digit of the fractional part, what is the value of the following expression:

$$\boldsymbol{d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10000} \times d_{100000} \times d_{1000000} \times d_{10000000} \times d_{100000000}}$$

(Inspired by problem 40 of Project Euler)

41 | Pandigital prime

We say that a number with n digits is pandigital if it contains all digits from 1 to n exactly once. For example, 2143 is pandigital and is also prime.

What is the largest pandigital prime number?

(Inspired by problem 41 of Project Euler)

42 | UN x UN + UN = DEUX

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation where letters represent numbers to be found.

Solve the cryptarithm below (give the answer as the number DEUX):

```
UN x UN + UN = DEUX
```

(Proposed by Le Coyote)

43 | Palindrome squares

A palindromic number reads the same backward as forward. A single-digit number is a palindrome.

Give the sum of the numbers whose square is a palindrome of at most 13 digits.

(Proposed by Le Coyote)

44 | Pentagon numbers

Pentagonal numbers follow the formula $P_n = n(3n-1)/2$. The first ten pentagonal numbers are:

$$1, 5, 12, 22, 35, 51, 70, 92, 117, 145, \ldots$$

We can see that $P_4 + P_7 = 22 + 70 = 92 = P_8$. However, their difference, $70 - 22 = 48$, is not pentagonal.

Find the two pentagonal numbers P_j and P_k, whose sum and difference are pentagonal, and whose $D = |P_k - P_j|$ is minimized. Give the value of D.

(Inspired by problem 44 of Project Euler)

45 Triangular, pentagonal, and hexagonal

Triangular, pentagonal, and hexagonal numbers follow these formulas:

Triangular $T_n = n(n+1)/2 : 1, 3, 6, 10, 15, \ldots$
Pentagonal $P_n = n(3n-1)/2 : 1, 5, 12, 22, 35, \ldots$
Hexagonal $H_n = n(2n-1) : 1, 6, 15, 28, 45, \ldots$

We can verify that $T_{285} = P_{165} = H_{143} = 40755$.

Find the next triangular number that is also pentagonal and hexagonal.

(Inspired by problem 45 of Project Euler)

46 | Another conjecture of Goldbach

Christian Goldbach conjectured that every odd composite number could be written as the sum of a prime number and twice a square number.

$$
\begin{aligned}
9 &= 7 + 2 \times 1^2 \\
15 &= 7 + 2 \times 2^2 \\
21 &= 3 + 2 \times 3^2 \\
25 &= 7 + 2 \times 3^2 \\
27 &= 19 + 2 \times 2^2 \\
33 &= 31 + 2 \times 1^2
\end{aligned}
$$

It turns out that this conjecture is false.

What is the smallest odd composite number that contradicts this conjecture?

(Inspired by problem 46 of Project Euler)

47 | Distinct prime factors

The first two consecutive numbers with two distinct prime factors are:

$$\begin{aligned} 14 &= 2 \times 7 \\ 15 &= 3 \times 5 \end{aligned}$$

The first three consecutive numbers with three distinct prime factors are:

$$\begin{aligned} 644 &= 2^2 \times 7 \times 23 \\ 645 &= 3 \times 5 \times 43 \\ 646 &= 2 \times 17 \times 19. \end{aligned}$$

Find the first four consecutive numbers with four distinct prime factors. What is the smallest of these numbers?

(Inspired by problem 47 of Project Euler)

48 $1^1 + 2^2 + 3^3 + \cdots$

$1^1 + 2^2 + 3^3 + \cdots + 10^{10} = 10405071317$.

Give the first 10 digits of the series $1^1 + 2^2 + 3^3 + \cdots + 2013^{2013}$.

(Inspired by problem 48 of Project Euler)

49 | A very special arithmetic progression…

The arithmetic progression 1487, 4817, 8147, with a common difference of 3330, is special for two reasons:

1. all 3 terms are prime;
2. all 3 terms are composed of the same digits

There is only one other arithmetic progression of 4-digit numbers with the same properties. Which one?
Give the first term multiplied by the common difference as your answer.

(Inspired by problem 49 of Project Euler)

50 | Hamming sequence

Consider the integers whose only prime divisors are 2, 3, and 5. Arrange them in ascending order. This is the Hamming sequence:

$$2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50, \ldots$$

What is the 2013$^{\text{th}}$ term in this sequence?

(Inspired by puzzle 5 from "Jeux et casse-tête à programmer" by J. Arsac)

51 | Swiss Cryptarithm

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.

Solve the cryptarithm below (give the sum obtained as the answer):

```
 ARGAND
+ EULER
-------
 GULDIN
```

Additional information:

- the number 2 does not appear;
- the number 0 appears.

(Proposed by Marie-Pierre Falissard)

52 | Multiples contain the same digits

We can see that the number 125874 and its double 251748 contain the same digits, but in a different order.

Find the smallest positive integer n such that n, $2n$, $3n$, $4n$, $5n$ and $6n$ contain the same digits.

(Inspired by problem 52 of Project Euler)

53 | Grid of sums

Consider a square grid with 3 rows and 3 columns. We want to fill this grid with whole numbers. To start with, we place the number 20 in one cell and the number 13 in another. The other cells are filled in one after the other, in an order to be defined. The number placed in a square must be the sum of the numbers in the surrounding squares (i.e., the squares that touch the square to be filled on one side or corner). The largest number placed is called the grid's filling number.

Where should the two starting numbers be placed, and in what order should the grid boxes be filled to obtain the largest possible filling number?

What is the largest filling number?

*(**Les défis mathématiques du Monde, épisode 3**)*

54 | Many 2s

Consider the multiplication:

`_ 2 _ _ _ x _ _ = 2222 _ _`

In this operation, we have written both 2s (there are no others).

Find the largest possible multiplicand.

(Lewis Carroll Trophy 2013 for high school students)

55 | Lychrel numbers

If we take 47, reverse it, and add the two numbers together ($47 + 74 = 121$), we get a palindrome, i.e., a natural number that is equal to itself when read from left to right or right to left. The same applies to 20: $20 + 02 = 22$.

Not all numbers produce palindromes so quickly. For example, it takes three iterations starting from 349: $349 + 943 = 1292$, $1292 + 2921 = 4213$, $4213 + 3124 = 7337$.

There are also numbers, such as 196, that will never produce a palindrome using this process. These numbers are called "Lychrel numbers".

Given that 10 677 is the first number requiring more than 50 iterations to produce a palindrome, how many Lychrel numbers are there below 10 000?

(Inspired by problem 55 of Project Euler)

56 Maximum sum of digits of a power

A gogol (10^{100}) is an enormous number: 1 followed by 100 zeros. However, the sum of its digits is only 1.

Considering natural numbers of the form ab, where $a < 250$ and $b < 250$, what is the maximum sum of digits that can be obtained?

(Inspired by problem 56 of Project Euler)

57 | Continued fraction

The square root of two can be represented by a continued fraction:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

The first four iterations give:

$$\begin{aligned}
&1 + 1/2 = 3/2 = 1.5 \\
&1 + 1/(2 + 1/2) = 7/5 = 1.4 \\
&1 + 1/(2 + 1/(2 + 1/2)) = 17/12 = 1.41666\cdots \\
&1 + 1/(2 + 1/(2 + 1/(2 + 1/2))) = 41/29 = 1.41379\cdots
\end{aligned}$$

The next three fractions are $99/70$, $239/169$, and $577/408$. The eighth fraction, 1393/985, is the first one where the numerator has more digits than the denominator.

Among the first 10 000 fractions, how many have a numerator with more digits than the denominator?

(Inspired by problem 57 of Project Euler)

58 | Prime numbers on a spiral

Starting from 1 and forming a spiral by turning counterclockwise, we obtain a square with a width of 7:

37	36	35	34	33	32	**31**
38	**17**	16	15	14	**13**	30
39	18	**5**	4	**3**	12	29
40	19	6	1	2	11	28
41	20	**7**	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

It is interesting to note that the odd squares are located on the diagonal line starting from 1 towards the lower right corner. It is even more interesting to note that 8 of the 13 numbers on the diagonals are prime numbers, representing approximately 62% of the prime numbers.
If we add another layer to the spiral, we will obtain a square with a width of 9.
Continuing this process, for what width of the spiral will there be less than 13% prime numbers on the diagonals for the first time?

(Inspired by problem 58 of Project Euler)

59 | Untouchable numbers

An untouchable number is a natural number that cannot be expressed as the sum of the proper divisors of an integer (divisors other than the integer itself).

For example, 9 is not untouchable, because 15 has proper divisors 1, 3, 5, and $1 + 3 + 5 = 9$.

On the other hand, 52 is untouchable because no integer has 52 as the sum of its proper divisors.

The first untouchable numbers are: 2, 5, 52, 88, 96, 120, 124, 146, 162, 188, …

If k is not untouchable, let $p(k)$ be the smallest integer whose sum of proper divisors is k. **What is the maximum value of $p(k)$ for k less than 666?**

(Proposed by Nicolas Quinodoz)

60 | Mass suicide

The 2013 members of a cult decided to commit suicide. To perform the funeral ritual, they formed a circle and numbered themselves from 1 to 2013.
They began counting, starting with number 1. Every 7th person would have to die. Thus, the first to die will be number 7, the second number 14, the third number 21, and so on.
You are a member of this cult, but you have no desire to die! Your task is to find the position in the circle that will allow you to be designated last, and thus escape death.

Which position will save you?

(Proposed by "problème de Josèphe")

61 No to isolation!

A digit is isolated if its left and right neighbors are different from itself. For example, in 776444, 6 is isolated, but the other digits are not.

On the other hand, the first three numbers that are not multiples of 10 whose squares contain no isolated digits are:

$$88^2 = 7744$$
$$74162^2 = 5500002244$$
$$105462^2 = 11122233444$$

What is the fourth?

(Proposed by Le Coyote)

62 | Cubic permutations

The digits of the cube 41063625 (345^3) can be permuted to produce two other cubes: 56623104 (384^3) and 66430125 (405^3). In fact, 41063625 is the smallest cube with this property.

Find the smallest cube for which exactly four permutations of its digits are cubes.

Note: the cube itself + 3 permutations of its digits.

(Inspired by problem 62 of Project Euler)

63 | x^n gives an n-digit number

The 5-digit number $16807 = 7^5$ is also a number raised to the power of 5. Similarly, the 9-digit number $134217728 = 8^9$ is a number raised to the power of 9.

How many positive integers with n digits are also numbers raised to the power of n?

(Inspired by problem 63 of Project Euler)

64 | Multiplicative persistence

Consider a positive integer, for example, 377. Multiply its digits: $3 \times 7 \times 7 = 147$. Do the same with the result 147: $1 \times 4 \times 7 = 28$. Repeat: $2 \times 8 = 16$. Again: $1 \times 6 = 6$. Once we reach a single-digit number, we stop.

377, 147, 28, 16, 6 is the "multiplicative sequence" of 377, and the "multiplicative persistence" p of 377 is the number of times the digits had to be multiplied before arriving at a single-digit number; here, $p = 4$.

We conjecture that p cannot exceed $11\ldots$

What is the smallest integer less than 1 000 000 with the largest multiplicative persistence p?

(Proposed by Le Coyote)

65 | Numbers of the year 2014

If we add the product of its four digits to the number 2014: $2014 + 2 \times 0 \times 1 \times 4$, we get 2014.

Find the sum of the other positive integers that add up to 2014 when you add the product of their digits.

(Individual quarterfinals of the 28th Mathematical and Logical Games Championship)

66 | Period in the decimal expansion

A unit fraction is a rational number written as a fraction where the numerator is 1 and the denominator is a positive integer.

$$\begin{aligned}
1/2 &= 0.5\\
1/3 &= 0.(3)\\
1/4 &= 0.25\\
1/5 &= 0.2\\
1/6 &= 0.1(6)\\
1/7 &= 0.(142857)\\
1/8 &= 0.125\\
1/9 &= 0.(1)\\
1/10 &= 0.1
\end{aligned}$$

$0.1(6)$ means $0.166666\cdots$, and $1/6$ has a period of 1 digit in its decimal expansion. We can see that $1/7$ has a period of 6 digits, while $1/2$ has no period.

Find the value of $d < 5000$, where $1/d$ has the longest period in its decimal expansion.

(Inspired by problem 26 of Project Euler)

67 | Stern's diatomic sequence

Stern's diatomic sequence is the result of the following small equations:

$$
\begin{aligned}
s_0 &= 0 \\
s_1 &= 1 \\
s_{2n} &= s_n \\
s_{2n+1} &= s_n + s_{n+1}
\end{aligned}
$$

What is the value of $s_{10\,000\,001}$?

(Proposed by Le Coyote)

68 | SIX^2 = TROIS

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.

$$\mathtt{SIX}^2 = \mathtt{TROIS}$$

What is the value of `TROIS`?

(Berloquin et al., "Voulez-vous jouer avec nous?", Ballard, 1975)

69 Euler's function

Two integers are relatively prime if their greatest common divisor is 1. For example, 8 and 15 are relatively prime.
Euler's function, $\varphi(n)$, is used to determine the number of integers smaller than n that are relatively prime to n. For example, since 1, 2, 4, 5, 7, and 8 are all smaller than 9 and relatively prime to 9, $\varphi(9) = 6$.

n	**Relatively Prime**	$\varphi(n)$	$n/\varphi(n)$
2	1	1	2
3	$1, 2$	2	1.5
4	$1, 3$	2	2
5	$1, 2, 3, 4$	4	1.25
6	$1, 5$	2	3
7	$1, 2, 3, 4, 5, 6$	6	$1.1666\cdots$
8	$1, 3, 5, 7$	4	2
9	$1, 2, 4, 5, 7, 8$	6	1.5
10	$1, 3, 7, 9$	4	2.5

We can see that $n/\varphi(n)$ is maximum for $n = 6$.

Find the value of $n < 1\,000\,000$ for which $n/\varphi(n)$ is maximum.

(Inspired by problem 69 of Project Euler)

70 Circular permutation

Take the number 102564. By moving the last digit completely to the left, we obtain a multiple (different from the original number).

Indeed, $4 \times 102564 = 410256$.

Add all 6-digit numbers that have this property.

(Proposed by Le Coyote)

71 | $567^2 = 321489$

$567^2 = 321489$

This equation uses all the numbers from 1 to 9, once each (except for the square).

What is the only other number that, when squared, has the same property?

(Proposed by Le Coyote)

72 | ...xxx

Of all integers less than 1 billion, how many are squares ending with exactly 3 identical digits?

For example, $213444 = 462^2$.

(Proposed by Le Coyote)

73 Palindromes and palindrome squares

An integer is a palindrome when it reads the same backward as forward. For example, 235532 and 636 are palindromes.

What is the largest 7-digit palindrome whose square is also a palindrome?

(Proposed by Le Coyote)

74 | Chain of factorial sums

The number 145 is well known for the property that the sum of the factorial of its digits equals 145:

$$1! + 4! + 5! = 1 + 24 + 120 = 145$$

Using this process, 169 produces the longest loop of numbers. It turns out that there are only three such loops:

$$\begin{aligned}
&169 \to 363601 \to 1454 \to 169\\
&871 \to 45361 \to 871\\
&872 \to 45362 \to 872
\end{aligned}$$

It is not difficult to prove that any starting number will eventually produce a loop. For example,

$$\begin{aligned}
&69 \to 363600 \to 1454 \to 169 \to 363601 (\to 1454)\\
&78 \to 45360 \to 871 \to 45361 (\to 871)\\
&540 \to 145 (\to 145)
\end{aligned}$$

69 produces a chain of 5 different terms.

What is the smallest number less than 5 million that produces the longest chain of different terms?

(Inspired by problem 74 of Project Euler)

75 | A crazy conversation. Although…

X and Y are two different integers greater than 1, with $X + Y < 100$.
Simon and Paul are two mathematicians. Simon knows the sum $S = X + Y$. Paul knows the product $P = X \times Y$. Simon knows that Paul knows the product, and Paul knows that Simon knows the sum.
A conversation ensues between the two mathematicians:

Paul says, "I can't find these numbers."
Simon says, "I was sure you couldn't find them. I can't find them either."
Paul says, "So, I found these numbers."
Simon says, "If you could find them, then I found them too."

What are these numbers? Give the answer as the product $X \times Y \times S \times P$.

(Hans Freudenthal, "Nieuw Archief Voor Wiskunde", Series 3, Volume 17, 1969, page 152)

76 | Fishing

The natural numbers are arranged as shown in the diagram below:

```
1st row:  1  2  6  7 15 16 ...
2nd row:  3  5  8 14 17 ...
3rd row:  4  9 13 18 ...
4th row: 10 12 ...
5th row: 11 ...
...     : ...
```

(Add as many rows as needed)

In which row is the number 20142014 located?

(Inspired by the 2011 CMP problem from the mathematics and physics circle of the Société jurassienne d'émulation)

77 | Sum of two squares

Two 4-digit numbers have the following property: $abcd = ab^2 + cd^2$.

$$1233 = 12^2 + 33^2$$
$$8833 = 88^2 + 33^2$$

What is the largest 8-digit number of this type: $abcdefgh = abcd^2 + efgh^2$?

(Proposed by Le Coyote)

78 | $abcd = a^b \times c^d$

What is the only number with the following pattern: $abcd = a^b \times c^d$?

Note (May 1, 2014): This is not a cryptarithm. Two different letters can represent the same digit.

(Henry Ernest Dudeney)

79 A maximum product

Take the first eight odd numbers 1, 3, 5, 7, 9, 11, 13 and 15, arranged in a certain order: x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, such that the product

$$p = (2 - x_1) \cdot (4 - x_2) \cdot (6 - x_3) \cdot (8 - x_4) \cdot (10 - x_5) \cdot (12 - x_6) \cdot (14 - x_7) \cdot (16 - x_8)$$

is as large as possible.

What is the maximum product?

(Puzzle 68 from the book "70 énigmes corrigées pour lycéens et plus, tome 1", by Jacques Lévy)

80 | Consecutive years with the same number of divisors

The year 2014 has a special feature: it has the same number of divisors as the previous year (2013) and the following year (2015).

How many years between 1 and 3000 (excluding these two years) have this feature?

(Proposed by Le Coyote)

81 | Mirror squares

Some square numbers have a curious property: their "mirror" is also a square. Furthermore, the square root of one of these two numbers is also the "mirror" of the square root of the other. For example:

$$169 = 13^2 \text{ and } 961 = 31^2$$
$$14884 = 122^2 \text{ and } 48841 = 221^2$$

What is the largest 9-digit square number with this property?

(Puzzle 15 from the book "70 énigmes corrigées pour lycéens et plus, tome 1", by Jacques Lévy)

82 Euler's indicator

Two numbers are coprime if their greatest common divisor is 1. For example, 8 and 9 are coprime. Euler's indicator is the function φ that gives the number of strictly positive integers less than or equal to n that are coprime with n.

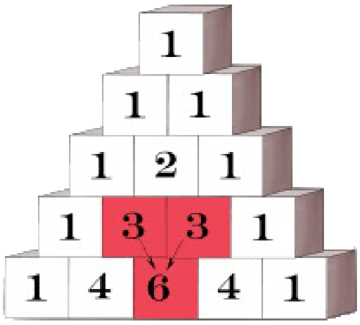
- $\varphi(1) = 1$, by definition.
- $\varphi(2) = 1$, because 1 is prime with all integers.
- $\varphi(3) = 2$, because 1 and 2 are prime with 3.
- $\varphi(4) = 2$, because 1 and 3 are prime with 4.
- $\varphi(5) = 4$, because 1, 2, 3, and 4 are prime with 5.
- $\varphi(6) = 2$, because 1 and 5 are coprime with 6.
- $\varphi(7) = 6$, because all numbers from 1 to 6 are coprime with 7.
- $\varphi(8) = 4$, because among the numbers from 1 to 8, only the four numbers 1, 3, 5, and 7 are coprime with 8.
- …

Find the smallest integer n such that $\varphi(n) = \varphi(n+1) = \varphi(n+2)$.

(Proposed by Le Coyote)

83 | Pascal's Triangle

To construct Pascal's triangle, place 1 at the top of the pyramid, then 1 and 1 below it, on either side. The ends of the lines are always 1, and the other numbers are the sum of the two numbers directly above them.



What is the smallest number greater than 1 that appears 8 times in Pascal's triangle?

(Proposed by Le Coyote)

84 A long journey

I travel a distance as follows:

Step 1: I move forward 1 centimeter.
Step 2: I turn left 90° and move forward 2 cm.
Step 3: I turn left 90° and move forward 3 cm.
…
Step N: I turn left 90° and move forward N centimeters.

At the end of a step, I find myself more than 65 kilometers from my starting point (as the crow flies). Furthermore, this distance is a whole number of centimeters.

What is the minimum length of the route (in centimeters)?

(Proposed by Jodyl, member of Défi Turing)

85 Numbers with different digits

There are 32 490 numbers with all different digits between 1 and 100 000, for example, 4, 72, 1 468, 53 920, etc.

What is the sum of these numbers?

(Proposed by Le Coyote)

86 Curious 2000^{th} term of a sequence

We have a sequence whose first two terms are: $U_1 = 3/2$ and $U_2 = 5/3$.

We calculate the next term in the series using the formula $U_n = 2003 - \dfrac{6002}{U_{n-1}} + \dfrac{4000}{U_{n-1}U_{n-2}}$.

What will be the 2000^{th} term in the series? Round to the nearest whole number.

P.S. No! 2000 is not the correct answer…

(Proposed by Jodyl, member of Défi Turing, based on an article by Jean-Michel Müller)

87 | Chaotic fate of role-players

One million players participate in a game that has 10 levels. At the beginning, all players are at level 1.

At the end of each round of play, each player rolls a balanced twelve-sided die, numbered from 1 to 12. The player advances one level if they roll a number higher than their current level. Otherwise, they remain at the same level.

Be careful! At level 10, "advancing one level" means returning to level 1!

After a certain amount of time, the number of players at each level tends to stabilize.

What will be the average number of players at level 10? We will round to the nearest whole number.

(Proposed by Jodyl, member of Défi Turing)

88 | Friday the 13th

2014 was a year with few Friday the 13ths, with only one in June.

Between January 1st 2001 and December 31th 2100, how many Friday the 13ths were there?

(Proposed by Le Coyote)

89 | Square cryptarithm

$ABCDCBA$ is a 7-digit number encoded using the 4 letters A, B, C and D. Each letter always represents the same digit, and two different letters always represent two different digits.

Give the largest square of the form $ABCDCBA$.

(FFJM – Swiss Final 2014, problem 18)

90 | Light!

276 lamps are numbered from 1 to 276.
To pass the time, 25 children take turns pressing the switches. The first child presses every switch. The second presses switches 2, 4, 6, etc. (all switches with a number that is a multiple of 2), the third presses switches 3, 6, 9, etc. The fourth presses all switches with a number that is a multiple of 4, and so on until the 25th child.
Before the first child presses the buttons, all the lights are off.

How many lights will be on after all 25 children have pressed the buttons?

(Proposed by Le Coyote)

91 | An almost true conjecture

A conjecture states that all odd numbers can be written in the form $p + 2 \times k^2$, where p is a prime number and k is a positive integer or zero.

$$\begin{aligned} 9 &= 7 + 2 \times 1^2 \\ 15 &= 7 + 2 \times 2^2 \\ 21 &= 3 + 2 \times 3^2 \\ 25 &= 7 + 2 \times 3^2 \\ 27 &= 19 + 2 \times 2^2 \\ 33 &= 31 + 2 \times 1^2 \end{aligned}$$

However, this conjecture is false.

Give the sum of the numbers between 2 and 100 000 that contradict this conjecture.

(A conjecture by Goldbach from 1752, disproved by M. A. Stern and his students in 1856.)

92 | Sum of consecutive integers equals 2014

The number 2014 can be written as a sum of consecutive positive integers.

For example: $502 + 503 + 504 + 505 = 2014$.

There are other possibilities.

Give the product of the first terms of all possible sums (including the example).

(FFJM - Swiss Final 2014, problem 16)

93 | Sums of prime numbers

It is possible to write 10 as a sum of prime numbers in exactly five ways:

$$\begin{aligned}
&7+3\\
&5+5\\
&5+3+2\\
&3+3+2+2\\
&2+2+2+2+2
\end{aligned}$$

What is the smallest integer that can be written as a sum of prime numbers in more than one million different ways?

(Problem 77 of Project Euler)

94 | 1 or 89?

A sequence of integers is created as follows: the next number in the list is obtained by adding the squares of the digits of the previous number.

Two examples:

$$44 \to 32(= 16 + 16) \to 13(= 9 + 4) \to 10(= 1 + 9) \to \mathbf{1} \to \mathbf{1}$$
$$85 \to \mathbf{89} \to 145 \to 42 \to 20 \to 4 \to 16 \to 37 \to 58 \to \mathbf{89}$$

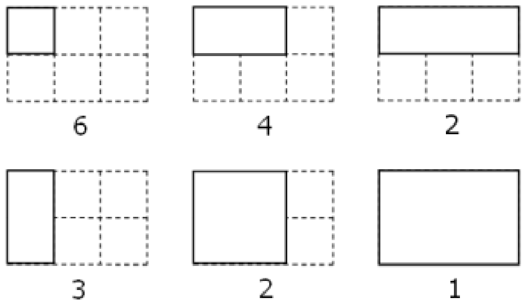
We can see that a sequence that reaches 1 or 89 will remain stuck in an infinite loop. The most incredible thing is that with any strictly positive starting number, any sequence will eventually reach 1 or 89.

How many starting numbers less than or equal to 5 million will reach 89?

(Problem 92 of Project Euler)

95 | Counting rectangles

In a 3×2 rectangular grid, there are 18 rectangles (squares are also counted as rectangles):



Find the area of the grid with the number of rectangles closest to 5 million.

(Problem 85 of Project Euler)

96 | $p^2 + q^3 + r^4$

28 is the smallest number that can be written as $p^2 + q^3 + r^4$, where p, q and r are prime numbers.

$$28 = 2^2 + 2^3 + 2^4$$
$$33 = 3^2 + 2^3 + 2^4$$
$$49 = 5^2 + 2^3 + 2^4$$
$$47 = 2^2 + 3^3 + 2^4$$

How many numbers less than or equal to 1 billion can be written in this way?

(Problem 87 of Project Euler)

97 | Even/odd cryptarithm

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.
In the multiplication below, `I` stands for an odd number and `P` for an even number. For example, `PPII` could be the number 4413 or 2873.

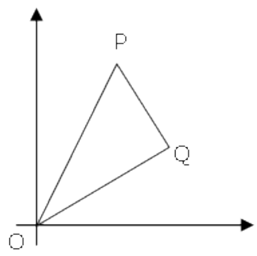
```
    I P P
x     P P
---------
  P I P P
  P I P
---------
  I I P P
```

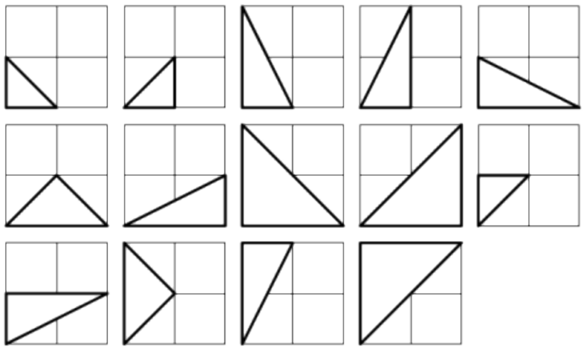
What is the value of this product (`IIPP`)?

(Problem 56 from "100 jeux de chiffres", by Pierre Berloquin)

98 | Right triangles with integer coordinates

The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ have integer coordinates and form a triangle OPQ with the origin. There are exactly 14 right-angled triangles OPQ for $0 \leq x_1, y_1, x_2, y_2 \leq 2$.





How many right-angled triangles OPQ are there for $0 \leq x_1, y_1, x_2, y_2 \leq 60$?

(Problem 91 of Project Euler)

99 | Lucky year

The terms 1, 2, 3 and 4 of a sequence are 13^0, 13^1, 13^2 and 2014, respectively. Each subsequent term in the sequence is always equal to the sum of the previous four terms.
The 5th term of the sequence is therefore equal to $1 + 13 + 169 + 2014 = 2197$.

How many digits does the 2014th term of the sequence have?

(Problem 18 from the semifinals of the 2014 Mathematical and Logical Games)

100 | Magic square

Complete the magic square below using only prime numbers between 2 and 100.[1]

		7
1		
	13	

Give the product of the 3 blue squares as your answer.

(Problem 70 from “100 jeux de chiffres”, by Pierre Berloquin)

[1] The sums of the three numbers in each row, each column, and both main diagonals are the same.

101 | Division by 11

Mathilde has just divided a number n by 11. The quotient, which is exact, is equal to the sum of the cubes of the digits of n.

For example: $6171/11 = 561 = 6^3 + 1^3 + 7^3 + 1^3$.

Give the sum of the numbers n that have this property.

(Inspired by problem 15 from the semifinals of the 2014 International Mathematical and Logical Games Championship)

102 | $1741725 = 1^7 + 7^7 + 4^7 + 1^7 + 7^7 + 2^7 + 5^7$

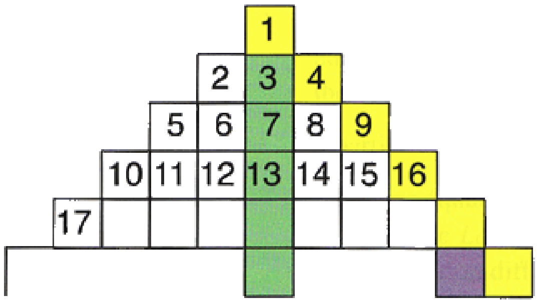
The number 1741725 is the smallest 7-digit number such that $1741725 = 1^7 + 7^7 + 4^7 + 1^7 + 7^7 + 2^7 + 5^7$.

Give the sum of all 7-digit numbers that have this property.

(Proposed by Le Coyote)

103 | Funny coordinates

We place the integers in a table according to the layout below:



- A row is designated by the number furthest to the left in that row.
- A column is designated by the topmost number in that column.
- The position of a number is identified by these two coordinates. For example, 15 is in position (10, 9).

What are the coordinates of 2014 in this table? Give the product of the two coordinates as your answer.

(Problem from the magazine Logimath)

104 Sequence of integers with the same number of divisors

The number 242 is the beginning of a sequence of four consecutive integers with the same number of divisors. Indeed, 242, 243, 244 and 245 each have 6 divisors:

$$242 : 1, 2, 11, 22, 121, 242$$
$$243 : 1, 3, 9, 27, 81, 243$$
$$244 : 1, 2, 4, 61, 122, 244$$
$$245 : 1, 5, 7, 35, 49, 245$$

Consider integers less than 50 000.

What is the smallest integer that is the beginning of the longest sequence of integers with the same number of divisors?

(Proposed by Le Coyote)

105 Arithmetic expressions

Let us calculate integers using exactly one of each digit from the set $\{1, 2, 3, 4\}$, with the four arithmetic operations $+$, $-$, $\times$, / and parentheses.

For example,

$$\begin{aligned} 8 &= (4 \times (1+3))/2 \\ 14 &= 4 \times (3 + 1/2) \\ 19 &= 4 \times (2+3) - 1 \\ 36 &= 3 \times 4 \times (2+1) \end{aligned}$$

Concatenating digits, such as $12 + 34$, is not allowed.

Using the set $\{1, 2, 3, 4\}$, it is possible to obtain 31 different numbers, with a maximum of 36; all numbers from 1 to 28 can be obtained.

Find the set of 4 distinct digits $a < b < c < d$, for which the longest string of consecutive integers from 1 to n can be obtained. Give the string $abcd$ as the answer.

(Problem 93 of Project Euler)

106 Two cryptarithms

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.

We know that `XYZ + AB = CDEF` and that `XYZ - AB = BGA`.

What is the value of the product `XYZ` $\times$ `AB`?

(Problem 49 from "100 jeux de chiffres", by Pierre Berloquin)

107 Pandigital square

We say that an integer is pandigital if it is composed of all the digits from 0 to 9. For example, 1023456789 is the smallest pandigital integer.

What is the smallest pandigital square?

(Proposed by Le Coyote)

108 | Happy palindrome year!

2015 is a palindrome number… in base 2. Indeed, $2015 = 11111011111_2$.

This is not that rare, since 2016 is also a palindrome number in base 2 and even in base 3 (provided you add zeros to the left). In fact, $2016 = 0000011111100000_2 = 002202200_3$.

If we consider bases from 2 to 16, how many years of the 3rd millennium (2001–3000) will be palindromic numbers in at least one of these bases? If necessary, add zeros to the left, regardless of the base. For example, 2020 will be considered a palindrome, because it could be written as 02020_{10}.

(Proposed by Le Coyote)

109 | Feline cryptarithm

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.

`CHAT + CHAT = MINOU`

Let x be the smallest possible value of `MINOU` and y the largest. What is the value of $x + y$?

(Problem 75 from "100 jeux de chiffres", by Pierre Berloquin)

110 | Probability: 50%

Two balls are drawn at random from an urn containing blue and red balls.
If the urn contains 21 colored balls, 15 blue and 6 red, the probability of drawing 2 blue balls is $P(\mathrm{BB}) = (15/21) \times (14/20) = 1/2$.
The next blue/red distribution that gives $P(\mathrm{BB}) = 1/2$ is 85 blue and 35 red: $P(\mathrm{BB}) = (85/120) \times (84/119) = 1/2$.

Find the first blue/red distribution where $P(\mathrm{BB}) = 1/2$, in an urn containing more than 10 billion balls. Give the number of blue balls as your answer.

(Problem 100 of Project Euler)

111 | Baron's Number

What is the only number $n > 1$ such that

$$n = d_k^{d_k} + d_{k-1}^{d_{k-1}} + \cdots + d_2^{d_2} + d_1^{d_1}$$

The d_i are the digits that make up the number n.
By convention, $0^0 = 1$.

*(**Wikipédia: Nombre de Münchhausen**)*

112 | Harmony

Fill in the boxes in the grid below with whole numbers, so that each of these ten newly filled boxes contains a number that is the average of the numbers contained in the four boxes with which it shares a side.

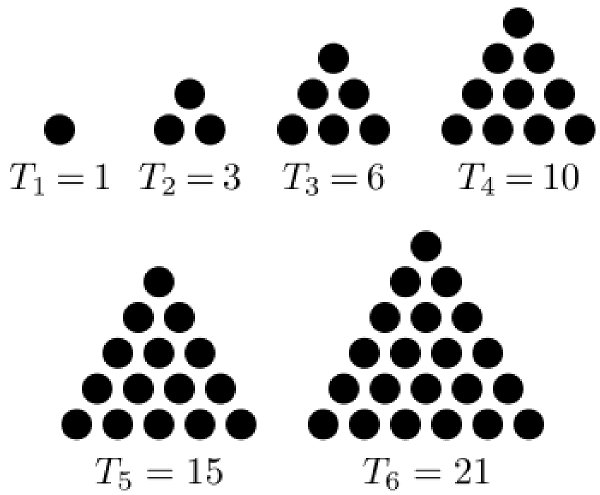
	4	7	1	2	1	
10						0
0						1
	10	2	7	1	0	

What is the product of the numbers in these 10 newly filled boxes (excluding any 0s)?

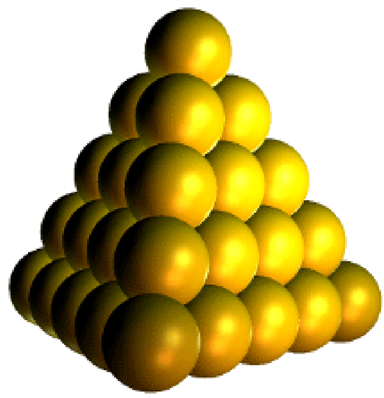
(Jeux & Stratégie No. 13, February 1982, p.60. Proposed by Over_score, member of Défi Turing)

113 | Triangular and tetrahedral numbers

A triangular number is a figurative number that can be represented by a triangle. The first triangular numbers are: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, …
A tetrahedral number is a figurative number that can be represented by a pyramid with a triangular base, i.e., a tetrahedron. The first tetrahedral numbers are: 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, …



The first six triangular numbers



The fifth tetrahedral number (35)

We can see that the numbers 1 and 10 are both triangular and tetrahedral.

What is the largest number that is both triangular and tetrahedral?

(Proposed by Le Coyote)

114 | 49^{th} natural Kaprekar number

A natural integer k with n digits is called a natural Kaprekar number if its square can be broken down into a right part with n digits and a left part with n or $n-1$ digits such that their sum is k.

45 is the 3rd natural Kaprekar number, because $45^2 = 2025$ and $20 + 25 = 45$.
297 is the 6th natural Kaprekar number, because $297^2 = 88209$ and $88 + 209 = 297$.

The first 15 natural Kaprekar numbers are: 1, 9, 45, 55, 99, 297, 703, 999, 2223, 2728, 4950, 5050, 7272, 7777, 9999.

What is the 49th natural Kaprekar number?

(Proposed by Le Coyote)

115 | Just four digits

Only two digits (0 and 1) are needed to write a correct multiplication corresponding to the configuration below.

$$
\begin{array}{cccccc}
 & & 1 & 0 & 1 & 0 \\
 & & \times & & 1 & 1 \\
\hline
 & & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 1 & 0 & \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

There are other solutions, such as 1111 × 11, for example, which use only 1s and 2s.

With the configuration below, this is no longer possible. Even three digits are not enough. With four digits, it is finally possible. There are only two multiplications that satisfy this condition.

$$
\begin{array}{ccccc}
 & \cdot & \cdot & \cdot & \cdot \\
 & \times & & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

What is the product of the results of these two multiplications?

(Jeux & Stratégie No. 13, February 1982, p.56. Proposed by Over_score, member of Défi Turing)

116 | Quadruplet of integers divisible by a cube

1375, 1376 and 1377 form the smallest triplet of consecutive integers that are all divisible by a cube other than 1. Indeed, 1375 is divisible by 125, 1376 by 8, and 1377 by 27.

What is the smallest quadruplet with this property? Give the smallest of the 4 numbers as your answer.

(Proposed by Le Coyote)

117 Almost prime numbers

A positive integer n is said to be "almost prime" if there are two distinct prime numbers p and q such that $n = pq$. Remember that 1 is not prime.

How many almost prime numbers smaller than 100 000 are there?

(Problem from July 25th of the "Calendrier mathématique 2014")

118 | Sequence of multiples

Let us consider the ten numbers: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12.
We want to order them so that, for any two consecutive numbers, one is always a multiple of the other.

For example: 6, 3, 9, 1, 5, 10, 2, 8, 4, 12.

How many orders satisfy this condition?

(Proposed by Le Coyote)

119 | Number of divisors

The number 80 has 10 divisors: 1, 2, 4, 5, 8, 10, 16, 20, 40 and 80.

Given that $50! = 50 \times 49 \times 48 \times 47 \times \cdots \times 3 \times 2 \times 1$, give the number of divisors of $50!$.

(Proposed by Le Coyote)

120 | Origin is in a triangle... or not

Three distinct points are placed randomly on the Cartesian plane, with $-1000 \leq x, y \leq 1000$, such that a triangle is formed.
Consider the following triangles ABC and DEF:

$$A(-340, 495), B(-153, -910), C(835, -947)$$
$$D(-175, 41), E(-421, -714), F(574, -645)$$

We can verify that the origin $(0, 0)$ is inside triangle ABC, but outside triangle DEF.

The file **120-fichier.txt**[1] (right click and "Save link as...") contains the coordinates of the vertices of ten thousand triangles.

How many of these triangles contain the origin?

Note: the first two lines of the file represent the triangles in the example given above.

(Problem 102 of Project Euler)

[1]This text file *120-fichier.txt* is also attached to this PDF.

121 How many magic squares?

To create a magic square, fill in the nine boxes in the square below with the numbers 1 to 9 so that the sum of the numbers in each four-box square is always the same.

Example:

9	1	8
2	7	3
6	4	5

$9 + 1 + 2 + 7 = 1 + 8 + 7 + 3 = 2 + 7 + 6 + 4 = 7 + 3 + 4 + 5 = 19.$

How many magic squares are there (ignoring symmetries)?

(Proposed by Le Coyote)

122 How many heterogeneous squares?

To obtain a heterogeneous square of order 3, fill in the nine boxes of the square below with the numbers 1 to 9 so that the eight sums of the three numbers in the rows, columns, and two diagonals are all different.

Example:

4	5	7
1	3	8
9	2	6

$4+5+7 \neq 1+3+8 \neq 9+2+6 \neq 4+1+9 \neq 5+3+2 \neq 7+8+6 \neq 4+3+6 \neq 7+3+9.$

How many heterogeneous squares of order 3 are there (not counting symmetries)?

(Based on problem 15 in the book "Le jardin du Sphinx" by Pierre Berloquin)

123 Antimagic squares

In an antimagic square, the sums of the rows, columns, and diagonals must not only all be different, as in a heterogeneous square (see problem 122). The sums must also be consecutive, for example from 29 to 38.

There are no antimagic squares of order 1, 2 or 3.

How many antimagic squares are there that complete the partially filled square below?

		14	13
16	15	4	2

(Based on problem 16 in the book "Le jardin du Sphinx" by Pierre Berloquin)

124 | Prime numbers everywhere!

1373 is an interesting prime number because if you look at all the substrings of at least two consecutive digits that make it up, you only see prime numbers.

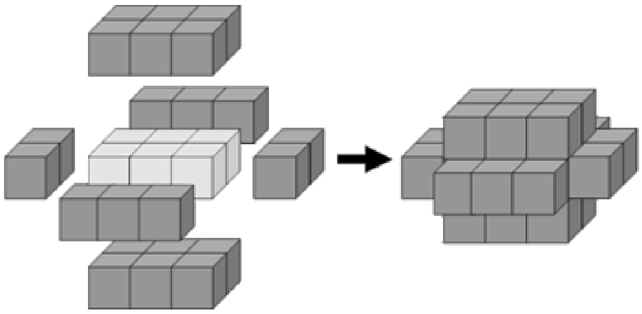
Indeed, 13, 37, 73, 137, 373 and 1373 are prime numbers.

What is the largest prime number less than 10 million that has this property?

(Proposed by Le Coyote)

125 Wrapping a parallelepiped

The minimum number of cubes needed to cover all visible faces of a rectangular parallelepiped measuring $3 \times 2 \times 1$ is 22.



If we then added a second layer to this solid, it would take 46 cubes to cover all visible faces, the third layer would require 78 cubes, and the fourth layer would require 118 cubes.
However, the first layer on a parallelepiped measuring $5 \times 1 \times 1$ also requires 22 cubes. Similarly, the first layer on parallelepipeds with dimensions $5 \times 3 \times 1$, $7 \times 2 \times 1$ and $11 \times 1 \times 1$ all require 46 cubes.
We will use the notation $C(n)$ to represent the number of rectangular parallelepipeds that contain n cubes in one of their layers. So, $C(22) = 2$, $C(46) = 4$, $C(78) = 5$, and $C(118) = 8$.
It turns out that 154 is the smallest value of n for which $C(n) = 10$.

Find the smallest value of n for which $C(n) = 100$.

(Problem 126 of Project Euler)

126 | Subtriples

Pierre de Fermat (1601?–1665) used the term "subdouble" to refer to an integer whose sum of proper divisors is equal to twice that number.

For example, the sum of the proper divisors of 120 is equal to:
$1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 + 30 + 40 + 60 = 240.$

Similarly, he uses the term "subtriple" to refer to an integer whose sum of proper divisors is equal to three times that number.

Give the sum of the subtriples less than 100 thin 000.

(Proposed by Le Coyote)

127 Perpetual calendar

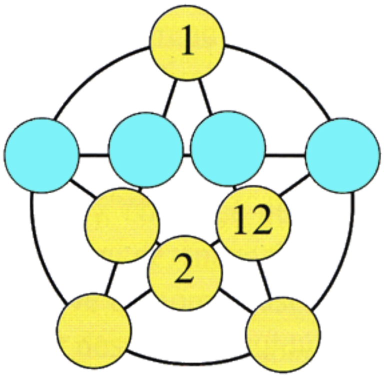
A perpetual calendar consists of 14 different calendars. This is because January 1st is one of seven days of the week, and a year may or may not be a leap year. So $7 \times 2 = 14$.
You will start your calendar collection in 2016. When the calendar for a given year is identical to one you already have, you will not buy it.

When you have completed your collection, what will be the sum of the years on your 14 calendars?

(Proposed by Le Coyote)

128 | Pentacle

We want to fill the ten discs in the figure below with different integers between 1 and 12, so that the five rows of four numbers and the circle of five numbers have the same sum.



Give the number formed by placing the four numbers on the blue circles side by side, read from left to right.

(Problem from the Euromath Cup)

129 | More than 4...

Let $F(n) = 1 + 2 + 3 + \cdots + n$ be the sum of the first n non-zero integers.
Before adding them up, I wanted to write all the integers from 1 to n on my computer. The only problem was that the "4" key on my keyboard wasn't working. So I had the idea of replacing all the "4"s with other values, for example with "3"s.
By doing so, it is clear that the sum obtained will be less than $F(n)$.
To correct this slightly, I decided to replace the "4" alternately with a "3" and then with a "5", and I noted $S(n)$ as the sum of the n numbers obtained.

For example, we have $S(6) = 1 + 2 + 3 + 3 + 5 + 6 = 20$ and $F(6) = 21$.
Similarly, $S(20) = 1 + 2 + 3 + 3 + 5 + \cdots + 12 + 13 + 15 + 15 + 16 + \cdots + 20 = F(20)$.
We also have $S(50) = 1256$ and $F(50) = 1275$.

What is the largest value of n, less than one million, for which we have $S(n) = F(n)$?

(Proposed by Olivier Rochoir)

130 | Optimal horizontal line

The file **130-fichier.txt**[1] contains the coordinates of 1000 points in the plane.

What is the height of the horizontal line that minimizes the sum of the distances to the points? If there are several lines, give the average of the heights.

(Proposed by Le Coyote)

[1]This text file *130-fichier.txt* is also attached to this PDF.

131 | Sum of the first inverted integers

Let $S(n) = 1 + 2 + 3 + \cdots + n$ be the sum of the first n non-zero integers.
Furthermore, let $I(n)$ be the sum of the first n inverted integers.

We then obtain, for example,
$S(14) = 1 + 2 + 3 + 4 + \cdots + 10 + 11 + 12 + 13 + 14$
and
$I(14) = 1 + 2 + 3 + 4 + \cdots + 01 + 11 + 21 + 31 + 41.$

It is easy to see that $X = 10$ is the first value for which $S(X) > I(X)$.

What will be the sum of the integers X less than one million that have the property $S(X) > I(X)$?

(Proposed by cspecial)

132 | Average of squares

The average of the squares of the integers from 1 to 5 is equal to $(1 + 4 + 9 + 16 + 25)/5 = 11$.
The average of the squares of the integers from 1 to 77 is equal to 2015.

How many positive integers strictly less than one billion are equal to the average of the squares of consecutive integers from 1 to a certain number?

Note: the average of a single number is equal to that number.

(1/4 of the 2014-15 FFJM individual final)

133 | Guru and his followers

A guru has locked his 33 followers in separate rooms in his huge mansion. The followers cannot communicate with each other.
The guru lives in a luxurious room in his mansion, where the "lamp of truth" is located. He proposes an initiation game to his followers. Every day, he will bring a follower, chosen at random in secret, to his room. There, the follower will be free to decide whether or not to turn on the lamp. The guru will never touch the lamp, which is off at the start of the game.
The stakes are as follows: if a follower, upon entering the guru's room, claims that all the followers have already been to this room at least once and is correct, the 33 followers will be freed. However, if they are wrong, all the followers will begin their journey to Venus…
Before the game begins and the followers are isolated in their rooms, they have a moment to develop a strategy. Their only means of communication will be the "lamp of truth".

The file **tirage.txt**[1] contains the results of 10 000 random draws by the guru (numbers from 1 to 33, indicating the follower's room). Line 1 contains the number of the follower called on the first day, line 2 the follower on the second day, and so on. This file obviously cannot be used to define a strategy, since the followers do not have access to it.

If they discover the right strategy, after how many days will the followers be freed?

(This puzzle is known on the web as "The (100) prisoners and the light bulb")

[1] This text file *tirage.txt* is also attached to this PDF.

134 | 2015 is (once again) a palindrome

The year 2015 has an interesting property: $2015 = 84^2 - 71^2 = 48^2 - 17^2$.
We will have to wait until 3024 to see the same property again: $3024 = 75^2 - 51^2 = 57^2 - 15^2$.
For the differences between the two squares, only two-digit numbers that are neither multiples of 11 nor multiples of 10 will be used.

Give the sum of the years in the 6th millennium that have this property.

(Proposed by Le Coyote)

135 Partition of an integer

There are exactly six ways to write 5 as a sum of strictly positive integers:

$$\begin{aligned}&4+1\\&3+2\\&3+1+1\\&2+2+1\\&2+1+1+1\\&1+1+1+1+1\end{aligned}$$

How many different ways can 200 be written as the sum of at least two strictly positive integers?

(Problem 76 of Project Euler)

136 | An orderly multiplication

A, B, C, D and E are five different numbers. There is only one multiplication

$$AB \times C = DE$$

where $A < B < C < D < E$.

Give the number $ABCDE$ as the answer.

(Problem 1 from "Le chat à six pattes et autres casse-tête" by Louis Thépault)

137 Go first!

To play the game, Sisyphus drew 106 squares on the ground, numbered from 0 to 105, and he has a token and a six-sided die (balanced).
Sisyphus starts the game by placing the token on square 0. He then rolls the die a series of times. When the die shows the value k, he moves the token forward k squares and:

- if he reaches or exceeds square number 100, Sisyphus wins;
- if he lands on a square whose number is a prime number less than 100, Sisyphus loses;
- in other cases, Sisyphus rolls the die again and continues the game.

What is the probability that Sisyphus will win?

Enter the last ten digits of the numerator of the irreducible fraction as your answer.

(Problem inspired by Problem III of the 2013 Concours Général des Lycées. Proposed by David Draï)

138 | a, e, i, o, u, a, e, i, …

The file **dico.txt**[1] contains 323 471 French words without accents. We would like to know how many of them contain vowels (except y) in the cyclic order a, e, i, o, u. Here are some of these words:

a, je, cage, coupable, hibou, émir, …

However, these words do not satisfy the constraint:

engagé, capable, embout, émirat, lynx, …

How many words in the dico.txt dictionary satisfy this constraint?

(Proposed by Le Coyote)

[1] This text file *dico.txt* is also attached to this PDF.

139 | How many darts?

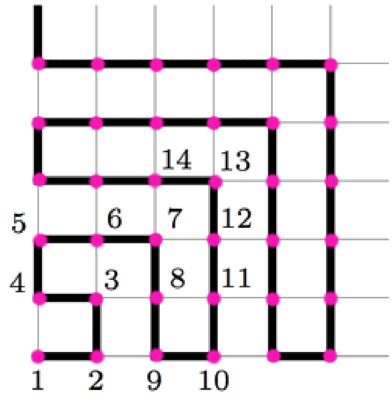
We throw darts at a grid of 100 squares by 100. Each square has the same probability of being hit.

How many darts must be thrown for the probability of hitting the same square multiple times to be greater than 1/2?

(Inspired by the birthday paradox)

140 | Snake

The points on a 100×100 grid are numbered from 1 to 10 000 following the path below.



What is the sum of the numbers on the diagonal line from the top left corner to the bottom right corner of the grid?

(Inspired by the March 4th problem from the 2015 Mathematical Calendar)

141 How many 6s are there?

Multiply the digits of a number greater than 9. If the result is a single-digit number, it is called the image of the original number. If it has more than one digit, repeat the operation until you get a single-digit number.

For example: $666 \to 216 \to 12 \to 2$.

How many numbers between 10 and 10 million have 6 as their image?

(Problem of July 30th from the 2015 Mathematical Calendar)

142 | Irregular numbers

Positive integers that are not divisible by any of their digits are called “irregular” numbers.

For example, 203, 547 and 998 are irregular.

How many irregular numbers are there below 1 million?

(Problem of July 17th from the 2015 Mathematical Calendar)

143 | Half of 5

On a board, we write all the positive integers 1, 2, 3, …, N, where N is a positive integer with 7 digits.

Find the largest possible value of N for which exactly $N/2$ numbers on the board contain at least one digit 5.

(Problem of April 30th from the 2015 Mathematical Calendar)

144 | 7 as the only king

How many six-digit numbers are there in which 7 appears only once and is the highest digit?

For example: 253753, 111172, 744252, etc.

(Problem of June 22nd from the 2015 Mathematical Calendar)

145 | Omnidextrous prime squares

A **prime** square of order 3 is a 3×3 square grid in which each cell contains a number between 0 and 9, these numbers forming prime numbers when read by columns or rows. No two columns or rows can be identical.

An **ambidextrous** prime square is a prime square that always contains prime numbers, whether the rows are read from right to left or left to right.

An **omnidextrous** prime square is an ambidextrous prime square in which the columns and diagonals contain prime numbers whether read from top to bottom or bottom to top. There is another constraint on these prime numbers: they cannot start with 0.

Here is an example of an omnidextrous prime square of order 3:

1	1	3
1	5	1
3	1	1

How many omnidextrous prime squares of order 3 are there?

(Problem 37 from the book "Aventures stratégiques et logiques" by Dennis E. Shasha)

146 | Sales representative's invitations

Thomas is a sales representative. He wants to invite one of his 10 clients to a restaurant each week, but he wants to do so based on how much he likes them.
He creates 10 cards, writes a score from 1 to 10 on each one, and ranks them in ascending order. He always invites the client corresponding to the top card. After dinner, he places this card behind a number of cards corresponding to its score.

Ranking for the first five weeks:

Week 1 $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$
Week 2 $[2, 1, 3, 4, 5, 6, 7, 8, 9, 10]$
Week 3 $[1, 3, 2, 4, 5, 6, 7, 8, 9, 10]$
Week 4 $[3, 1, 2, 4, 5, 6, 7, 8, 9, 10]$
Week 5 $[1, 2, 4, 3, 5, 6, 7, 8, 9, 10]$

In which week will the customer with a score of 10 be in 4th place in the ranking for the first time?

(Inspired by "Les invitations du représentant", Jeux & Stratégie 22, p.54)

147 | Culinary cryptarithm

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found. No number begins with a 0.

Solve the cryptarithm below, knowing that `FOC` is a square:

`CUIRE + EN + POELE = FRIRE`

Give the sum obtained (`FRIRE`) as your answer.

(Jouer Jeux Mathématiques magazine)

148 | Finish it!

All the numbers in this multiplication, except for six, have been replaced by dots.

```
      . . 1
    × . 2 .
-----------
    . 3 . .
  . 4 . .
. 5 . .
-----------
6 . . . . .
```

What is the number in the last line (the one that starts with 6)?

(Jeux & Stratégie 6, December 1980, p.55)

149 | Magic division

When dividing 100 000 000 by an integer, it can happen that the divisor, the quotient, and the remainder are composed of the same digits. More precisely, all the digits of the divisor appear at least once in the quotient and in the remainder. Furthermore, no other digits appear.

Here is one possibility: $100\,000\,000/91\,810 = 1089$, remainder 18 910. The 1, 8, 9 and 0 appear in all three numbers.

What is the other possibility? Give the divisor as your answer.

(Proposed by Le Coyote)

150 | No black squares

The numbers 4, 7, 19 and 37 have a remarkable property:

- Every natural number can be written as the sum of at most 4 integer squares (Lagrange's theorem)
- Every natural number can be written as the sum of at most 7 integer cubes (with the exception of 17 integers, all less than 455, which require 8 or 9…)
- Any natural number can be written as the sum of at most 19 fourth powers
- Any natural number can be written as the sum of at most 37 fifth powers.

Fill in the grid below:

	a	b	c	d	e	f
A						
B						
C						
D						
E						
F						

Horizontally

A. Fifth power

B. Fifth power

C. Fifth power

D. Fifth power

E. Fifth power

F. Divisible by all integers strictly less than its fifth root and has strictly more divisors than all integers less than it.

Vertically

a. Divisible by 19

b. Divisible by 37 and divisible by the cube of a prime number

c. Divisible by 7

d. Prime number

e. Divisible by 4 and divisible by the cube of a prime number

f. Divisible by 19

Write down the number formed by the digits in the yellow diagonal as the answer.

(Mathematics and Logical Games Championship. Improved and proposed by David Draï.)

151 | Happy new year 2016!

2016 is the area of a right triangle, whose three sides form a Pythagorean triple.

What are the lengths of sides a, b and c? Give your answer as the number obtained by juxtaposing a, b and c, with $a < b < c$.

(Proposed by Le Coyote)

152 | Sum of the digits of its cube

What is the largest integer equal to the sum of the digits of its cube?

(Euromath Regional Cup)

153 | $DIX^2 + UN^2 = CENTUN$

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found.

Solve the cryptarithm below (give the sum obtained as the answer):

$$DIX^2 + UN^2 = CENTUN$$

("Les maths au carré", by Marie-Pierre Falissard, problem 2)

154 | Stable cube

When you cube 10, you get 1 000. The cube of 10 can only be written using 0s and 1s, which are the digits of 10. The same applies to 100 and all powers of 10.

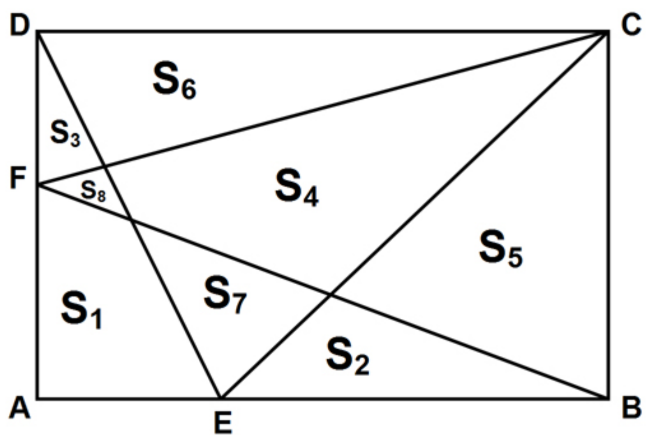
What is the smallest integer $n > 1$ that is not a power of 10 and such that n^3 can be written using the same digits as n?

(Proposed by Le Coyote)

155 | Prime partitions of a rectangle

$ABCD$ is a rectangle such that $AB \geq AD$. AB, AD, AE and AF are integer lengths.

- A configuration is said to be prime when the 8 areas S_i $(i = 1, 2, ..., 8)$ are integers and prime to each other (globally and not pairwise).
- Two prime configurations G and G' are said to be equivalent if: $\{S'_1, S'_2, S'_3, S'_4, S'_5, S'_6, S'_7, S'_8\} = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ (we are referring to sets).
- Among a set of equivalent prime configurations, we agree to call the main configuration the one for which AB is minimal (which amounts to choosing the one with the smallest perimeter or the one closest to a square…).



Examples

1. $(AB, AD, AE, AF) = (10, 6, 5, 3)$ is the smallest principal configuration, with: $(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) = (10, 3, 3, 16, 12, 12, 2, 2)$, and a total area of 60.
2. $(21, 16, 14, 8)$ is a main configuration that gives $(70, 8, 21, 99, 48, 63, 6, 21)$, and a total area of 336.

Let N be the number of principal configurations when $AB \leq 100$. These N configurations give only $(N - k)$ distinct total areas, because k of them are obtained twice. Let P be the sum of these k areas.

What is the value of $N \times P$?

(Proposed by Le David Draï)

156 | Final power

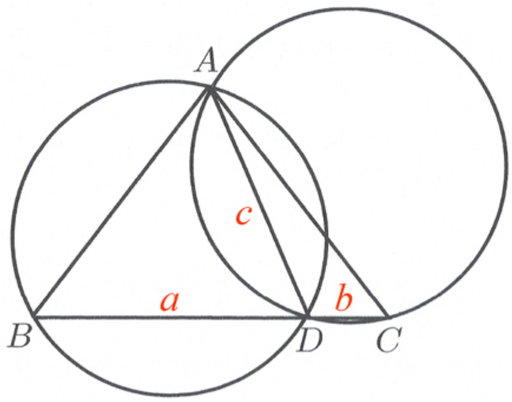
It is said that 7 is a final power for 3, because $3^7 = 2187$ (the last digit is 7).

What is the smallest final power for 18? In other words, what is the smallest integer n such that the decimal representation of 18^n ends with that of n?

(Mathematics and Logical Games Championship)

157 | Triangles with integer area

Let a, b, c and n be four integers such that $b < a \leq n$ and $c \leq n$.
In a triangle ABC (ABC is oriented in the positive trigonometric direction), let D be a point on side $[BC]$ such that: $BD = a$, $DC = b$ and $DA = c$.
We know that the radius of the circle circumscribing triangle ABD is equal to the radius of the circle circumscribing triangle ADC.



If $n = 200$, how many non-homothetic configurations exist such that the area of triangle ABC is an integer?

(Proposed by Le David Draï)

158 Cube the digits

Let's add the cubes of the digits that make up the number 2016. We get $2^3 + 0^3 + 1^3 + 6^3 = 225$. Let's repeat this process with the digits of the result. We get 141, then successively 66, 432, 99, 1458, 702, 351, 153, 153, …

What is the sum of the years in the 3rd millennium for which this process leads to the number 153?

(FFJM individual quarterfinals 2016)

159 | Prime coding

Here is a multiplication problem where the numbers are replaced by asterisks.

```
      * * *
  x     * *
  -------
    * * * *
  * * * *
  ---------
  * * * * *
```

It turns out that each asterisk is a single-digit prime number, i.e., 2, 3, 5 or 7.

What is the result of the multiplication?

(La Recherche magazine)

160 | Magic snail

Write a number from 1 to 3 in certain boxes of the grid so that:

- each number appears once, and only once, in each row and column;
- the first 1 (given in the figure) is at the entrance;
- when you follow the snail to the center, the numbers you encounter are in the order 1, 2, 3, 1, 2, 3 … 1, 2, 3.

If you number the squares of the snail as shown in the drawing, what is the sum of the 15 products "square number × square content"?

The 10 empty squares will not be taken into account.

(Quarterfinals of the Mathematical and Logical Games Championship)

161 | Fortune teller

A fortune teller uses five white cards numbered from 2 to 6 and four red cards numbered from 3 to 6. She places all the cards on the table, alternating colors systematically. Each card must have a number that is divisible by (other than 1) at least one of its two neighbors (at the ends, its neighbor).

For example: 245563463.

Following this rule, how many different numbers can be formed?

(FFJM, individual quarterfinals 2016, problem 12)

162 | Palindrome

Multiply three consecutive strictly positive even numbers together. The result is a palindrome.

What is the smallest palindrome obtained in this way?

(Proposed by Le Coyote. Inspired by a problem from the 30th Mathematical and Logical Games Championship)

163 | Prime-indexed sequence

A sequence is defined by this recurrence relation:

$$P_{n+1} = P_{n-1} + P_{n-2}, \text{with } P_0 = 3, P_1 = 0 \text{ and } P_2 = 2.$$

The first 18 terms of this sequence are: 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119.

We note that: $P_2 = 2$, $P_3 = 3$, $P_5 = 5$, $P_7 = 7$, $P_{11} = 22 = 2 \times 11$, $P_{13} = 39 = 3 \times 13$, $P_{17} = 119 = 7 \times 17$.

Now, 2, 3, 5, 7, 11, 13 and 17 form the beginning of the sequence of prime numbers. The other terms are not multiples of their index.

One might conjecture that if P_n is a multiple of n, then n is a prime number. However, this conjecture is false.

What is the smallest non-prime integer $n > 18$ such that P_n is a multiple of n?

("Suite de Perrin". Proposed by Vincent Bernigole)

164 | The ninth

By removing the 0 from 405, we get 45, which is equal to $405/9$.

We are interested in strictly positive integers n, less than 10 000 000, which, when their 0(s) are removed, give $n/9$.

What is the sum of the integers that have this property?

(Proposed by Le Coyote. Inspired by a problem from the 30th Mathematical and Logical Games Championship)

165 | Words of the year

The file **dico.txt**[1] contains 323 471 French words without accents. By assigning each letter its position in the alphabet (a $= 1$, b $= 2$, c $= 3$, …, z $= 26$), then multiplying all the letters in a word together, we obtain a number.

For example, "chaud"$= 3 \times 8 \times 1 \times 21 \times 4 = 2016$.

If we apply this process to all the words in the dico.txt file, which year of the third millennium will occur most often? Give your answer as the product of the year and the number of occurrences.

(Proposed by Le Coyote)

[1] This text file *dico.txt* is also attached to this PDF.

166 | N-ary numbers

An n-ary number is an integer whose digits add up to n. Example: 5, 32, 11111 and 20021 are 5-ary numbers.

If we consider integers less than or equal to 1 000 000, what is the most common value of n? Give the number of times this value of n appears as your answer.

(Proposed by Le Coyote. Inspired by a problem from the 30th Mathematical and Logical Games Championship)

167 Between 2 and 5

Find the smallest positive integer N such that there are exactly 25 integers x satisfying the inequality $2 \leq N/x \leq 5$.

(Problem of August 20th in the 2015 Mathematical Calendar)

168 | All even or all odd

What is the sum of all squares less than 1 000 000 that are written with either all even or all odd digits?

Note: 1, 4 and 9 will be included in the sum. The digit 0 is considered even.

(Proposed by Le Coyote. Inspired by a problem from the 30th Mathematical and Logical Games Championship)

169 | Seating arrangements

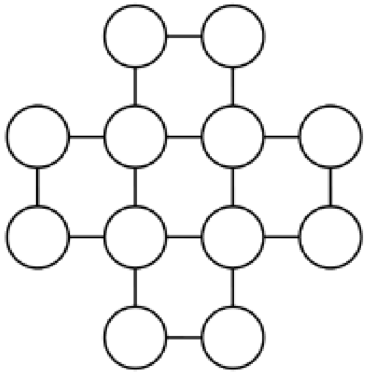
Eleven couples are going to sit around a table. We want to alternate men and women, and we also want no husband to sit next to his wife.

How many seating arrangements satisfy these two conditions?

(Lucas' households)

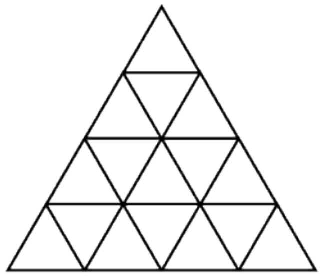
170 | Numbers in a Cross

In how many ways can the integers from 1 to 12 be arranged so that the sum of the numbers at the vertices of each square is always the same?



(Proposed by David Draï. Heavily inspired by the problem from February 11th in the 2015 Mathematical Calendar)

171 | Rhombuses



Let $T(n)$ be the number of rhombuses contained in an equilateral triangle with side length n.

In the figure opposite, $n = 4$ and $T(4) = 21$.

What is the smallest value of n such that T(n) is a multiple of 10^6?

(Proposed by David Draï. Heavily inspired by the problem of February 17th from the 2015 Mathematical Calendar)

172 | Seven fun

An integer is 7-fun if the sum of its digits is divisible by 7. A 7-fun pair consists of two consecutive 7-fun integers.

For example, 69999 and 70000 form a 7-fun pair.

Find all 7-fun pairs for integers less than one million. Give the sum of the smallest integers in these pairs.

(FFJM, Belgian final 2015, problem 11)

173 | Eight-Parade

On a circle, place eight points A, B, C, D, E, F, G and H so that $ABCDEFGH$ forms a regular octagon. Assign them the numbers 0, 2, 0, 6, 2, 0, 1 and 6, respectively.
Starting with the number 0, without lifting the pencil, draw a path of seven different segments that indicates the date June 2, 2016: 0-2-0-6-2-0-1-6.
This forms six angles whose vertices are on the circle. The sum of the six angles cannot be less than a certain value M (in degrees), which is actually reached for some paths.
Finally, we associate an 8-digit number with each path: each digit corresponds to the rank of the letter in the alphabet. For example, we associate the number 62385178 with the path $FBCHEAGH$.
Let P be the sum of the numbers associated with all routes having a minimum angle sum M.

What is the value of $M \times P$?

(Proposed by David Draï. Heavily inspired by problem 16705, "Le Huit-Parade" published in Tangente No. 167, from the Swiss Final of the Mathematical and Logical Games Championship)

174 | Let's place n points on a circle...

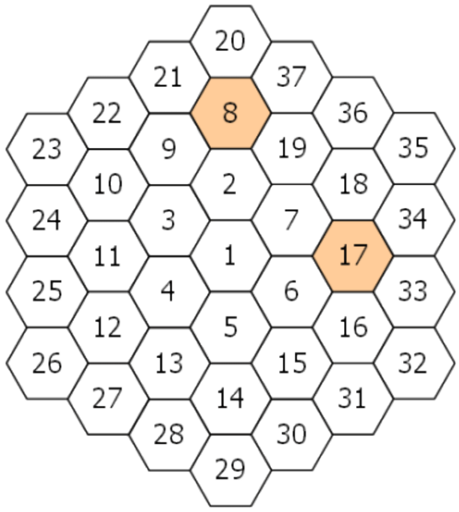
We place n points randomly on a circle, and we denote by $p(n)$ the probability that these n points all belong to the same semicircle.

What is the smallest integer value of $1/p(n)$ greater than 10^4?

(Proposed by David Draï)

175 | Hexagonal tiles

A hexagonal tile numbered 1 is surrounded by a ring of six tiles numbered 2 to 7, with the first tile in the ring placed at noon and turning counterclockwise.
New rings are added in the same way, with the following rings numbered 8 to 19, 20 to 37, 38 to 61, and so on.
The diagram opposite shows the first three rings.



By calculating the difference between the tile numbered n and each of its six neighbors, we will call $\mathrm{DP}(n)$ the number of these differences that are prime numbers.
For example, around tile number 8, the differences are 12, 29, 11, 6, 1 and 13. Therefore, $\mathrm{DP}(8) = 3$.
Similarly, around tile number 17, the differences are 1, 17, 16, 1, 11 and 10, hence $\mathrm{DP}(17) = 2$.
It can be shown that the maximum value of $\mathrm{DP}(n)$ is 3.
When all the tiles for which $\mathrm{DP}(n) = 3$ are listed in ascending order to form a sequence, the 10^{th} tile has the number 271.

Find the number of the 2016^{th} tile in this sequence.

(Problem 128 of Project Euler)

176 | Sea races

Four boats are competing in a regatta. The regatta consists of seven races.
At the end of each race, each crew is awarded one point if they finish the race, plus one point for each boat that finishes behind them.
There are never any ties in a race, but to break any ties in the total points, the rule stipulates that one crew is "ahead" of another if, over the seven races, it finished ahead of the other more often.

At the end of such a regatta, it was found that:

- all boats finished all races,
- crews A, B, and C are tied on points,
- crew A is "ahead" of B, B is "ahead" of C, and C is "ahead" of A!
- the winning crew D finished in all possible places.

Let $S_1, \ldots, S_k (k > 1)$ be the possible total scores for crew D. A regatta is defined as an (ordered) list of the seven rankings in the seven races.

Example: $(ABCD)$, $(BCDA)$, $(CDAB)$, $(DABC)$, $(ACBD)$, $(ADBC)$, $(CABD)$ is a regatta. There are therefore $(4!)^7 = 4\,586\,471\,424$ regattas.
Let N_i be the number of regattas that satisfy all the constraints and for which the total score of crew D is S_i.

What is the sum of $N_i \times S_i$, for i ranging from 1 to k?

(Proposed by David Draï, based on a problem entitled "Courses en mer" published in issue 168 of Tangente)

177 | So many 0s…

$n!$ means $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.

How many 0s does $n!$ end with for $n = 100\,000\,000$?

(Proposed by Over_score)

178 | T9

Keypad cell phones offer a text input method called T9. The 26 letters of the alphabet are distributed across keys 2 to 9, as shown in the figure opposite.

To enter a word, simply enter the corresponding sequence of numbers. For example, “Turing” will be coded as 887464.
Of course, several words may correspond to the same sequence. For example, the sequence 36724368 encodes the six words “dopaient”, “doraient”, “dosaient”, “enragent”, “enraient”, “foraient”.

The file **dico.txt**[1] contains 323 471 unaccented French words.

Using this dictionary, find the T9 sequence that encodes the most French words.

(Proposed by Le Coyote)

[1] This text file *dico.txt* is also attached to this PDF.

179 | How many grains?

A cryptarithm is a numerical and logical puzzle consisting of a mathematical equation in which letters represent numbers to be found. Two different letters represent two different numbers, and two different numbers are always represented by two different letters. No non-zero number begins with a zero.

```
  GRAIN
+ GRAIN
+ GRAIN
+  ...
-------
= SABLE
```

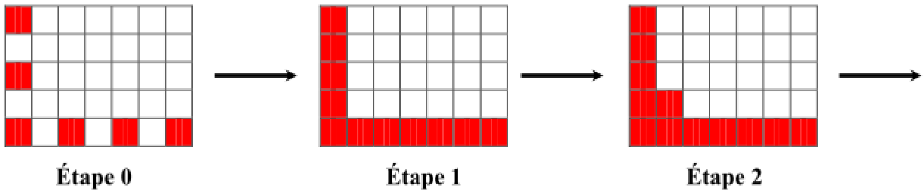
What is the maximum number of `GRAIN` that can be added together so that the cryptarithm has at least one solution? Give the sum of the possible `SABLE` values for this maximum number of `GRAIN` as your answer.

(From "121 rapidos et autres énigmes mathématiques")

180 | Contagion

At the beginning (step 0), before launching a computer program, a certain number of squares in a 5×7 grid are “infected”.
Then, the computer simulates contagion.
Step by step, each uninfected square adjacent on one side to exactly two infected squares is infected in turn (unfortunately, the infected squares remain infected!).
There is a nice demonstration* of the fact that at least 6 non-adjacent squares must be “infected” in pairs at the beginning for all 35 squares in the grid to be infected after a certain number of steps.
In this case, total infection—if it occurs—necessarily takes place in at most 29 steps.

Example[1]



The grid above allows total contamination in 10 steps.
Let $N(k)$ be the number of starting grids with exactly 6 non-adjacent contaminated squares, leading to total contamination in k steps.

What is the sum of $k \times N(k)$, for $k = 1, \ldots, 29$?

* It is easy to see that during the contamination process, the total perimeter of the contaminated area remains unchanged. Since the perimeter of the grid is 24 and that of a square is 4, at least $24/4 = 6$ (non-adjacent) squares are needed to achieve complete contamination.

(Proposed by David Draï)

[1] “Étape” means “Step”.

181 | Pandigital endpoints

The Fibonacci sequence is defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 1$ and $F_2 = 1$.
It turns out that F_{541}, which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9-pandigital (all digits from 1 to 9 are present, but not necessarily in order).
F_{2749}, which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9-pandigital.

Let F_k be the first Fibonacci number for which the first nine digits and the last nine digits are 1-9-pandigital. Find k.

(Problem 104 of Project Euler)

182 | Reversible numbers

Certain positive integers n have the property that the sum $[n + \mathrm{mirror}(n)]$ consists entirely of odd digits.[1]
For example, $36 + 63 = 99 \wedge 409 + 904 = 1313$. We will call these numbers "reversible"; therefore, 36, 63, 409 and 904 are reversible.
Zeros are not allowed in the sum, nor at the end of the number n. For example, 10 is not a reversible number, even though $10 + 01 = 11$.
There are 120 reversible numbers less than one thousand.

How many reversible numbers are there less than ten billion?

(Problem 145 of Project Euler)

[1] $\mathrm{mirror}(n)$ means to reverse the (decimal) digits of the number n.

183 | All 7s

This division is correct. All the 7s that appear are given. All other numbers have been erased.

```
_ 7 _ _ 7 _ | _ 7
            |----------
  _ _       | _ _ _ _ _
    7 _     |
    _ _ 7   |
        _ _ |
          _ |
```

There are several solutions…

Give the sum of all possible dividends.

(Lewis Carroll Trophy)

184 | Heterographs

The file **dico.txt**[1] contains 323 471 French words without accents.
A heterograph is a word in which each letter appears at most once. Here are some examples of such words:

a, je, cage, clapoter, hibou, émir, va-nu-pieds (hyphens are not letters), …

However, these words do not satisfy the constraint:

enragé (accents do not count), capable, …

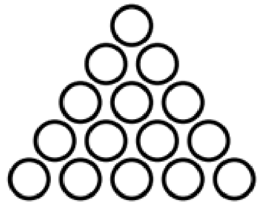
How many words in the dico.txt dictionary are heterograms?

(Proposed by Le Coyote)

[1] This text file *dico.txt* is also attached to this PDF.

185 | Pyramid

The number contained in each circle is the sum of the numbers contained in the two circles directly below it.
All numbers are strictly positive integers that are distinct from each other.



Assuming the leftmost number is less than the rightmost number, what numbers must be placed on the bottom row to minimize the sum of the 15 numbers?

The answer will be the concatenation of these five numbers read from left to right.

(Proposed by David Draï)

186 | Gemellograms

The file **dico.txt**[1] contains 323 471 French words without accents.
We will call a "gemellogram" (a word invented for this purpose) a word in which each letter appears exactly twice. Here are a few examples of such words:

joujou, ionisons, même or mémé (accents do not count), kif-kif (hyphens do not count)…

However, these words do not satisfy the constraint:

maman, passe-passe

How many words in the dico.txt dictionary are gemellograms?

(Proposed by Le Coyote)

[1] This text file *dico.txt* is also attached to this PDF.

187 | Divisibility

Two different integers are randomly drawn from the interval $[1, 10000]$.

What is the probability that the larger number is a multiple of the smaller number?

When calculating this probability, you will obtain a repeating decimal. Give the period[1] as your answer.

(Proposed by Le Coyote)

[1] The repeating decimal digits.

188 | Sum of cubes

Let S be the set of strictly positive integers that can be decomposed into a sum of cubes of odd integers, each pair of which is distinct.

Examples: 125 and 2568 are in S because $5^3 = 125$ and $1^3 + 3^3 + 7^3 + 1^3 = 2568$.

For any integer i in $\{0, 1, ..., 287\}$, we call s_i the smallest element of S such that $s_i = i[288]$ (i.e., the remainder of the Euclidean division of s_i by 288 is i).
We therefore have $s_0 = 17568$, $s_1 = 1$, $s_2 = 3746$, …

What is the sum of the s_i values for i ranging from 0 to 287?

P.S. We chose 288 because $2016 = 7 \times 288$.

(Proposed by David Draï)

189 | Sum of integer fractions

In the equation below, x, y, and n are positive integers.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$$

For $n=4$, there are exactly 3 distinct solutions:

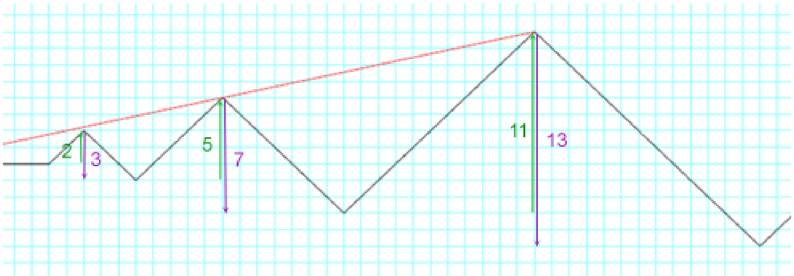
$$\frac{1}{5}+\frac{1}{20}=\frac{1}{4}$$
$$\frac{1}{6}+\frac{1}{12}=\frac{1}{4}$$
$$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

What is the smallest value of n for which the number of distinct solutions is greater than or equal to 2016?

(Problem 108 of Project Euler)

190 | Mountains of prime numbers

"Mountains" with slopes of exactly 45° and heights governed by prime numbers pn follow one another to form a mountain range. The height of the left side of the k^{th} mountain is p_{2k-1}, while that of the right side is p_{2k}. The first mountains in this range are shown below.



Tenzing sets out to climb these mountains one after the other, starting with the lowest. At the top of each peak, he looks back and counts how many previously conquered peaks he can see.
In the example below, the line of sight from the third peak, drawn in red, shows that he can only see the second peak. Similarly, from the 9th peak, he can only see three peaks: the 5th, 7th and 8th.
Let $P(k)$ be the number of peaks visible when looking back from the k^{th} peak. We have $P(3) = 1$ and $P(9) = 3$. Furthermore, the sum of $P(k)$, for k ranging from 1 to 100, is 227.

What is the sum of $P(k)$, for k ranging from 1 to 10 000?

(Proposed by David Draï. Inspired by problem 569 pf Project Euler)

191 | Bouncing numbers

When reading a positive integer from left to right, if no digit is larger than the digit to its left, it is called an "ascending number." Example: 134468.

Similarly, if no digit is larger than the digit to its right, it is called a "descending number." Example: 66420.

We will call a positive integer that is neither ascending nor descending a "bouncing number." For example, 155349.

Clearly, there cannot be bouncing numbers less than 100, but slightly more than half of the numbers less than 1000 are bouncing numbers (525). In fact, the smallest number for which the proportion of bouncing numbers first reaches 50% is 538.

Bounced numbers become increasingly frequent, and the proportion of bouncing numbers exceeds 90% starting at 21780.

Find the smallest positive integer for which the proportion of bouncing numbers first reaches or exceeds 99.9%.

(Problem 112 of Project Euler)

192 | Squares everywhere

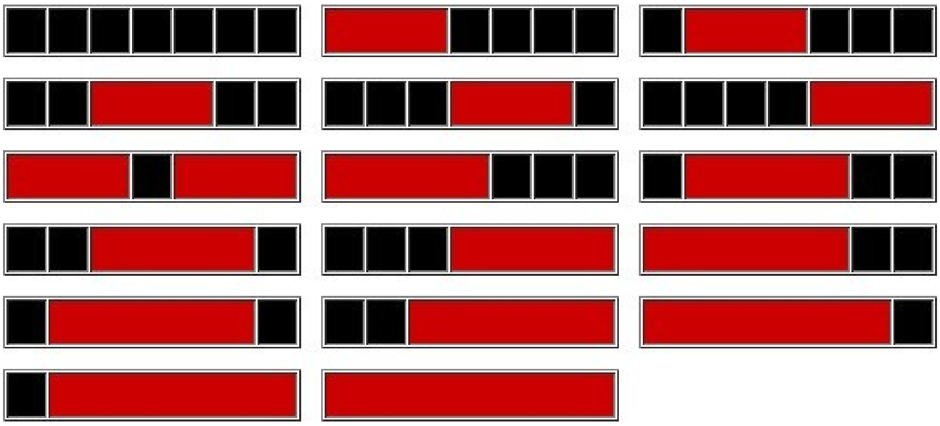
Let $n = x + y + z$, where $x > y > z > 0$ are integers.

Find the smallest n such that $x + y$, $x - y$, $x + z$, $x - z$, $y + z$ and $y - z$ are all squares.

(Problem 142 of Project Euler)

193 | Santa Claus' Gifts

A sled measuring seven units in length contains red packages with a minimum length of 3 units placed on it. Two packages must be separated by at least one unit in length. There are exactly 17 ways to fill the sled:



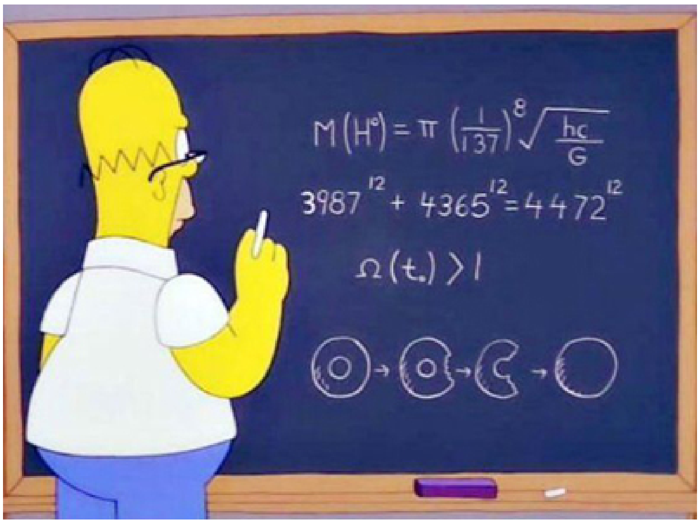
How many ways can a sled measuring 60 units in length be filled?

NOTE: Although the above example does not lend itself to this, it is permissible to mix package sizes. For example, on a sleigh measuring eight units in length, you can use the arrangement 3-(1)-4 where (1) is an empty space.

(Problem 114 of Project Euler)

194 | Homer's theorem

In episode 2 of season 10 of The Simpsons (1998), entitled "Homer's Last Invention", the following table can be seen:



The line $3987^{12} + 4365^{12} = 4472^{12}$ is a false counterexample to Fermat's theorem. Indeed, the left-hand side is $63976656349698612616236230953154487896987106$, while the right-hand side is $63976656348486725806862358322168575784124416$, a difference of $1211886809373872630985912112862690$. However, the relative error is small enough (1.9×10^{-11}) that a standard calculator would consider these two terms to be equal. Note that the relative difference was calculated by dividing the difference by the left-hand term.

At the start of 2017, we would like to find the triplet (a, b, c), with a, b and c all different, such that $a^{17} + b^{17} = c^{17}$ has the smallest relative error, with the numbers a and b between 100 and 10 000. Give your answer as the product $a \times b$.

(Inspired by a post by Dr. Goulu: "« Contre-exemples » au théorème de Fermat-Wiles")

195 | 987654321

In this puzzle, you have to insert $+$ or $-$ signs between some of the numbers from 9 to 1 in order to obtain an exact equal sum.
With seven addition and subtraction signs, you can write, for example: $9+8+76+5+4-3+2-1=100$. But there are 14 other ways to write 100, and it is possible to use fewer signs. For example, with only four signs, we get:

$$98-76+54+3+21=100.$$

We want to write as many consecutive natural numbers as possible in this way, using a minimum number of symbols. Here is a possible start:

- $9+8-76-5+43+21=0$
- $98-76-54+32+1=1$
- $9+87-65+4-32-1=2$
- etc.

Let A be the first integer that cannot be written in this way. The sequence will go from 0 to $A-1$.
Let B be the total number of operation signs used to write all integers less than A.
Let C be the integer less than A that can be obtained with the fewest operational signs.
Let $D_1, D_2, \ldots, D_n$ be the integers less than A that can only be written in one way (in the sense that 100 can be written in 15 ways…).

What is the value of $A \times B \times C \times D_1 \times D_2 \times \cdots \times D_n$?

(Proposed by David Draï, based on "The Numerology of Dr Matrix", Martin Gardner, Simon and Schuster, 1967)

196 | Odd-digit numbers

Consider all five-digit numbers that can be composed using each odd digit exactly once: 13579, 13597, 13759, …, 97531.

What is the sum of all these numbers?

(www.enygmatik.com)

197 I am the one who is

Find the positive integers less than one billion where the first digit is the number of 0s in the number, the second is the number of 1s in the number, and so on until the last digit of the number.
The first of these numbers is 1210.

What is the sum of all these numbers?

(www.prise2tete.fr)

198 | Forbidden squares

There are infinite sets of natural numbers such that the sum of any number of their (distinct) elements is never the square of an integer. This is the case for $\{2, 8, 32, ..., 2^{2k+1}, ...\}$, for example.

A natural way to construct such a set step by step is as follows (by recurrence):
We construct the strictly increasing sequence $(u_n)_{n\geq 1}$ defined by the fact that for any non-zero integer n, u_n is the smallest natural number such that the sum of the elements of any non-empty subset of $(u_1, u_2, ..., u_n)$ is never a square of an integer.

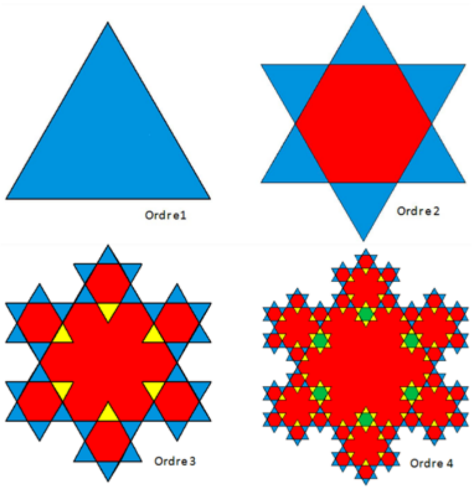
What is the sum of the first 14 terms of this sequence?

(Proposed by David Draï, based on "Carrés interdits" in "121 rapidos et autres énigmes mathématiques", Michel Criton, POLE, 2007)

199 | Snowflakes

An n-order snowflake is formed by superimposing an equilateral triangle (rotated 180 degrees) on each equilateral triangle of the same size in an $n-1$ order snowflake. An 1-order snowflake is an equilateral triangle.

Some areas of the snowflake are superimposed several times. In the image opposite, blue represents areas of thickness 1, red those of thickness 2, yellow those of thickness 3, and so on…



For a snowflake of order n, we call $A(n)$ the number of blue triangles and $B(n)$ the number of yellow triangles.
Let $G(n)$ be the greatest common divisor (GCD) of $A(n)$ and $B(n)$.

For example, $A(3) = 30$, $B(3) = 6$ and $G(3) = 6$.
$A(11) = 3\,027\,630$, $B(11) = 19\,862\,070$ and $G(11) = 30$.

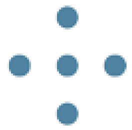
Furthermore, $G(500) = 186$.

What is the sum of $G(n)$ for n ranging from 3 to 10 000?

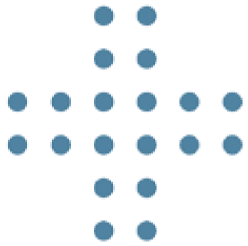
(Proposed by David Draï, based on problem 570 of Project Euler)

200 | Cross squares

A single square can be formed whose four vertices are points on this cross with side length 1.



Similarly, 21 squares can be formed with this cross with side length 2.



To the side length n of the cross, we associate the number of squares $C(n)$ that can be formed using the points of the cross as vertices. Thus, $C(1) = 1$ and $C(2) = 21$.

What is the sum of the $C(n)$ for n ranging from 1 to 20?

(Proposed by David Draï, based on the April 8th Problem from the 2014 Mathematical Calendar)

201 Lucky numbers

Lucky number — A natural number determined in 1956 by Polish mathematician Stanislaw Ulam (1909-1984) by applying the principle of Eratosthenes' sieve.

We start by removing the even numbers. Since 3 remains after 1, which is considered lucky, the third number out of three among those remaining is removed. Next, the smallest number not yet removed is 7. The seventh number out of seven among those remaining is then removed, and so on, with the smallest remaining number always indicating the rank of the numbers to be removed.

There are 23 lucky numbers between 1 and 100:
1 3 7 9 13 15 21 25 31 33 37 43 49 51 63 67 69 73 75 79 87 93 99.

How many lucky numbers are there in the range $[1, 100\,000]$?

(Proposed by Le Coyote)

202 | Twins

Write the numbers $1, 2, \ldots, 37$ on the first row of a table with 28 rows and 37 columns. Then, on the second row, write $38, \ldots, 74$, and so on (from left to right).
Also write the numbers $1, 2, \ldots, 28$ in the first column. Then, in the second column, write $29, \ldots, 56$, and so on (from top to bottom).
In some cells, the two numbers will be identical. We will call them twins.

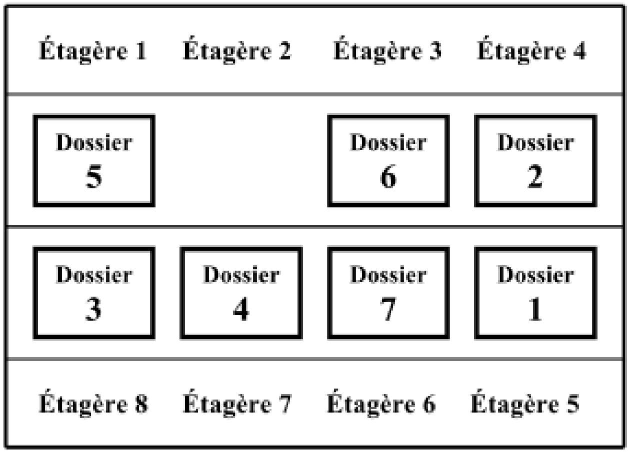
What is the sum of the twin numbers? Each twin number is counted only once.

(2017 Mathematical Calendar, Presses universitaires de Strasbourg)

203 | Maya's mess

Maya, the secretary, has messed up all the files.

Each file number should correspond to the number of the shelf it is on. We need to put them back in place quickly before Mr. Boss arrives![1]



But be careful! The files are very heavy, and Maya can only move one at a time, pushing it to a neighboring shelf (to the left, right, front, or back) as long as that shelf is empty. Maya found the most economical solution (in terms of time and energy) since she managed to do it in a minimum number of moves, A.

Let p be the integer part of $(A+1)/2$. In the initial situation (see diagram opposite, top view), the number 50621743 can be read by following the ascending order of the shelves and assigning the number 0 to the empty shelf.

Similarly, what 8-digit number B can be read after the p-th move? We will enter the value of the product $A \times B$.

(Proposed by David Draï, inspired by Problem No. 1 of January 19th, 1997, from the Mathematical Games of "Monde" by Élisabeth BUSSER and Gilles COHEN — Éditions POLE)

[1] "Étagère" means "Shelf"; "Dossier" means "File".

204 | Square sums

How many natural numbers less than or equal to 10^{13} have a sum of digits that is the square of an integer (including 0, 1, 4 and 9)?

For example, the number 27 satisfies this property because $2 + 7 = 9 = 3^2$.

(Proposed by David Draï, based on the September 23rd puzzle from the 2016 Mathematical Calendar)

205 | Parallelograms

We draw two parallel lines in a first direction, then three parallel lines in a second direction (different from the first), then four parallel lines in a third direction (different from the previous two).
At this stage, we have drawn a total of 9 lines and we can see 27 parallelograms drawn on the figure. We then continue constructing the figure using the same process.

What is the minimum number of lines that must be drawn for the number of parallelograms drawn to be a multiple of one million?

(Proposed by David Draï, inspired by a problem from the Mathematical and Logical Games Championship)

206 Prime numbers in arithmetic progression

For any integer n greater than 1, we define the "primorial" of n, denoted $P(n)$, as the product of all prime numbers less than or equal to n. Thus, $P(10) = 210$.

- $5 \to 11 \to 17 \to 23 \to 29$ is the longest arithmetic sequence of prime numbers less than 100 (ratio = 6)
- $7 \to 157 \to 307 \to 457 \to 607 \to 757 \to 907$ is the longest arithmetic sequence of prime numbers less than 1000 (ratio = 150)
- $199 \to 409 \to 619 \to 829 \to 1\,039 \to 1\,249 \to 1\,459 \to 1\,669 \to 1\,879 \to 2\,089$ is the longest arithmetic sequence of prime numbers less than 10 000 (common difference = 210).

Regarding the ratios of these arithmetic sequences, it has been proven that if the sequence is of length k, then the ratio is a multiple of $P(k)$, unless k is prime and the sequence begins at k.
Since 2004, we have known that for any integer n, there is at least one arithmetic sequence of prime numbers greater than or equal to n. But to date, we do not know of any such sequences longer than 26 …[1]

What is the longest arithmetic sequence of prime numbers less than 1 million, given that its first term is neither 13 nor 17? We will calculate the sum of the terms of this sequence.

(Proposed by David Draï)

[1] *(Commented by **drai.david**, 03/27/2017)* The idea initially came to me from the article "Les aiguilles tournent, le mystère demeure" by Vincent Borrelli and Jean-Luc Rullière, published in: Dossier Pour la science n°91 (April–June 2016): "Quand les Maths prennent forme". I then supplemented my information at: **https://fr.wikipedia.org/wiki/Théorème_de_Green-Tao**. And there are a host of results related to this problem on this page: **http://primerecords.dk/aprecords.htm**.

207 | Regions

By drawing a line, we divide the plane into two regions.
By drawing another line intersecting the first, the plane is divided into four regions.
By drawing a third line intersecting the first two at two different points, we obtain seven regions.
Let R_n be the sequence of the number of regions of the plane defined by n intersecting lines 2 by 2 and non-concurrent 3 by 3. Thus, for example, $R_3 = 7$.

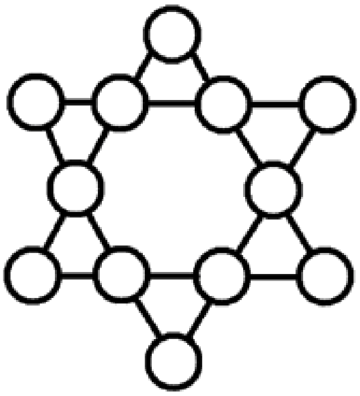
It seems that the probability that a randomly chosen prime number does not divide any element of R_n is very close to $1/2$.
Let $S(k)$ be the sum of prime numbers less than or equal to k that do not divide any element of R_n.
We are given $S(10) = 3 + 5 = 8$, $S(100) = 638$ and $S(1000) = 38\,582$.

What is the value of $S(10\,000)$?

(Proposed by David Draï, based on "100 jeux de logique", Larousse, 2014)

208 | Star of David

How many ways can the integers from 1 to 12 be placed in the circles so that the sum of the numbers in each straight line is the same?



(Problem of June 19th from the 2015 Mathematical Calendar)

209 | Sum or double?

Jean writes the numbers 1 and then 2 on the board.
He then writes a number that is either double the previous number or the sum of the two previous numbers.

Example of a possible sequence: 1, 2, 3, 6, 12, 18, ...

If Jean writes 50 numbers following this rule and the last number written is odd, what is the largest number that can be written?

(Proposed by David Draï, based on the September 19th puzzle from the 2016 Mathematical Calendar)

210 | Cyclic numbers

A cyclic number of length n is a natural number whose circular permutations of digits correspond to the first $(n-1)$ multiples of the number. The best known (and smallest) is 142 857, of length 6, because:

$$\begin{aligned} 142\,857 \times 2 &= 285\,714 \\ 142\,857 \times 3 &= 428\,571 \\ 142\,857 \times 4 &= 571\,428 \\ 142\,857 \times 5 &= 714\,285 \\ 142\,857 \times 6 &= 857\,142 \end{aligned}$$

If zeros are not allowed at the beginning of numbers, then 142 857 is the only cyclic number in base 10. However, if they are allowed, the list of cyclic numbers continues as follows:

0588235294117647 (16 digits); 052631578947368421 (18 digits); …

A prime number p is said to be long if the period of the decimal expansion of $1/p$ is of length $p-1$. For example: $1/7 = 0.142857142857\cdots$ The period (142857) is of length 6, so 7 is a long prime.

Note: The period of $1/p$ always begins with the decimal expansion's tenths digit.

Now, extraordinarily, the list of cyclic numbers coincides exactly with the list of periods of long primes! Indeed, the second long prime is 17, and we have $1/17 = 0.0588235294117647\,0588235294117647\cdots$

Let $S(n)$ be the sum of all cyclic numbers with at most n digits. Thus, $S(10) = 142\,857$ and $S(20) = 53\,219\,814\,241\,628\,925$.

What are the last 10 digits of $S(5000)$?

(Proposed by David Draï)

211 All primes!

Let p and q be two prime numbers less than 900 billion.

If $p+6$, $p+10$, $q+4$, $q+10$ and $p+q+1$ are all primes, what is the largest value that $p+q$ can take?

(Proposed by David Draï, based on the June 11th puzzle from the 2015 Mathematical Calendar)

212 Triangles from father to son

Consider the following geometric transformation: given an initial triangle (the "father triangle"), take the feet of the bisectors of the old triangle as the vertices of the new triangle (the "son triangle").
This transformation has the particularity of attenuating the differences between the angles of the initial triangle. Thus, if we iterate the process, two triangles that are initially different in shape will, via this transformation, lead to triangles that are increasingly "similar" and close to the common equilateral limit shape. This fact, far from being obvious, was only proven in 2006.
On the other hand, it is clear that the sequence of iterated triangles converges to a point G. However, this point cannot be described explicitly (except in the case of the equilateral triangle, of course!).
Let us now consider the simplest non-trivial case: let ABC be an initial isosceles triangle with right angle at A, and G the point of convergence of the sequence of iterated triangles.

What is the value of AG/AB, rounded to 10^{-9}? Enter the first 9 decimal places of the result obtained.

(Proposed by David Draï, based on the article "Le triangle: une porte d'entrée vers le chaos" (Grégoire Nicollier), published in Dossier Pour la science n°91 (April–June 2016): "Quand les Maths prennent forme")

213 | Keith numbers

A Keith number is an n-digit number K with the following property: starting with numbers composed of each of the n digits of K, a Fibonacci-like sequence is formed by calculating the sum of the last n numbers in the sequence to determine the next number. If this sequence at some point yields the number K, that number is called a Keith number.

Example: $K = 197$
$1, 9, 7, 17(= 1 + 9 + 7), 33(= 9 + 7 + 17), 57(= 7 + 17 + 33), 107(= 17 + 33 + 57), 197(= 33 + 57 + 107)$

197 is therefore a Keith number. The length of the sequence generated by 197 is 8.

Let all Keith numbers between 10 and $1\,000\,000$ be given. Find the sum of all these numbers multiplied by the length of their sequence.

(Proposed by Le Coyote)

214 Highly divisible pandigitals

A 10-pandigital integer consists of 10 different digits. For example: 1234567890 or 7605483291.

Give the sum of the 10-pandigital integers that are divisible by all integers less than 19.

(Proposed by Le Coyote)

215 | The mystery triangle

Given a triangle, let's take the symmetrical point of each vertex in relation to the opposite side or its extension. This gives us the vertices of a new triangle: the "reflected" triangle.
If we repeat this process, we obtain a sequence of triangles whose evolution depends heavily on the shape of the initial triangle, hence the interest in studying it.
Any triangle whose reflected triangle is similar to it is called a "fixed point" of this geometric transformation. Thus, there are only four types of "fixed points" for this transformation:

- Flattened triangles: it can also be noted that isosceles triangles with a main angle of 120° produce flattened triangles.
- The equilateral triangle, an "attractive" fixed point.
- the "heptagonal" triangle, whose angles are equal to fractions of $1/7$, $2/7$, and $4/7$ of 180°, and which is obtained from a regular heptagon by taking the first, second and fourth vertices.
- The "mystery" triangle, whose three (different) angles are probably not rational fractions of 180°.

However, we do have two clues about this last triangle: like the "heptagonal" triangle, it has an obtuse angle and is a "repulsive" fixed point, meaning that a triangle with a very similar shape produces a reflected triangle with a less similar shape.

What is the value in degrees, rounded to the nearest 10^{-5}, of each of the two acute angles of the "mystery" triangle? Enter the product of these two values, omitting the decimal point. The expected result is the exact product of two approximate values, not an approximate value of the best possible product.

(Proposed by David Draï. Based on the article "Le triangle: une porte d'entrée vers le chaos" (Grégoire Nicollier), published in Dossier Pour la science n°91 (April–June 2016): "Quand les Maths prennent forme")

216 | Square palindromes

A palindrome is a number that reads the same from left to right as it does from right to left, such as 12321.

We calculate the sum of the squares of the natural numbers, in order: $0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \cdots$
We stop the calculation as soon as, after adding the square of a palindromic number with at least two digits, we obtain a sum that is also a palindromic number.

What will this sum be?

(2017 individual quarterfinals of the 31st FFJM Championship)

217 | Beetles

We have a ruler graduated in units of 1000. The graduations are numbered from left to right, from 0 to 1000.
On each graduation of the ruler, corresponding to a prime number, we place a beetle (which is reduced to a point of zero dimension).
Each beetle moves at a speed of one unit per second. The first beetle (placed on 2) moves to the right, the second (placed on 3) to the left, the third (placed on 5) to the right, and so on, alternately, until the last beetle placed on 997, which moves to the left.
When a beetle encounters another beetle, both beetles change direction and continue to travel along the ruler, still at the same speed of one unit per second.
When a beetle reaches the end of the ruler, it falls off and moves away from it.

How many seconds will it take for all the beetles to fall off the ruler?

(Proposed by Le Coyote)

218 | 2017

We are looking for integers whose square root has the digits 2017 (in that order) as the first digits after the decimal point. For example, the square root of 10 858 is 104.2017274···

How many integers between 1 and 10^{10} satisfy this condition?

(Proposed by Le Coyote)

219 | Palindromes of averages

The pair of numbers $(32, 98)$ has an interesting feature: their arithmetic mean (65) and geometric mean (56) can be derived from each other by reversing the order of their digits.

There are two pairs (a, b) and (c, d), composed of different 4-digit numbers, that have the same property.

Give your answer as $ab + cd$.

(Proposed by Le Coyote)

220 | Squares as sums of consecutive integers

Let F be the function that, for any strictly positive integer n, associates the smallest natural number whose square can be written as the sum of n strictly positive consecutive integers, if it exists, and 0 otherwise.

Examples
$F(1) = 1$, because $1^2 = 1$;
$F(2) = 3$, because $3^2 = 4 + 5$;
$F(3) = 3$, because $3^2 = 2 + 3 + 4$;
$F(4) = 0$, because no square can be written as the sum of 4 consecutive integers;
$F(6) = 9$, because $9^2 = 11 + 12 + 13 + 14 + 15 + 16$;
$F(8) = 6$, because $6^2 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$;
$F(18) = 15$, because $15^2 = 4 + 5 + \cdots + 21$.

The sum of $F(n)$ for n ranging from 1 to 1000 is 448 612.

What is the sum of $F(n)$ for n ranging from 1 to 1 million?

(Proposed by David Draï. Freely inspired by the problem of January 14th from the 2015 Mathematical Calendar)

221 Square + constant = square

Let F be the function that, for any non-zero integer n, associates the sum of natural integers x such that $x^2 + n$ is a square integer.

Examples
$F(98) = 0$, because there is no natural integer x such that $x^2 + 98$ is a square.
$F(99) = 49 + 15 + 1 = 65$, because $49^2 + 99$, $15^2 + 99$ and $1^2 + 99$ are the only three squares (50^2, 18^2 and 10^2) of the form $x^2 + 99$.

What is the sum of $F(n)$ for n ranging from 1 to 1 million?

(Proposed by David Draï, based on the March 14 problem from the 2017 Mathematical Calendar)

222 | The row of lamps

Consider a row of 8 000 lamps. Initially, only the one on the far left is lit.
Then, every second, the following operation is performed: each lamp changes state (lit or unlit) if the one to its left was lit a second earlier. The leftmost lamp remains on at all times. This operation is instantaneous. The process stops when the lamp on the far right turns on for the first time.

How many lamps are on at that point?

(Mathematical and Logic Games Championship)

223 | Dividing the disk

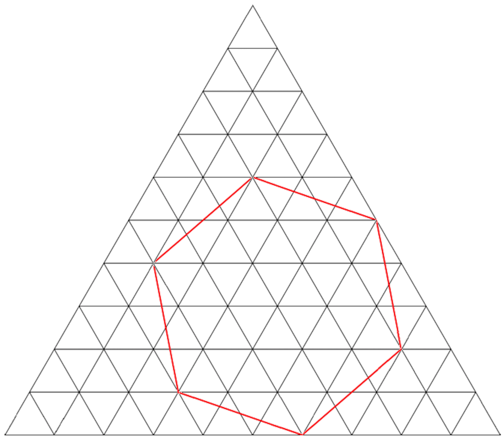
When two points are placed on a circle, connecting these two points divides the disk into two regions.
With three points, we obtain four regions.
With four points, we have a maximum of eight regions.
With five points, we get a maximum of sixteen regions.
But with six points or more, we leave the powers of two behind. With 6 points, we get a maximum of 31 regions, and with 7 points, 57 regions.

How many regions can we get with a maximum of 100 points?

(Mathematical Carnival, Martin Gardner, Vintage, 1977)

224 Counting hexagons

An equilateral triangle with sides of length $n > 2$ is divided into n^2 equilateral triangles with sides of length 1, as shown in the diagram below, where $n = 10$.



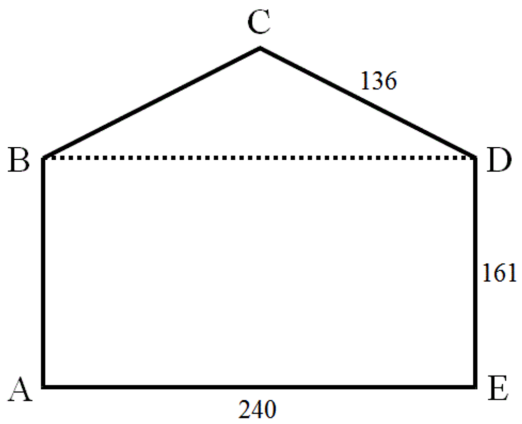
The vertices of these triangles form a triangular network of $(n+1)(n+2)/2$ points. Let $H(n)$ be the number of regular hexagons that can be formed by connecting 6 of these points.
For example, $H(3) = 1$, $H(6) = 12$ and $H(20) = 966$.

What is the sum of all $H(n)$ for n ranging from 3 to 1 000?

(Proposed by David Draï, inspired by Project Euler 577)

225 Heron's envelopes

An envelope is a pentagon composed of an isosceles triangle (the flap) placed above a rectangle. Here is an example of an envelope with sides of integer lengths:



Note that to form an envelope, the height from C of triangle BCD must be less than the height AB of rectangle $ABDE$. Furthermore, AE may well be less than or equal to AB.
In the envelope in the example above, not only are all the sides integers, but so are all the diagonals (AC, AD, BD, BE and CE). We will call an envelope with such properties a Heron envelope.
Let $S(p)$ be the sum of the perimeters of all Heron envelopes with a perimeter less than or equal to p.
We are given: $S(500) = 488$, $S(1000) = 5386$, $S(1500) = 11\,460$, $S(2000) = 25\,282$ and $S(2500) = 43\,864$.

What is the value of $S(25\,000)$?

(Proposed by David Draï, inspired by Project Euler 583)

226 | Cubes and factorials

6 cubes divide $3! \times 5! \times 7!$: $1, 8, 27, 64, 216, 1728$.

How many cubes divide $n = 3! \times 5! \times 7! \times 9! \times \cdots \times 4999!$?

We will take the twelve digits before the zeros that end the number n.

(Proposed by David Draï)

227 | Distinct differences

Consider a grid with 3 rows and 13 columns.
In the first row, the integers from 1 to 13 are arranged in ascending order.
In the second row, these same integers are arranged in any order.
In the third row, write the absolute value of the difference between the two integers in each column.
Let N be the number of ways to fill the second row so that 13 distinct integers appear in the third row.
Let $N(k)$ be the number of solutions for which the integer k in the first and second rows appear in the same column.
Thus, N is equal to the sum of $N(k)$ for k ranging from 1 to 13.

What is the sum of $k \times N(k)$ for k ranging from 1 to 13?

(Proposed by David Draï)

228 | Hyperball

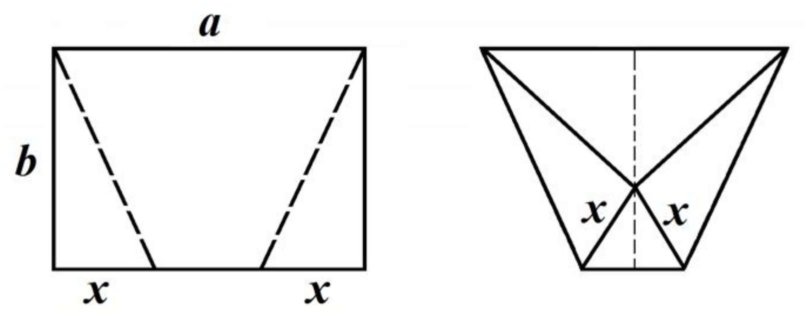
Let $T(r)$ be the number of quadruplets of integers (x, y, z, t) such that: $x^2 + y^2 + z^2 + t^2 \leq r^2$.
In other words, $T(r)$ is the number of points with integer coordinates contained in a 4-dimensional hyperball with radius r.
Given that: $T(0) = 1, T(1) = 9, T(2) = 89, T(5) = 3121$ and $T(100) = 493\,490\,641$.

What is the value of T(500)?

(Proposed by David Draï, based on Project Euler 596)

229 | Folding the corners

a and b are two non-zero natural numbers such that $a < 2b$.
Let's fold the corners of the rectangle in the figure on the left so that the two lower vertices coincide. This gives us the figure on the right.



How many pairs of integers (a, b) exist with $b \leq 10^6$, such that x is also an integer?

(Proposed by David Draï, freely inspired by the July 27th puzzle from the 2017 Mathematical Calendar)

230 | Random sort

A deck of n cards, numbered from 1 to n, is shuffled randomly so that each permutation is equally likely. These cards must be sorted in ascending order using the following technique:

1. We observe the sequence of cards. If it is already sorted, then there is no need to continue the action. Otherwise, if there are sequences of cards arranged in ascending order without any "gaps", then these sequences form groups of cards that are stapled together.
   For example, with 7 cards initially in the order 4 1 2 3 7 5 6, cards 1, 2 and 3 will be stapled together, as will cards 5 and 6 (this gives us a group of 3 cards, a group of 2 cards, and 2 isolated cards).
2. The cards are then "shuffled" by throwing them into the air (the stapled cards obviously remain together). The cards (or stapled card packs) are then picked up at random. We assume that all possible picks are equally likely, despite the fact that some cards are alone and others are grouped together.
3. Steps 1 and 2 are repeated until all the cards are sorted.

Let $S(n)$ be the average number of tosses required to sort all the cards (this is therefore an expectation). Since the order is checked before the first toss, we have $S(1) = 0$.
We also give $S(2) = 1$, $S(3) = 7/3$, $S(4) = 47/13$ and $S(5) = 4213/871$.

What is the value of $S(10)$? Enter the first 11 digits of the answer, omitting the decimal point.

(Proposed by David Draï, based on Project Euler 595)

231 Greatest proper divisor

Let F be the function that, for any strictly positive integer n, associates the number of integers p whose greatest proper divisor (i.e., distinct from p) is n.
For example, $F(7) = 4$, because 7 is the greatest proper divisor of 14, 21, 35 and 49.
We also have $F(2017) = 306$.

What is the sum of $F(n)$ for n ranging from 2 to 1 million?

(Proposed by David Draï, freely inspired by the January 9th problem from the 2017 Mathematical Calendar)

232 | Sums of consecutive odd numbers

Let F be the function that, for any strictly positive integer n, associates the number of ways to write n as the sum of at least two strictly positive consecutive odd integers.

Examples:

- $F(64) = 3$ because $64 = 31 + 33 = 13 + 15 + 17 + 19 = 1 + 3 + \cdots + 15$
- $F(360) = 6$
- $F(4725) = 11$.

What is the sum of $F(n)$ for n ranging from 1 to 1 million?

(Proposed by David Draï, freely inspired by the January 2nd problem from the 2017 Mathematical Calendar)

233 | Quadrilateral with maximum area

Let F be the function that, for any triplet of integers (a, b, p), associates the maximum area of a quadrilateral with diagonals of length a and b and perimeter p, where $0 < b \leq a$ and $2a < p < a + b + \sqrt{a^2 + b^2}$ *.

Here are some values of F:

$$
\begin{aligned}
&F(3, 1, 7) = 3/2, \\
&F(3, 2, 7) = \sqrt{429}/8, \\
&F(3, 2, 8) = 3, \\
&F(14, 11, 32) = 45, \\
&F(19, 14, 42) = 70, \\
&F(24, 15, 50) = 70.
\end{aligned}
$$

What is the sum of all distinct integer values of F obtained with: $1000 \leq b < a \leq 1200$, $2400 < p \leq 2800$ and $\mathrm{GCD}(a, b, p) = 1$?

* Beyond this value (i.e., for $a + b + \sqrt{a^2 + b^2} < p \leq 2(a + b)$, the limit value for the perimeter of a quadrilateral with given diagonals a and b), the function F is ill-defined because the maximum area is then obtained for limit configurations where the quadrilateral becomes a triangle, with the diagonals having a common endpoint.

(Proposed by David Draï)

234 | Number of divisors

Let $D(x)$ be the number of divisors of x. Let F be the function that, for any non-zero natural number n, associates:

- the smallest natural number x such that $D(x^2)/D(x) = n$, if this x exists;
- 0 otherwise.

Example: $F(3)$
144 has 15 divisors, while 144^2 has 45, and $45/15 = 3$. Since 144 is the smallest integer with this property, $F(3) = 144$.

What is the sum of $F(n)$ for n ranging from 1 to 22? Enter the twelve digits on the rightmost of the result.

(Proposed by David Draï)

235 | Hitori

Rules of the game

Blacken certain squares in the grid so that:

- in each row and column, the remaining letters are all different;
- two squares adjacent on one side cannot both be blackened;
- the remaining squares must form a single block.

D	B	A	E	C
A	D	C	B	B
C	E	D	B	B
A	E	B	E	E
A	A	E	C	D

D	B	A	E	C
	D	C		B
C	E	D	B	
A		B		E
	A	E	C	D

Solve this grid:

A	B	E	F	H	E	D	C
F	B	A	G	G	H	C	A
H	G	F	B	D	C	G	E
D	D	E	A	F	F	H	G
C	E	G	G	A	D	C	F
H	A	G	D	C	E	F	D
B	H	D	C	C	G	D	A
D	C	H	F	E	C	A	B

Going through the grid from left to right and top to bottom, we assign the k-th prime number to box k (so the top right corner has the number $p_8 = 19$ and the bottom right corner has the number $p_{64} = 311$). **What is the product of the shaded boxes? Enter the result modulo 10^{13}.**

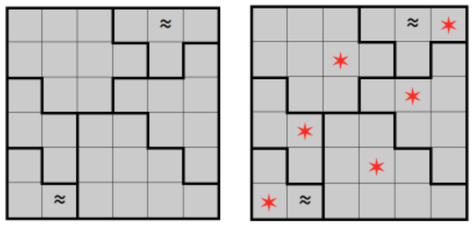
(Proposed by David Draï)

236 | Star system

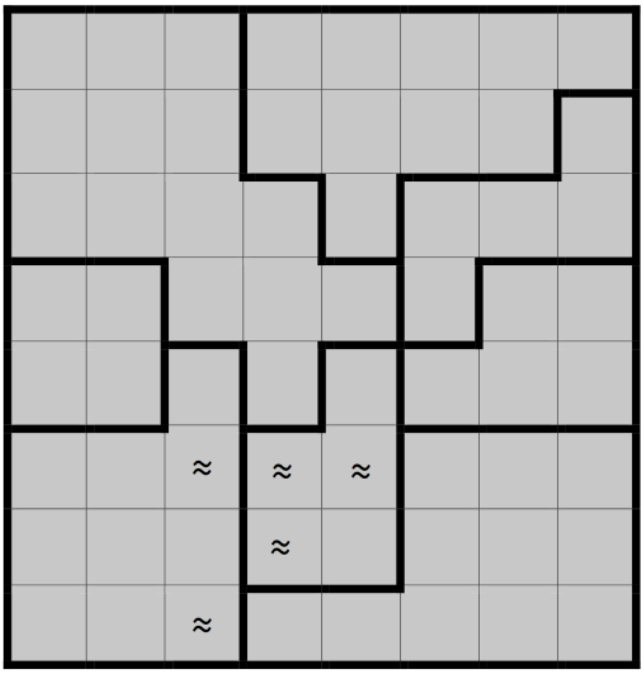
Rules of the game

Find the position of the stars.

- There is one star per row, per column, and per region outlined by a thicker line.
- Furthermore, the stars do not touch each other, even diagonally.
- The waves indicate squares that do not contain a star.



Solve this grid:



Going through the grid from left to right and top to bottom, we assign the k-th prime number to box k (so the top right corner has the number $p_8 = 19$ and the bottom right corner has the number $p_{64} = 311$). **What is the product of the boxes containing stars?**

(Proposed by David Draï)

237 Sandwich numbers

A "sandwich" number is a number with $3n$ digits whose middle section (the number written with the n middle digits) is equal to the sum of the first section (the number written with the first n digits) and the last section (the number written with the last n digits).

For example, 203818 is a sandwich number because $20 + 18 = 38$.

What is the smallest sandwich number that is a multiple of 2018 and contains the sequence of consecutive digits 2, 0, 1, 8? (203818 is not suitable because 2, 0, 1 and 8 are not consecutive).

(Proposed by Le Coyote, based on a problem by Raymond Bloch in Tangente no. 179, p.49)

238 | Barrier of indivisibility

For any positive integer n, we define the function F by $F(n) = k$ where k is the smallest positive integer such that $n + k$ is not divisible by $k + 1$.

Examples

1. 13 is divisible by 1, 14 is divisible by 2, 15 is divisible by 3, 16 is divisible by 4, but 17 is NOT divisible by 5. Therefore, $F(13) = 4$.
2. 120 is divisible by 1, but 121 is NOT divisible by 2. Therefore, $F(120) = 1$.

We define $P(s, N)$ as the number of integers n ($1 < n < N$) for which $F(n) = s$.
Thus $P(3, 14) = 1$ and $P(6, 10^6) = 14286$.

What is the sum, for i ranging from 1 to 1000, of $P\left(i, 10^i\right)$? Enter the last 12 digits of the result.

(Proposed by David Draï, based on Project Euler 601)

239 | Factorial decomposition

Consider the number 48.
There are five pairs of integers a and b ($a \leq b$) such that $a \times b = 48$: $(1, 48)$, $(2, 24)$, $(3, 16)$, $(4, 12)$ and $(6, 8)$.
We can see that 6 and 8 each have 4 divisors. Therefore, among the five pairs, one of them is composed of two integers with the same number of divisors.
Let $C(n)$ be the number of pairs of positive integers a and b such that $a \times b = n$, $a \leq b$, and such that a and b have the same number of divisors.
Thus $C(48) = 1$.

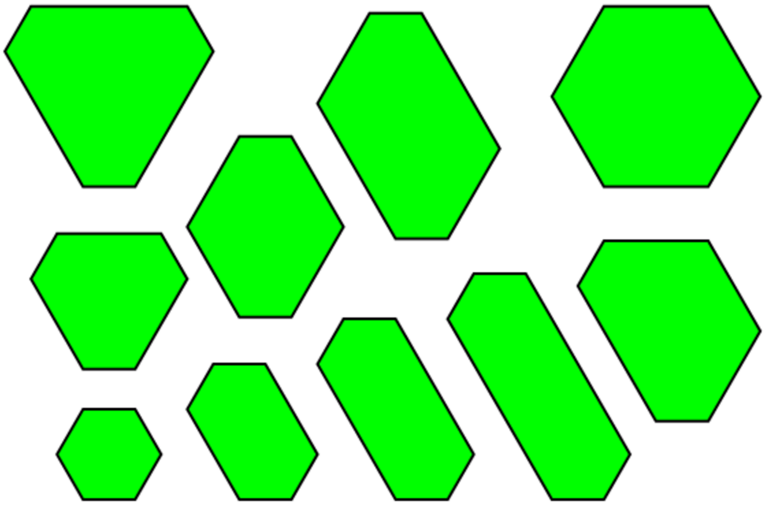
We also have:
$C(9!) = 5$: $(384, 945)$, $(420, 864)$, $(480, 756)$, $(540, 672)$ and $(560, 648)$.
$C(10!) = 3$: $(1680, 2160)$, $(1800, 2016)$ and $(1890, 1920)$.

What is the value of $C(40!)$?

(Proposed by David Draï, based on Project Euler 598)

240 | Equiangular hexagons

Let $H(n)$ be the number of distinct convex equiangular hexagons with integer sides and a perimeter less than or equal to n.
Two hexagons are distinct if and only if they are not isometric.
We are given: $H(6) = 1$, $H(12) = 10$, $H(100) = 31\,248$.



Equiangular hexagons with a perimeter less than or equal to 12

What is the value of $H(300)$?

(Proposed by David Draï, based on Project Euler 600)

241 The bags of counterfeit coins

Among eight bags, numbered from 1 to 8 and each containing 100 coins, six contain gold coins (a real gold coin weighs 10 grams) and two contain counterfeit coins, indistinguishable from the real ones except by their weight.
The counterfeit coins in one of the bags weigh 11 grams each, and the counterfeit coins in the other bag weigh 12 grams.
You have a graduated scale that can weigh any mass up to 10 kg to the nearest gram.

You must find, in a single weighing, and by taking the minimum number of coins from the bags, which bag contains the 11-gram coins and which bag contains the 12-gram coins.

What will be the sum, in grams, of the 56 distinct masses that can be read on the scale?

(Proposed by David Draï, based on a problem from "Algorithmes, jeux et stratégies" by Élisabeth Busser and Gilles Cohen, Éditions Pole, 2013)

242 | Mirror, my beautiful mirror

Let mirror be the function that, for any positive integer n, associates the number obtained by reading n from right to left.

Examples: $\text{mirror}(12) = 21$; $\text{mirror}(340) = 43$.

Let n be a positive integer with k digits ($k > 1$).
Let F be the function that, for any n, associates the number of integers p_i with k digits such that $p_i + n = \text{mirror}(p_i)$.

Examples

1. 36 is a 2-digit number.
   $F(36) = 5$ because there are only 5 two-digit integers such that $p_i + 36 = \text{mirror}(p_i)$: 15, 26, 37, 48 and 59.
   Indeed: $15 + 36 = 51$; $26 + 36 = 62$; … ; $59 + 36 = 95$.
2. However, $F(37) = 0$.

What is the sum of $F(n)$ for n ranging from 10 to 10^{12}?

(Proposed by David Draï, freely inspired by the April 25th puzzle from the 2017 Mathematical Calendar)

243 | For the n-th time!

Let n be a strictly positive integer with p digits.
Let F be the function that, for any n, associates the number obtained by the following process: starting from n, we write p numbers, such that each of them is the sum of the p-th powers of the digits of the previous one.

Example: $213 \to 2^3 + 1^3 + 3^3 = 36 \to 3^3 + 6^3 = 243 \to 2^3 + 4^3 + 3^3 = 99$. Thus, $F(213) = 99$.

Let G be the function that, for any strictly positive integer p, associates the largest value of $F(n)$ obtained when n describes the set of p-digit integers.

Examples:
$G(1) = 9$, because $F(9) = 9^1 = 9$.
$G(2) = 145$, because $F(58) = F(77) = F(85) = 145$.
Indeed, 58 and 85 give $5^2 + 8^2 = 89$ and 77 gives $7^2 + 7^2 = 98$ after one iteration.
And since it is impossible to obtain 99 after one iteration from a 2-digit integer, $8^2 + 9^2 = 145$ is indeed the largest integer that can be obtained in the second iteration from a 2-digit number.

Let H be the function that, for any strictly positive integer p, associates the smallest p-digit antecedent of $G(p)$ with F.

Examples: $H(1) = 9$ and $H(2) = 58$.

What are the sums of $G(p)$ and $H(p)$ for p ranging from 1 to 11? Enter the sum of the two results.

(Proposed by David Draï, freely inspired by the May 8th puzzle from the 2017 Mathematical Calendar)

244 | Sum of cubes

Choose a number, calculate the sum of the cubes of its digits, and write down the result, which will be the second number. Repeat the same operation with this second number and continue until you come across a number that has already been written down.

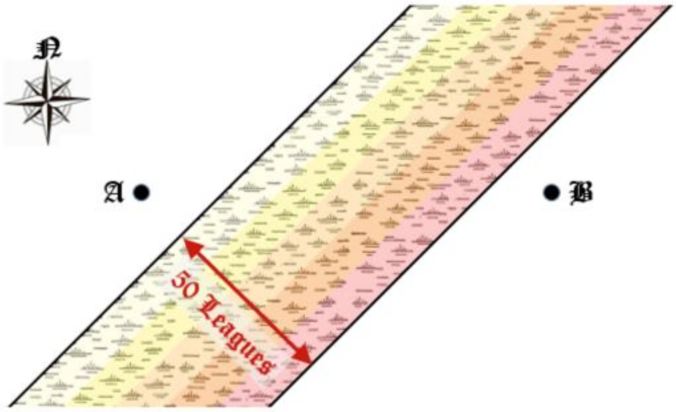
Examples
Starting with 1: 1 1 (sequence length 2)
Starting with 2: 2 8 512 134 92 737 713 371 371 (sequence length 9)
Starting with 2029: 2029 745 532 160 217 352 160 (sequence length 7)

What is the smallest number less than 1 million that produces the longest sequence?

(Proposed by Le Coyote. Inspired by a problem from the quarterfinals of the 32nd Mathematical and Logic Games Championship)

245 | Marsh

Frodo and Sam must travel 100 leagues eastward to get from point A to point B.
On normal terrain, they can cover 10 leagues per day; under these conditions, the journey would therefore take 10 days.
But their path is crossed by a long marsh that stretches exactly from southwest to northeast, and walking through this marsh will slow them down.
The swamp is 50 leagues wide at all points, and the middle of segment $[AB]$ is located in the middle of the swamp (i.e., 25 leagues from each "shore").
Here is a map of the region:



The swamp consists of five distinct regions, each 10 leagues wide, as shown by the different colors on the map.
The strip closest to point A is a relatively easy swamp; it can be crossed at a speed of 9 leagues per day.
However, each strip becomes progressively more difficult to negotiate, with speeds decreasing successively to 8, 7, 6 and finally 5 leagues per day for the last region of the swamp.
Then the marsh ends abruptly and the terrain returns to normal, with a speed of 10 leagues per day.
If Frodo and Sam were to travel in a straight line eastward from A to B, they would travel exactly 100 leagues, and the journey would take approximately 13.4738 days. However, this time can be reduced if they agree to deviate from the "straight path".

Determine the shortest possible time to travel from A to B.

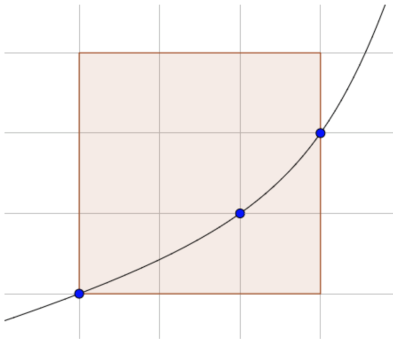
Give your answer in days, rounded to 10 decimal places. The decimal point should be omitted.

(Project Euler 607)

246 | Convex path

Let $F(N)$ be the maximum number of grid nodes in an $N \times N$ square through which the graph of a strictly increasing and strictly convex function can pass.

The figure shows the graphical representation of a function passing through 3 nodes in a 3×3 square.



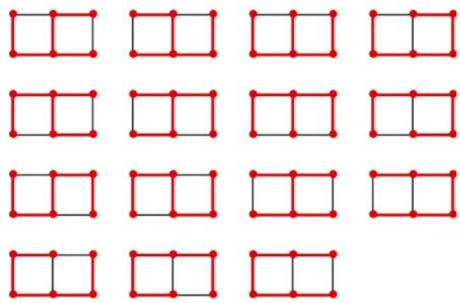
We are given $F(1) = 2$, $F(3) = 3$ (figure opposite), $F(9) = 6$, $F(11) = 7$, $F(100) = 30$ and $F(50\,000) = 1898$.

What is the value of $F(10^{18})$?

(Proposed by David Draï. Project Euler 604)

247 | Covering trees

A grid of six squares is drawn with black outline strokes. You want to highlight certain edges in red so that any vertex of the squares that make up the grid can be connected to any other vertex in a single way by following a red line.



1. How many different ways can the red lines be drawn?

The drawing opposite shows the 15 ways to color a two-by-two grid.

2. Same question with a two-by-two grid.

The answer should be the product of the two results.

(Proposed by David Draï. Affaire de logique — Problem 1004)

248 | Moving target

On a grid of n squares by 1, a red target is placed randomly on one of the squares with the same red background. The target is therefore invisible.
After each shot, the target moves randomly to an adjacent square.
The shooter is an expert who never misses the target square, **and adopts a strategy that minimizes the maximum number of shots, which we will denote by $C(n)$**. (Note: this strategy does not necessarily minimize the expected value!)
If the target is hit, the square changes color and the target becomes visible, signaling to the shooter that they have hit it and that the game is over.
Let $E(n)$ be the average number of shots needed to hit the target.

Given that $E(1) = 1$, $E(2) = E(3) = 3/2$, $E(4) = 37/16$ and $E(5) = 19/6$.

What is the value of $E(10)$?

Enter the product of the numerator and denominator of the irreducible fraction obtained.

Note: the calculation of $E(n)$ will be based solely on all possible paths of the target comprising exactly $C(n)$ movements, as if the target always moved $C(n)$ times, whether it was hit or not.

(Proposed by David Draï. Inspired by Problem 1024 "Cibles mouvantes" from "Affaire de logique")

249 | 99 times larger

Let E be the set of natural integers n satisfying the following property:

If we write a 1 to the left and a 1 to the right of n, we obtain the number $99 \times n$.

The set E seems to have the following two properties (verified for all n in E less than 10^{2500} ...):

1. all elements of E are congruent to A modulo 10^P (where P is the largest integer for which this property remains true);
2. the numbers of digits of the elements of E form an arithmetic sequence with common difference R and first term U.

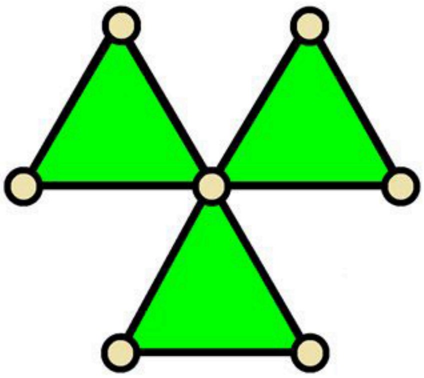
What is the value of $A \times P \times R \times U$?

Enter the last 12 digits of the result.

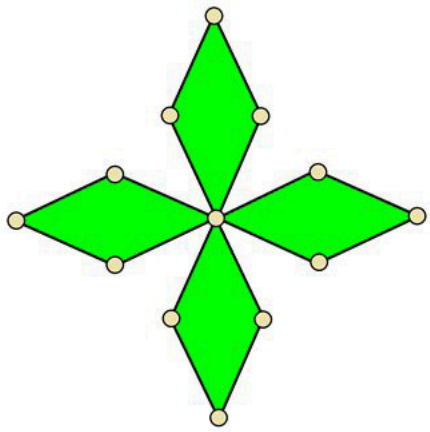
(Proposed by David Draï. Based on "Affaire de logique" — Problem 1018)

250 | Flowers with pentagonal petals

Consider n polygons with n sides, arranged in a star shape, with a common vertex, as shown in the figures below.



$n = 3$



$n = 4$

1. When $n = 3$, the integers from 1 to 7 can be placed in the circles in 144 different ways so that the sums of the numbers at the vertices of the three triangles are equal;
2. In the case where $n = 4$, the integers from 1 to 13 can be placed in 311 040 different ways so that the sums of the numbers at the vertices of the four quadrilaterals are equal.

When $n = 5$, how many different ways can the integers from 1 to 21 be placed so that the sums of the numbers at the vertices of the five pentagons are equal?

(Proposed by David Draï. Inspired by the May 24th puzzle from the 2017 Mathematical Calendar)

251 | Neighbors

The numbers 1 to 25 are placed in a square grid so that each number, except 1 and 2, is the sum of two of its neighbors (in the grid below, 1 has eight neighbors).

Example

	1	2	3	4	5
A	19	11	15	20	21
B	13	6	5	4	17
C	23	7	1	3	14
D	16	9	8	2	12
E	25	24	18	10	22

It turns out that the number 18 is the only number that, when placed in the center of the grid, allows for a unique filling, except for symmetries and rotations.
Thus, entering an additional value in a cell breaks any symmetry and ensures the uniqueness of the grid.

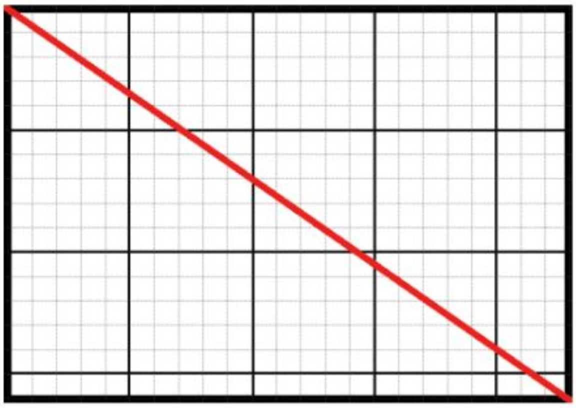
We place 18 in the center and 7 in cell A2. We number the cells from left to right and top to bottom from 1 to 25.

What is the sum of the 25 products “number in a cell × cell number”?

(Proposed by David Draï)

252 | On the tiles, the chord…

Let a, b and c be three strictly positive integers such that $a > b > c$.
A rectangular piece measuring a cm in length and b cm in width is tiled with squares measuring c cm on each side.
A straight chord connects two opposite corners of the piece, as shown in the figure below. Any cuts are always along the bottom and/or right side of the figure.



$a = 23$, $b = 16$ et $c = 5$

Let $F(a, b, c)$ be the number of tiles that the chord crosses.
For example: $F(23, 16, 5) = 8$ and $F(4500, 2400, 30) = 220$.

What is the sum of $F(a, b, c)$ for $4400 \leq a \leq 4600$, $2300 \leq b \leq 2500$ and $10 \leq c \leq 60$?

(Proposed by David Draï. Freely inspired by the July 27th puzzle from the 2017 Mathematical Calendar)

253 | Paper clips

A stationer has chains of paper clips attached to each other.
During a “manipulation”, he detaches a paper clip at both ends and obtains that paper clip as well as two new smaller chains. A manipulation cannot involve a paper clip at the end of a chain.

1. What is the longest initial chain that allows the stationer, in 10 manipulations performed beforehand, to provide a customer with any number of paper clips between 1 and the length of the chain, without having to perform any additional manipulations in front of them, but simply by combining a certain number of the sub-chains obtained?

The stationer receives a chain made up of 10 000 paper clips in three different colors: the first is red, the second blue, and the third green. He notices that by removing one in four paper clips (starting with the fourth: the 4th, the 8th, the 12th, etc.), the sequence of the 2500 paper clips removed is identical to the sequence of the first 2500 paper clips in the initial chain, and the sequence of the remaining 7500 is identical to that of the first 7500 paper clips in the initial chain.

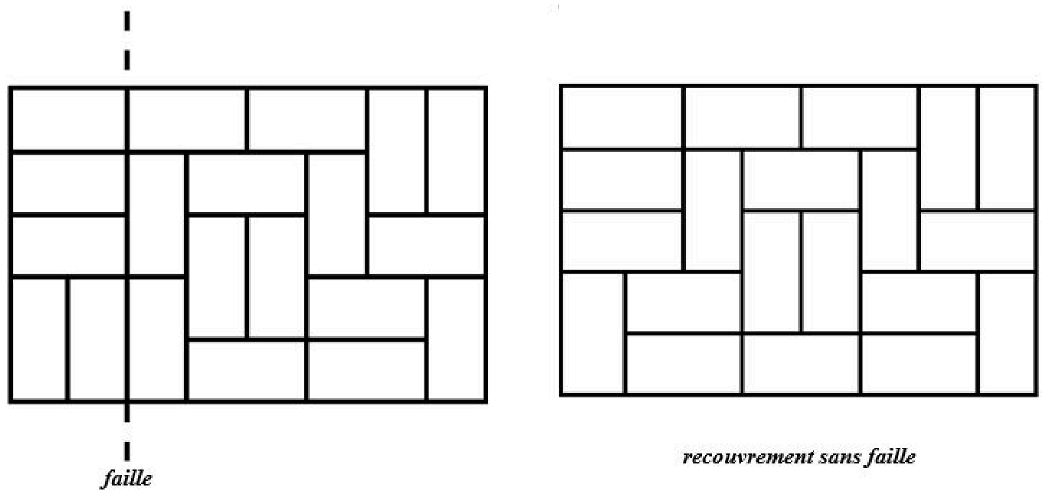
2. How many red, blue and green paper clips are there in the chain?

Enter the sum of the result of **1** and the product of the 3 results of **2**.

(Proposed by David Draï, based on Affaire de logique — Problem 1009)

254 | Flawless rectangles

When covering an entire rectangle with whole sides and an even area with 2×1 dominoes, a “flaw” may occur, i.e., a line of division crossing the rectangle from side to side, horizontally or vertically.[1]



Let A be the total number of possible coverings of a rectangle measuring 8×5.
Let B be the number of flawless coverings of this rectangle.

What is the value of $A \times B$?

(Proposed by David Draï. Inspired by Problem 6 of the Mathematical Games of “Monde”, February 25th, 1997)

[1] “faille” means “flaw”; “recouvrement sans faille” means “flawless covering”.

255 Self-referential table

In this context,

- there are … times the number 9 outside of line 1;
- there are … times the number 8 outside of line 2;
- there are … times the number 7 outside of line 3;
- there are … times the number 6 outside of line 4;
- there are … times the number 5 outside of line 5;
- there are … times the number 4 outside of line 6;
- there are … times the number 3 outside of line 7;
- there are … times the number 2 outside of line 8;
- there are … times the number 1 outside of line 9.

Fill in the "blanks" with numbers (between 0 and 9) so that all the statements are true.
Enter the number obtained by concatenating the 9 digits read from top to bottom.

(Proposed by David Draï. Inspired by Problem 3 of the Mathematical Games of "Monde", February 4th, 1997)

256 Postman always rings twice

Let $A = \{a_1, a_2, \ldots, a_n\}$ be the largest subset of the set $E = \{2, 3, 4, \ldots, N-1\}$ such that:

- the product P of the factorials of all a_k by the factorial of N is a square;
- P is as large as possible.

What is the value of P if $N = 99$? Enter the ten digits before the zeros that end the decimal representation of P.

(Proposed by David Draï. Inspired by Problem 1028 from the Mathematical Games of "Monde")