Haar wavelet

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Signal space

Recall the Hilbert space

$$L^2 = \{f(t) : \int_{-\infty}^{+\infty} |f(t)|^2 dt < +\infty\}$$

with inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt$$

and norm

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\infty}^{+\infty} |f(t)|^2 dt}$$

Scaling function (father wavelet)

定义

The scaling function (father wavelet) of Haar wavelet is the box function

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

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- $\int_{-\infty}^{+\infty} \phi(t)dt = 1$.
- ϕ_k is a shift (translation) of ϕ by k.
- Step function

$$s(t) = a_k, \ k \le t \le k+1 \Leftrightarrow s(t) = \sum_k a_k \phi(t-k).$$

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Space V_0

- $V_0 = \{s(t) : s(t) = \sum_k a_k \phi(t-k), \quad \sum_{-\infty}^{+\infty} |a_k|^2 < +\infty\} \subset L^2.$

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- $\{\phi(t-k) : k = 0, \pm 1, \cdots\}$ form an ON basis for V_0.
- approximate signals $f(t) \in L^2(\mathbb{R})$
  - project them on V_0
  - expand projection in terms of the translated scaling functions.

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Consider $\phi(2t-k)$

- Define

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- Functions $\{\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k) : k = 0, \pm 1, \cdots\}$ form an ON basis for V_j, $\phi_{jk}(t) = \frac{1}{\sqrt{2}}[\phi_{j+1,2k}(t) + \phi_{j+1,2k+1}(t)]$.

Subspace

- $V_0 \subset V_1 \subset \cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$.

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引理

1. $s(t) \in V_0 \Leftrightarrow s(2^j t) \in V_j$
2. $s(t) \in V_j \Leftrightarrow s(2^{-j} t) \in V_0$.

Orthogonal complement of V_0 in V_1

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- Denote $W_0 := V_0^{\perp} = \{s \in V_1 : (s, f) = 0$ for all $f \in V_0\}$,

$$V_1 = V_0 \bigoplus W_0.$$

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- From dilation equation

$$\begin{aligned}\phi(t-k) =& \phi(2t-2k) + \phi(2t-2k-1) \\ =& 2^{-1/2}(\phi_{1,2k}(t) + \phi_{1,2k+1}(t))\end{aligned}$$

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get

$$(\phi_{0k}, \phi_{1l}) = 2^{1/2}\int_{-\infty}^{+\infty} \phi(t-k)\phi(2t-l)dt = \begin{cases} 2^{-1/2} & \text{if } l = 2k, 2k+1, \\ 0 & \text{otherwise.}\end{cases}$$

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定义 (Haar wavelet (mother wavelet))

We call $w(t) = \phi(2t) - \phi(2t-1)$ is the Haar wavelet or mother wavelet.

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引理

For fixed j,

$$(w_{jk}, w_{jk'}) = \delta_{kk'}, \quad (\phi_{jk}, w_{jk'}) = 0$$

where $k, k' = 0, \pm1, \pm2, \cdots.$

ON basis of W_j

定理

Let W_j be the orthogonal complement of V_j in V_{j+1}:

$$V_j \bigoplus W_j = V_{j+1}.$$

The wavelets $\{w_{jk}(t) k = 0, \pm 1, \cdots\}$ form an ON basis for W_j.

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$$s(t) = \sum_k a_k \phi_{j+1,k}(t)$$

and $(s, \phi_{jn}) = 0, \ \forall n$.

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Hence

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$$s = \sum_{k=0}^{j} w_k + s_0,\ w_k \in W_k,\ s_0 \in V_0.$$

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- $(w_{jk}, w_{j'k'}) = 0$ if $j \neq j'$.

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C form an ON set of $W_j$omplement

引理

$(w_{jk}, w_{j'k'}) = \delta_{jj'}\delta_{kk'}$ *for* $j, j', \pm k, \pm k' = 0, 1, \cdots$

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定理

$$L^2(\mathbb{R}) = V_0 \bigoplus \sum_{\ell=0}^{+\infty} \bigoplus W_\ell = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus \cdots,$$

so that each $f(t) \in L^2(\mathbb{R})$ can be written uniquely in the form

$$f = f_0 + \sum_{\ell=0}^{+\infty} w_\ell, \quad w_\ell \in W_\ell, f_0 \in V_0.$$

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Proof

Based on the theory of Hilbert spaces, it is sufficient to show: for $\forall f \in L^2(\mathbb{R})$ and $\forall \epsilon > 0$, there eixsts an integer j_ϵ and a step function $s = \sum_k a_k \phi_{jk} \in V_j$ such that

$$\|f - s\| < \epsilon.$$

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Since the space of step functions $S^2_{[-\infty,+\infty]}$ is dense in $L^2(\mathbb{R})$,

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ON basis for $L^2(\mathbb{R})$

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推论

$$L^2(\mathbb{R}) = \sum_{l=-\infty}^{+\infty} \bigoplus W_l = \cdots W_{-1} \bigoplus W_0 \bigoplus W_1 \bigoplus \cdots,$$

so that each $f(t) \in L^2(\mathbb{R})$ can be written uniquely in the form

$$f = \sum_{l=-\infty}^{+\infty} w_l, \quad w_l \in W_l.$$

In particular, $\{w_{jk} : j, k = 0, \pm 1, \pm 2, \cdots\}$ is an ON basis for $L^2(\mathbb{R})$.

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- Recall relations

$$\begin{cases} \phi_{j-1,k}(t) = \frac{1}{\sqrt{2}}(\phi_{j,2k}(t) + \phi_{j,2k+1}(t)) \\ w_{j-1,k}(t) = \frac{1}{\sqrt{2}}(\phi_{j,2k}(t) - \phi_{j,2k+1}(t)) \end{cases}$$

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Fundamental recursion

- $$\begin{cases} \text{Average (lowpass)} & a_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} + a_{j,2k+1}) \\ \text{Differences (highpass)} & b_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} - a_{j,2k+1}) \end{cases}$$

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Synthesis recursion

$$
\begin{array}{ccccccccccc}
a_{j-2,k} & \leftarrow & (\downarrow 2) & \mathbf{C}^T & & & & & & & \\
 & & & & \nwarrow & & & & & & \\
 & & & & & a_{j-1,k} & \leftarrow & (\downarrow 2) & \mathbf{C}^T & & \\
 & & & & \swarrow & & & & & \nwarrow & \\
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\begin{array}{ccccccccccc}
 & & \mathbf{C} & (\uparrow 2) & \leftarrow & a_{j-1,k} & \leftarrow & (\downarrow 2) & \mathbf{C}^T & & \\
 & \swarrow & & & & & & & & \nwarrow & \\
a_{jk} & & & \mathit{Synthesis} & & & & \mathit{Analysis} & & & a_{jk} \\
\mathit{Output} & \nwarrow & & & & & & & & \swarrow & \mathit{Input.} \\
 & & \mathbf{D} & (\uparrow 2) & \leftarrow & b_{j-1,k} & \leftarrow & (\downarrow 2) & \mathbf{D}^T & &
\end{array}
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Matalb complementation db2

- $t = \text{linspace}(0, 1, 2 \wedge 10);$
  $x = 2 * \text{rand}(1, 2 \wedge 10) - 0.5;$
  $a = \exp(-t. \wedge 2). * \sin(100 * t. \wedge 2);$
  $a = f + x;$
  $[cA, cD] = \text{dwt}(a, 'db2');$
  $\text{subplot}(2, 2, 1)$
  $\text{plot}(a)$
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  $\text{plot}(cA)$
  $\text{subplot}(2, 2, 4)$
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  $\text{plot}(cD)$
- $s = \text{idwt}(cA, cD,' db2');$
  $\text{subplot}(1, 2, 1)$
  $\text{plot}(a)$
  $\text{subplot}(1, 2, 2)$
  $\text{plot}(s)$