Linear Filters and Convolution

Babak Taati

Linear Time(Shift) Invariant (LTI) Systems

The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.

Linear Time(Shift) Invariant (LTI) Systems

The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.

A Linear Time Invariant system has two properties:

1. Time Invariance.
2. Linearity

Linear Time(Shift) Invariant (LTI) Systems

The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.

A Linear Time Invariant system has two properties:

1. Time Invariance.
2. Linearity

<u>Time Invariance:</u> Same inputs produce the same outputs at all times. In other words if an input f(t) produces g(t) today, f(t) will also produce g(t) tomorrow.
Or f(t-a) would produce g(t-a)

Linear Time(Shift) Invariant (LTI) Systems

The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.

A Linear Time Invariant system has two properties:

1. Time Invariance.
2. Linearity

<u>Time Invariance:</u> Same inputs produce the same outputs at all times. In other words if an input f(t) produces g(t) today, f(t) will also produce g(t) tomorrow.
Or f(t-a) would produce g(t-a)

i.e. if

Linear Time(Shift) Invariant (LTI) Systems

The Linear Filters that we are studying in the course belong to a class of systems known as Linear Time Invariant (LTI) systems. Traditionally they have been studied for signals that vary with time. But since we are currently dealing with single images, which vary with space, such filters for images are termed as Linear Shift Invariant (LSI) Filters.

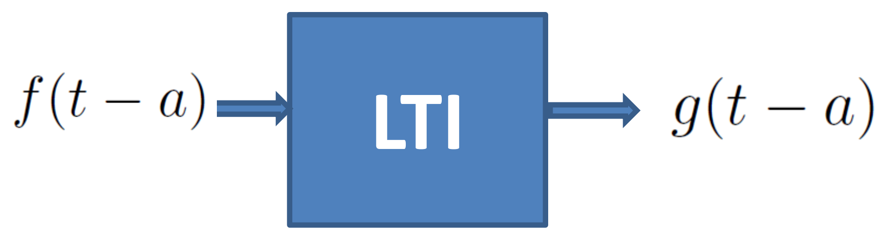
A Linear Time Invariant system has two properties:

1. Time Invariance.
2. Linearity

Time Invariance: Same inputs produce the same outputs at all times. In other words if an input f(t) produces g(t) today, f(t) will also produce g(t) tomorrow.
Or f(t-a) would produce g(t-a)

i.e. if

Then,



Linear Time(Shift) Invariant (LTI) Systems

<u>Linearity:</u> Linearity is a combination of two properties in turn:

Linear Time(Shift) Invariant (LTI) Systems

Linearity: Linearity is a combination of two properties in turn:

i) If $f(t)$ produces an output $g(t)$,
then for a linear system,
$a.f(t)$ produces an output $a.g(t)$

(This property is called HOMOGENEITY)

Linear Time(Shift) Invariant (LTI) Systems

Linearity: Linearity is a combination of two properties in turn:

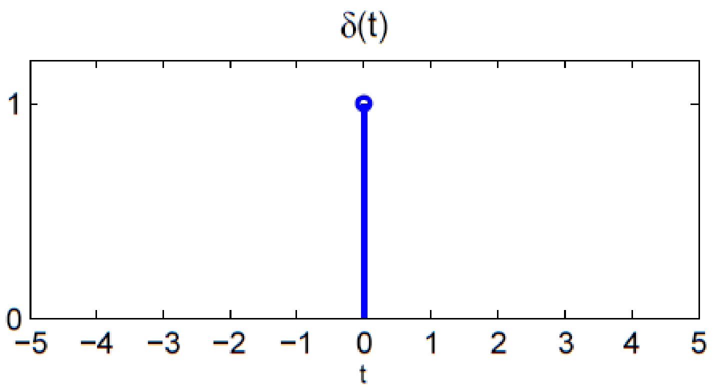
i) If $f(t)$ produces an output $g(t)$,
then for a linear system,
$a.f(t)$ produces an output $a.g(t)$

(This property is called HOMOGENEITY)

i) If $f_1(t)$ produces $g_1(t)$, *AND* $f_2(t)$ produces $g_2(t)$,
then for a linear system,
$f_1(t)+ f_2(t)$ *produces* $g_1(t)+ g_2(t)$.

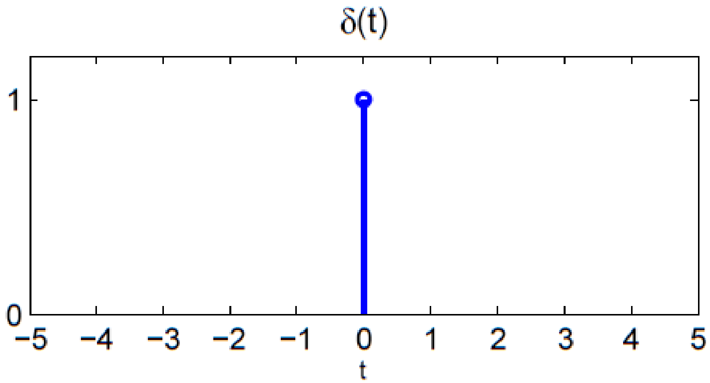
(This property is called SUPERPOSITION)

Impulse Functions

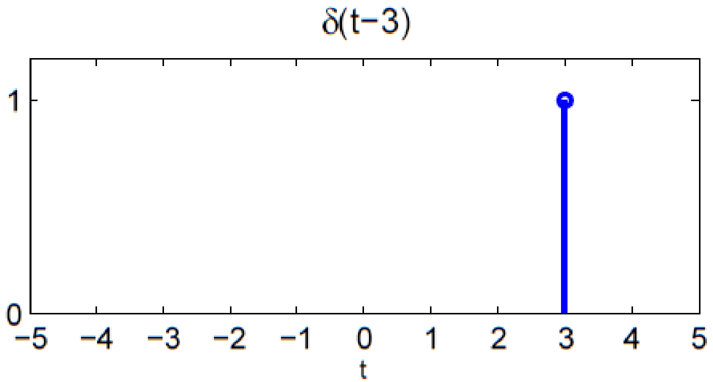


Impulse Centered at the Origin

Impulse Functions

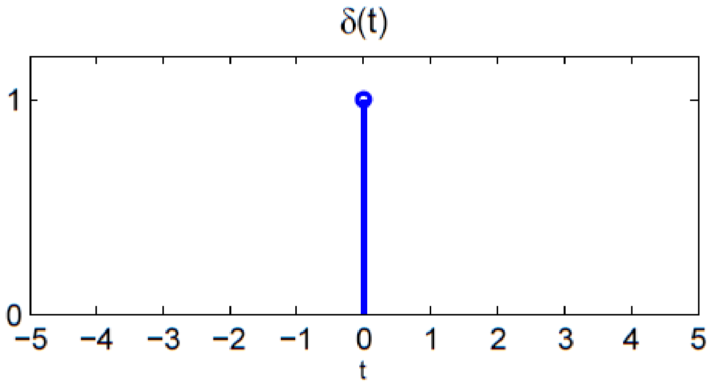


Impulse Centered at the Origin

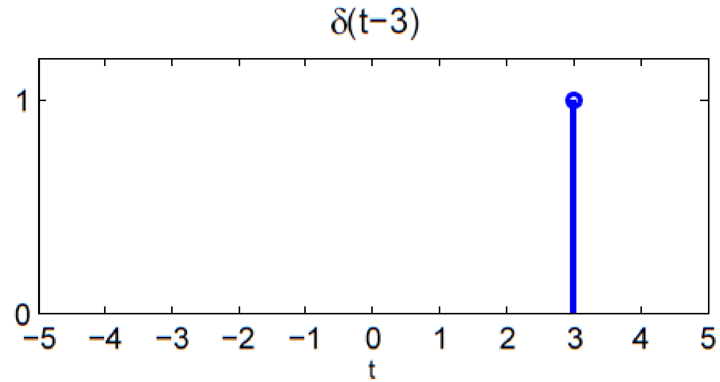


Shifted Impulse (Right Shifted)

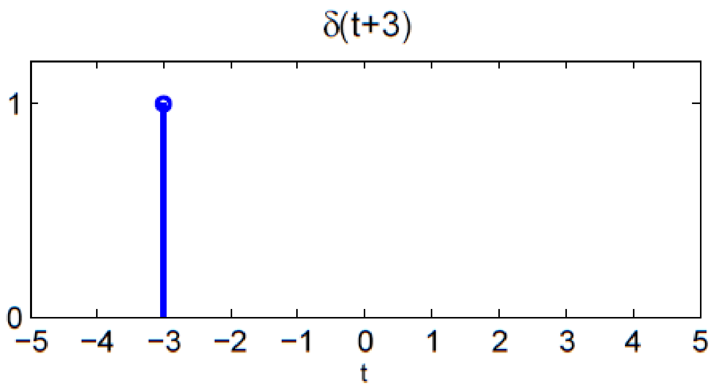
Impulse Functions



Impulse Centered at the Origin

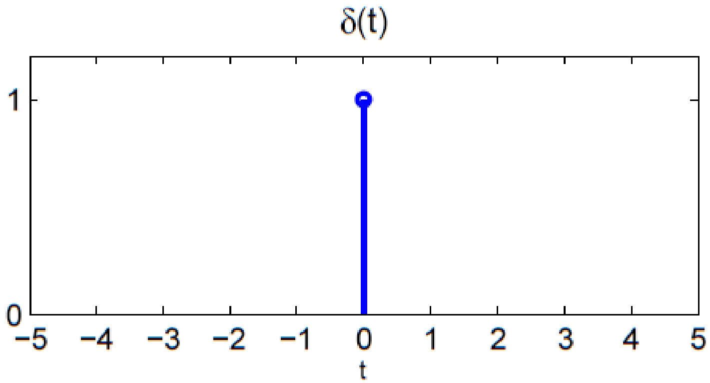


Shifted Impulse (Right Shifted)

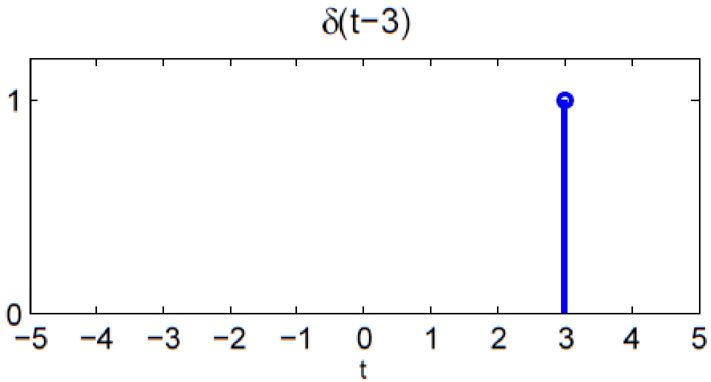


Shifted Impulse (Left Shifted)

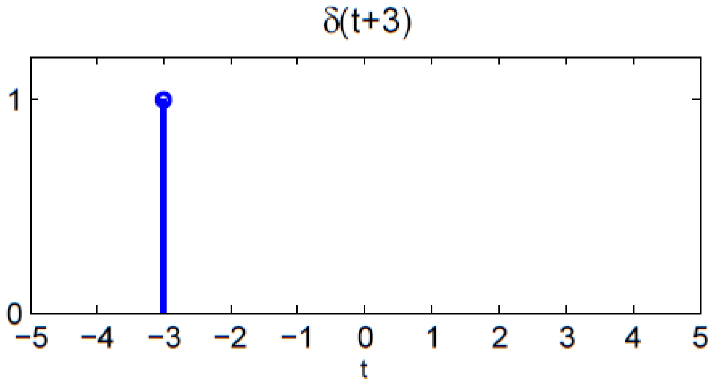
Impulse Functions



Impulse Centered at the Origin



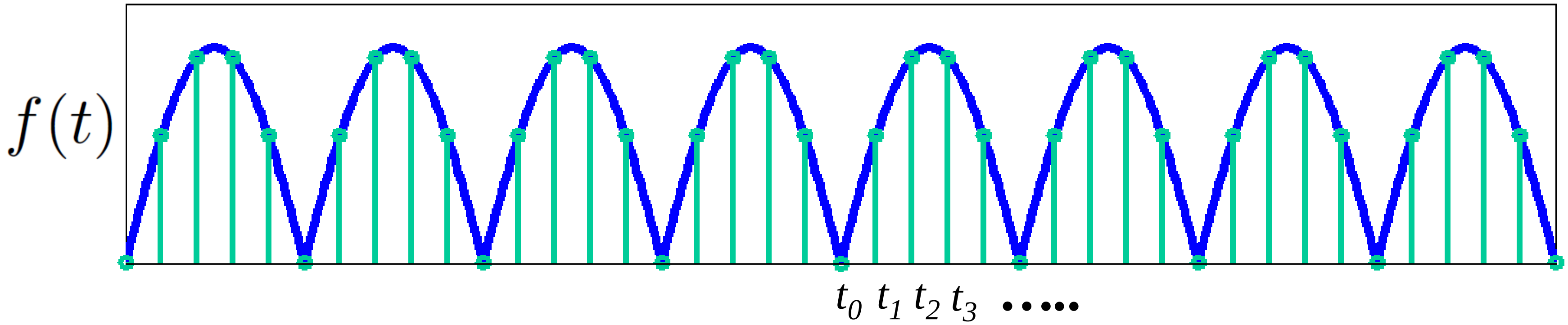
Shifted Impulse (Right Shifted)



Shifted Impulse (Left Shifted)

To figure out the direction of the shift, imagine this as a shift in the origin, and see where the argument of the impulse function is zero, i.e t-3 is zero for t=+3, and t+3 is zero for t=-3

Arbitrary Function as a weighted sum of shifted impulses



$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

Impulse Response

- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)

Impulse Response

- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)
- This is because the impulse response captures the complete behavior of the system.

Impulse Response

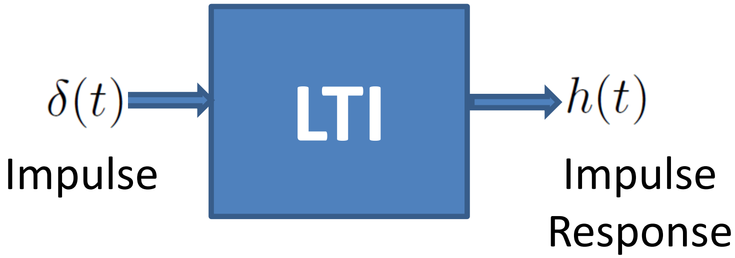
- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)
- This is because the impulse response captures the complete behavior of the system.
- Why?

Impulse Response

- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)
- This is because the impulse response captures the complete behavior of the system.
- Why?
- Because, if we know the response to an impulse, we can use shift invariance, and linearity (homogeneity and superposition) to compute the output to arbitrary inputs.

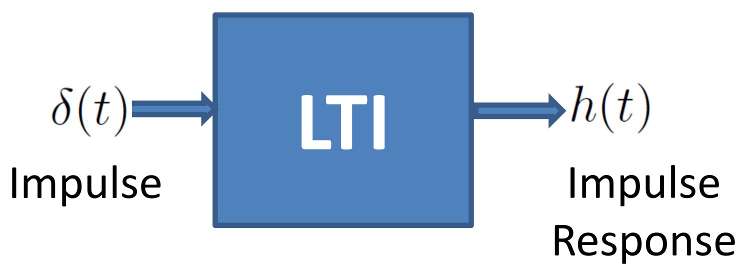
Impulse Response

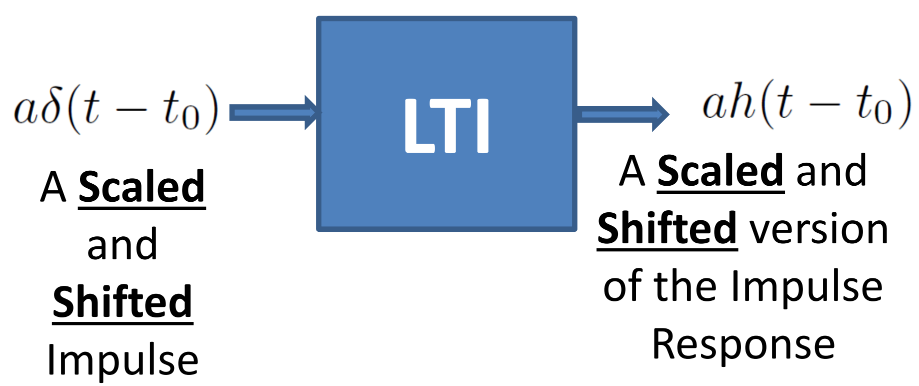
- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)
- This is because the impulse response captures the complete behavior of the system.
- Why?
- Because, if we know the response to an impulse, we can use shift invariance, and linearity (homogeneity and superposition) to compute the output to arbitrary inputs.



Impulse Response

- An LTI system is usually described in terms of its impulse response (i.e. output in response to an impulse input)
- This is because the impulse response captures the complete behavior of the system.
- Why?
- Because, if we know the response to an impulse, we can use shift invariance, and linearity (homogeneity and superposition) to compute the output to arbitrary inputs.





Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given: $\delta(t) \text{ produces } h(t)$

Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given: $\delta(t)$ produces $h(t)$

$$\implies \delta(t - t_i) \text{ produces } h(t - t_i) \boxed{\text{(Using Shift Invariance)}}$$

Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given: $\delta(t)$ produces $h(t)$

$$\implies \delta(t - t_i) \text{ produces } h(t - t_i) \boxed{\text{(Using Shift Invariance)}}$$

$$\implies f(t_i)\delta(t - t_i) \text{ produces } f(t_i)h(t - t_i) \boxed{\text{(Using Homogeneity)}}$$

Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given: $\delta(t)$ produces $h(t)$

$$\implies \delta(t - t_i) \text{ produces } h(t - t_i) \ \boxed{\text{(Using Shift Invariance)}}$$

$$\implies f(t_i)\delta(t - t_i) \text{ produces } f(t_i)h(t - t_i) \ \boxed{\text{(Using Homogeneity)}}$$

$$\implies f(t) \text{ produces } \sum_i f(t_i)h(t - t_i) \ \boxed{\text{(Using Superposition)}}$$

Given Impulse Response, Computing output to arbitrary function

- If the impulse response of a system (filter) is h(t), i.e. it produces an output h(t) in response to δ(t), what would be its output in response to the following function as an input:

$$f(t) = \sum_i f(t_i)\delta(t - t_i)$$

- Given: $\delta(t)$ produces $h(t)$

$$\implies \delta(t - t_i) \text{ produces } h(t - t_i) \; \boxed{\text{(Using Shift Invariance)}}$$

$$\implies f(t_i)\delta(t - t_i) \text{ produces } f(t_i)h(t - t_i) \; \boxed{\text{(Using Homogeneity)}}$$

$$\implies f(t) \text{ produces } \boxed{\textstyle\sum_i f(t_i)h(t - t_i)} \; \boxed{\text{(Using Superposition)}}$$

<u>This sum is the convolution of f(t) and h(t), written as f(t)*h(t), and IS the output of the filter, i.e. the output is the correlation of the *flipped* h(t) with f(t)</u>

Convolution between f(t) and h(t)

$$f(t) * h(t) = \sum_i f(t_i)h(t - t_i)$$

- *i.e., the concepts of convolution, flipping one signal, and then taking its correlation with the input to get the output are NOT handed to us by fiat. Rather they naturally emerge from the properties of Linearity and Time/Shift Invariance.*

- ***Homework**: Extend these concepts for 2D images and material covered in Lecture 3.*