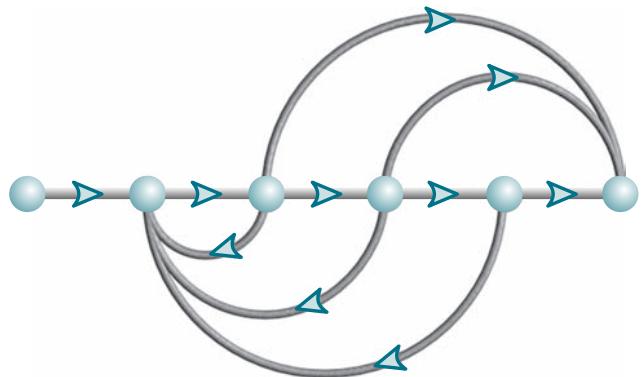


Modeling in the Frequency Domain

2



Chapter Learning Outcomes

After completing this chapter, the student will be able to:

- Find the Laplace transform of time functions and the inverse Laplace transform (Sections 2.1–2.2)
- Find the transfer function from a differential equation and solve the differential equation using the transfer function (Section 2.3)
- Find the transfer function for linear, time-invariant electrical networks (Section 2.4)
- Find the transfer function for linear, time-invariant translational mechanical systems (Section 2.5)
- Find the transfer function for linear, time-invariant rotational mechanical systems (Section 2.6)
- Find the transfer functions for gear systems with no loss and for gear systems with loss (Section 2.7)
- Find the transfer function for linear, time-invariant electromechanical systems (Section 2.8)
- Produce analogous electrical and mechanical circuits (Section 2.9)
- Linearize a nonlinear system in order to find the transfer function (Sections 2.10–2.11)

Case Study Learning Outcomes

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to find the transfer function of each subsystem.
- Given a model of a human leg or a nonlinear electrical circuit, you will be able to linearize the model and then find the transfer function.

2.1 Introduction

In Chapter 1, we discussed the analysis and design sequence that included obtaining the system's schematic and demonstrated this step for a position control system. To obtain a schematic, the control systems engineer must often make many simplifying assumptions in order to keep the ensuing model manageable and still approximate physical reality.

The next step is to develop mathematical models from schematics of physical systems. We will discuss two methods: (1) transfer functions in the frequency domain and (2) state equations in the time domain. These topics are covered in this chapter and in Chapter 3, respectively. As we proceed, we will notice that in every case the first step in developing a mathematical model is to apply the fundamental physical laws of science and engineering. For example, when we model electrical networks, Ohm's law and Kirchhoff's laws, which are basic laws of electric networks, will be applied initially. We will sum voltages in a loop or sum currents at a node. When we study mechanical systems, we will use Newton's laws as the fundamental guiding principles. Here we will sum forces or torques. From these equations we will obtain the relationship between the system's output and input.

In Chapter 1 we saw that a differential equation can describe the relationship between the input and output of a system. The form of the differential equation and its coefficients are a formulation or description of the system. Although the differential equation relates the system to its input and output, it is not a satisfying representation from a system perspective. Looking at Eq. (1.2), a general, n th-order, linear, time-invariant differential equation, we see that the system parameters, which are the coefficients, as well as the output, $c(t)$, and the input, $r(t)$, appear throughout the equation.

We would prefer a mathematical representation such as that shown in Figure 2.1(a), where the input, output, and system are distinct and separate parts. Also, we would like to represent conveniently the interconnection of several subsystems. For example, we would like to represent *cascaded* interconnections, as shown

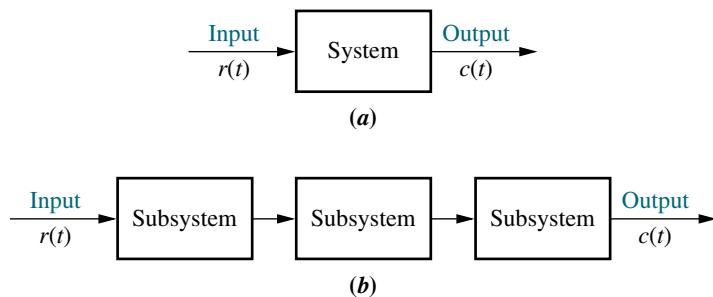


FIGURE 2.1 **a.** Block diagram representation of a system; **b.** block diagram representation of an interconnection of subsystems

Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

in Figure 2.1(b), where a mathematical function, called a transfer function, is inside each block, and block functions can easily be combined to yield Figure 2.1 (a) for ease of analysis and design. This convenience cannot be obtained with the differential equation.

2.2 Laplace Transform Review

A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the groundwork for the Laplace transform, with which we can represent the input, output, and system as separate entities. Further, their interrelationship will be simply algebraic. Let us first define the Laplace transform and then show how it simplifies the representation of physical systems (Nilsson, 1996).

The Laplace transform is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt \quad (2.1)$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. (2.1) exists, we can find a function, $F(s)$, that is called the *Laplace transform* of $f(t)$.¹

The notation for the lower limit means that even if $f(t)$ is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges. Thus, we can find the Laplace transform of impulse functions. This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at $t = 0$. Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity. Using the Laplace transform we need only know the initial conditions before the discontinuity. See Kailath (1980) for a more detailed discussion.

The inverse Laplace transform, which allows us to find $f(t)$ given $F(s)$, is

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = f(t)u(t) \quad (2.2)$$

where

$$\begin{aligned} u(t) &= 1 & t > 0 \\ &= 0 & t < 0 \end{aligned}$$

is the unit step function. Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Using Eq. (2.1), it is possible to derive a table relating $f(t)$ to $F(s)$ for specific cases. Table 2.1 shows the results for a representative sample of functions. If we use the tables, we do not have to use Eq. (2.2), which requires complex integration, to find $f(t)$ given $F(s)$.

¹The Laplace transform exists if the integral of Eq. (2.1) converges. The integral will converge if $\int_{0-}^{\infty} |f(t)|e^{-\sigma_1 t} dt < \infty$. If $|f(t)| < M e^{\sigma_2 t}$, $0 < t < \infty$, the integral will converge if $\infty > \sigma_1 > \sigma_2$. We call σ_2 the *abscissa of convergence*, and it is the smallest value of σ , where $s = \sigma + j\omega$, for which the integral exists.

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s + a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

In the following example we demonstrate the use of Eq. (2.1) to find the Laplace transform of a time function.

Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st} dt = \int_0^\infty Ae^{-at}e^{-st} dt = A \int_0^\infty e^{-(s+a)t} dt \\ &= -\frac{A}{s+a} e^{-(s+a)t} \Big|_{t=0}^\infty = \frac{A}{s+a} \end{aligned} \quad (2.3)$$

In addition to the Laplace transform table, Table 2.1, we can use Laplace transform theorems, listed in Table 2.2, to assist in transforming between $f(t)$ and $F(s)$. In the next example, we demonstrate the use of the Laplace transform theorems shown in Table 2.2 to find $f(t)$ given $F(s)$.

Example 2.2

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

¹For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

²For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a *partial-fraction expansion*. If $F_1(s) = N(s)/D(s)$, where the order of $N(s)$ is less than the order of $D(s)$, then a partial-fraction expansion can be made. If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \quad (2.4)$$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence,

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5} \quad (2.5)$$

Taking the inverse Laplace transform, using Item 1 of Table 2.1, along with the differentiation theorem (Item 7) and the linearity theorem (Item 3 of Table 2.2), we obtain

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathcal{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right] \quad (2.6)$$

Using partial-fraction expansion, we will be able to expand functions like $F(s) = 2/(s^2 + s + 5)$ into a sum of terms and then find the inverse Laplace transform for each term. We will now consider three cases and show for each case how an $F(s)$ can be expanded into partial fractions.

Case 1. Roots of the Denominator of $F(s)$ Are Real and Distinct An example of an $F(s)$ with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)} \quad (2.7)$$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called *residues*, form the numerators. Hence,

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \quad (2.8)$$

To find K_1 , we first multiply Eq. (2.8) by $(s+1)$, which isolates K_1 . Thus,

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)} \quad (2.9)$$

Letting s approach -1 eliminates the last term and yields $K_1 = 2$. Similarly, K_2 can be found by multiplying Eq. (2.8) by $(s+2)$ and then letting s approach -2 ; hence, $K_2 = -2$.

Each component part of Eq. (2.8) is an $F(s)$ in Table 2.1. Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2e^{-2t})u(t) \quad (2.10)$$

In general, then, given an $F(s)$ whose denominator has real and distinct roots, a partial-fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2) \cdots (s+p_m) \cdots (s+p_n)} \\ &= \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)} \end{aligned} \quad (2.11)$$

can be made if the order of $N(s)$ is less than the order of $D(s)$. To evaluate each residue, K_i , we multiply Eq. (2.11) by the denominator of the corresponding partial fraction. Thus, if we want to find K_m , we multiply Eq. (2.11) by $(s+p_m)$ and get

$$\begin{aligned} (s+p_m)F(s) &= \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2) \cdots (s+p_m) \cdots (s+p_n)} \\ &= (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{(s+p_2)} + \cdots + K_m + \cdots \\ &\quad + (s+p_m)\frac{K_n}{(s+p_n)} \end{aligned} \quad (2.12)$$

If we let s approach $-p_m$, all terms on the right-hand side of Eq. (2.12) go to zero except the term K_m , leaving

$$\left. \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2) \cdots (s+p_m) \cdots (s+p_n)} \right|_{s \rightarrow -p_m} = K_m \quad (2.13)$$

The following example demonstrates the use of the partial-fraction expansion to solve a differential equation. We will see that the Laplace transform reduces the task of finding the solution to simple algebra.

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \quad (2.14)$$

SOLUTION: Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-) = 0$ and $\dot{y}(0-) = 0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \quad (2.15)$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} \quad (2.16)$$

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} \quad (2.17)$$

where, from Eq. (2.13),

$$K_1 = \left. \frac{32}{(s+4)(s+8)} \right|_{s=0} = 1 \quad (2.18a)$$

$$K_2 = \left. \frac{32}{s(s+8)} \right|_{s=-4} = -2 \quad (2.18b)$$

$$K_3 = \left. \frac{32}{s(s+4)} \right|_{s=-8} = 1 \quad (2.18c)$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \quad (2.19)$$

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \quad (2.20)$$

MATLAB
ML

Students who are using MATLAB should now run ch2p1 through ch2p8 in Appendix B. This is your first MATLAB exercise. You will learn how to use MATLAB to (1) represent polynomials, (2) find roots of polynomials, (3) multiply polynomials, and (4) find partial-fraction expansions. Finally, Example 2.3 will be solved using MATLAB.

TryIt 2.1

Use the following MATLAB and Control System Toolbox statement to form the linear, time-invariant (LTI) transfer function of Eq. (2.22).

```
F=zpk([], [-1 -2 -2], 2)
```

The $u(t)$ in Eq. (2.20) shows that the response is zero until $t = 0$. Unless otherwise specified, all inputs to systems in the text will not start until $t = 0$. Thus, output responses will also be zero until $t = 0$. For convenience, we will leave off the $u(t)$ notation from now on. Accordingly, we write the output response as

$$y(t) = 1 - 2e^{-4t} + e^{-8t} \quad (2.21)$$

Case 2. Roots of the Denominator of $F(s)$ Are Real and Repeated An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (2.22)$$

The roots of $(s+2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a *multiple root of multiplicity 2*.

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \quad (2.23)$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying Eq. (2.23) by $(s+2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \quad (2.24)$$

Letting s approach -2 , $K_2 = -2$. To find K_3 we see that if we differentiate Eq. (2.24) with respect to s ,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3 \quad (2.25)$$

K_3 is isolated and can be found if we let s approach -2 . Hence, $K_3 = -2$.

Each component part of Eq. (2.23) is an $F(s)$ in Table 2.1; hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \quad (2.26)$$

If the denominator root is of higher multiplicity than 2, successive differentiation would isolate each residue in the expansion of the multiple root.

TryIt 2.2

Use the following MATLAB statements to help you get Eq. (2.26).

```
numf=2;
denf=poly([-1 -2 -2]);
[k,p,k]=residue...
(numf,denf)
```

In general, then, given an $F(s)$ whose denominator has real and repeated roots, a partial-fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} \\ &= \frac{N(s)}{(s + p_1)^r (s + p_2) \cdots (s + p_n)} \\ &= \frac{K_1}{(s + p_1)^r} + \frac{K_2}{(s + p_1)^{r-1}} + \cdots + \frac{K_r}{(s + p_1)} \\ &\quad + \frac{K_{r+1}}{(s + p_2)} + \cdots + \frac{K_n}{(s + p_n)} \end{aligned} \quad (2.27)$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ and the repeated roots are of multiplicity r at $-p_1$. To find K_1 through K_r for the roots of multiplicity greater than unity, first multiply Eq. (2.27) by $(s + p_1)^r$ getting $F_1(s)$, which is

$$\begin{aligned} F_1(s) &= (s + p_1)^r F(s) \\ &= \frac{(s + p_1)^r N(s)}{(s + p_1)^r (s + p_2) \cdots (s + p_n)} \\ &= K_1 + (s + p_1) K_2 + (s + p_1)^2 K_3 + \cdots + (s + p_1)^{r-1} K_r \\ &\quad + \frac{K_{r+1}(s + p_1)^r}{(s + p_2)} + \cdots + \frac{K_n(s + p_1)^r}{(s + p_n)} \end{aligned} \quad (2.28)$$

Immediately, we can solve for K_1 if we let s approach $-p_1$. We can solve for K_2 if we differentiate Eq. (2.28) with respect to s and then let s approach $-p_1$. Subsequent differentiation will allow us to find K_3 through K_r . The general expression for K_1 through K_r for the multiple roots is

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \rightarrow -p_1} \quad i = 1, 2, \dots, r; \quad 0! = 1 \quad (2.29)$$

Case 3. Roots of the Denominator of $F(s)$ Are Complex or Imaginary An example of $F(s)$ with complex roots in the denominator is

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} \quad (2.30)$$

This function can be expanded in the following form:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2s + 5} \quad (2.31)$$

K_1 is found in the usual way to be $\frac{3}{5}$. K_2 and K_3 can be found by first multiplying Eq. (2.31) by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5} \right) s^2 + \left(K_3 + \frac{6}{5} \right) s + 3 \quad (2.32)$$

TryIt 2.3

Use the following MATLAB and Control System Toolbox statement to form the LTI transfer function of Eq. (2.30).

```
F = tf([3], [1 2 5 0])
```

Balancing coefficients, $(K_2 + \frac{3}{5}) = 0$ and $(K_3 + \frac{6}{5}) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5} \quad (2.33)$$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\mathcal{L}[Ae^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2} \quad (2.34)$$

Similarly,

$$\mathcal{L}[Be^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2 + \omega^2} \quad (2.35)$$

Adding Eqs. (2.34) and (2.35), we get

$$\mathcal{L}[Ae^{-at}\cos \omega t + Be^{-at}\sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2} \quad (2.36)$$

TryIt 2.4

Use the following MATLAB and Symbolic Math Toolbox statements to get Eq. (2.38) from Eq. (2.30).

```
syms s
f=ilaplace...
(3/(s*(s^2+2*s+5)));
pretty(f)
```

We now convert the last term of Eq. (2.33) to the form suggested by Eq. (2.36) by completing the squares in the denominator and adjusting terms in the numerator without changing its value. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{(s+1) + (1/2)(2)}{(s+1)^2 + 2^2} \quad (2.37)$$

Comparing Eq. (2.37) to Table 2.1 and Eq. (2.36), we find

$$f(t) = \frac{3}{5} - \frac{3}{5} e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \quad (2.38)$$

In order to visualize the solution, an alternate form of $f(t)$, obtained by trigonometric identities, is preferable. Using the amplitudes of the cos and sin terms, we factor out $\sqrt{1^2 + (1/2)^2}$ from the term in parentheses and obtain

$$f(t) = \frac{3}{5} - \frac{3}{5} \sqrt{1^2 + (1/2)^2} e^{-t} \left(\frac{1}{\sqrt{1^2 + (1/2)^2}} \cos 2t + \frac{1/2}{\sqrt{1^2 + (1/2)^2}} \sin 2t \right) \quad (2.39)$$

Letting $1/\sqrt{1^2 + (1/2)^2} = \cos \phi$ and $(1/2)/\sqrt{1^2 + (1/2)^2} = \sin \phi$,

$$f(t) = \frac{3}{5} - \frac{3}{5} \sqrt{1^2 + (1/2)^2} e^{-t} (\cos \phi \cos 2t + \sin \phi \sin 2t) \quad (2.40)$$

or

$$f(t) = 0.6 - 0.671 e^{-t} \cos(2t - \phi) \quad (2.41)$$

where $\phi = \arctan 0.5 = 26.57^\circ$. Thus, $f(t)$ is a constant plus an exponentially damped sinusoid.

In general, then, given an $F(s)$ whose denominator has complex or purely imaginary roots, a partial-fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s^2 + as + b) \dots} \\ &= \frac{K_1}{(s + p_1)} + \frac{(K_2s + K_3)}{(s^2 + as + b)} + \dots \end{aligned} \quad (2.42)$$

can be made if the order of $N(s)$ is less than the order of $D(s)$, p_1 is real, and $(s^2 + as + b)$ has complex or purely imaginary roots. The complex or imaginary roots are expanded with $(K_2s + K_3)$ terms in the numerator rather than just simply K_i , as in the case of real roots. The K_i 's in Eq. (2.42) are found through balancing the coefficients of the equation after clearing fractions. After completing the squares on $(s^2 + as + b)$ and adjusting the numerator, $(K_2s + K_3)/(s^2 + as + b)$ can be put into the form shown on the right-hand side of Eq. (2.36).

Finally, the case of purely imaginary roots arises if $a = 0$ in Eq. (2.42). The calculations are the same.

Another method that follows the technique used for the partial-fraction expansion of $F(s)$ with real roots in the denominator can be used for complex and imaginary roots. However, the residues of the complex and imaginary roots are themselves complex conjugates. Then, after taking the inverse Laplace transform, the resulting terms can be identified as

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos \theta \quad (2.43)$$

and

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin \theta \quad (2.44)$$

For example, the previous $F(s)$ can also be expanded in partial fractions as

$$\begin{aligned} F(s) &= \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + j2)(s + 1 - j2)} \\ &= \frac{K_1}{s} + \frac{K_2}{s + 1 + j2} + \frac{K_3}{s + 1 - j2} \end{aligned} \quad (2.45)$$

Finding K_2 ,

$$K_2 = \frac{3}{s(s + 1 - j2)} \Big|_{s \rightarrow -1 - j2} = -\frac{3}{20}(2 + j1) \quad (2.46)$$

Similarly, K_3 is found to be the complex conjugate of K_2 , and K_1 is found as previously described. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{20} \left(\frac{2 + j1}{s + 1 + j2} + \frac{2 - j1}{s + 1 - j2} \right) \quad (2.47)$$

from which

$$\begin{aligned} f(t) &= \frac{3}{5} - \frac{3}{20} \left[(2 + j1)e^{-(1+j2)t} + (2 - j1)e^{-(1-j2)t} \right] \\ &= \frac{3}{5} - \frac{3}{20} e^{-t} \left[4 \left(\frac{e^{j2t} + e^{-j2t}}{2} \right) + 2 \left(\frac{e^{j2t} + e^{-j2t}}{2j} \right) \right] \end{aligned} \quad (2.48)$$

TryIt 2.5

Use the following MATLAB statements to help you get Eq. (2.47).

```
numf=3
denf=[1 2 5 0]
[k,p,k]=residue...
(numf,denf)
```

Using Eqs. (2.43) and (2.44), we get

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) = 0.6 - 0.671e^{-t} \cos(2t - \phi) \quad (2.49)$$

where $\phi = \arctan 0.5 = 26.57^\circ$.

Symbolic Math
SM

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch2spl and ch2sp2 in Appendix F at www.wiley.com/college/nise. You will learn how to construct symbolic objects and then find the inverse Laplace and Laplace transforms of frequency and time functions, respectively. The examples in Case 2 and Case 3 in this section will be solved using the Symbolic Math Toolbox.

Skill-Assessment Exercise 2.1

PROBLEM: Find the Laplace transform of $f(t) = te^{-5t}$.

ANSWER: $F(s) = 1/(s+5)^2$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.2

PROBLEM: Find the inverse Laplace transform of $F(s) = 10/[s(s+2)(s+3)^2]$.

ANSWER: $f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$

The complete solution is at www.wiley.com/college/nise.

2.3 The Transfer Function

In the previous section we defined the Laplace transform and its inverse. We presented the idea of the partial-fraction expansion and applied the concepts to the solution of differential equations. We are now ready to formulate the system representation shown in Figure 2.1 by establishing a viable definition for a function that algebraically relates a system's output to its input. This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation. The function will also allow us to *algebraically* combine mathematical representations of subsystems to yield a total system representation.

Let us begin by writing a general n th-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \quad (2.50)$$

where $c(t)$ is the output, $r(t)$ is the input, and the a_i 's, b_i 's, and the form of the differential equation represent the system. Taking the Laplace transform of both sides,

$$\begin{aligned} & a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition} \\ & \quad \text{terms involving } c(t) \\ & = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition} \\ & \quad \text{terms involving } r(t) \end{aligned} \quad (2.51)$$

Equation (2.51) is a purely algebraic expression. If we assume that *all initial conditions are zero*, Eq. (2.51) reduces to

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s) \quad (2.52)$$

Now form the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$:

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)} \quad (2.53)$$

Notice that Eq. (2.53) separates the output, $C(s)$, the input, $R(s)$, and the system, the ratio of polynomials in s on the right. We call this ratio, $G(s)$, the *transfer function* and evaluate it with *zero initial conditions*.

The transfer function can be represented as a block diagram, as shown in Figure 2.2, with the input on the left, the output on the right, and the system transfer function inside the block. Notice that the denominator of the transfer function is identical to the characteristic polynomial of the differential equation. Also, we can find the output, $C(s)$ by using

$$C(s) = R(s)G(s) \quad (2.54)$$

Let us apply the concept of a transfer function to an example and then use the result to find the response of the system.

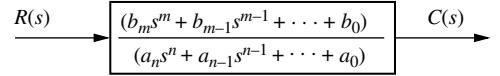


FIGURE 2.2 Block diagram of a transfer function

Example 2.4

Transfer Function for a Differential Equation

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \quad (2.55)$$

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \quad (2.56)$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \quad (2.57)$$

MATLAB
ML

Symbolic Math
SM

Students who are using MATLAB should now run ch2p9 through ch2p12 in Appendix B. You will learn how to use MATLAB to create transfer functions with numerators and denominators in polynomial or factored form. You will also learn how to convert between polynomial and factored forms. Finally, you will learn how to use MATLAB to plot time functions.

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch2sp3 in Appendix F at www.wiley.com/college/nise. You will learn how to use the Symbolic Math Toolbox to simplify the input of complicated transfer functions as well as improve readability. You will learn how to enter a symbolic transfer function and convert it to a linear, time-invariant (LTI) object as presented in Appendix B, ch2p9.

Example 2.5

TryIt 2.6

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.60).

```
syms s
C=1/(s*(s+2))
C=ilaplace(C)
```

TryIt 2.7

Use the following MATLAB statements to plot Eq. (2.60) for t from 0 to 1 s at intervals of 0.01 s.

```
t = 0:0.01:1;
plot...
(t,(1/2-1/2*exp(-2*t)))
```

System Response from the Transfer Function

PROBLEM: Use the result of Example 2.4 to find the response, $c(t)$ to an input, $r(t) = u(t)$, a unit step, assuming zero initial conditions.

SOLUTION: To solve the problem, we use Eq. (2.54), where $G(s) = 1/(s + 2)$ as found in Example 2.4. Since $r(t) = u(t)$, $R(s) = 1/s$, from Table 2.1. Since the initial conditions are zero,

$$C(s) = R(s)G(s) = \frac{1}{s(s + 2)} \quad (2.58)$$

Expanding by partial fractions, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s + 2} \quad (2.59)$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \quad (2.60)$$

Skill-Assessment Exercise 2.3

PROBLEM: Find the transfer function, $G(s) = C(s)/R(s)$, corresponding to the differential equation $\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$.

ANSWER: $G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.4

PROBLEM: Find the differential equation corresponding to the transfer function,

$$G(s) = \frac{2s + 1}{s^2 + 6s + 2}$$

ANSWER: $\frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 2c = 2\frac{dr}{dt} + r$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.5

PROBLEM: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s + 4)(s + 8)}$$

ANSWER: $c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$

The complete solution is at www.wiley.com/college/nise.



In general, a physical system that can be represented by a linear, time-invariant differential equation can be modeled as a transfer function. The rest of this chapter will be devoted to the task of modeling individual subsystems. We will learn how to represent electrical networks, translational mechanical systems, rotational mechanical systems, and electromechanical systems as transfer functions. As the need arises, the reader can consult the Bibliography at the end of the chapter for discussions of other types of systems, such as pneumatic, hydraulic, and heat-transfer systems (*Cannon, 1967*).

2.4 Electrical Network Transfer Functions

In this section, we formally apply the transfer function to the mathematical modeling of electric circuits including passive networks and operational amplifier circuits. Subsequent sections cover mechanical and electromechanical systems.

Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.² Table 2.3 summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are Kirchhoff's laws. We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero. From these relationships we can write the differential equations for the circuit. Then we can take the Laplace transforms of the differential equations and finally solve for the transfer function.

² *Passive* means that there is no internal source of energy.

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Simple Circuits via Mesh Analysis

Transfer functions can be obtained using Kirchhoff's voltage law and summing voltages around loops or meshes.³ We call this method *loop* or *mesh analysis* and demonstrate it in the following example.

Example 2.6

Transfer Function—Single Loop via the Differential Equation

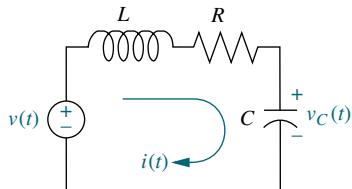


FIGURE 2.3 RLC network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION: In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = v(t) \quad (2.61)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L \frac{d^2q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = v(t) \quad (2.62)$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \quad (2.63)$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC \frac{d^2v_C(t)}{dt^2} + RC \frac{dv_C(t)}{dt} + v_C(t) = v(t) \quad (2.64)$$

³ A particular loop that resembles the spaces in a screen or fence is called a *mesh*.

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \quad (2.65)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad (2.66)$$

as shown in Figure 2.4.

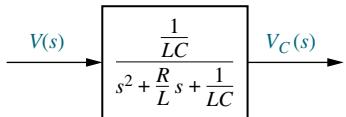


FIGURE 2.4 Block diagram of series RLC electrical network

Let us now develop a technique for simplifying the solution for future problems. First, take the Laplace transform of the equations in the voltage-current column of Table 2.3 assuming zero initial conditions.

For the capacitor,

$$V(s) = \frac{1}{Cs} I(s) \quad (2.67)$$

For the resistor,

$$V(s) = RI(s) \quad (2.68)$$

For the inductor,

$$V(s) = LsI(s) \quad (2.69)$$

Now define the following transfer function:

$$\frac{V(s)}{I(s)} = Z(s) \quad (2.70)$$

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current. But, unlike resistance, this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function *impedance*. The impedance for each of the electrical elements is shown in Table 2.3.

Let us now demonstrate how the concept of impedance simplifies the solution for the transfer function. The Laplace transform of Eq. (2.61), assuming zero initial conditions, is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \quad (2.71)$$

Notice that Eq. (2.71), which is in the form

$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}] \quad (2.72)$$

suggests the series circuit shown in Figure 2.5. Also notice that the circuit of Figure 2.5 could have been obtained immediately from the circuit of Figure 2.3 simply by replacing each element with its impedance. We call this altered circuit the *transformed circuit*. Finally, notice that the transformed circuit leads immediately to Eq. (2.71) if we add impedances in series as we add resistors in series. Thus, rather than writing the differential equation first and then taking the

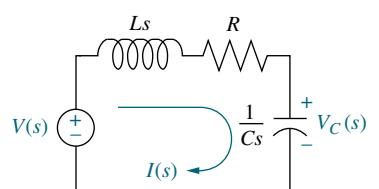


FIGURE 2.5 Laplace-transformed network

Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit. We summarize the steps as follows:

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

We now redo Example 2.6 using the transform methods just described and bypass the writing of the differential equation.

Example 2.7

Transfer Function—Single Loop via Transform Methods

PROBLEM: Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.

SOLUTION: Using Figure 2.5 and writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain

$$\left(Ls + R + \frac{1}{Cs} \right) I(s) = V(s) \quad (2.73)$$

Solving for $I(s)/V(s)$,

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \quad (2.74)$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s) \frac{1}{Cs} \quad (2.75)$$

Solving Eq. (2.75) for $I(s)$, substituting $I(s)$ into Eq. (2.74), and simplifying yields the same result as Eq. (2.66).

Simple Circuits via Nodal Analysis

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. We call this method *nodal analysis*. We now demonstrate this principle by redoing Example 2.6 using Kirchhoff's current law and the transform methods just described to bypass writing the differential equation.

Example 2.8

Transfer Function—Single Node via Transform Methods

PROBLEM: Repeat Example 2.6 using nodal analysis and without writing a differential equation.

SOLUTION: The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure 2.5. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. From Eq. (2.70), each $I(s) = V(s)/Z(s)$. Hence,

$$\frac{V_C(s)}{I/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \quad (2.76)$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the series resistor and inductor. Solving Eq. (2.76) for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as Eq. (2.66).

Simple Circuits via Voltage Division

Example 2.6 can be solved directly by using voltage division on the transformed network. We now demonstrate this technique.

Example 2.9

Transfer Function—Single Loop via Voltage Division

PROBLEM: Repeat Example 2.6 using voltage division and the transformed circuit.

SOLUTION: The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s) \quad (2.77)$$

Solving for the transfer function, $V_C(s)/V(s)$, yields the same result as Eq. (2.66).

Review Examples 2.6 through 2.9. Which method do you think is easiest for this circuit?

The previous example involves a simple, single-loop electrical network. Many electrical networks consist of multiple loops and nodes, and for these circuits we must write and solve simultaneous differential equations in order to find the transfer function, or solve for the output.

Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.

4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Let us look at an example.

Example 2.10

Transfer Function—Multiple Loops

PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

SOLUTION: The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions. The result is shown in Figure 2.6(b). The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,

$$R_1 I_1(s) + L s I_1(s) - L s I_2(s) = V(s) \quad (2.78)$$

Around Mesh 2, where $I_2(s)$ flows,

$$L s I_2(s) + R_2 I_2(s) + \frac{1}{C s} I_2(s) - L s I_1(s) = 0 \quad (2.79)$$

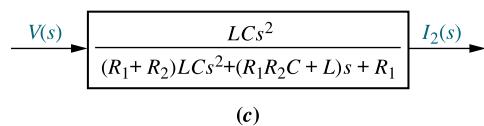
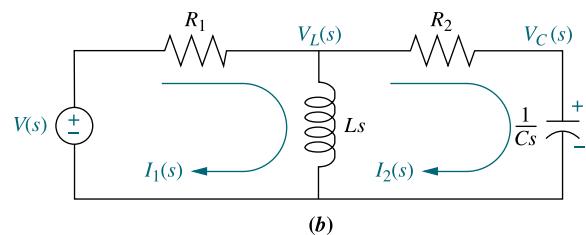
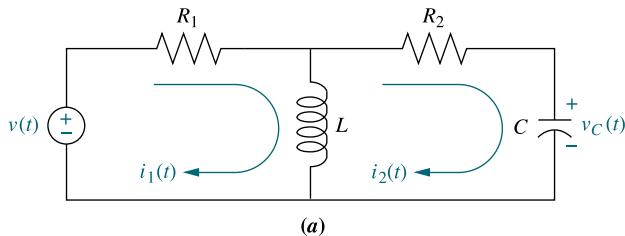


FIGURE 2.6 a. Two-loop electrical network;
b. transformed two-loop electrical network;
c. block diagram

Combining terms, Eqs. (2.78) and (2.79) become simultaneous equations in $I_1(s)$ and $I_2(s)$:

$$(R_1 + Ls)I_1(s) - LsI_2(s) = V(s) \quad (2.80a)$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs} \right) I_2(s) = 0 \quad (2.80b)$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve Eq. (2.80) for $I_2(s)$.⁴ Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \quad (2.81)$$

where

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs} \right) \end{vmatrix}$$

Forming the transfer function, $G(s)$, yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1} \quad (2.82)$$

as shown in Figure 2.6(c).

We have succeeded in modeling a physical network as a transfer function: The network of Figure 2.6(a) is now modeled as the transfer function of Figure 2.6(c). Before leaving the example, we notice a pattern first illustrated by Eq. (2.72). The form that Eq. (2.80) take is

$$\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{array} \right] I_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{array} \right] I_2(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{array} \right] \quad (2.83a)$$

$$- \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{array} \right] I_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{array} \right] I_2(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{array} \right] \quad (2.83b)$$

Recognizing the form will help us write such equations rapidly; for example, mechanical equations of motion (covered in Sections 2.5 and 2.6) have the same form.

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch2sp4 in Appendix F at www.wiley.com/college/nise, where Example 2.10 is solved. You will learn how to use the Symbolic Math Toolbox to solve simultaneous equations using Cramer's rule. Specifically, the Symbolic Math Toolbox will be used to solve for the transfer function in Eq. (2.82) using Eq. (2.80).

Symbolic Math
SM

⁴ See Appendix G (Section G.4) at www.wiley.com/college/nise for Cramer's rule.

Complex Circuits via Nodal Analysis

Often, the easiest way to find the transfer function is to use nodal analysis rather than mesh analysis. The number of simultaneous differential equations that must be written is equal to the number of nodes whose voltage is unknown. In the previous example we wrote simultaneous mesh equations using Kirchhoff's voltage law. For multiple nodes we use Kirchhoff's current law and sum currents flowing from each node. Again, as a convention, currents flowing from the node are assumed to be positive, and currents flowing into the node are assumed to be negative.

Before progressing to an example, let us first define *admittance*, $Y(s)$, as the reciprocal of impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \quad (2.84)$$

When writing nodal equations, it can be more convenient to represent circuit elements by their admittance. Admittances for the basic electrical components are shown in Table 2.3. Let us look at an example.

Example 2.11

Transfer Function—Multiple Nodes

PROBLEM: Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure 2.6(b). Use nodal analysis.

SOLUTION: For this problem, we sum currents at the nodes rather than sum voltages around the meshes. From Figure 2.6(b) the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0 \quad (2.85a)$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0 \quad (2.85b)$$

Rearranging and expressing the resistances as conductances,⁵ $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain,

$$\left(G_1 + G_2 + \frac{1}{Ls} \right) V_L(s) - G_2 V_C(s) = V(s) G_1 \quad (2.86a)$$

$$-G_2 V_L(s) + (G_2 + Cs) V_C(s) = 0 \quad (2.86b)$$

Solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2)s^2 + \frac{G_1 G_2 L + C}{LC}s + \frac{G_2}{LC}} \quad (2.87)$$

as shown in Figure 2.7.

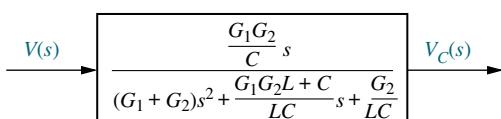


FIGURE 2.7 Block diagram of the network of Figure 2.6

⁵In general, admittance is complex. The real part is called conductance and the imaginary part is called susceptance. But when we take the reciprocal of resistance to obtain the admittance, a purely real quantity results. The reciprocal of resistance is called conductance.

Another way to write node equations is to replace voltage sources by current sources. A voltage source presents a constant voltage to any load; conversely, a current source delivers a constant current to any load. Practically, a current source can be constructed from a voltage source by placing a large resistance in series with the voltage source. Thus, variations in the load do not appreciably change the current, because the current is determined approximately by the large series resistor and the voltage source. Theoretically, we rely on *Norton's theorem*, which states that a voltage source, $V(s)$, in series with an impedance, $Z_s(s)$, can be replaced by a current source, $I(s) = V(s)/Z_s(s)$, in parallel with $Z_s(s)$.

In order to handle multiple-node electrical networks, we can perform the following steps:

1. Replace passive element values with their admittances.
2. Replace all sources and time variables with their Laplace transform.
3. Replace transformed voltage sources with transformed current sources.
4. Write Kirchhoff's current law at each node.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Let us look at an example.

Example 2.12

Transfer Function—Multiple Nodes with Current Sources

PROBLEM: For the network of Figure 2.6, find the transfer function, $V_C(s)/V(s)$, using nodal analysis and a transformed circuit with current sources.

SOLUTION: Convert all impedances to admittances and all voltage sources in series with an impedance to current sources in parallel with an admittance using Norton's theorem.

Redrawing Figure 2.6(b) to reflect the changes, we obtain Figure 2.8, where $G_1 = 1/R_1$, $G_2 = 1/R_2$, and the node voltages—the voltages across the inductor and the capacitor—have been identified as $V_L(s)$ and $V_C(s)$, respectively. Using the general relationship, $I(s) = Y(s)V(s)$, and summing currents at the node $V_L(s)$,

$$G_1 V_L(s) + \frac{1}{Ls} V_L(s) + G_2 [V_L(s) - V_C(s)] = V(s)G_1 \quad (2.88)$$

Summing the currents at the node $V_C(s)$ yields

$$Cs V_C(s) + G_2 [V_C(s) - V_L(s)] = 0 \quad (2.89)$$

Combining terms, Eqs. (2.88) and (2.89) become simultaneous equations in $V_C(s)$ and $V_L(s)$, which are identical to Eq. (2.86) and lead to the same solution as Eq. (2.87).

An advantage of drawing this circuit lies in the form of Eq. (2.86) and its direct relationship to Figure 2.8, namely

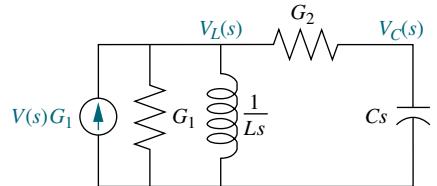


FIGURE 2.8 Transformed network ready for nodal analysis

$$\left[\begin{array}{c} \text{Sum of admittances} \\ \text{connected to Node 1} \end{array} \right] V_L(s) - \left[\begin{array}{c} \text{Sum of admittances} \\ \text{common to the two} \\ \text{nodes} \end{array} \right] V_C(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{currents at Node 1} \end{array} \right] \quad (2.90a)$$

$$- \left[\begin{array}{c} \text{Sum of admittances} \\ \text{common to the two} \\ \text{nodes} \end{array} \right] V_L(s) + \left[\begin{array}{c} \text{Sum of admittances} \\ \text{connected to Node 2} \end{array} \right] V_C(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{currents at Node 2} \end{array} \right] \quad (2.90b)$$

A Problem-Solving Technique

In all of the previous examples, we have seen a repeating pattern in the equations that we can use to our advantage. If we recognize this pattern, we need not write the equations component by component; we can sum impedances around a mesh in the case of mesh equations or sum admittances at a node in the case of node equations. Let us now look at a three-loop electrical network and write the mesh equations by inspection to demonstrate the process.

Example 2.13

Mesh Equations via Inspection

PROBLEM: Write, but do not solve, the mesh equations for the network shown in Figure 2.9.

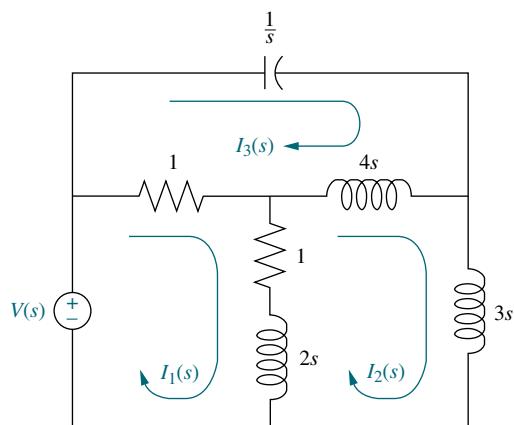


FIGURE 2.9 Three-loop electrical network

SOLUTION: Each of the previous problems has illustrated that the mesh equations and nodal equations have a predictable form. We use that knowledge to solve this three-loop problem. The equation for Mesh 1 will have the following form:

$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{array} \right] I_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and} \\ \text{Mesh 2} \end{array} \right] I_2(s) \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and} \\ \text{Mesh 3} \end{array} \right] I_3(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{array} \right] \end{aligned} \quad (2.91)$$

Similarly, Meshes 2 and 3, respectively, are

$$-\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and} \\ \text{Mesh 2} \end{array} \right] I_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{array} \right] I_2(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and} \\ \text{Mesh 3} \end{array} \right] I_3(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{array} \right] \quad (2.92)$$

and

$$\begin{aligned} -\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and} \\ \text{Mesh 3} \end{array} \right] I_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and} \\ \text{Mesh 3} \end{array} \right] I_2(s) \\ + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 3} \end{array} \right] I_3(s) = \left[\begin{array}{c} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 3} \end{array} \right] \end{aligned} \quad (2.93)$$

Substituting the values from Figure 2.9 into Eqs. (2.91) through (2.93) yields

$$+(2s+2)I_1(s) - (2s+1)I_2(s) \quad - I_3(s) = V(s) \quad (2.94a)$$

$$-(2s+1)I_1(s) + (9s+1)I_2(s) \quad - 4sI_3(s) = 0 \quad (2.94b)$$

$$-I_1(s) \quad - 4sI_2(s) + (4s+1+\frac{1}{s})I_3(s) = 0 \quad (2.94c)$$

which can be solved simultaneously for any desired transfer function, for example, $I_3(s)/V(s)$.

TryIt 2.8

Use the following MATLAB and Symbolic Math Toolbox statements to help you solve for the electrical currents in Eq. (2.94).

```
syms s I1 I2 I3 V
A=[(2*s+2) - (2*s+1)...
    -1
    -(2*s+1) (9*s+1)...
    -4*s
    -1 -4*s...
    (4*s+1+1/s)];
B=[I1; I2; I3];
C=[V; 0; 0];
B=inv(A)*C;
pretty(B)
```

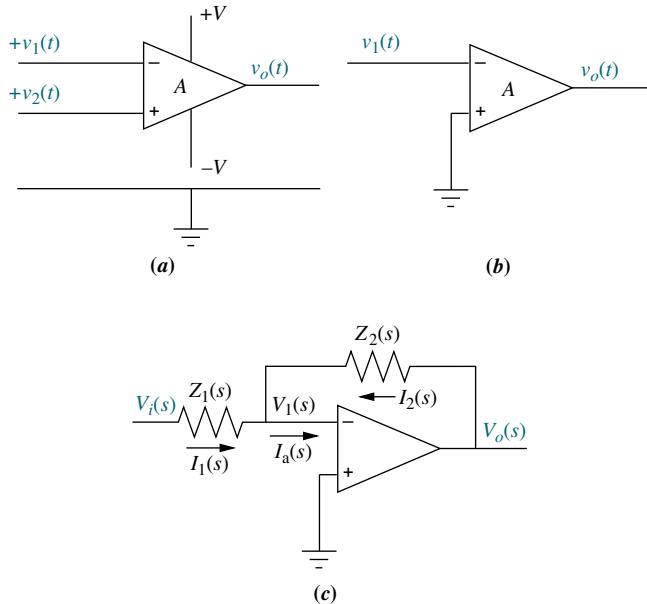


FIGURE 2.10 **a.** Operational amplifier; **b.** schematic for an inverting operational amplifier; **c.** inverting operational amplifier configured for transfer function realization. Typically, the amplifier gain, A , is omitted.

Passive electrical circuits were the topic of discussion up to this point. We now discuss a class of active circuits that can be used to implement transfer functions. These are circuits built around an operational amplifier.

Operational Amplifiers

An *operational amplifier*, pictured in Figure 2.10(a), is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $V_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

The output, $v_o(t)$, is given by

$$v_o(t) = A(v_2(t) - v_1(t)) \quad (2.95)$$

Inverting Operational Amplifier

If $v_2(t)$ is grounded, the amplifier is called an *inverting operational amplifier*, as shown in Figure 2.10(b). For the inverting operational amplifier, we have

$$v_o(t) = -Av_1(t) \quad (2.96)$$

If two impedances are connected to the inverting operational amplifier as shown in Figure 2.10(c), we can derive an interesting result if the amplifier has the characteristics mentioned in the beginning of this subsection. If the input impedance to the amplifier is high, then by Kirchhoff's current law, $I_a(s) = 0$ and $I_1(s) = -I_2(s)$.

Also, since the gain A is large, $v_1(t) \approx 0$. Thus, $I_1(s) = V_i(s)/Z_1(s)$, and $-I_2(s) = -V_o(s)/Z_2(s)$. Equating the two currents, $V_o(s)/Z_2(s) = -V_i(s)/Z_1(s)$, or the transfer function of the inverting operational amplifier configured as shown in Figure 2.10(c) is

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \quad (2.97)$$

Example 2.14

Transfer Function—Inverting Operational Amplifier Circuit

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.11.

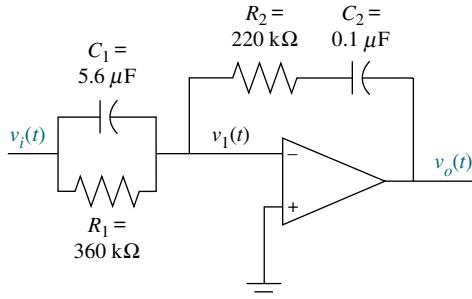


FIGURE 2.11 Inverting operational amplifier circuit for Example 2.14

SOLUTION: The transfer function of the operational amplifier circuit is given by Eq. (2.97). Since the admittances of parallel components add, $Z_1(s)$ is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1} \quad (2.98)$$

For $Z_2(s)$ the impedances add, or

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \quad (2.99)$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.97) and simplifying, we get

$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s} \quad (2.100)$$

The resulting circuit is called a PID controller and can be used to improve the performance of a control system. We explore this possibility further in Chapter 9.

Noninverting Operational Amplifier

Another circuit that can be analyzed for its transfer function is the noninverting operational amplifier circuit shown in Figure 2.12. We now derive the transfer

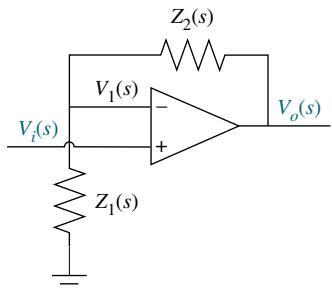


FIGURE 2.12 General noninverting operational amplifier circuit

function. We see that

$$V_o(s) = A(V_i(s) - V_1(s)) \quad (2.101)$$

But, using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s) \quad (2.102)$$

Substituting Eq. (2.102) into Eq. (2.101), rearranging, and simplifying, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))} \quad (2.103)$$

For large A , we disregard unity in the denominator and Eq. (2.103) becomes

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}} \quad (2.104)$$

Let us now look at an example.

Example 2.15

Transfer Function—Noninverting Operational Amplifier Circuit

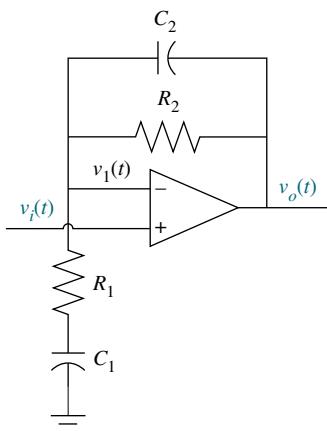


FIGURE 2.13 Noninverting operational amplifier circuit for Example 2.15

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.13.

SOLUTION: We find each of the impedance functions, $Z_1(s)$ and $Z_2(s)$, and then substitute them into Eq. (2.104). Thus,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} \quad (2.105)$$

and

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)} \quad (2.106)$$

Substituting Eqs. (2.105) and (2.106) into Eq. (2.104) yields

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1} \quad (2.107)$$

Skill-Assessment Exercise 2.6

PROBLEM: Find the transfer function, $G(s) = V_L(s)/V(s)$, for the circuit given in Figure 2.14. Solve the problem two ways—mesh analysis and nodal analysis. Show that the two methods yield the same result.

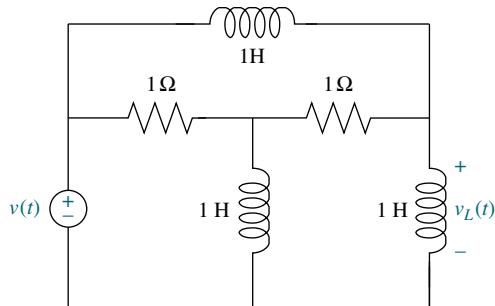


FIGURE 2.14 Electric circuit for Skill-Assessment Exercise 2.6

ANSWER: $V_L(s)/V(s) = (s^2 + 2s + 1)/(s^2 + 5s + 2)$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.7

PROBLEM: If $Z_1(s)$ is the impedance of a $10 \mu\text{F}$ capacitor and $Z_2(s)$ is the impedance of a $100 \text{k}\Omega$ resistor, find the transfer function, $G(s) = V_o(s)/V_i(s)$, if these components are used with (a) an inverting operational amplifier and (b) a noninverting amplifier as shown in Figures 2.10(c) and 2.12, respectively.

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ANSWER: $G(s) = -s$ for an inverting operational amplifier; $G(s) = s + 1$ for a noninverting operational amplifier.

The complete solution is at www.wiley.com/college/nise.

In this section, we found transfer functions for multiple-loop and multiple-node electrical networks, as well as operational amplifier circuits. We developed mesh and nodal equations, noted their form, and wrote them by inspection. In the next section we begin our work with mechanical systems. We will see that many of the concepts applied to electrical networks can also be applied to mechanical systems via analogies—from basic concepts to writing the describing equations by inspection. This revelation will give you the confidence to move beyond this textbook and study systems not covered here, such as hydraulic or pneumatic systems.

2.5 Translational Mechanical System Transfer Functions

We have shown that electrical networks can be modeled by a transfer function, $G(s)$, that algebraically relates the Laplace transform of the output to the Laplace transform of the input. Now we will do the same for mechanical systems. In this section we concentrate on translational mechanical systems. In the next section we extend the concepts to rotational mechanical systems. Notice that the end product, shown in Figure 2.2, will be mathematically indistinguishable from an electrical network. Hence, an electrical network can be interfaced to a mechanical system by cascading their transfer functions, provided that one system is not loaded by the other.⁶

⁶The concept of loading is explained further in Chapter 5.

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
Viscous damper	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N·s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

Mechanical systems parallel electrical networks to such an extent that there are analogies between electrical and mechanical components and variables. Mechanical systems, like electrical networks, have three passive, linear components. Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipator is analogous to electrical resistance. Let us take a look at these mechanical elements, which are shown in Table 2.4. In the table, K , f_v , and M are called *spring constant*, *coefficient of viscous friction*, and *mass*, respectively.

We now create analogies between electrical and mechanical systems by comparing Tables 2.3 and 2.4. Comparing the force-velocity column of Table 2.4 to the voltage-current column of Table 2.3, we see that mechanical force is analogous to electrical voltage and mechanical velocity is analogous to electrical current. Comparing the force-displacement column of Table 2.4 with the voltage-charge column of Table 2.3 leads to the analogy between the mechanical displacement and electrical charge. We also see that the spring is analogous to the capacitor, the viscous damper is analogous to the resistor, and the mass is analogous to the inductor. Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations. If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations. We, however, will use this model for mechanical systems so that we can write equations directly in terms of displacement.

Another analogy can be drawn by comparing the force-velocity column of Table 2.4 to the current-voltage column of Table 2.3 in reverse order. Here the analogy is between force and current and between velocity and voltage. Also, the

spring is analogous to the inductor, the viscous damper is analogous to the resistor, and the mass is analogous to the capacitor. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical differential equations are analogous to nodal equations. We will discuss these analogies in more detail in Section 2.9.

We are now ready to find transfer functions for translational mechanical systems. Our first example, shown in Figure 2.15(a), is similar to the simple *RLC* network of Example 2.6 (see Figure 2.3). The mechanical system requires just one differential equation, called the *equation of motion*, to describe it. We will begin by assuming a positive direction of motion, for example, to the right. This assumed positive direction of motion is similar to assuming a current direction in an electrical loop. Using our assumed direction of positive motion, we first draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it. Next we use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero. Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function. An example follows.

Example 2.16

Transfer Function—One Equation of Motion

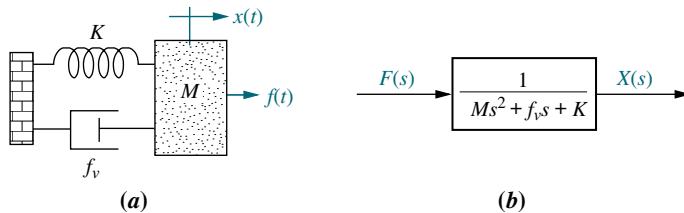


FIGURE 2.15 a. Mass, spring, and damper system; b. block diagram

PROBLEM: Find the transfer function, $X(s)/F(s)$, for the system of Figure 2.15(a).

SOLUTION: Begin the solution by drawing the free-body diagram shown in Figure 2.16(a). Place on the mass all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

We now write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in Figure 2.16(a):

$$M \frac{d^2x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t) \quad (2.108)$$

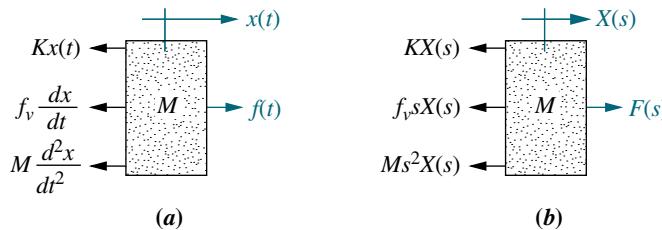


FIGURE 2.16 a. Free-body diagram of mass, spring, and damper system; b. transformed free-body diagram

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v s X(s) + KX(s) = F(s) \quad (2.109)$$

or

$$(Ms^2 + f_v s + K)X(s) = F(s) \quad (2.110)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K} \quad (2.111)$$

which is represented in Figure 2.15(b).

Now can we parallel our work with electrical networks by circumventing the writing of differential equations and by defining impedances for mechanical components? If so, we can apply to mechanical systems the problem-solving techniques learned in the previous section. Taking the Laplace transform of the force-displacement column in Table 2.4, we obtain for the spring,

$$F(s) = KX(s) \quad (2.112)$$

for the viscous damper,

$$F(s) = f_v s X(s) \quad (2.113)$$

and for the mass,

$$F(s) = Ms^2 X(s) \quad (2.114)$$

If we define impedance for mechanical components as

$$Z_M(s) = \frac{F(s)}{X(s)} \quad (2.115)$$

and apply the definition to Eqs. (2.112) through (2.114), we arrive at the impedances of each component as summarized in Table 2.4 (*Raven, 1995*).⁷

Replacing each force in Figure 2.16(a) by its Laplace transform, which is in the format

$$F(s) = Z_M(s)X(s) \quad (2.116)$$

we obtain Figure 2.16(b), from which we could have obtained Eq. (2.109) immediately without writing the differential equation. From now on we use this approach.

⁷Notice that the impedance column of Table 2.4 is not a direct analogy to the impedance column of Table 2.3, since the denominator of Eq. (2.115) is displacement. A direct analogy could be derived by defining mechanical impedance in terms of velocity as $F(s)/V(s)$. We chose Eq. (2.115) as a convenient definition for writing the equations of motion in terms of displacement, rather than velocity. The alternative, however, is available.

Finally, notice that Eq. (2.110) is of the form

$$[\text{Sum of impedances}]X(s) = [\text{Sum of applied forces}] \quad (2.117)$$

which is similar, but not analogous, to a mesh equation (see footnote 7).

Many mechanical systems are similar to multiple-loop and multiple-node electrical networks, where more than one simultaneous differential equation is required to describe the system. In mechanical systems, the number of equations of motion required is equal to the number of *linearly independent* motions. Linear independence implies that a point of motion in a system can still move if all other points of motion are held still. Another name for the number of linearly independent motions is the number of *degrees of freedom*. This discussion is not meant to imply that these motions are not coupled to one another; in general, they are. For example, in a two-loop electrical network, each loop current depends on the other loop current, but if we open-circuit just one of the loops, the other current can still exist if there is a voltage source in that loop. Similarly, in a mechanical system with two degrees of freedom, one point of motion can be held still while the other point of motion moves under the influence of an applied force.

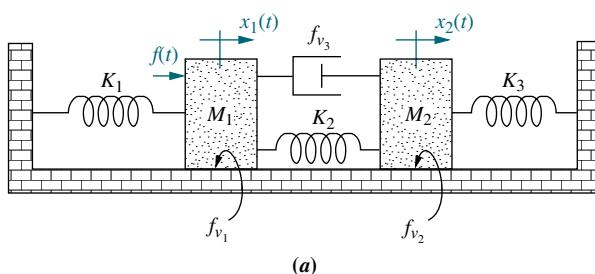
In order to work such a problem, we draw the free-body diagram for each point of motion and then use superposition. For each free-body diagram we begin by holding all other points of motion still and finding the forces acting on the body due only to its own motion. Then we hold the body still and activate the other points of motion one at a time, placing on the original body the forces created by the adjacent motion.

Using Newton's law, we sum the forces on each body and set the sum to zero. The result is a system of simultaneous equations of motion. As Laplace transforms, these equations are then solved for the output variable of interest in terms of the input variable from which the transfer function is evaluated. Example 2.17 demonstrates this problem-solving technique.

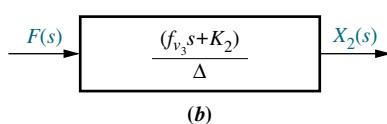
Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).



(a)



(b)

FIGURE 2.17 **a.** Two-degrees-of-freedom translational mechanical system;⁸ **b.** block diagram

Virtual Experiment 2.1 Automobile Suspension

Put theory into practice exploring the dynamics of another two degree of freedom system—an automobile suspension system driving over a bumpy road demonstrated with the Quanser Active Suspension System modeled in LabVIEW.



Virtual experiments are found on WileyPLUS.

⁸Friction shown here and throughout the book, unless otherwise indicated, is viscous friction. Thus, f_{v1} and f_{v2} are not Coulomb friction, but arise because of a viscous interface.

SOLUTION: The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free-body diagrams. For example, the forces on M_1 are due to (1) its own motion and (2) the motion of M_2 transmitted to M_1 through the system. We will consider these two sources separately.

If we hold M_2 still and move M_1 to the right, we see the forces shown in Figure 2.18(a). If we hold M_1 still and move M_2 to the right, we see the forces shown in Figure 2.18(b). The total force on M_1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure 2.18(c). For M_2 , we proceed in a similar fashion: First we move M_2 to the right while holding M_1 still; then we move M_1 to the right and hold M_2 still. For each case we evaluate the forces on M_2 . The results appear in Figure 2.19.

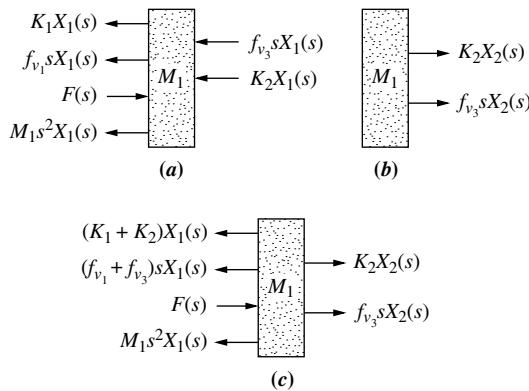


FIGURE 2.18 a. Forces on M_1 due only to motion of M_1 ; b. forces on M_1 due only to motion of M_2 ; c. all forces on M_1

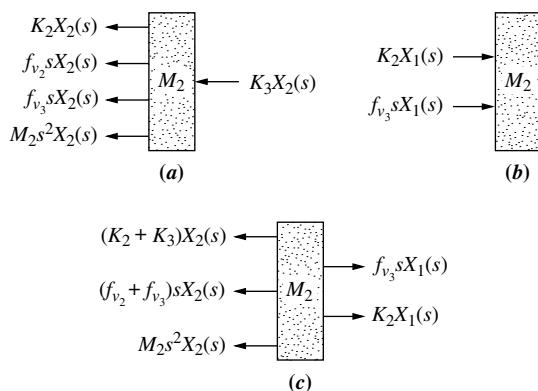


FIGURE 2.19 a. Forces on M_2 due only to motion of M_2 ; b. forces on M_2 due only to motion of M_1 ; c. all forces on M_2

The Laplace transform of the equations of motion can now be written from Figures 2.18(c) and 2.19(c) as

$$[M_1s^2(f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s) \quad (2.118a)$$

$$-(f_{v_3}s + K_2)X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0 \quad (2.118b)$$

From this, the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta} \quad (2.119)$$

as shown in Figure 2.17(b) where

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

Notice again, in Eq. (2.118), that the form of the equations is similar to electrical mesh equations:

$$\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right] \quad (2.120a)$$

$$- \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{array} \right] X_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{array} \right] \quad (2.120b)$$

The pattern shown in Eq. (2.120) should now be familiar to us. Let us use the concept to write the equations of motion of a three-degrees-of-freedom mechanical network by inspection, without drawing the free-body diagram.

Example 2.18

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.

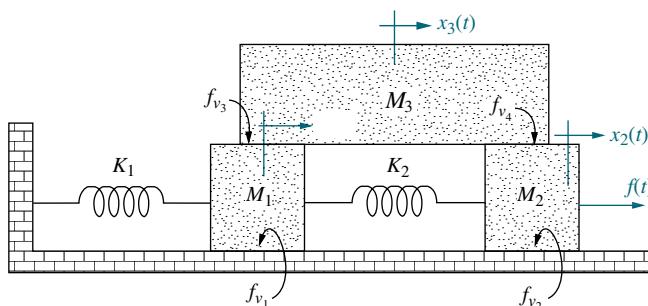


FIGURE 2.20 Three-degrees-of-freedom translational mechanical system

SOLUTION: The system has three degrees of freedom, since each of the three masses can be moved independently while the others are held still. The form of the equations will be similar to electrical mesh equations. For M_1 ,

$$\begin{aligned} \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_2(s) \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_3 \end{array} \right] X_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right] \end{aligned} \quad (2.121)$$

Similarly, for M_2 and M_3 , respectively,

$$\begin{aligned} - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{array} \right] X_2(s) \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_2 \text{ and } x_3 \end{array} \right] X_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{array} \right] \\ - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_3 \end{array} \right] X_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_2 \text{ and } x_3 \end{array} \right] X_2(s) \\ + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_3 \end{array} \right] X_3(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_3 \end{array} \right] \end{aligned} \quad (2.122)$$

M_1 has two springs, two viscous dampers, and mass associated with its motion. There is one spring between M_1 and M_2 and one viscous damper between M_1 and M_3 . Thus, using Eq. (2.121),

$$[M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v_3}s X_3(s) = 0 \quad (2.124)$$

Similarly, using Eq. (2.122) for M_2 ,

$$-K_2 X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}s X_3(s) = F(s) \quad (2.125)$$

and using Eq. (2.123) for M_3 ,

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3s^2 + (f_{v_3} + f_{v_4})s]X_3(s) = 0 \quad (2.126)$$

Equations (2.124) through (2.126) are the equations of motion. We can solve them for any displacement, $X_1(s)$, $X_2(s)$, or $X_3(s)$, or transfer function.

Skill-Assessment Exercise 2.8

PROBLEM: Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

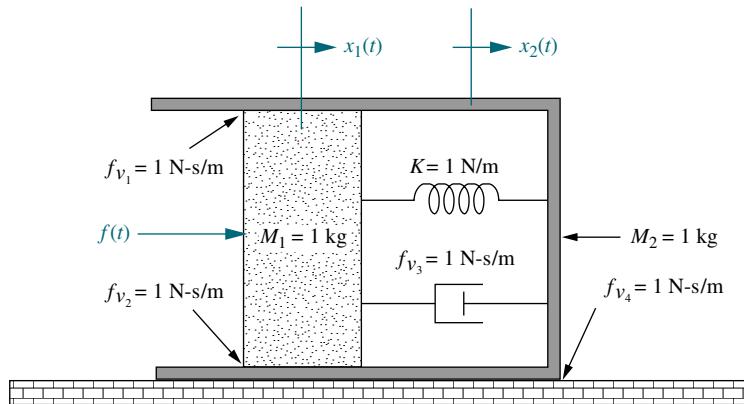


FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

ANSWER: $G(s) = \frac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$

The complete solution is at www.wiley.com/college/nise.

2.6 Rotational Mechanical System Transfer Functions

Having covered electrical and translational mechanical systems, we now move on to consider rotational mechanical systems. Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. Table 2.5 shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance
Spring	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
Viscous damper	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
Inertia	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton-meters/radian), D – N-m-s/rad (newton-meters-seconds/radian). J – kg-m²(kilograms-meters² – newton-meters-seconds²/radian).

components look the same as translational symbols, but they are undergoing rotation and not translation.

Also notice that the term associated with the mass is replaced by inertia. The values of K , D , and J are called *spring constant*, *coefficient of viscous friction*, and *moment of inertia*, respectively. The impedances of the mechanical components are also summarized in the last column of Table 2.5. The values can be found by taking the Laplace transform, assuming zero initial conditions, of the torque-angular displacement column of Table 2.5.

The concept of degrees of freedom carries over to rotational systems, except that we test a point of motion by *rotating* it while holding still all other points of motion. The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.

Writing the equations of motion for rotational systems is similar to writing them for translational systems; the only difference is that the free-body diagram consists of torques rather than forces. We obtain these torques using superposition. First, we rotate a body while holding all other points still and place on its free-body diagram all torques due to the body's own motion. Then, holding the body still, we rotate adjacent points of motion one at a time and add the torques due to the adjacent motion to the free-body diagram. The process is repeated for each point of motion. For each free-body diagram, these torques are summed and set equal to zero to form the equations of motion.

Two examples will demonstrate the solution of rotational systems. The first one uses free-body diagrams; the second uses the concept of impedances to write the equations of motion by inspection.

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

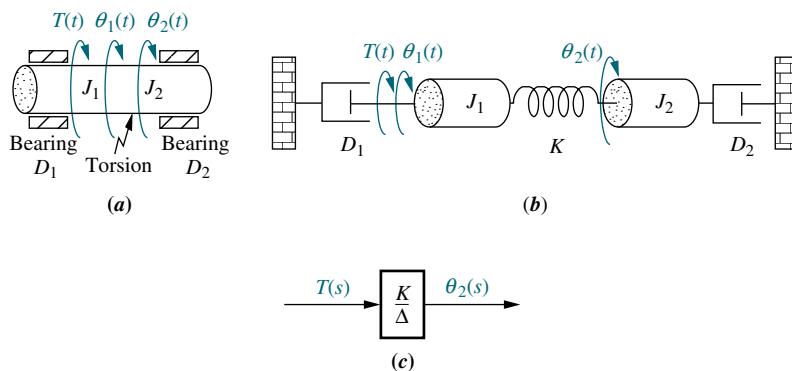


FIGURE 2.22 a. Physical system; b. schematic; c. block diagram

SOLUTION: First, obtain the schematic from the physical system. Even though torsion occurs throughout the rod in Figure 2.22(a),⁹ we approximate the system by assuming that the torsion acts like a spring concentrated at one particular point in the rod, with an inertia J_1 to the left and an inertia J_2 to the right.¹⁰ We also assume that the damping inside the flexible shaft is negligible. The schematic is shown in Figure 2.22(b). There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system.

Next, draw a free-body diagram of J_1 , using superposition. Figure 2.23(a) shows the torques on J_1 if J_2 is held still and J_1 rotated. Figure 2.23(b) shows the torques on J_1 if J_1 is held still and J_2 rotated. Finally, the sum of Figures 2.23(a) and 2.23(b) is shown in Figure 2.23(c), the final free-body diagram for J_1 . The same process is repeated in Figure 2.24 for J_2 .

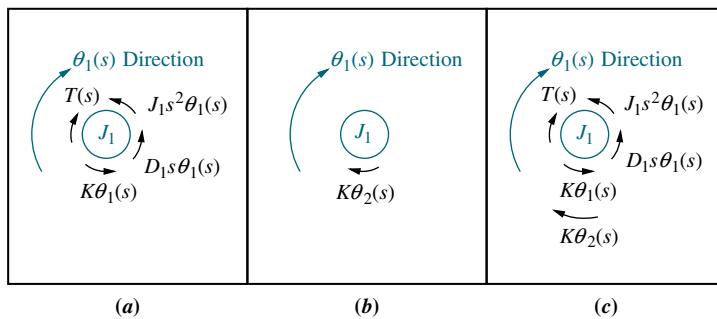
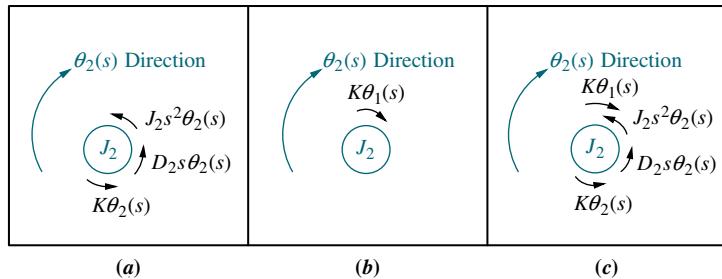


FIGURE 2.23 a. Torques on J_1 due only to the motion of J_1 ; b. torques on J_1 due only to the motion of J_2 ; c. final free-body diagram for J_1

⁹In this case the parameter is referred to as a *distributed* parameter.

¹⁰The parameter is now referred to as a *lumped* parameter.

FIGURE 2.24 a. Torques on J_2 due only to the motion of J_2 ; b. torques on J_2 due only to the motion of J_1 ; c. final free-body diagram for J_2



Summing torques respectively from Figures 2.23(c) and 2.24(c) we obtain the equations of motion,

$$(J_1 s^2 + D_1 s + K) \theta_1(s) - K \theta_2(s) = T(s) \quad (2.127a)$$

$$-K \theta_1(s) + (J_2 s^2 + D_2 s + K) \theta_2(s) = 0 \quad (2.127b)$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \quad (2.128)$$

as shown in Figure 2.22(c), where

$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

Notice that Eq. (2.127) have that now well-known form

$$\left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{array} \right] \theta_1(s) - \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{array} \right] \quad (2.129a)$$

$$- \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{array} \right] \theta_1(s) + \left[\begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{array} \right] \theta_2(s) = \left[\begin{array}{c} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{array} \right] \quad (2.129b)$$

Example 2.20

Equations of Motion By Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

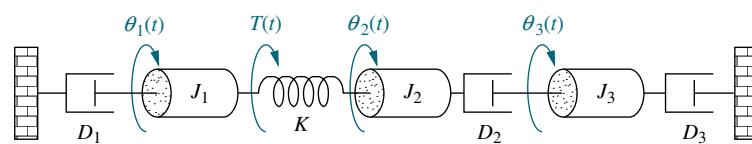


FIGURE 2.25 Three-degrees-of-freedom rotational system

SOLUTION: The equations will take on the following form, similar to electrical mesh equations:

$$\begin{bmatrix} \text{Sum of impedances connected to the motion at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of impedances between } \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) \\ - \begin{bmatrix} \text{Sum of impedances between } \theta_1 \text{ and } \theta_3 \end{bmatrix} \theta_3(s) = \begin{bmatrix} \text{Sum of applied torques at } \theta_1 \end{bmatrix} \quad (2.130a)$$

$$-\begin{bmatrix} \text{Sum of impedances between } \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of impedances connected to the motion at } \theta_2 \end{bmatrix} \theta_2(s) \\ - \begin{bmatrix} \text{Sum of impedances between } \theta_2 \text{ and } \theta_3 \end{bmatrix} \theta_3(s) = \begin{bmatrix} \text{Sum of applied torques at } \theta_2 \end{bmatrix} \quad (2.130b)$$

$$-\begin{bmatrix} \text{Sum of impedances between } \theta_1 \text{ and } \theta_3 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of impedances between } \theta_2 \text{ and } \theta_3 \end{bmatrix} \theta_2(s) \\ + \begin{bmatrix} \text{Sum of impedances connected to the motion at } \theta_3 \end{bmatrix} \theta_3(s) = \begin{bmatrix} \text{Sum of applied torques at } \theta_3 \end{bmatrix} \quad (2.130c)$$

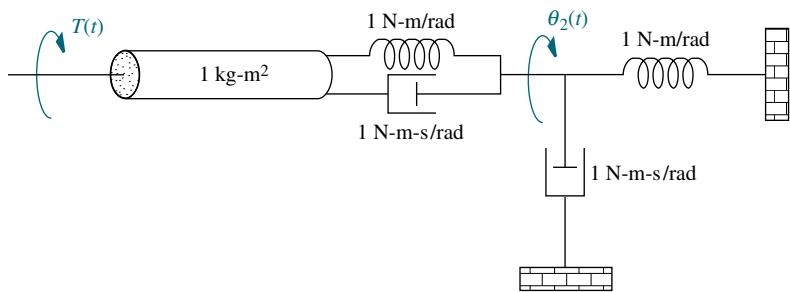
Hence,

$$\begin{aligned} (J_1 s^2 + D_1 s + K) \theta_1(s) & -K \theta_2(s) & -0 \theta_3(s) = T(s) \\ -K \theta_1(s) + (J_2 s^2 + D_2 s + K) \theta_2(s) & & -D_2 s \theta_3(s) = 0 \\ -0 \theta_1(s) & -D_2 s \theta_2(s) + (J_3 s^2 + D_3 s + D_2 s) \theta_3(s) = 0 \end{aligned} \quad (2.131a, b, c)$$

Skill-Assessment Exercise 2.9

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure 2.26.

FIGURE 2.26 Rotational mechanical system for Skill-Assessment Exercise 2.9



ANSWER: $G(s) = \frac{1}{2s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

2.7 Transfer Functions for Systems with Gears

Now that we are able to find the transfer function for rotational systems, we realize that these systems, especially those driven by motors, are rarely seen without associated gear trains driving the load. This section covers this important topic.

Gears provide mechanical advantage to rotational systems. Anyone who has ridden a 10-speed bicycle knows the effect of gearing. Going uphill, you shift to provide more torque and less speed. On the straightaway, you shift to obtain more speed and less torque. Thus, gears allow you to match the drive system and the load—a trade-off between speed and torque.

For many applications, gears exhibit *backlash*, which occurs because of the loose fit between two meshed gears. The drive gear rotates through a small angle before making contact with the meshed gear. The result is that the angular rotation of the output gear does not occur until a small angular rotation of the input gear has occurred. In this section, we idealize the behavior of gears and assume that there is no backlash.

The linearized interaction between two gears is depicted in Figure 2.27. An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$. Let us now find the relationship between the rotation of Gear 1, $\theta_1(t)$, and Gear 2, $\theta_2(t)$.

From Figure 2.27, as the gears turn, the distance traveled along each gear's circumference is the same. Thus,

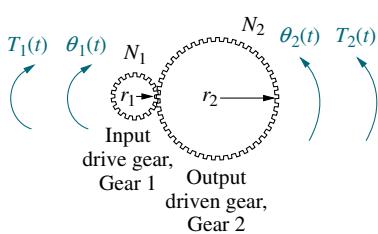


FIGURE 2.27 A gear system

$$r_1\theta_1 = r_2\theta_2 \quad (2.132)$$

or

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} \quad (2.133)$$

since the ratio of the number of teeth along the circumference is in the same proportion as the ratio of the radii. We conclude that the ratio of the angular displacement of the gears is inversely proportional to the ratio of the number of teeth.

What is the relationship between the input torque, T_1 , and the delivered torque, T_2 ? If we assume the gears are *lossless*, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2.¹¹ Since the translational energy of force times displacement becomes the rotational energy of torque times angular displacement,

$$T_1\theta_1 = T_2\theta_2 \quad (2.134)$$

Solving Eq. (2.134) for the ratio of the torques and using Eq. (2.133), we get

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} \quad (2.135)$$

Thus, the torques are directly proportional to the ratio of the number of teeth. All results are summarized in Figure 2.28.

Let us see what happens to mechanical impedances that are driven by gears. Figure 2.29(a) shows gears driving a rotational inertia, spring, and viscous damper. For clarity, the gears are shown by an end-on view. We want to represent Figure 2.29(a) as an equivalent system at θ_1 without the gears. In other words, can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?

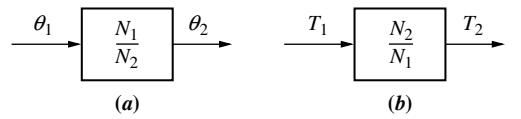


FIGURE 2.28 Transfer functions for **a.** angular displacement in lossless gears and **b.** torque in lossless gears

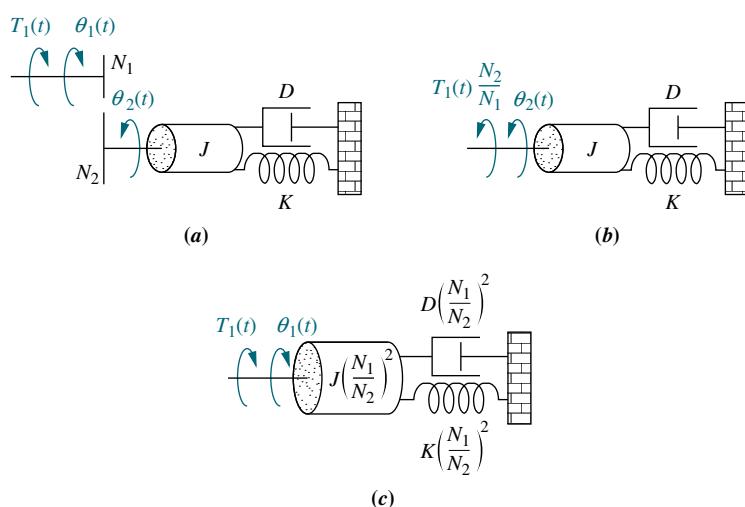


FIGURE 2.29 **a.** Rotational system driven by gears; **b.** equivalent system at the output after reflection of input torque; **c.** equivalent system at the input after reflection of impedances

¹¹This is equivalent to saying that the gears have negligible inertia and damping.

From Figure 2.28(b), T_1 can be reflected to the output by multiplying by N_2/N_1 . The result is shown in Figure 2.29(b), from which we write the equation of motion as

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s) \frac{N_2}{N_1} \quad (2.136)$$

Now convert $\theta_2(s)$ into an equivalent $\theta_1(s)$, so that Eq. (2.136) will look as if it were written at the input. Using Figure 2.28(a) to obtain $\theta_2(s)$ in terms of $\theta_1(s)$, we get

$$(Js^2 + Ds + K) \frac{N_1}{N_2} \theta_1(s) = T_1(s) \frac{N_2}{N_1} \quad (2.137)$$

After simplification,

$$\left[J \left(\frac{N_1}{N_2} \right)^2 s^2 + D \left(\frac{N_1}{N_2} \right)^2 s + K \left(\frac{N_1}{N_2} \right)^2 \right] \theta_1(s) = T_1(s) \quad (2.138)$$

which suggests the equivalent system at the input and without gears shown in Figure 2.29(c). Thus, the load can be thought of as having been reflected from the output to the input.

Generalizing the results, we can make the following statement: *Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio*

$$\left(\frac{\text{Number of teeth of gear on destination shaft}}{\text{Number of teeth of gear on source shaft}} \right)^2$$

where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft. The next example demonstrates the application of the concept of reflected impedances as we find the transfer function of a rotational mechanical system with gears.

Example 2.21

Transfer Function—System with Lossless Gears

PROBLEM: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure 2.30(a).

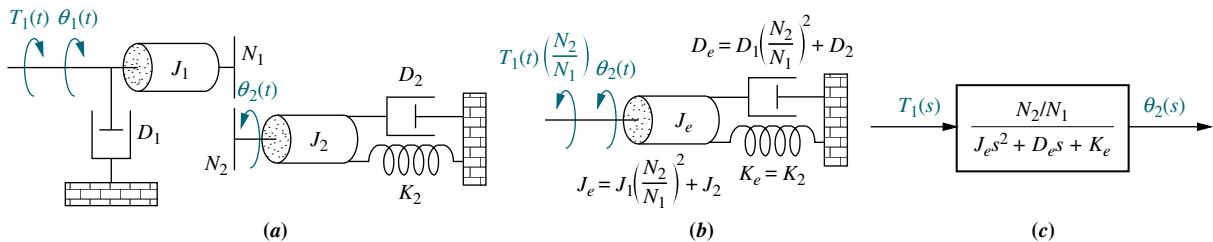


FIGURE 2.30 **a.** Rotational mechanical system with gears; **b.** system after reflection of torques and impedances to the output shaft; **c.** block diagram

SOLUTION: It may be tempting at this point to search for two simultaneous equations corresponding to each inertia. The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears. Thus, there is only one degree of freedom and hence one equation of motion.

Let us first reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output as shown in Figure 2.30(b), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1) . The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e) \theta_2(s) = T_1(s) \frac{N_2}{N_1} \quad (2.139)$$

where

$$J_e = J_1 \left(\frac{N_2}{N_1} \right)^2 + J_2; \quad D_e = D_1 \left(\frac{N_2}{N_1} \right)^2 + D_2; \quad K_e = K_2$$

Solving for $\theta_2(s)/T_1(s)$, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e} \quad (2.140)$$

as shown in Figure 2.30(c).

In order to eliminate gears with large radii, a *gear train* is used to implement large gear ratios by cascading smaller gear ratios. A schematic diagram of a gear train is shown in Figure 2.31. Next to each rotation, the angular displacement relative to θ_1 has been calculated. From Figure 2.31,

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6} \theta_1 \quad (2.141)$$

For gear trains, we conclude that the equivalent gear ratio is the product of the individual gear ratios. We now apply this result to solve for the transfer function of a system that does not have lossless gears.

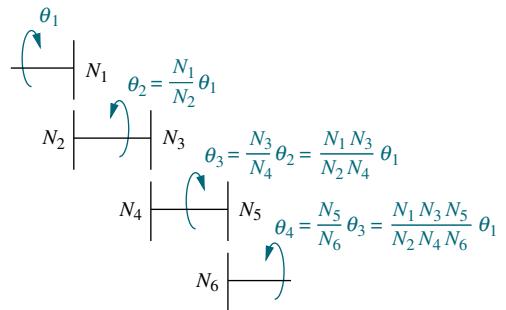
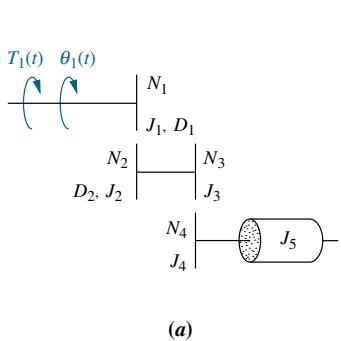


FIGURE 2.31 Gear train

Example 2.22

Transfer Function—Gears with Loss

PROBLEM: Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure 2.32(a).



$$\begin{aligned} & T_1(t) \quad \theta_1(t) \\ & \left| \begin{array}{c} N_1 \\ J_1, D_1 \end{array} \right. \\ & \left| \begin{array}{c} N_2 \\ D_2, J_2 \end{array} \right. \left| \begin{array}{c} N_3 \\ J_3 \end{array} \right. \\ & \left| \begin{array}{c} N_4 \\ J_4 \end{array} \right. \left| \begin{array}{c} N_5 \\ J_5 \end{array} \right. \end{aligned}$$

$$\begin{aligned} & J_e = J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2} \right)^2 + (J_4 + J_5) \left(\frac{N_1 N_3}{N_2 N_4} \right)^2 \\ & D_e = D_1 + D_2 \left(\frac{N_1}{N_2} \right)^2 \end{aligned}$$

$$\xrightarrow{T_1(s)} \boxed{\frac{1}{J_e s^2 + D_e s}} \xrightarrow{\theta_1(s)}$$

FIGURE 2.32
a. System using a gear train; **b.** equivalent system at the input; **c.** block diagram

SOLUTION: This system, which uses a gear train, does not have lossless gears. All of the gears have inertia, and for some shafts there is viscous friction. To solve the problem, we want to reflect all of the impedances to the input shaft, θ_1 . The gear ratio is not the same for all impedances. For example, D_2 is reflected only through one gear ratio as $D_2(N_1/N_2)^2$, whereas J_4 plus J_5 is reflected through two gear ratios as $(J_4 + J_5)[(N_3/N_4)(N_1/N_2)]^2$. The result of reflecting all impedances to θ_1 is shown in Figure 2.32(b), from which the equation of motion is

$$(J_e s^2 + D_e s) \theta_1(s) = T_1(s) \quad (2.142)$$

where

$$J_e = J_1 + (J_2 + J_3) \left(\frac{N_1}{N_2} \right)^2 + (J_4 + J_5) \left(\frac{N_1 N_3}{N_2 N_4} \right)^2$$

and

$$D_e = D_1 + D_2 \left(\frac{N_1}{N_2} \right)^2$$

From Eq. (2.142), the transfer function is

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s} \quad (2.143)$$

as shown in Figure 2.32(c).

Skill-Assessment Exercise 2.10

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system with gears shown in Figure 2.33.

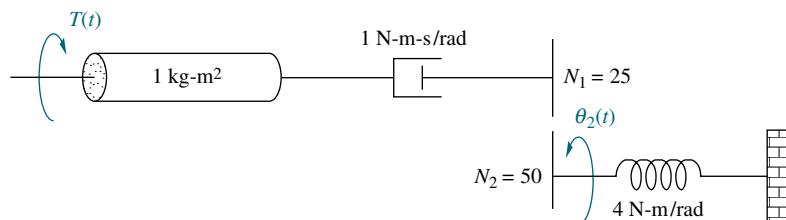


FIGURE 2.33 Rotational mechanical system with gears for Skill-Assessment Exercise 2.10

ANSWER: $G(s) = \frac{1/2}{s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

2.8 Electromechanical System Transfer Functions

In the last section we talked about rotational systems with gears, which completed our discussion of purely mechanical systems. Now, we move to systems that are hybrids of electrical and mechanical variables, the *electromechanical systems*. We have seen one application of an electromechanical system in Chapter 1, the antenna azimuth position control system. Other applications for systems with electromechanical components are robot controls, sun and star trackers, and computer tape and disk-drive position controls. An example of a control system that uses electromechanical components is shown in Figure 2.34.

A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input. We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor (*Mablekos, 1980*). The motor's schematic is shown in Figure 2.35(a), and the transfer function we will derive appears in Figure 2.35(b).

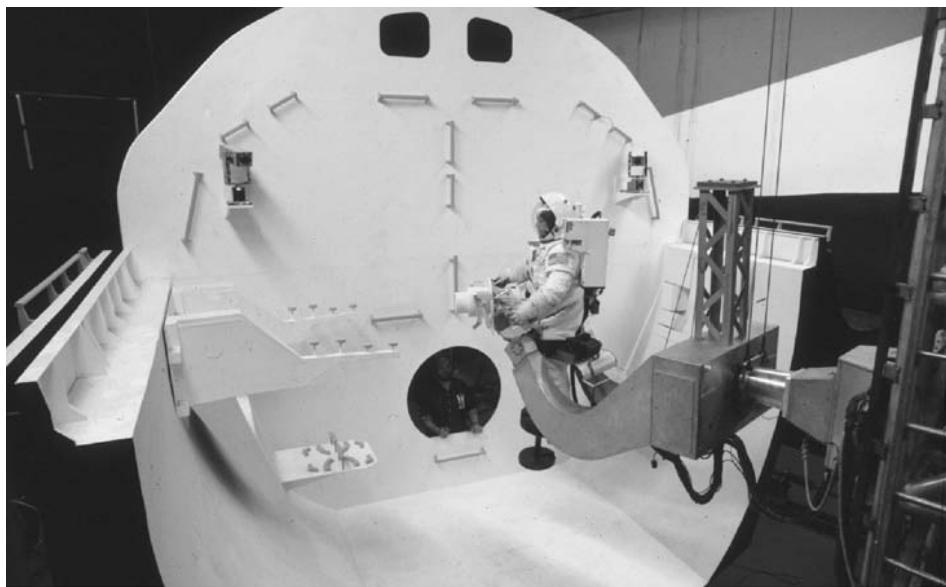


FIGURE 2.34 NASA flight simulator robot arm with electromechanical control system components.

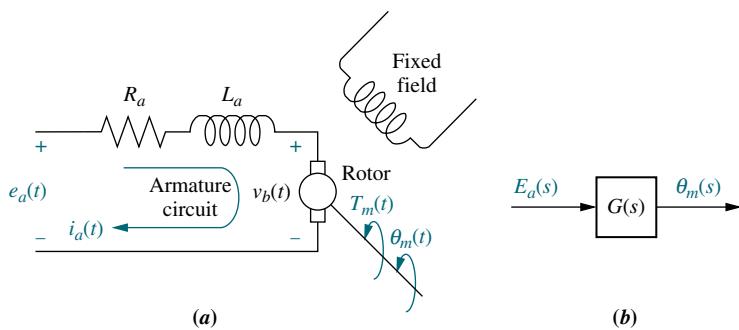


FIGURE 2.35 DC motor: **a.** schematic;¹² **b.** block diagram

¹² See Appendix I at www.wiley.com/college/nise for a derivation of this schematic and its parameters.

In Figure 2.35(a) a magnetic field is developed by stationary permanent magnets or a stationary electromagnet called the *fixed field*. A rotating circuit called the *armature*, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $F = Bl i_a(t)$, where B is the magnetic field strength and l is the length of the conductor. The resulting torque turns the *rotor*, the rotating member of the motor.

There is another phenomenon that occurs in the motor: A conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to $e = Blv$, where e is the voltage and v is the velocity of the conductor normal to the magnetic field. Since the current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed. Thus,

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \quad (2.144)$$

We call $v_b(t)$ the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant; and $d\theta_m(t)/dt = \omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s \theta_m(s) \quad (2.145)$$

The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$, is found by writing a loop equation around the Laplace transformed armature circuit (see Figure 3.5(a)):

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s) \quad (2.146)$$

The torque developed by the motor is proportional to the armature current; thus,

$$T_m(s) = K_t I_a(s) \quad (2.147)$$

where T_m is the torque developed by the motor, and K_t is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of K_t is equal to the value of K_b . Rearranging Eq. (2.147) yields

$$I_a(s) = \frac{1}{K_t} T_m(s) \quad (2.148)$$

To find the transfer function of the motor, we first substitute Eqs. (2.145) and (2.148) into (2.146), yielding

$$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s) \quad (2.149)$$

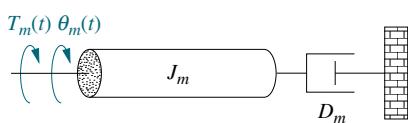


FIGURE 2.36 Typical equivalent mechanical loading on a motor

Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to separate the input and output variables and obtain the transfer function, $\theta_m(s)/E_a(s)$.

Figure 2.36 shows a typical equivalent mechanical loading on a motor. J_m is the equivalent inertia at the armature and includes both the armature inertia and, as we will see later, the load inertia reflected to the armature.

D_m is the equivalent viscous damping at the armature and includes both the armature viscous damping and, as we will see later, the load viscous damping reflected to the armature. From Figure 2.36,

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s) \quad (2.150)$$

Substituting Eq. (2.150) into Eq. (2.149) yields

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \quad (2.151)$$

If we assume that the armature inductance, L_a , is small compared to the armature resistance, R_a , which is usual for a dc motor, Eq. (2.151) becomes

$$\left[\frac{R_a}{K_t} (J_m s + D_m) + K_b \right] s\theta_m(s) = E_a(s) \quad (2.152)$$

After simplification, the desired transfer function, $\theta_m(s)/E_a(s)$, is found to be

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t / (R_a J_m)}{s \left[s + \frac{1}{J_m} (D_m + \frac{K_t K_b}{R_a}) \right]} \quad (2.153)^{13}$$

Even though the form of Eq. (2.153) is relatively simple, namely

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s + \alpha)} \quad (2.154)$$

the reader may be concerned about how to evaluate the constants.

Let us first discuss the mechanical constants, J_m and D_m . Consider Figure 2.37, which shows a motor with inertia J_a and damping D_a at the armature driving a load consisting of inertia J_L and damping D_L . Assuming that all inertia and damping values shown are known, J_L and D_L can be reflected back to the armature as some equivalent inertia and damping to be added to J_a and D_a , respectively. Thus, the equivalent inertia, J_m , and equivalent damping, D_m , at the armature are

$$J_m = J_a + J_L \left(\frac{N_1}{N_2} \right)^2; \quad D_m = D_a + D_L \left(\frac{N_1}{N_2} \right)^2 \quad (2.155)^{14}$$

Now that we have evaluated the mechanical constants, J_m and D_m , what about the electrical constants in the transfer function of Eq. (2.153)? We will show that these constants can be obtained through a *dynamometer* test of the motor, where a dynamometer measures the torque and speed of a motor under the condition of a

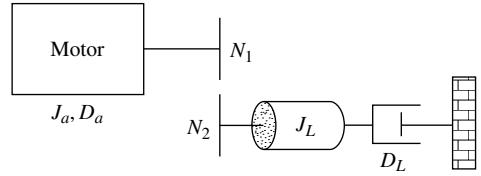


FIGURE 2.37 DC motor driving a rotational mechanical load

¹³The units for the electrical constants are K_t = N-m-A (newton-meters/ampere), and K_b = V-s/rad (volt-seconds/radian).

¹⁴If the values of the mechanical constants are not known, motor constants can be determined through laboratory testing using transient response or frequency response data. The concept of transient response is covered in Chapter 4; frequency response is covered in Chapter 10.

constant applied voltage. Let us first develop the relationships that dictate the use of a dynamometer.

Substituting Eqs. (2.145) and (2.148) into Eq. (2.146), with $L_a = 0$, yields

$$\frac{R_a}{K_t} T_m(s) + K_b s \theta_m(s) = E_a(s) \quad (2.156)$$

Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t} T_m(t) + K_b \omega_m(t) = e_a(t) \quad (2.157)$$

where the inverse Laplace transform of $s\theta_m(s)$ is $d\theta_m(t)/dt$ or, alternately, $\omega_m(t)$.

If a dc voltage, e_a , is applied, the motor will turn at a constant angular velocity, ω_m , with a constant torque, T_m . Hence, dropping the functional relationship based on time from Eq. (2.157), the following relationship exists when the motor is operating at steady state with a dc voltage input:

$$\frac{R_a}{K_t} T_m + K_b \omega_m = e_a \quad (2.158)$$

Solving for T_m yields

$$T_m = -\frac{K_b K_t}{R_a} \omega_m + \frac{K_t}{R_a} e_a \quad (2.159)$$

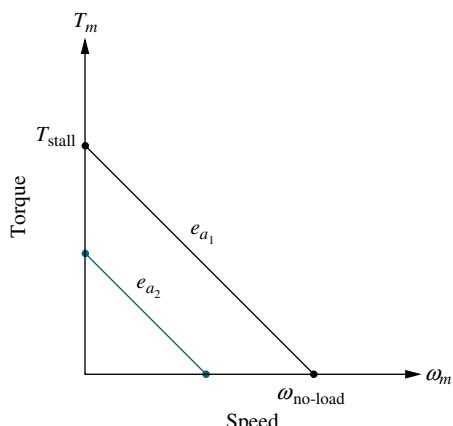


FIGURE 2.38 Torque-speed curves with an armature voltage, e_a , as a parameter

Equation (2.159) is a straight line, T_m vs. ω_m , and is shown in Figure 2.38. This plot is called the *torque-speed curve*. The torque axis intercept occurs when the angular velocity reaches zero. That value of torque is called the *stall torque*, T_{stall} . Thus,

$$T_{\text{stall}} = \frac{K_t}{R_a} e_a \quad (2.160)$$

The angular velocity occurring when the torque is zero is called the *no-load speed*, $\omega_{\text{no-load}}$. Thus,

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \quad (2.161)$$

The electrical constants of the motor's transfer function can now be found from Eqs. (2.160) and (2.161) as

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} \quad (2.162)$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} \quad (2.163)$$

The electrical constants, K_t/R_a and K_b , can be found from a dynamometer test of the motor, which would yield T_{stall} and $\omega_{\text{no-load}}$ for a given e_a .

Example 2.23

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION: Begin by finding the mechanical constants, J_m and D_m , in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

$$J_m = J_a + J_L \left(\frac{N_1}{N_2} \right)^2 = 5 + 700 \left(\frac{1}{10} \right)^2 = 12 \quad (2.164)$$

and the total damping at the armature of the motor is

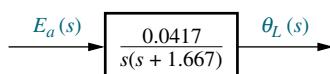
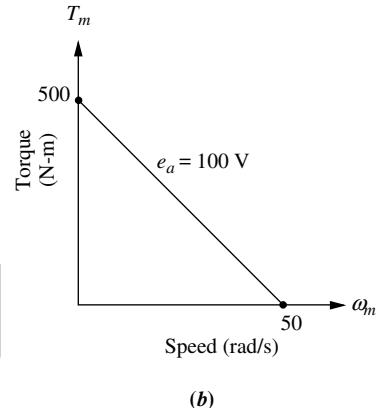
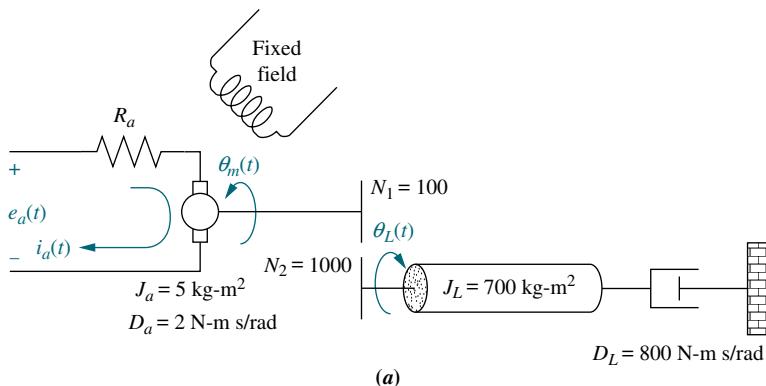
$$D_m = D_a + D_L \left(\frac{N_1}{N_2} \right)^2 = 2 + 800 \left(\frac{1}{10} \right)^2 = 10 \quad (2.165)$$

Now we will find the electrical constants, K_t/R_a and K_b . From the torque-speed curve of Figure 2.39(b),

$$T_{\text{stall}} = 500 \quad (2.166)$$

$$\omega_{\text{no-load}} = 50 \quad (2.167)$$

$$e_a = 100 \quad (2.168)$$



(c)

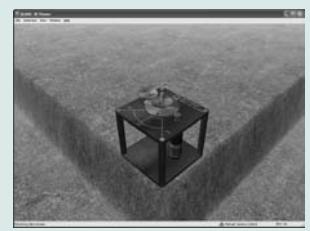
FIGURE 2.39 **a.** DC motor and load; **b.** torque-speed curve; **c.** block diagram

Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \quad (2.169)$$

Virtual Experiment 2.2 Open-Loop Servo Motor

Put theory into practice exploring the dynamics of the Quanser Rotary Servo System modeled in LabVIEW. It is particularly important to know how a servo motor behaves when using them in high-precision applications such as hard disk drives.



Virtual experiments are found on WileyPLUS.

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2 \quad (2.170)$$

Substituting Eqs. (2.164), (2.165), (2.169), and (2.170) into Eq. (2.153) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s \left\{ s + \frac{1}{12} [10 + (5)(2)] \right\}} = \frac{0.417}{s(s + 1.667)} \quad (2.171)$$

In order to find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s + 1.667)} \quad (2.172)$$

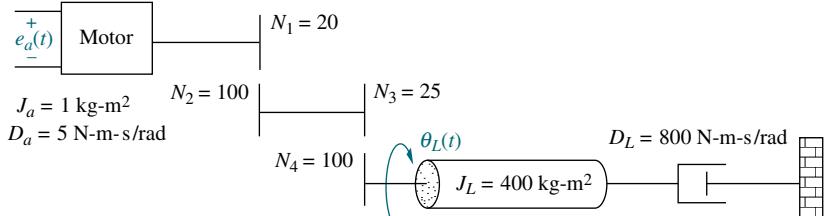
as shown in Figure 2.39(c).

Skill-Assessment Exercise 2.11

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Control Solutions

FIGURE 2.40 Electro-mechanical system for Skill-Assessment Exercise 2.11

PROBLEM: Find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, for the motor and load shown in Figure 2.40. The torque-speed curve is given by $T_m = -8\omega_m + 200$ when the input voltage is 100 volts.



ANSWER: $G(s) = \frac{1/20}{s[s + (15/2)]}$

The complete solution is at www.wiley.com/college/nise.

2.9 Electric Circuit Analogs

In this section, we show the commonality of systems from the various disciplines by demonstrating that the mechanical systems with which we worked can be represented by equivalent electric circuits. We have pointed out the similarity between the equations resulting from Kirchhoff's laws for electrical systems and the equations of motion of mechanical systems. We now show this commonality even more convincingly by producing electric circuit equivalents for mechanical systems. The variables of the electric circuits behave exactly as the analogous

variables of the mechanical systems. In fact, converting mechanical systems to electrical networks before writing the describing equations is a problem-solving approach that you may want to pursue.

An electric circuit that is analogous to a system from another discipline is called an electric circuit *analog*. Analogs can be obtained by comparing the describing equations, such as the equations of motion of a mechanical system, with either electrical mesh or nodal equations. When compared with mesh equations, the resulting electrical circuit is called a *series analog*. When compared with nodal equations, the resulting electrical circuit is called a *parallel analog*.

Series Analog

Consider the translational mechanical system shown in Figure 2.41(a), whose equation of motion is

$$(Ms^2 + f_v s + K)X(s) = F(s) \quad (2.173)$$

Kirchhoff's mesh equation for the simple series *RLC* network shown in Figure 2.41(b) is

$$\left(Ls + R + \frac{1}{Cs} \right) I(s) = E(s) \quad (2.174)$$

As we previously pointed out, Eq. (2.173) is not directly analogous to Eq. (2.174) because displacement and current are not analogous. We can create a direct analogy by operating on Eq. (2.173) to convert displacement to velocity by dividing and multiplying the left-hand side by s , yielding

$$\frac{Ms^2 + f_v s + K}{s} s X(s) = \left(Ms + f_v + \frac{K}{s} \right) V(s) = F(s) \quad (2.175)$$

Comparing Eqs. 2.174 and 2.175, we recognize the sum of impedances and draw the circuit shown in Figure 2.41(c). The conversions are summarized in Figure 2.41(d).

When we have more than one degree of freedom, the impedances associated with a motion appear as series electrical elements in a mesh, but

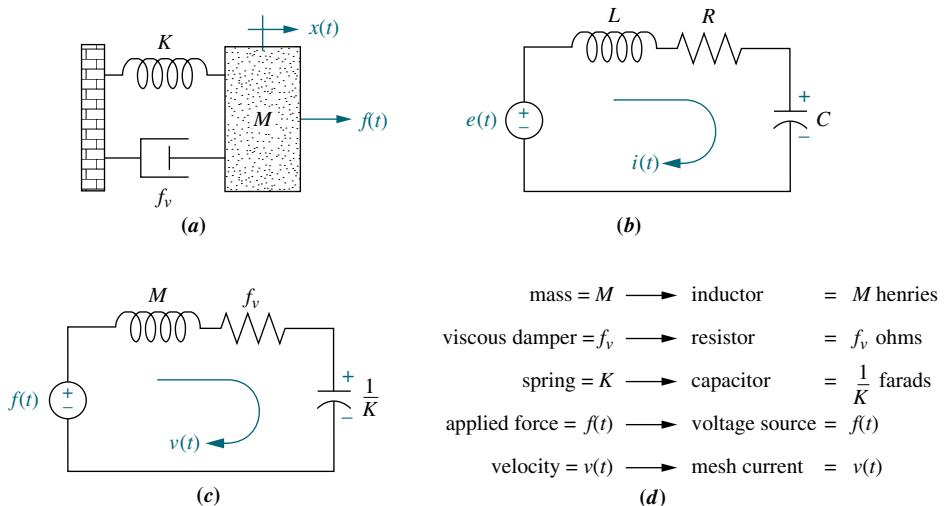


FIGURE 2.41 Development of series analog: **a**, mechanical system; **b**, desired electrical representation; **c**, series analog; **d**, parameters for series analog

the impedances between adjacent motions are drawn as series electrical impedances between the two corresponding meshes. We demonstrate with an example.

Example 2.24

Converting a Mechanical System to a Series Analog

PROBLEM: Draw a series analog for the mechanical system of Figure 2.17(a).

SOLUTION: Equations (2.118) are analogous to electrical mesh equations after conversion to velocity. Thus,

$$\left[M_1 s + \left(f_{v_1} + f_{v_3} \right) + \frac{(K_1 + K_2)}{s} \right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s} \right) V_2(s) = F(s) \quad (2.176a)$$

$$- \left(f_{v_3} + \frac{K_2}{s} \right) V_1(s) + \left[M_2 s + \left(f_{v_2} + f_{v_3} \right) + \frac{(K_2 + K_3)}{s} \right] V_2(s) = 0 \quad (2.176b)$$

Coefficients represent sums of electrical impedance. Mechanical impedances associated with M_1 form the first mesh, where impedances between the two masses are common to the two loops. Impedances associated with M_2 form the second mesh. The result is shown in Figure 2.42, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2 , respectively.

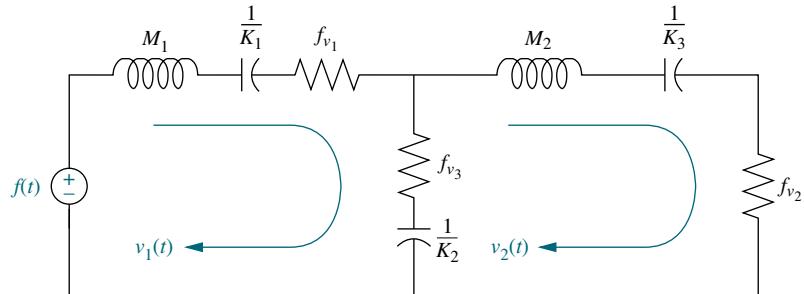


FIGURE 2.42 Series analog of mechanical system of Figure 2.17(a)

Parallel Analog

A system can also be converted to an equivalent parallel analog. Consider the translational mechanical system shown in Figure 2.43(a), whose equation of motion is given by Eq. (2.175). Kirchhoff's nodal equation for the simple parallel *RLC* network shown in Figure 2.43(b) is

$$\left(Cs + \frac{1}{R} + \frac{1}{Ls} \right) E(s) = I(s) \quad (2.177)$$

Comparing Eqs. (2.175) and (2.177), we identify the sum of admittances and draw the circuit shown in Figure 2.43(c). The conversions are summarized in Figure 2.43(d).

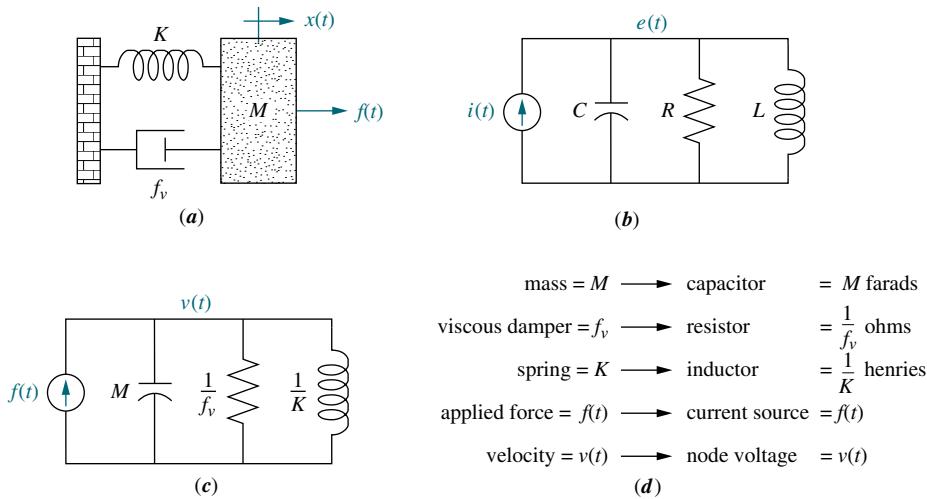


FIGURE 2.43 Development of parallel analog: **a.** mechanical system; **b.** desired electrical representation; **c.** parallel analog; **d.** parameters for parallel analog

When we have more than one degree of freedom, the components associated with a motion appear as parallel electrical elements connected to a node, but the components of adjacent motions are drawn as parallel electrical elements between two corresponding nodes. We demonstrate with an example.

Example 2.25

Converting a Mechanical System to a Parallel Analog

PROBLEM: Draw a parallel analog for the mechanical system of Figure 2.17(a).

SOLUTION: Equation (2.176) is also analogous to electrical node equations. Coefficients represent sums of electrical admittances. Admittances associated with M_1 form the elements connected to the first node, where mechanical admittances between the two masses are common to the two nodes. Mechanical admittances associated with M_2 form the elements connected to the second node. The result is shown in Figure 2.44, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2 , respectively.

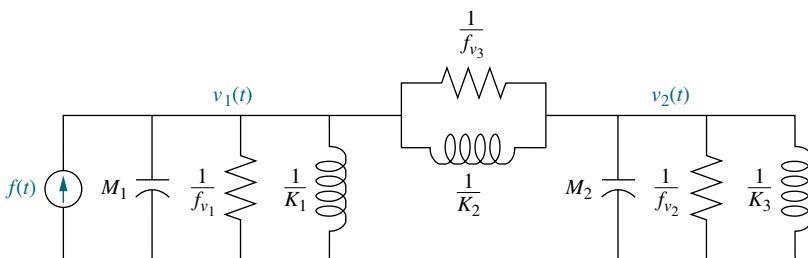


FIGURE 2.44 Parallel analog of mechanical system of Figure 2.17(a)

Skill-Assessment Exercise 2.12

PROBLEM: Draw a series and parallel analog for the rotational mechanical system of Figure 2.22.

ANSWER: The complete solution is at www.wiley.com/college/nise.

2.10 Nonlinearities

The models thus far are developed from systems that can be described approximately by linear, time-invariant differential equations. An assumption of *linearity* was implicit in the development of these models. In this section, we formally define the terms *linear* and *nonlinear* and show how to distinguish between the two. In Section 2.11, we show how to approximate a nonlinear system as a linear system so that we can use the modeling techniques previously covered in this chapter (Hsu, 1968).

A linear system possesses two properties: superposition and homogeneity. The property of *superposition* means that the output response of a system to the sum of inputs is the sum of the responses to the individual inputs. Thus, if an input of $r_1(t)$ yields an output of $c_1(t)$ and an input of $r_2(t)$ yields an output of $c_2(t)$, then an input of $r_1(t) + r_2(t)$ yields an output of $c_1(t) + c_2(t)$. The property of *homogeneity* describes the response of the system to a multiplication of the input by a scalar. Specifically, in a linear system, the property of homogeneity is demonstrated if for an input of $r_1(t)$ that yields an output of $c_1(t)$, an input of $Ar_1(t)$ yields an output of $Ac_1(t)$; that is, multiplication of an input by a scalar yields a response that is multiplied by the same scalar.

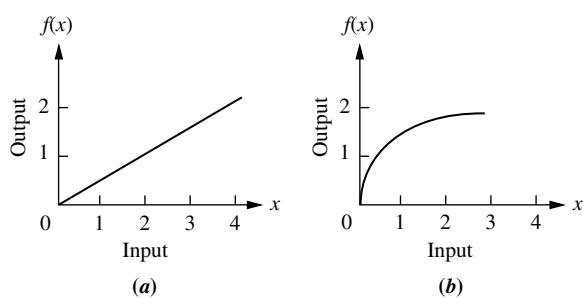


FIGURE 2.45 **a.** Linear system; **b.** nonlinear system

We can visualize linearity as shown in Figure 2.45. Figure 2.45(a) is a linear system where the output is always $\frac{1}{2}$ the input, or $f(x) = 0.5x$, regardless of the value of x . Thus each of the two properties of linear systems applies. For example, an input of 1 yields an output of $\frac{1}{2}$ and an input of 2 yields an output of 1. Using superposition, an input that is the sum of the original inputs, or 3, should yield an output that is the sum of the individual outputs, or 1.5. From Figure 2.45(a), an input of 3 does indeed yield an output of 1.5.

To test the property of homogeneity, assume an input of 2, which yields an output of 1. Multiplying this input by 2 should yield an output of twice as much, or 2. From

Figure 2.45(a), an input of 4 does indeed yield an output of 2. The reader can verify that the properties of linearity certainly do not apply to the relationship shown in Figure 2.45(b).

Figure 2.46 shows some examples of physical nonlinearities. An electronic amplifier is linear over a specific range but exhibits the nonlinearity called *saturation* at high input voltages. A motor that does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone*. Gears that do not fit tightly exhibit a nonlinearity called *backlash*: The input moves over a small range

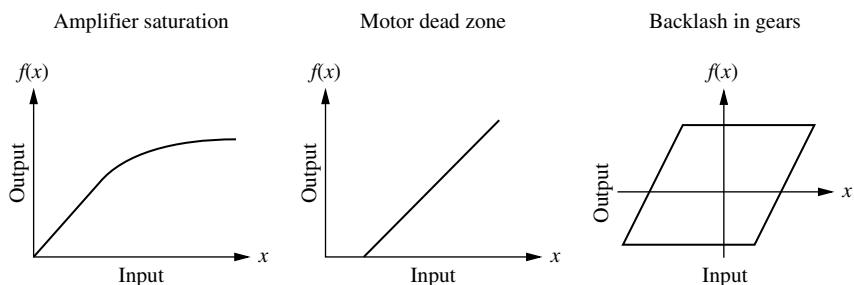


FIGURE 2.46 Some physical nonlinearities

without the output responding. The reader should verify that the curves shown in Figure 2.46 do not fit the definitions of linearity over their entire range. Another example of a nonlinear subsystem is a phase detector, used in a phase-locked loop in an FM radio receiver, whose output response is the sine of the input.

A designer can often make a linear approximation to a nonlinear system. Linear approximations simplify the analysis and design of a system and are used as long as the results yield a good approximation to reality. For example, a linear relationship can be established at a point on the nonlinear curve if the range of input values about that point is small and the origin is translated to that point. Electronic amplifiers are an example of physical devices that perform linear amplification with small excursions about a point.

2.11 Linearization

The electrical and mechanical systems covered thus far were assumed to be linear. However, if any nonlinear components are present, we must linearize the system before we can find the transfer function. In the last section, we defined and discussed nonlinearities; in this section, we show how to obtain linear approximations to nonlinear systems in order to obtain transfer functions.

The first step is to recognize the nonlinear component and write the nonlinear differential equation. When we linearize a nonlinear differential equation, we linearize it for small-signal inputs about the steady-state solution when the small-signal input is equal to zero. This steady-state solution is called *equilibrium* and is selected as the second step in the linearization process. For example, when a pendulum is at rest, it is at equilibrium. The angular displacement is described by a nonlinear differential equation, but it can be expressed with a linear differential equation for small excursions about this equilibrium point.

Next we linearize the nonlinear differential equation, and then we take the Laplace transform of the linearized differential equation, assuming zero initial conditions. Finally, we separate input and output variables and form the transfer function. Let us first see how to linearize a function; later, we will apply the method to the linearization of a differential equation.

If we assume a nonlinear system operating at point A , $[x_0, f(x_0)]$ in Figure 2.47, small changes in the input can be related to changes in the output about the point by way of the slope of the curve at the point A . Thus, if the slope of the curve at point A is m_a , then small excursions of the input about point A , δ_x , yield small changes in the output, $\delta f(x)$, related by the slope at point A . Thus,

$$[f(x) - f(x_0)] \approx m_a(x - x_0) \quad (2.178)$$

from which

$$\delta f(x) \approx m_a \delta x \quad (2.179)$$

and

$$f(x) \approx f(x_0) + m_a(x - x_0) \approx f(x_0) + m_a \delta x \quad (2.180)$$

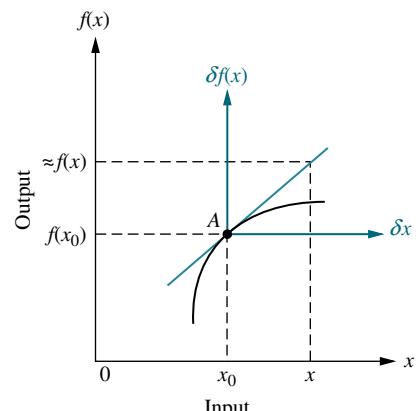


FIGURE 2.47 Linearization about point A

This relationship is shown graphically in Figure 2.47, where a new set of axes, δ_x and $\delta f(x)$, is created at the point A , and $f(x)$ is approximately equal to $f(x_0)$, the ordinate of the new origin, plus small excursions, $m_a\delta x$, away from point A . Let us look at an example.

Example 2.26

Linearizing a Function

PROBLEM: Linearize $f(x) = 5 \cos x$ about $x = \pi/2$.

SOLUTION: We first find that the derivative of $f(x)$ is $df/dx = (-5 \sin x)$. At $x = \pi/2$, the derivative is -5 . Also $f(x_0) = f(\pi/2) = 5 \cos(\pi/2) = 0$. Thus, from Eq. (2.180), the system can be represented as $f(x) = -5 \delta x$ for small excursions of x about $\pi/2$. The process is shown graphically in Figure 2.48, where the cosine curve does indeed look like a straight line of slope -5 near $\pi/2$.

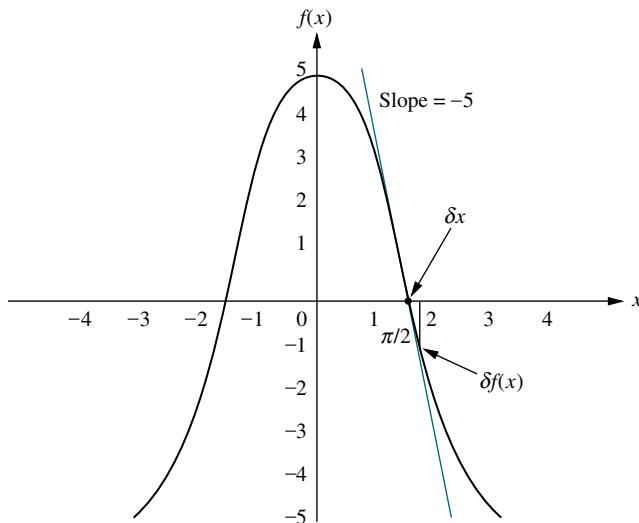


FIGURE 2.48 Linearization of $5 \cos x$ about $x = \pi/2$

The previous discussion can be formalized using the Taylor series expansion, which expresses the value of a function in terms of the value of that function at a particular point, the excursion away from that point, and derivatives evaluated at that point. The Taylor series is shown in Eq. (2.181).

$$f(x) = f(x_0) + \frac{df}{dx} \Big|_{x=x_0} \frac{(x-x_0)}{1!} + \frac{d^2f}{dx^2} \Big|_{x=x_0} \frac{(x-x_0)^2}{2!} + \dots \quad (2.181)$$

For small excursions of x from x_0 , we can neglect higher-order terms. The resulting approximation yields a straight-line relationship between the change in $f(x)$ and the excursions away from x_0 . Neglecting the higher-order terms in Eq. (2.181), we get

$$f(x) - f(x_0) \approx \frac{df}{dx} \Big|_{x=x_0} (x - x_0) \quad (2.182)$$

or

$$\delta f(x) \approx m|_{x=x_0} \delta x \quad (2.183)$$

which is a linear relationship between $\delta f(x)$ and δx for small excursions away from x_0 . It is interesting to note that Eqs. (2.182) and (2.183) are identical to Eqs. (2.178) and (2.179), which we derived intuitively. The following examples illustrate linearization. The first example demonstrates linearization of a differential equation, and the second example applies linearization to finding a transfer function.

Example 2.27

Linearizing a Differential Equation

PROBLEM: Linearize Eq. (2.184) for small excursions about $x = \pi/4$.

$$\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + \cos x = 0 \quad (2.184)$$

SOLUTION: The presence of the term $\cos x$ makes this equation nonlinear. Since we want to linearize the equation about $x = \pi/4$, we let $x = \delta x + \pi/4$, where δx is the small excursion about $\pi/4$, and substitute x into Eq. (2.184):

$$\frac{d^2(\delta x + \frac{\pi}{4})}{dt^2} + 2 \frac{d(\delta x + \frac{\pi}{4})}{dt} + \cos(\delta x + \frac{\pi}{4}) = 0 \quad (2.185)$$

But

$$\frac{d^2(\delta x + \frac{\pi}{4})}{dt^2} = \frac{d^2\delta x}{dt^2} \quad (2.186)$$

and

$$\frac{d(\delta x + \frac{\pi}{4})}{dt} = \frac{d\delta x}{dt} \quad (2.187)$$

Finally, the term $\cos(\delta x + (\pi/4))$ can be linearized with the truncated Taylor series. Substituting $f(x) = \cos(x)$, $f(x_0) = f(\pi/4) = \cos(\pi/4)$, and $(x - x_0) = \delta x$ into Eq. (2.182) yields

$$\cos(\delta x + \frac{\pi}{4}) - \cos(\frac{\pi}{4}) = \frac{d \cos x}{dx} \Big|_{x=\frac{\pi}{4}} \delta x = -\sin(\frac{\pi}{4}) \delta x \quad (2.188)$$

Solving Eq. (2.188) for $\cos(\delta x + (\pi/4))$, we get

$$\cos(\delta x + \frac{\pi}{4}) = \cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4}) \delta x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \delta x \quad (2.189)$$

Substituting Eqs. (2.186), (2.187), and (2.189) into Eq. (2.185) yields the following linearized differential equation:

$$\frac{d^2\delta x}{dt^2} + 2 \frac{d\delta x}{dt} - \frac{\sqrt{2}}{2} \delta x = -\frac{\sqrt{2}}{2} \quad (2.190)$$

This equation can now be solved for δx , from which we can obtain $x = \delta x + (\pi/4)$.

Even though the nonlinear Eq. (2.184) is homogeneous, the linearized Eq. (2.190) is not homogeneous. Eq. (2.190) has a forcing function on its right-hand side. This additional term can be thought of as an input to a system represented by Eq. (2.184).

Another observation about Eq. (2.190) is the negative sign on the left-hand side. The study of differential equations tells us that since the roots of the characteristic equation are positive, the homogeneous solution grows without bound instead of diminishing to zero. Thus, this system linearized around $x = \pi/4$ is not stable.

Example 2.28

Transfer Function—Nonlinear Electrical Network

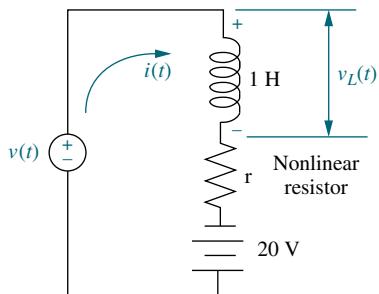


FIGURE 2.49 Nonlinear electrical network

PROBLEM: Find the transfer function, $V_L(s)/V(s)$, for the electrical network shown in Figure 2.49, which contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where i_r and v_r are the resistor current and voltage, respectively. Also, $v(t)$ in Figure 2.49 is a small-signal source.

SOLUTION: We will use Kirchhoff's voltage law to sum the voltages in the loop to obtain the nonlinear differential equation, but first we must solve for the voltage across the nonlinear resistor. Taking the natural log of the resistor's current-voltage relationship, we get $v_r = 10 \ln \frac{1}{2} i_r$. Applying Kirchhoff's voltage law around the loop, where $i_r = i$, yields

$$L \frac{di}{dt} + 10 \ln \frac{1}{2} i - 20 = v(t) \quad (2.191)$$

Next, let us evaluate the equilibrium solution. First, set the small-signal source, $v(t)$, equal to zero. Now evaluate the steady-state current. With $v(t) = 0$, the circuit consists of a 20 V battery in series with the inductor and nonlinear resistor. In the steady state, the voltage across the inductor will be zero, since $v_L(t) = L di/dt$ and di/dt is zero in the steady state, given a constant battery source. Hence, the resistor voltage, v_r , is 20 V. Using the characteristics of the resistor, $i_r = 2e^{0.1v_r}$, we find that $i_r = i = 14.78$ amps. This current, i_0 , is the equilibrium value of the network current. Hence $i = i_0 + \delta i$. Substituting this current into Eq. (2.191) yields

$$L \frac{d(i_0 + \delta i)}{dt} + 10 \ln \frac{1}{2} (i_0 + \delta i) - 20 = v(t) \quad (2.192)$$

Using Eq. (2.182) to linearize $\ln \frac{1}{2} (i_0 + \delta i)$, we get

$$\ln \frac{1}{2} (i_0 + \delta i) - \ln \frac{1}{2} i_0 = \frac{d(\ln \frac{1}{2} i)}{di} \Big|_{i=i_0} \delta i = \frac{1}{i} \Big|_{i=i_0} \delta i = \frac{1}{i_0} \delta i \quad (2.193)$$

or

$$\ln \frac{1}{2} (i_0 + \delta i) = \ln \frac{i_0}{2} + \frac{1}{i_0} \delta i \quad (2.194)$$

Substituting into Eq. (2.192), the linearized equation becomes

$$L \frac{d\delta i}{dt} + 10 \left(\ln \frac{i_0}{2} + \frac{1}{i_0} \delta i \right) - 20 = v(t) \quad (2.195)$$

Letting $L = 1$ and $i_0 = 14.78$, the final linearized differential equation is

$$\frac{d\delta i}{dt} + 0.677\delta i = v(t) \quad (2.196)$$

Taking the Laplace transform with zero initial conditions and solving for $\delta i(s)$, we get

$$\delta i(s) = \frac{V(s)}{s + 0.677} \quad (2.197)$$

But the voltage across the inductor about the equilibrium point is

$$v_L(t) = L \frac{d}{dt}(i_0 + \delta i) = L \frac{d\delta i}{dt} \quad (2.198)$$

Taking the Laplace transform,

$$V_L(s) = L s \delta i(s) = s \delta i(s) \quad (2.199)$$

Substituting Eq. (2.197) into Eq. (2.199) yields

$$V_L(s) = s \frac{V(s)}{s + 0.677} \quad (2.200)$$

from which the final transfer function is

$$\frac{V_L(s)}{V(s)} = \frac{s}{s + 0.677} \quad (2.201)$$

for small excursions about $i = 14.78$ or, equivalently, about $v(t) = 0$.

Skill-Assessment Exercise 2.13

PROBLEM: Find the linearized transfer function, $G(s) = V(s)/I(s)$, for the electrical network shown in Figure 2.50. The network contains a nonlinear resistor whose voltage-current relationship is defined by $i_r = e^{v_r}$. The current source, $i(t)$, is a small-signal generator.

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ANSWER: $G(s) = \frac{1}{s + 2}$

The complete solution is at www.wiley.com/college/nise.

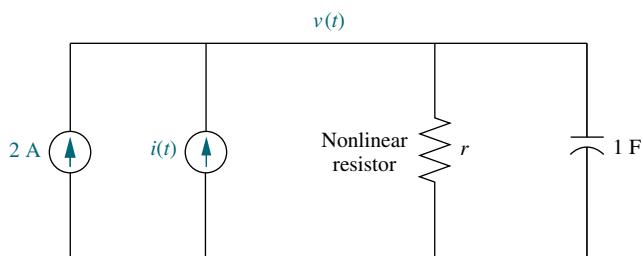


FIGURE 2.50 Nonlinear electric circuit for Skill-Assessment Exercise 2.13

Case Studies

Antenna Control: Transfer Functions

This chapter showed that physical systems can be modeled mathematically with transfer functions. Typically, systems are composed of subsystems of different types, such as electrical, mechanical, and electromechanical.

The first case study uses our ongoing example of the antenna azimuth position control system to show how to represent each subsystem as a transfer function.

PROBLEM: Find the transfer function for each subsystem of the antenna azimuth position control system schematic shown on the front endpapers. Use Configuration 1.

SOLUTION: First, we identify the individual subsystems for which we must find transfer functions; they are summarized in Table 2.6. We proceed to find the transfer function for each subsystem.

TABLE 2.6 Subsystems of the antenna azimuth position control system

Subsystem	Input	Output
Input potentiometer	Angular rotation from user, $\theta_i(t)$	Voltage to preamp, $v_i(t)$
Preamp	Voltage from potentiometers, $v_e(t) = v_i(t) - v_0(t)$	Voltage to power amp, $v_p(t)$
Power amp	Voltage from preamp, $v_p(t)$	Voltage to motor, $e_a(t)$
Motor	Voltage from power amp, $e_a(t)$	Angular rotation to load, $\theta_0(t)$
Output potentiometer	Angular rotation from load, $\theta_0(t)$	Voltage to preamp, $v_0(t)$

Input Potentiometer; Output Potentiometer

Since the input and output potentiometers are configured in the same way, their transfer functions will be the same. We *neglect* the dynamics for the potentiometers and simply find the relationship between the output voltage and the input angular displacement. In the center position the output voltage is zero. Five turns toward either the positive 10 volts or the negative 10 volts yields a voltage change of 10 volts. Thus, the transfer function, $V_i(s)/\theta_i(s)$, for the potentiometers is found by dividing the voltage change by the angular displacement:

$$\frac{V_i(s)}{\theta_i(s)} = \frac{10}{10\pi} = \frac{1}{\pi} \quad (2.202)$$

Preamplifier; Power Amplifier

The transfer functions of the amplifiers are given in the problem statement. Two phenomena are *neglected*. First, we *assume* that saturation is never reached. Second, the dynamics of the preamplifier are *neglected*, since its speed of response is typically much greater than that of the power amplifier. The transfer functions of both amplifiers are given in the problem statement and are the ratio of the Laplace transforms of the output voltage divided by the input voltage. Hence, for the

preamplifier,

$$\frac{V_p(s)}{V_e(s)} = K \quad (2.203)$$

and for the power amplifier,

$$\frac{E_a(s)}{V_p(s)} = \frac{100}{s + 100} \quad (2.204)$$

Motor and Load

The motor and its load are next. The transfer function relating the armature displacement to the armature voltage is given in Eq. (2.153). The equivalent inertia, J_m , is

$$J_m = J_a + J_L \left(\frac{25}{250} \right)^2 = 0.02 + 1 \frac{1}{100} = 0.03 \quad (2.205)$$

where $J_L = 1$ is the load inertia at θ_0 . The equivalent viscous damping, D_m , at the armature is

$$D_m = D_a + D_L \left(\frac{25}{250} \right)^2 = 0.01 + 1 \frac{1}{100} = 0.02 \quad (2.206)$$

where D_L is the load viscous damping at θ_0 . From the problem statement, $K_t = 0.5$ N-m/A, $K_b = 0.5$ V-s/rad, and the armature resistance $R_a = 8$ ohms. These quantities along with J_m and D_m are substituted into Eq. (2.153), yielding the transfer function of the motor from the armature voltage to the armature displacement, or

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t / (R_a J_m)}{s \left[s + \frac{1}{J_m} \left(D_m + \frac{K_t K_b}{R_a} \right) \right]} = \frac{2.083}{s(s + 1.71)} \quad (2.207)$$

To complete the transfer function of the motor, we multiply by the gear ratio to arrive at the transfer function relating load displacement to armature voltage:

$$\frac{\theta_0(s)}{E_a(s)} = 0.1 \frac{\theta_m(s)}{E_a(s)} = \frac{0.2083}{s(s + 1.71)} \quad (2.208)$$

The results are summarized in the block diagram and table of block diagram parameters (Configuration 1) shown on the front endpapers.

CHALLENGE: We now give you a problem to test your knowledge of this chapter's objectives: Referring to the antenna azimuth position control system schematic shown on the front endpapers, evaluate the transfer function of each subsystem. Use Configuration 2. Record your results in the table of block diagram parameters shown on the front endpapers for use in subsequent chapters' case study challenges.

Transfer Function of a Human Leg

In this case study we find the transfer function of a biological system. The system is a human leg, which pivots from the hip joint. In this problem, the component of weight is nonlinear, so the system requires linearization before the evaluation of the transfer function.

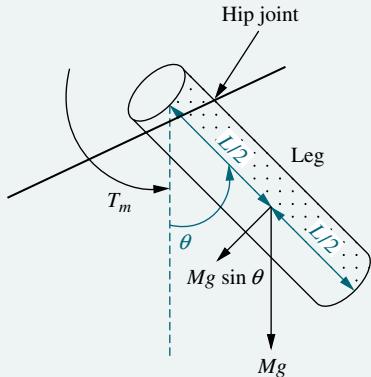


FIGURE 2.51 Cylinder model of a human leg.

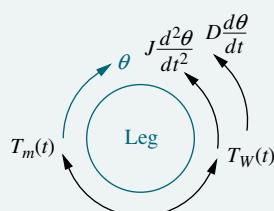


FIGURE 2.52 Free-body diagram of leg model

PROBLEM: The transfer function of a human leg relates the output angular rotation about the hip joint to the input torque supplied by the leg muscle. A simplified model for the leg is shown in Figure 2.51. The model *assumes* an applied muscular torque, $T_m(t)$, viscous damping, D , at the hip joint, and inertia, J , around the hip joint.¹⁵ Also, a component of the weight of the leg, Mg , where M is the mass of the leg and g is the acceleration due to gravity, creates a nonlinear torque. If we *assume* that the leg is of uniform density, the weight can be applied at $L/2$, where L is the length of the leg (Milsum, 1966). Do the following:

- Evaluate the nonlinear torque.
- Find the transfer function, $\theta(s)/T_m(s)$, for small angles of rotation, where $\theta(s)$ is the angular rotation of the leg about the hip joint.

SOLUTION: First, calculate the torque due to the weight. The total weight of the leg is Mg acting vertically. The component of the weight in the direction of rotation is $Mg \sin \theta$. This force is applied at a distance $L/2$ from the hip joint. Hence the torque in the direction of rotation, $T_W(t)$, is $Mg(L/2) \sin \theta$. Next, draw a free-body diagram of the leg, showing the applied torque, $T_m(t)$, the torque due to the weight, $T_W(t)$, and the opposing torques due to inertia and viscous damping (see Figure 2.52).

Summing torques, we get

$$J \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + Mg \frac{L}{2} \sin \theta = T_m(t) \quad (2.209)$$

We linearize the system about the equilibrium point, $\theta = 0$, the vertical position of the leg. Using Eq. (2.182), we get

$$\sin \theta - \sin 0 = (\cos 0)\delta\theta \quad (2.210)$$

from which, $\sin \theta = \delta\theta$. Also, $J d^2\theta/dt^2 = J d^2\delta\theta/dt^2$ and $D d\theta/dt = D d\delta\theta/dt$. Hence Eq. (2.209) becomes

$$J \frac{d^2\delta\theta}{dt^2} + D \frac{d\delta\theta}{dt} + Mg \frac{L}{2} \delta\theta = T_m(t) \quad (2.211)$$

Notice that the torque due to the weight approximates a spring torque on the leg. Taking the Laplace transform with zero initial conditions yields

$$\left(Js^2 + Ds + Mg \frac{L}{2} \right) \delta\theta(s) = T_m(s) \quad (2.212)$$

from which the transfer function is

$$\frac{\delta\theta(s)}{T_m(s)} = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{MgL}{2J}} \quad (2.213)$$

¹⁵For emphasis, J is not around the center of mass, as we previously assumed for inertia in mechanical rotation.

for small excursions about the equilibrium point, $\theta = 0$.

CHALLENGE: We now introduce a case study challenge to test your knowledge of this chapter's objectives. Although the physical system is different from a human leg, the problem demonstrates the same principles: linearization followed by transfer function evaluation.

Given the nonlinear electrical network shown in Figure 2.53, find the transfer function relating the output nonlinear resistor voltage, $V_r(s)$, to the input source voltage, $V(s)$.

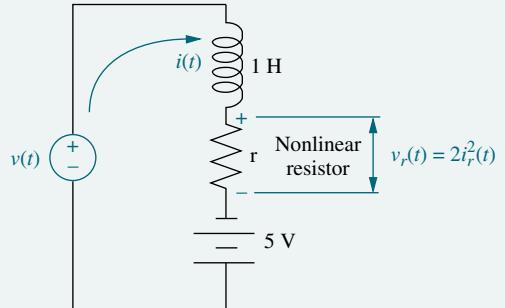


FIGURE 2.53 Nonlinear electric circuit

Summary

In this chapter, we discussed how to find a mathematical model, called a *transfer function*, for linear, time-invariant electrical, mechanical, and electromechanical systems. The transfer function is defined as $G(s) = C(s)/R(s)$, or the ratio of the Laplace transform of the output to the Laplace transform of the input. This relationship is algebraic and also adapts itself to modeling interconnected subsystems.

We realize that the physical world consists of more systems than we illustrated in this chapter. For example, we could apply transfer function modeling to hydraulic, pneumatic, heat, and even economic systems. Of course, we must assume these systems to be linear, or make linear approximations, in order to use this modeling technique.

Now that we have our transfer function, we can evaluate its response to a specified input. System response will be covered in Chapter 4. For those pursuing the state-space approach, we continue our discussion of modeling in Chapter 3, where we use the time domain rather than the frequency domain.

Review Questions

1. What mathematical model permits easy interconnection of physical systems?
2. To what classification of systems can the transfer function be best applied?
3. What transformation turns the solution of differential equations into algebraic manipulations?
4. Define the transfer function.
5. What assumption is made concerning initial conditions when dealing with transfer functions?
6. What do we call the mechanical equations written in order to evaluate the transfer function?
7. If we understand the form the mechanical equations take, what step do we avoid in evaluating the transfer function?
8. Why do transfer functions for mechanical networks look identical to transfer functions for electrical networks?

9. What function do gears perform?
10. What are the component parts of the mechanical constants of a motor's transfer function?
11. The motor's transfer function relates armature displacement to armature voltage. How can the transfer function that relates load displacement and armature voltage be determined?
12. Summarize the steps taken to linearize a nonlinear system.

Problems

1. Derive the Laplace transform for the following time functions: [Section: 2.2]

- a. $u(t)$
- b. $tu(t)$
- c. $\sin \omega t u(t)$
- d. $\cos \omega t u(t)$

2. Using the Laplace transform pairs of Table 2.1 and the Laplace transform theorems of Table 2.2, derive the Laplace transforms for the following time functions: [Section: 2.2]

- a. $e^{-at} \sin \omega t u(t)$
- b. $e^{-at} \cos \omega t u(t)$
- c. $t^3 u(t)$

3. Repeat Problem 18 in Chapter 1, using Laplace transforms. Assume that the forcing functions are zero prior to $t = 0-$. [Section: 2.2]

4. Repeat Problem 19 in Chapter 1, using Laplace transforms. Use the following initial conditions for each part as follows: (a) $x(0) = 4$, $x'(0) = -4$; (b) $x(0) = 4$, $x'(0) = 1$; (c) $x(0) = 2$, $x'(0) = 3$, where $x'(0) = \frac{dx}{dt}(0)$. Assume that the forcing functions are zero prior to $t = 0-$. [Section: 2.2]

5. Use MATLAB and the Symbolic Math Toolbox to find the Laplace transform of the following time functions: [Section: 2.2]

- a. $f(t) = 8t^2 \cos(3t + 45^\circ)$
- b. $f(t) = 3te^{-2t} \sin(4t + 60^\circ)$

6. Use MATLAB and the Symbolic Math Toolbox to find the inverse

Laplace transform of the following frequency functions: [Section: 2.2]

a. $G(s) = \frac{(s^2 + 3s + 10)(s + 5)}{(s + 3)(s + 4)(s^2 + 2s + 100)}$

b. $G(s) = \frac{s^3 + 4s^2 + 2s + 6}{(s + 8)(s^2 + 8s + 3)(s^2 + 5s + 7)}$

7. A system is described by the following differential equation:

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = \frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x$$

Find the expression for the transfer function of the system, $Y(s)/X(s)$. [Section: 2.3]

8. For each of the following transfer functions, write the corresponding differential equation. [Section: 2.3]

a. $\frac{X(s)}{F(s)} = \frac{7}{s^2 + 5s + 10}$

b. $\frac{X(s)}{F(s)} = \frac{15}{(s + 10)(s + 11)}$

c. $\frac{X(s)}{F(s)} = \frac{s + 3}{s^3 + 11s^2 + 12s + 18}$

9. Write the differential equation for the system shown in Figure P2.1. [Section: 2.3]

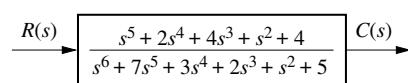


FIGURE P2.1

10. Write the differential equation that is mathematically equivalent to the

block diagram shown in Figure P2.2. Assume that $r(t) = 3t^3$. [Section: 2.3]

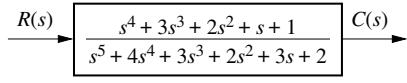


FIGURE P2.2

11. A system is described by the following differential equation: [Section 2.3]

$$\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 3x = 1$$

with the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$. Show a block diagram of the system, giving its transfer function and all pertinent inputs and outputs. (Hint: the initial conditions will show up as added inputs to an effective system with zero initial conditions.)

12. Use MATLAB to generate the transfer function: [Section: 2.3] MATLAB
ML

$$G(s) = \frac{5(s+15)(s+26)(s+72)}{s(s+55)(s^2+5s+30)(s+56)(s^2+27s+52)}$$

in the following ways:

- a. the ratio of factors;
- b. the ratio of polynomials.

13. Repeat Problem 12 for the following transfer function: [Section: 2.3] MATLAB
ML

$$G(s) = \frac{s^4 + 25s^3 + 20s^2 + 15s + 42}{s^5 + 13s^4 + 9s^3 + 37s^2 + 35s + 50}$$

14. Use MATLAB to generate the partial-fraction expansion of the following function: [Section: 2.3]

$$F(s) = \frac{10^4(s+5)(s+70)}{s(s+45)(s+55)(s^2+7s+110)(s^2+6s+95)}$$

15. Use MATLAB and the Symbolic Math Toolbox to input and form LTI objects in polynomial and factored form for the following frequency functions: [Section: 2.3] Symbolic Math
SM

a. $G(s) = \frac{45(s^2 + 37s + 74)(s^3 + 28s^2 + 32s + 16)}{(s+39)(s+47)(s^2 + 2s + 100)(s^3 + 27s^2 + 18s + 15)}$

b. $G(s) = \frac{56(s+14)(s^3 + 49s^2 + 62s + 53)}{(s^3 + 81s^2 + 76s + 65)(s^2 + 88s + 33)(s^2 + 56s + 77)}$

16. Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each network shown in Figure P2.3. [Section: 2.4]

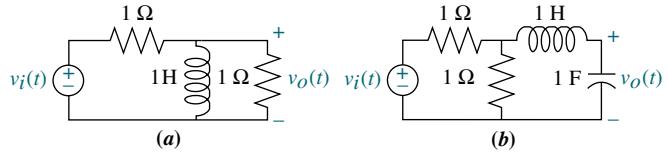


FIGURE P2.3

17. Find the transfer function, $G(s) = V_L(s)/V(s)$, for each network shown in Figure P2.4. [Section: 2.4]

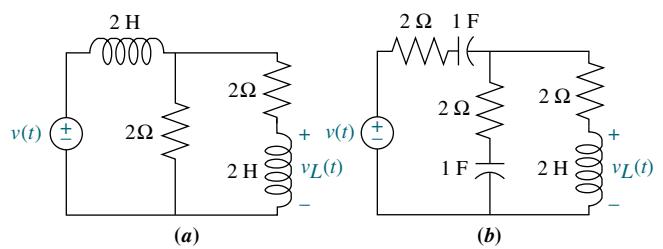


FIGURE P2.4

18. Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each network shown in Figure P2.5. Solve the problem using mesh analysis. [Section: 2.4]

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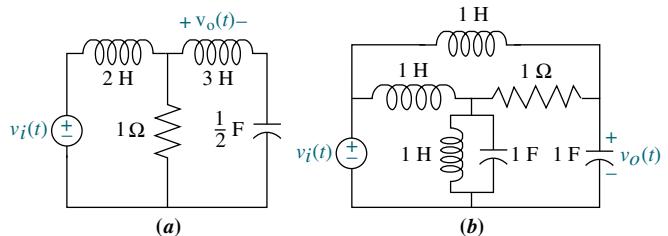


FIGURE P2.5

19. Repeat Problem 18 using nodal equations. [Section: 2.4]

20. a. Write, but do not solve, the mesh and nodal equations for the network of Figure P2.6. [Section: 2.4]

- b. Use MATLAB, the Symbolic Math Toolbox, and the equations found in part a to solve for the transfer function, $G(s) = V_o(s)/V(s)$. Use both the

mesh and nodal equations and show that either set yields the same transfer function. [Section: 2.4]

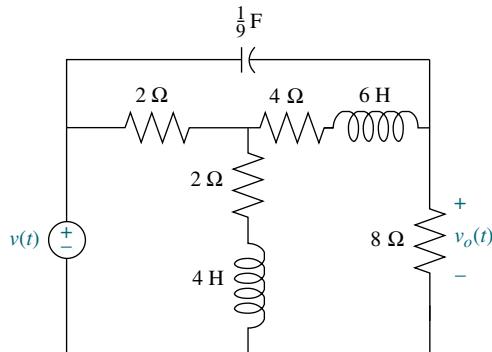
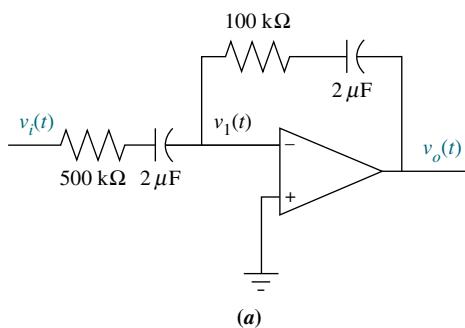


FIGURE P2.6

21. Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each operational amplifier circuit shown in Figure P2.7. [Section: 2.4]



(a)

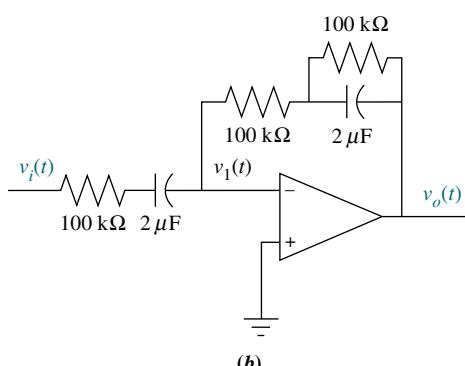
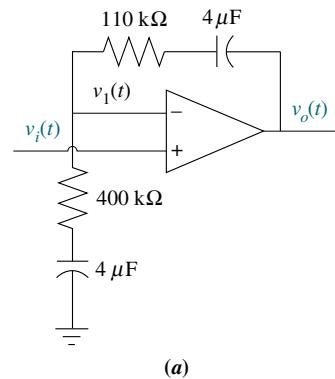
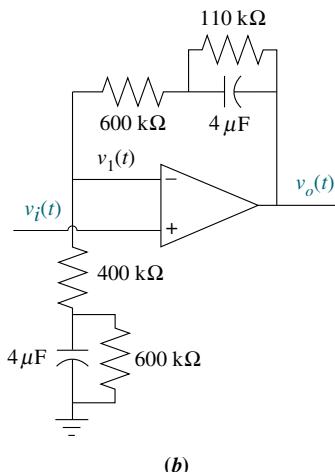


FIGURE P2.7

22. Find the transfer function, $G(s) = V_o(s)/V_i(s)$, for each operational amplifier circuit shown in Figure P2.8. [Section: 2.4]



(a)



(b)

FIGURE P2.8

23. Find the transfer function, $G(s) = X_1(s)/F(s)$, for the translational mechanical system shown in Figure P2.9. [Section: 2.5]

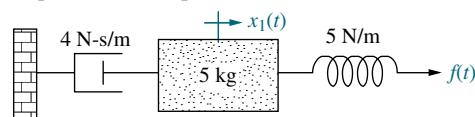


FIGURE P2.9

24. Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical network shown in Figure P2.10. [Section: 2.5]

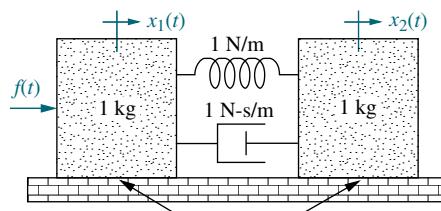


FIGURE P2.10

25. Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure P2.11. (Hint: place a zero mass at $x_2(t)$). [Section: 2.5]

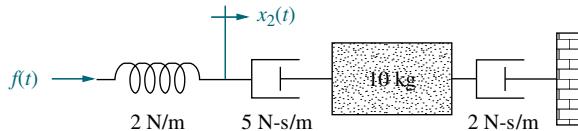


FIGURE P2.11

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26. For the system of Figure P2.12 find the transfer function, $G(s) = X_1(s)/F(s)$. [Section: 2.5]

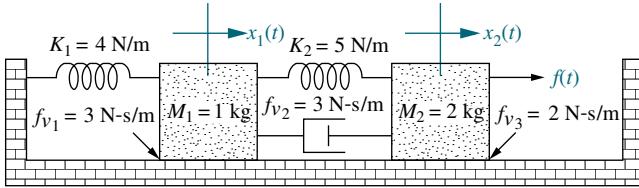


FIGURE P2.12

27. Find the transfer function, $G(s) = X_3(s)/F(s)$, for the translational mechanical system shown in Figure P2.13. [Section: 2.5]

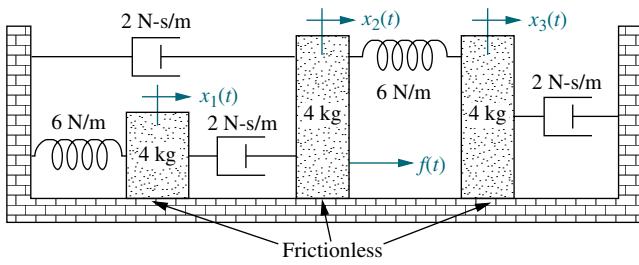
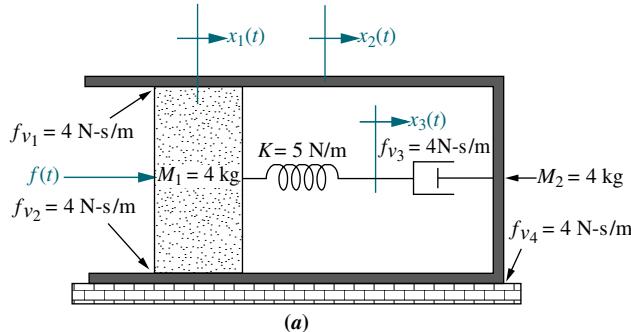


FIGURE P2.13

28. Find the transfer function, $X_3(s)/F(s)$, for each system shown in Figure P2.14. [Section: 2.5]



(a)

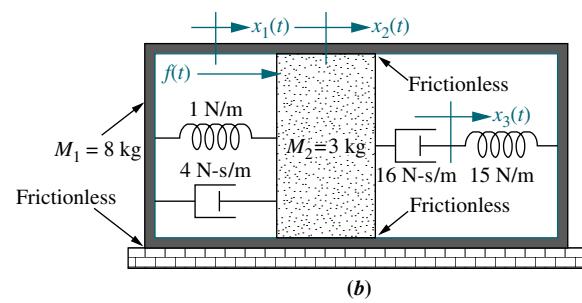


FIGURE P2.14

29. Write, but do not solve, the equations of motion for the translational mechanical system shown in Figure P2.15. [Section: 2.5]

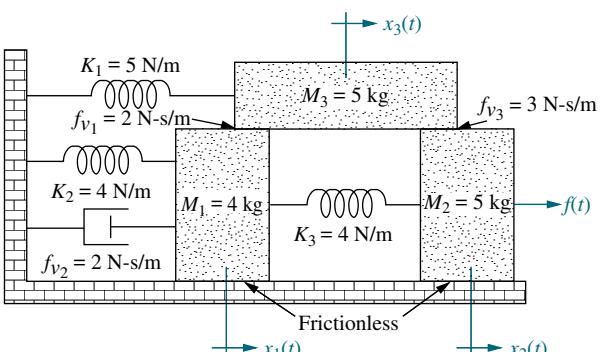
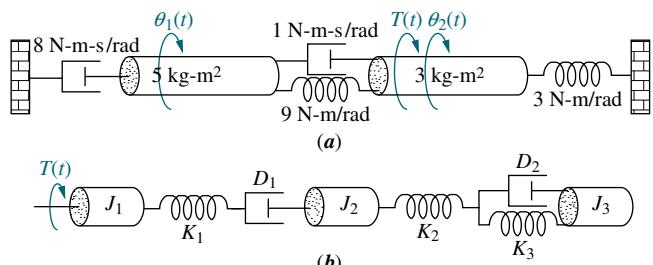


FIGURE P2.15

30. For each of the rotational mechanical systems shown in Figure P2.16, write, but do not solve, the equations of motion. [Section: 2.6]



(a)

(b)

FIGURE P2.16

31. For the rotational mechanical system shown in Figure P2.17, find the transfer function $G(s) = \theta_2(s)/T(s)$ [Section: 2.6]

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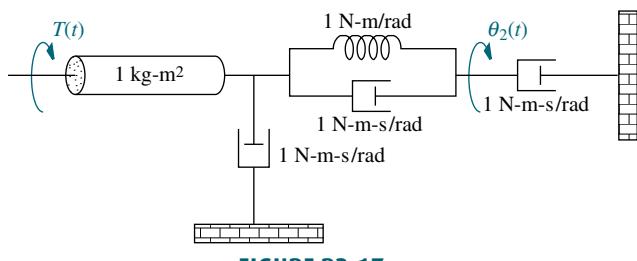


FIGURE P2.17

32. For the rotational mechanical system with gears shown in Figure P2.18, find the transfer function, $G(s) = \theta_3(s)/T(s)$. The gears have inertia and bearing friction as shown. [Section: 2.7]

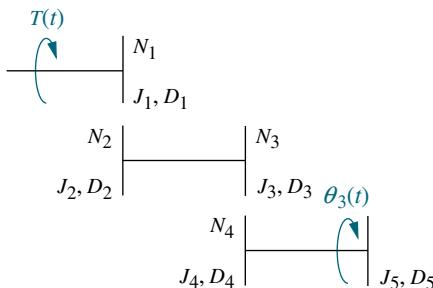


FIGURE P2.18

33. For the rotational system shown in Figure P2.19, find the transfer function, $G(s) = \theta_2(s)/T(s)$. [Section: 2.7]

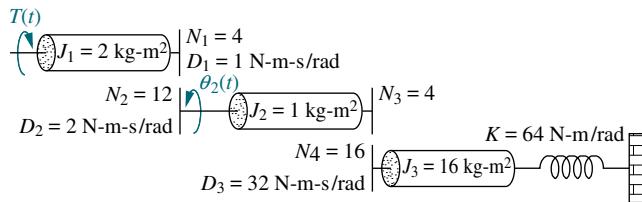


FIGURE P2.19

34. Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure P2.20. [Section: 2.7]

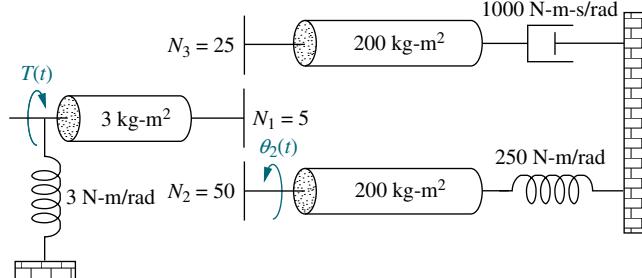


FIGURE P2.20

35. Find the transfer function, $G(s) = \theta_4(s)/T(s)$, for the rotational system shown in Figure P2.21. [Section: 2.7]

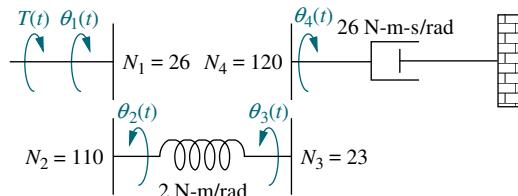


FIGURE P2.21

36. For the rotational system shown in Figure P2.22, find the transfer function, $G(s) = \theta_L(s)/T(s)$. [Section: 2.7]

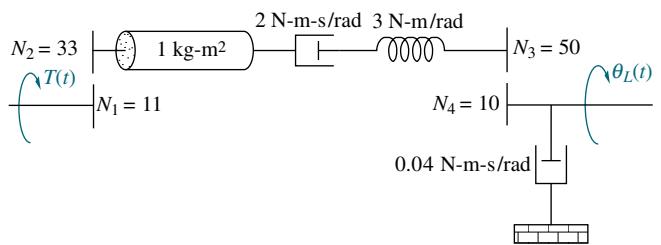


FIGURE P2.22

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37. For the rotational system shown in Figure P2.23, write the equations of motion from which the transfer function, $G(s) = \theta_1(s)/T(s)$, can be found. [Section: 2.7]

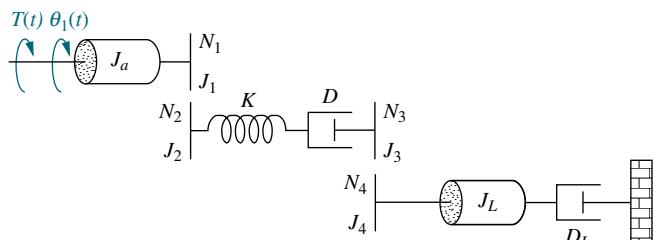


FIGURE P2.23

38. Given the rotational system shown in Figure P2.24, find the transfer function, $G(s) = \theta_6(s)/\theta_1(s)$. [Section: 2.7]

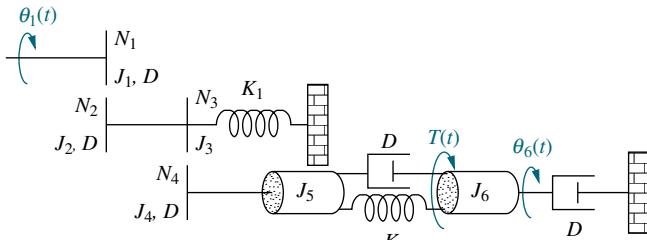


FIGURE P2.24

39. In the system shown in Figure P2.25, the inertia, J , of radius, r , is constrained to move only about the stationary axis A . A viscous damping force of translational value f_v exists between the bodies J and M . If an external force, $f(t)$, is applied to the mass, find the transfer function, $G(s) = \theta(s)/F(s)$. [Sections: 2.5; 2.6]

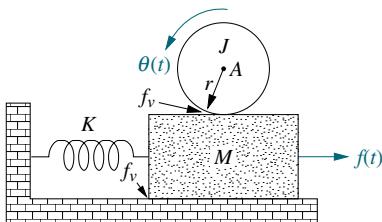


FIGURE P2.25

40. For the combined translational and rotational system shown in Figure P2.26, find the transfer function, $G(s) = X(s)/T(s)$. [Sections: 2.5; 2.6; 2.7]

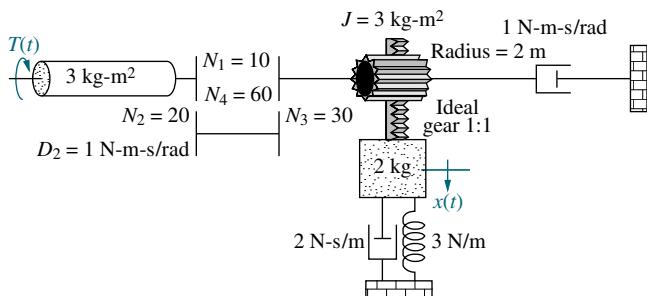


FIGURE P2.26

41. Given the combined translational and rotational system shown in Figure P2.27, find the transfer function, $G(s) = X(s)/T(s)$. [Sections: 2.5; 2.6]

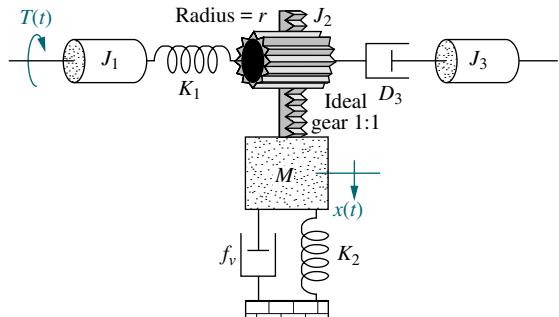


FIGURE P2.27

42. For the motor, load, and torque-speed curve shown in Figure P2.28, find the transfer function, $G(s) = \theta_L(s)/E_a(s)$. [Section: 2.8]

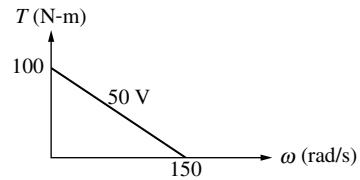
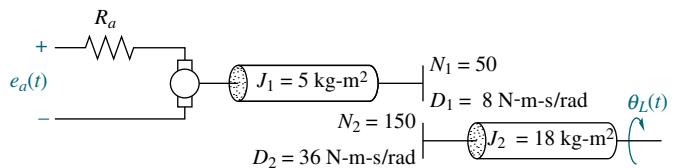


FIGURE P2.28

43. The motor whose torque-speed characteristics are shown in Figure P2.29 drives the load shown in the diagram. Some of the gears have inertia. Find the transfer function, $G(s) = \theta_2(s)/E_a(s)$. [Section: 2.8]

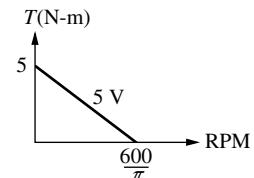
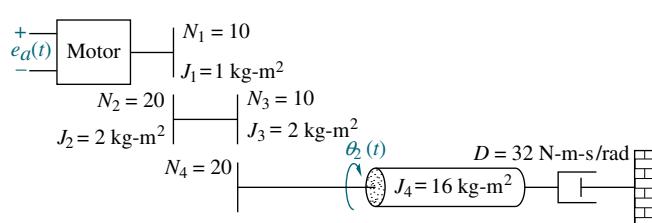


FIGURE P2.29

- 44.** A dc motor develops 55 N-m of torque at a speed of 600 rad/s when 12 volts are applied. It stalls out at this voltage with 100 N-m of torque. If the inertia and damping of the armature are 7 kg-m² and 3 N-m-s/rad, respectively, find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, of this motor if it drives an inertia load of 105 kg-m² through a gear train, as shown in Figure P2.30. [Section: 2.8]

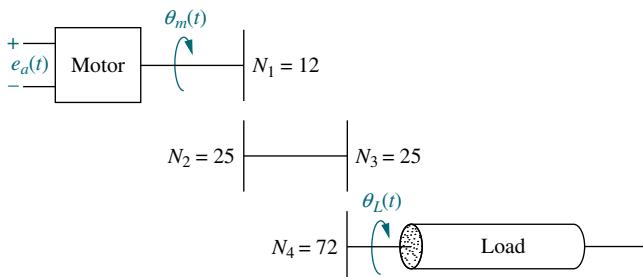


FIGURE P2.30

- 45.** In this chapter, we derived the transfer function of a dc motor relating the angular displacement output to the armature voltage input. Often we want to control the output torque rather than the displacement. Derive the transfer function of the motor that relates output torque to input armature voltage. [Section: 2.8]

- 46.** Find the transfer function, $G(s) = X(s)/E_a(s)$, for the system shown in Figure P2.31. [Sections: 2.5–2.8]

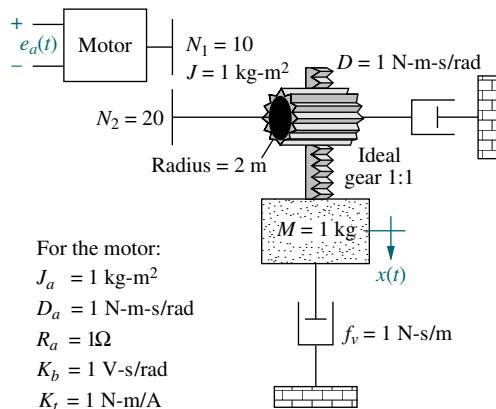


FIGURE P2.31

- 47.** Find the series and parallel analogs for the translational mechanical system shown in Figure 2.20 in the text. [Section: 2.9]

- 48.** Find the series and parallel analogs for the rotational mechanical systems shown in Figure P2.16(b) in the problems. [Section: 2.9]

- 49.** A system's output, c , is related to the system's input, r , by the straight-line relationship, $c = 5r + 7$. Is the system linear? [Section: 2.10]

- 50.** Consider the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = f(x)$$

where $f(x)$ is the input and is a function of the output, x . If $f(x) = \sin x$, linearize the differential equation for small excursions. [Section: 2.10]

a. $x = 0$

b. $x = \pi$

- 51.** Consider the differential equation

$$\frac{d^3x}{dt^3} + 10\frac{d^2x}{dt^2} + 31\frac{dx}{dt} + 30x = f(x)$$

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where $f(x)$ is the input and is a function of the output, x . If $f(x) = e^{-x}$, linearize the differential equation for x near 0. [Section: 2.10]

- 52.** Many systems are *piecewise* linear. That is, over a *large* range of variable values, the system can be described linearly. A system with amplifier saturation is one such example. Given the differential equation

$$\frac{d^2x}{dt^2} + 17\frac{dx}{dt} + 50x = f(x)$$

assume that $f(x)$ is as shown in Figure P2.32. Write the differential equation for each of the following ranges of x : [Section: 2.10]

a. $-\infty < x < -3$

b. $-3 < x < 3$

c. $3 < x < \infty$

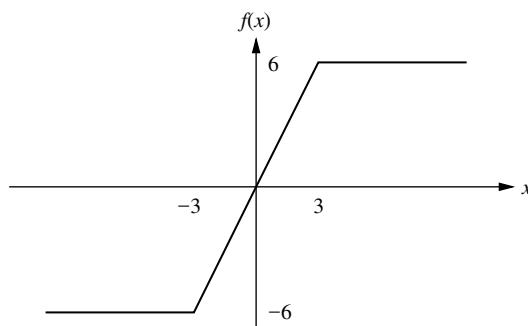


FIGURE P2.32

- 53.** For the translational mechanical system with a nonlinear spring shown in Figure P2.33, find the transfer function, $G(s) = X(s)/F(s)$, for small excursions around $f(t) = 1$. The spring is defined by $x_s(t) = 1 - e^{-f_s(t)}$, where $x_s(t)$ is the spring displacement and $f_s(t)$ is the spring force. [Section: 2.10]

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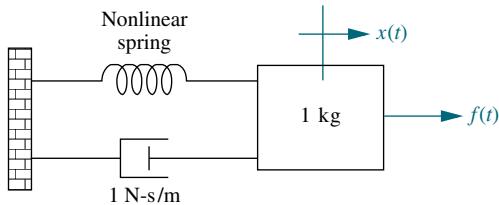


FIGURE P2.33

- 54.** Consider the restaurant plate dispenser shown in Figure P2.34, which consists of a vertical stack of dishes supported by a compressed spring. As each plate is removed, the reduced weight on the dispenser causes the remaining plates to rise. Assume that the mass of the system minus the top plate is M , the viscous friction between the piston and the sides of the cylinder is f_v , the spring constant is K , and the weight of a single plate is W_D . Find the transfer function, $Y(s)/F(s)$, where $F(s)$ is the step reduction in force felt when the top plate is removed, and $Y(s)$ is the vertical displacement of the dispenser in an upward direction.

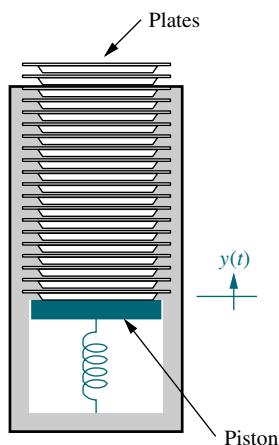


FIGURE P2.34 Plate dispenser

- 55.** Each inner ear in a human has a set of three nearly perpendicular semicircular canals of about 0.28 mm in diameter filled with fluid. Hair-cell transducers

that deflect with skull movements and whose main purpose is to work as attitude sensors as well as help us maintain our sense of direction and equilibrium are attached to the canals. As the hair cells move, they deflect a waterproof flap called the *cupula*. It has been shown that the skull and cupula movements are related by the following equation (Millsom, 1966):

$$J\ddot{\phi} + b\dot{\phi} + k\phi = (aJ)\ddot{\psi}$$

where

J = moment of inertia of the fluid in the thin tube (constant)

b = torque per unit relative angular velocity (constant)

k = torque per unit relative angular displacement (constant)

a = constant

$\phi(t)$ = angular deflection of the cupula (output)

$\ddot{\psi}(t)$ = skull's angular acceleration (input)

Find the transfer function $\frac{\Phi(s)}{\ddot{\Psi}(s)}$.

- 56.** Diabetes is an illness that has risen to epidemic proportions, affecting about 3% of the total world population in 2003. A differential equation model that describes the total population size of diabetics is

$$\frac{dC(t)}{dt} = -(\lambda + \mu + \delta + \gamma + \nu)C(t) + \lambda N(t)$$

$$\frac{dN(t)}{dt} = -(\nu + \delta)C(t) - \mu N(t) + I(t)$$

with the initial conditions $C(0) = C_0$ and $N(0) = N_0$ and

$I(t)$ = the system input: the number of new cases of diabetes

$C(t)$ = number of diabetics with complications

$N(t)$ = the system output: the total number of diabetics with and without complications

μ = natural mortality rate (constant)

λ = probability of developing a complication (constant)

δ = mortality rate due to complications (constant)

ν = rate at which patients with complications become severely disabled (constant)

γ = rate at which complications are cured (constant)

Assume the following values for parameters: $\nu = \delta = 0.05/\text{yr}$, $\mu = 0.02/\text{yr}$, $\gamma = 0.08/\text{yr}$, $\lambda = 0.7$, with initial conditions $C_0 = 47,000,500$ and $N_0 = 61,100,500$. Assume also that diabetic incidence is constant $I(t) = I = 6 \times 10^6$ (Boutayeb, 2004).

- Draw a block diagram of the system showing the output $N(s)$, the input $I(s)$, the transfer function, and the initial conditions.
- Use any method to find the analytic expression for $N(t)$ for $t \geq 0$.

- 57.** The circuit shown in Figure P2.35(a) is excited with the pulse shown in Figure P2.35(b).

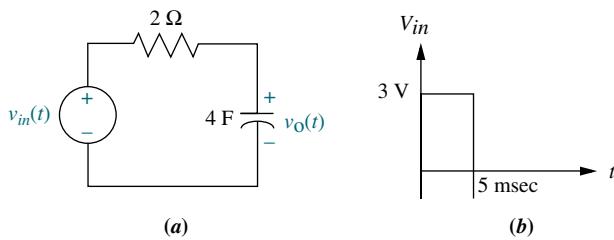


FIGURE P2.35

The Laplace transform can be used to calculate $v_o(t)$ in two different ways: The “exact” method is performed by writing $v_i(t) = 3[u(t) - u(t - 0.005)]$, from which we use the Laplace transform to obtain

$$V_{in}(s) = 3 \frac{1 - e^{-0.005s}}{s}$$

(Hint: look at Item 5 in Table 2.2, the time shift theorem.) In the second approach the pulse is approximated by an impulse input having the same area (energy) as the original input. From Figure P2.35(b): $v_{in}(t) \approx (3\text{ V})(5\text{ msec}) \delta(t) = 0.015\delta(t)$. In this case, $V_{in}(s) = 0.015$. This approximation can be used as long as the width of the pulse of Figure P2.35(b) is much smaller than the circuit’s smallest time constant. (Here, $\tau = RC = (2\Omega)(4\text{ F}) = 8\text{ sec} \gg 5\text{ msec}$.)

- Assuming the capacitor is initially discharged, obtain an analytic expression for $v_o(t)$ using both methods.
 - Plot the results of both methods using any means available to you, and compare both outputs. Discuss the differences.
- 58.** In a magnetic levitation experiment a metallic object is held up in the air suspended under an electromagnet. The vertical displacement of the object can

be described by the following nonlinear differential equation (Galvão, 2003):

$$m \frac{d^2H}{dt^2} = mg - k \frac{I^2}{H^2}$$

where

m = mass of the metallic object

g = gravity acceleration constant

k = a positive constant

H = distance between the electromagnet and the metallic object (output signal)

I = electromagnet’s current (input signal)

- Show that a system’s equilibrium will be achieved when $H_0 = I_0 \sqrt{\frac{k}{mg}}$.
- Linearize the equation about the equilibrium point found in Part a and show that the resulting transfer function obtained from the linearized differential equation can be expressed as

$$\frac{\delta H(s)}{\delta I(s)} = - \frac{a}{s^2 - b^2}$$

with $a > 0$. Hint: to perform the linearization, define $\delta H = H(t) - H_0$ and $\delta I = I(t) - I_0$; substitute into the original equation. This will give

$$m \frac{d^2(H_0 + \delta H)}{dt^2} = mg - k \frac{(I_0 + \delta I)^2}{(H_0 + \delta H)^2} = \gamma$$

Now get a first-order Taylor’s series approximation on the right-hand side of the equation. Namely, calculate

$$m \frac{d^2\delta H}{dt^2} = \frac{\partial \gamma}{\partial \delta H} \Big|_{\delta H=0, \delta I=0} \delta H + \frac{\partial \gamma}{\partial \delta I} \Big|_{\delta H=0, \delta I=0} \delta I$$

- 59.** Figure P2.36 shows a quarter-car model commonly used for analyzing suspension systems. The car’s tire is considered to act as a spring without damping, as shown. The parameters of the model are (Lin, 1997)

M_b = car’s body mass

M_w = wheel’s mass

K_a = strut’s spring constant

K_t = tire’s spring constant

f_v = strut’s damping constant

r = road disturbance (input)

x_s = car’s vertical displacement

x_w = wheel’s vertical displacement

Obtain the transfer function from the road disturbance to the car's vertical displacement $\frac{X_s(s)}{R(s)}$.

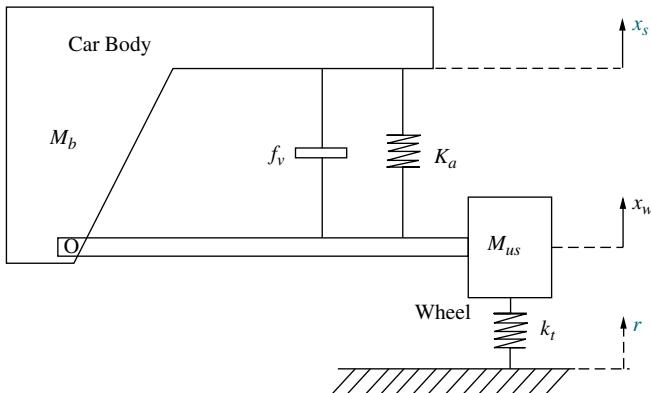
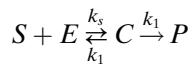


FIGURE P2.36 Quarter-car model used for suspension design.
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60. Enzymes are large proteins that biological systems use to increase the rate at which reactions occur. For example, food is usually composed of large molecules that are hard to digest; enzymes break down the large molecules into small nutrients as part of the digestive process. One such enzyme is amylase, contained in human saliva. It is commonly known that if you place a piece of uncooked pasta in your mouth its taste will change from paper-like to sweet as amylase breaks down the carbohydrates into sugars. Enzyme breakdown is often expressed by the following relation:



In this expression a substrate (S) interacts with an enzyme (E) to form a combined product (C) at a rate k_1 . The intermediate compound is reversible and gets disassociated at a rate k_{-1} . Simultaneously some of the compound is transformed into the final product (P) at a rate k_2 . The kinetics describing this reaction are known as the Michaelis-Menten equations and consist of four nonlinear differential equations. However, under some conditions these equations can be simplified. Let E_0 and S_0 be the initial concentrations of enzyme and substrate, respectively. It is generally accepted that under some energetic conditions or when the enzyme concentration is very big ($E_0 \gg S_0$),

the kinetics for this reaction are given by

$$\frac{dS}{dt} = k_\psi (\tilde{K}_s C - S)$$

$$\frac{dC}{dt} = k_\psi (S - \tilde{K}_M C)$$

$$\frac{dP}{dt} = k_2 C$$

where the following constant terms are used (Schnell, 2004) :

$$k_\psi = k_1 E_0$$

$$\tilde{K}_s = \frac{k-1}{k_\psi}$$

and

$$\tilde{K}_M = \tilde{K}_s + \frac{k_2}{k_\psi}$$

- a. Assuming the initial conditions for the reaction are $S(0) = S_0$, $E(0) = E_0$, $C(0) = P(0) = 0$, find the Laplace transform expressions for S , C , and P : $\mathcal{L}\{S\}$, $\mathcal{L}\{C\}$, and $\mathcal{L}\{P\}$, respectively.
- b. Use the final theorem to find $S(\infty)$, $C(\infty)$, and $P(\infty)$.
- 61. Humans are able to stand on two legs through a complex feedback system that includes several sensory inputs—equilibrium and visual along with muscle actuation. In order to gain a better understanding of the workings of the postural feedback mechanism, an individual is asked to stand on a platform to which sensors are attached at the base. Vibration actuators are attached with straps to the individual's calves. As the vibration actuators are stimulated, the individual sways and movements are recorded. It was hypothesized that the human postural dynamics are analogous to those of a cart with a balancing standing pole attached (inverted pendulum). In that case, the dynamics can be described by the following two equations:

$$J \frac{d^2\theta}{dt^2} = mgl \sin \theta(t) + T_{\text{bal}} + T_d(t)$$

$$T_{\text{bal}}(t) = -mgl \sin \theta(t) + kJ\theta(t) - \eta J\dot{\theta}(t)$$

$$-\rho J \int_0^t \theta(t) dt$$

where m is the individual's mass; l is the height of the individual's center of gravity; g is the gravitational constant; J is the individual's equivalent moment of inertia; η , ρ , and k are constants given by the body's postural control system; $\theta(t)$ is the

individual's angle with respect to a vertical line; $T_{\text{bal}}(t)$ is the torque generated by the body muscles to maintain balance; and $T_d(t)$ is the external torque input disturbance. Find the transfer function $\frac{\Theta(s)}{T_d(s)}$ (Johansson, 1988).

- 62.** Figure P2.37 shows a crane hoisting a load. Although the actual system's model is highly nonlinear, if the rope is considered to be stiff with a fixed length L , the system can be modeled using the following equations:

$$\begin{aligned} m_L \ddot{x}_{La} &= m_L g \phi \\ m_T \ddot{x}_T &= f_T - m_L g \phi \\ x_{La} &= x_T - x_L \\ x_L &= L \phi \end{aligned}$$

where m_L is the mass of the load, m_T is the mass of the cart, x_T and x_L are displacements as defined in the figure, ϕ is the rope angle with respect to the vertical, and f_T is the force applied to the cart (Marttinen, 1990).

- a. Obtain the transfer function from cart velocity to rope angle $\frac{\Phi(s)}{V_T(s)}$.
- b. Assume that the cart is driven at a constant velocity V_0 and obtain an expression for the resulting $\phi(t)$. Show that under this condition, the load will sway with a frequency $\omega_0 = \sqrt{\frac{g}{L}}$.
- c. Find the transfer function from the applied force to the cart's position, $\frac{X_T(s)}{F_T(s)}$.
- d. Show that if a constant force is applied to the cart, its velocity will increase without bound as $t \rightarrow \infty$.

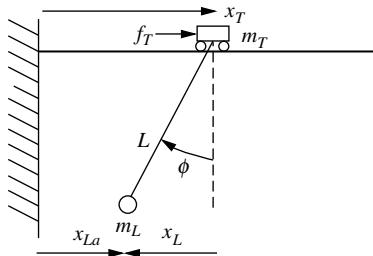


FIGURE P2.37 (© 1990 IEEE)

- 63.** In 1978, Malthus developed a model for human growth population that is also commonly used to model bacterial growth as follows. Let $\dot{N}(t)$ be the population density observed at time t . Let K be the

rate of reproduction per unit time. Neglecting population deaths, the population density at a time $t + \Delta t$ (with small Δt) is given by

$$N(t + \Delta t) \approx N(t) + KN(t)\Delta t$$

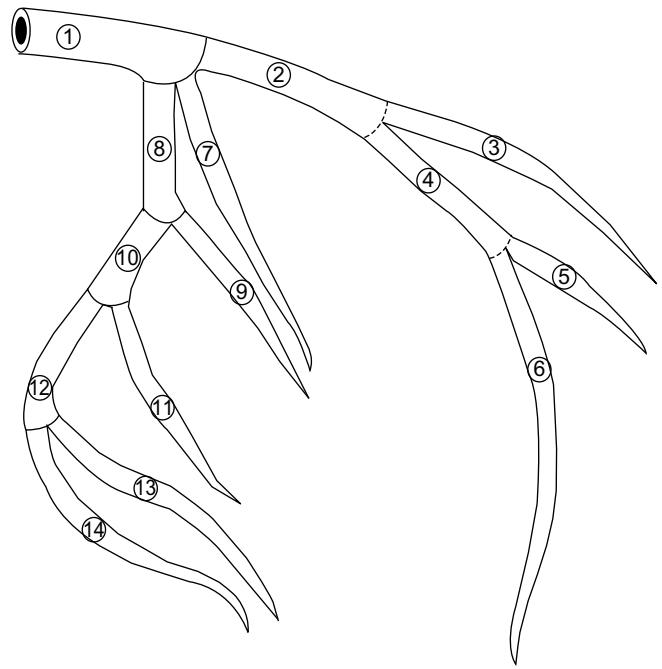
which also can be written as

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = KN(t)$$

Since $N(t)$ can be considered to be a very large number, letting $\Delta t \rightarrow 0$ gives the following differential equation (Edelstein-Keshet, 2005):

$$\frac{dN(t)}{dt} = KN(t)$$

- a. Assuming an initial population $N(0) = N_0$, solve the differential equation by finding $N(t)$.
- b. Find the time at which the population is double the initial population.
- 64.** Blood vessel blockages can in some instances be diagnosed through noninvasive techniques such as the use of sensitive microphones to detect flow acoustic anomalies. In order to predict the sound properties of the left coronary artery, a model has been developed that partitions the artery into 14 segments, as shown in Figure P2.38(a).



(a)

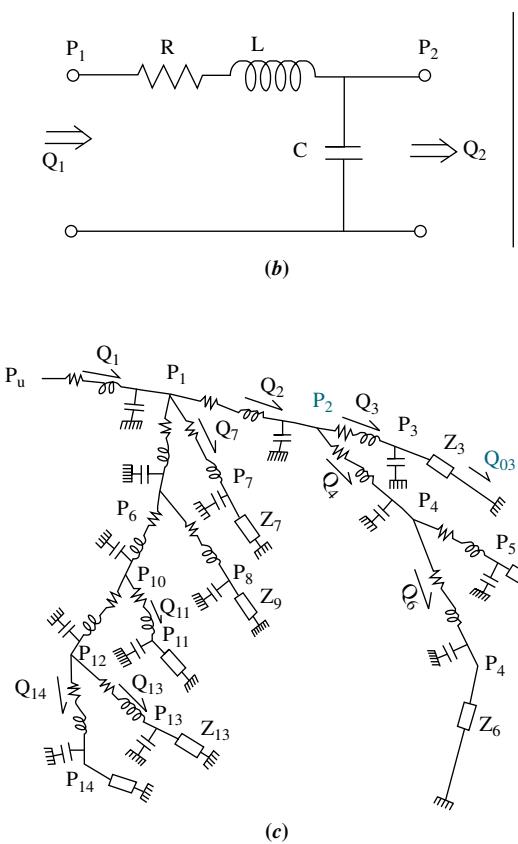


FIGURE P2.38 (© 1990 IEEE)

Each segment is then modeled through the analogous electrical circuit of Figure P2.38(b), resulting in the total model shown in Figure P2.38(c), where eight terminal resistances (Z) have been added. In the electrical model, pressure is analogous to voltage and blood flow is analogous to current. As an example, for Segment 3 it was experimentally verified that $R_3 = 4176 \Omega$, $C_3 = 0.98 \mu\text{F}$, $L_3 = 140.6 \text{ H}$, and $Z_3 = 308,163 \Omega$ (Wang, 1990).

- a. For Segment 3, find the transfer function from input pressure to blood flow through Z_3 , $\frac{Q_{03}(s)}{P_2(s)}$.
- b. It is well known in circuit analysis that if a constant input is applied to a circuit such as the one of Figure P2.38(b), the capacitor can be substituted by an open circuit and the inductor can be substituted by a short circuit as time approaches infinity. Use this fact to calculate the flow through Z_3 after a constant unit pressure pulse is applied and time approaches infinity.

- c. Verify the result obtained in Part b using the transfer function obtained in Part a and applying the final value theorem.

65. In order to design an underwater vehicle that has the characteristics of both a long-range transit vehicle (torpedo-like) and a highly maneuverable low-speed vehicle (boxlike), researchers have developed a thruster that mimics that of squid jet locomotion (Krieg, 2008). It has been demonstrated there that the average normalized thrust due to a command

step input, $U(s) = \frac{T_{ref}}{s}$, is given by:

$$T(t) = T_{ref}(1 - e^{-\lambda t}) + a \sin(2\pi f t)$$

where T_{ref} is the reference or desired thrust, λ is the system's damping constant, a is the amplitude of the oscillation caused by the pumping action of the actuator, f is the actuator frequency, and $T(t)$ is the average resulting normalized thrust. Find the thruster's transfer function $\frac{T(s)}{U(s)}$. Show all steps.

66. The Gompertz growth model is commonly used to model tumor cell growth. Let $v(t)$ be the tumor's volume, then

$$\frac{dv(t)}{dt} = \lambda e^{-\alpha t} v(t)$$

where λ and α are two appropriate constants (Edelstein-Keshet, 2005).

- a. Verify that the solution to this equation is given by $v(t) = v_0 e^{\lambda/\alpha(1-e^{-\alpha t})}$, where v_0 is the initial tumor volume.
- b. This model takes into account the fact that when nutrients and oxygen are scarce at the tumor's core, its growth is impaired. Find the final predicted tumor volume (let $t \rightarrow \infty$).
- c. For a specific mouse tumor, it was experimentally found that $\lambda = 2.5$ days, $\alpha = 0.1$ days with $v_0 = 50 \times 10^{-3} \text{ mm}^3$ (Chignola, 2005). Use any method available to make a plot of $v(t)$ vs. t .
- d. Check the result obtained in Part b with the results from the graph from Part c.

PROGRESSIVE ANALYSIS AND DESIGN PROBLEMS

67. **High-speed rail pantograph.** Problem 21 in Chapter 1 discusses active control of a pantograph mechanism for high-speed rail systems. The diagram for the pantograph and catenary coupling is shown in Figure P2.39(a). Assume the simplified model shown in Figure P2.39(b), where the catenary is represented by the spring, K_{ave} (O'Connor, 1997).

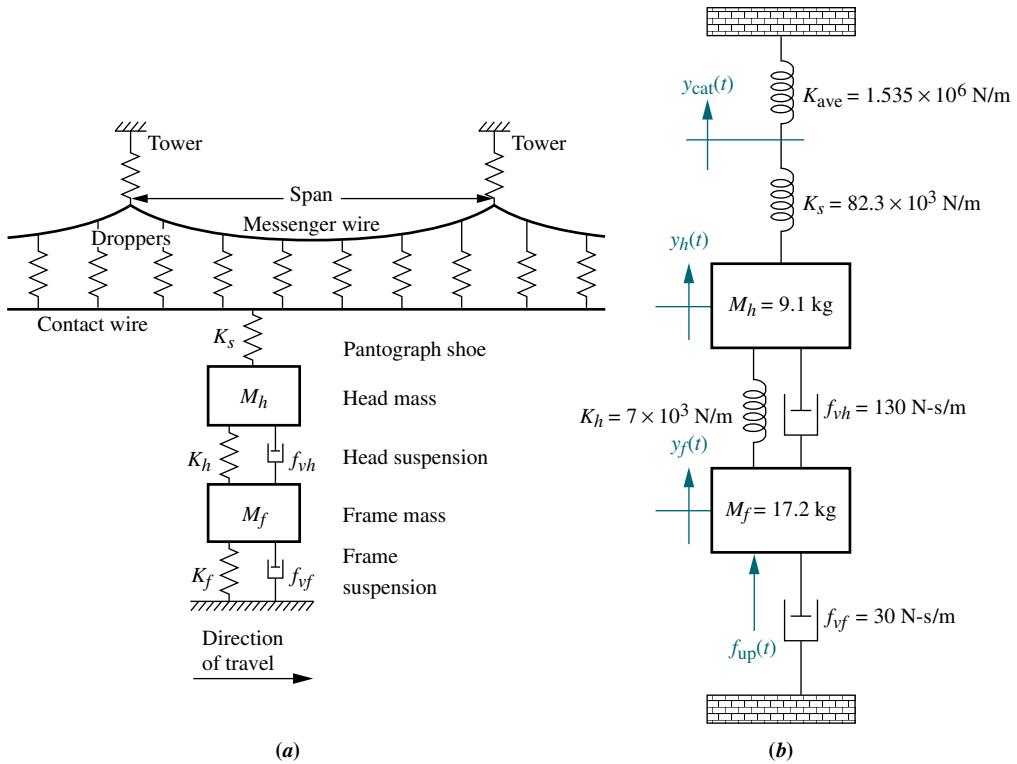


FIGURE P2.39 a. Coupling of pantograph and catenary; b. simplified representation showing the active-control force (Reprinted with permission of ASME.)

- Find the transfer function, $G_1(s) = Y_{cat}(s)/F_{up}(s)$, where $y_{cat}(t)$ is the catenary displacement and $f_{up}(t)$ is the upward force applied to the pantograph under active control.
 - Find the transfer function $G_2(s) = Y_h(s)/F_{up}(s)$, where $y_h(t)$ is the pantograph head displacement.
 - Find the transfer function, $G(s) = (Y_h(s) - Y_{cat}(s))/F_{up}(s)$.
68. **Control of HIV/AIDS.** HIV inflicts its damage by infecting healthy CD4 + T cells (a type of white blood cell) that are necessary to fight infection. As the virus embeds in a T cell and the immune system produces more of these cells to fight the infection, the virus propagates in an opportunistic fashion. As we now develop a simple HIV model, refer to Figure P2.40. Normally T cells are produced at a rate s and die at a rate d . The HIV virus is present in the bloodstream in the infected individual. These viruses in the bloodstream, called *free viruses*, infect healthy T cells at a rate β . Also, the viruses reproduce through the T cell multiplication process or otherwise at a rate k . Free viruses die at a rate c . Infected T cells die at a rate μ .

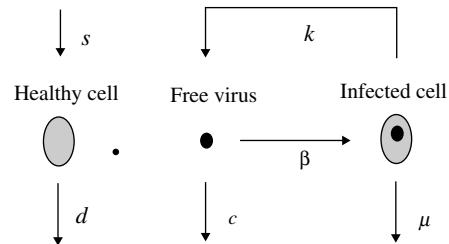


FIGURE P2.40 (© 2004 IEEE)

A simple mathematical model that illustrates these interactions is given by the following equations (Craig, 2004):

$$\frac{dT}{dt} = s - dT - \beta T v$$

$$\frac{dT^*}{dt} = \beta T v - \mu T^*$$

$$\frac{dv}{dt} = k T^* - c v$$

where

- T = number of healthy T cells
- T^* = number of infected T cells
- v = number of free viruses

- The system is nonlinear; thus linearization is necessary to find transfer functions as you will do in subsequent chapters. The nonlinear nature of this model can be seen from the above equations. Determine which of these equations are linear, which are nonlinear, and explain why.
- The system has two equilibrium points. Show that these are given by

$$(T_0, T_0^*, v_0) = \left(\frac{s}{d}, 0, 0 \right)$$

and

$$(T_0, T_0^*, v_0) = \left(\frac{c\mu}{\beta k}, \frac{s}{\mu} - \frac{cd}{\beta k}, \frac{sk}{c\mu} - \frac{d}{\beta} \right)$$

- Hybrid vehicle.** Problem 23 in Chapter 1 discusses the cruise control of serial, parallel, and split-power hybrid electric vehicles (HEVs). The functional block diagrams developed for these HEVs indicated that the speed of a vehicle depends upon the balance between the motive forces (developed by the gasoline engine and/or the electric motor) and running resistive forces. The resistive forces include the aerodynamic drag, rolling resistance, and climbing resistance. Figure P2.41 illustrates the running resistances for a car moving uphill (*Bosch, 2007*).

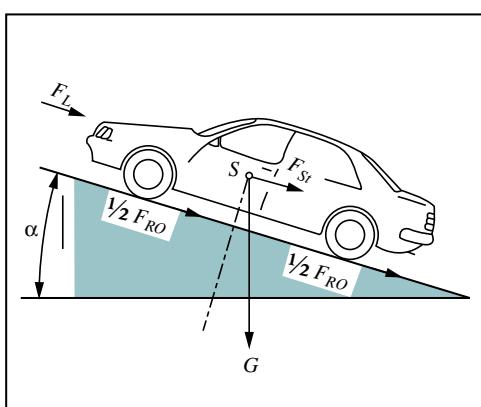


FIGURE P2.41 Running resistances

The total running resistance, F_w , is calculated as $F_w = F_{Ro} + F_L + F_{St}$, where F_{Ro} is the rolling resistance, F_L is the aerodynamic drag, and F_{St} is the climbing resistance. The aerodynamic drag is proportional to the square of the sum of car velocity, v , and the head-wind velocity, v_{hw} , or $v + v_{hw}$. The other two resistances are functions of car weight, G , and the gradient of the road (given by the gradient angle, α), as seen from the following equations:

$$F_{Ro} = fG \cos \alpha = fmg \cos \alpha$$

where

- f = coefficient of rolling resistance,
- m = car mass, in kg,
- g = gravitational acceleration, in m/s^2 .

$$F_L = 0.5\rho C_w A(v + v_{hw})^2$$

where

- ρ = air density, in kg/m^3 ,
- C_w = coefficient of aerodynamic drag,
- A = largest cross-section of the car, in kg/m^2 .
- $F_{St} = G \sin \alpha = mg \sin \alpha$

The motive force, F , available at the drive wheels is:

$$F = \frac{T i_{tot}}{r} \eta_{tot} = \frac{P \eta_{tot}}{v}$$

where

- T = motive torque,
- P = motive power,
- i_{tot} = total transmission ratio,
- r = tire radius,
- η_{tot} = total drive-train efficiency.

The surplus force, $F - F_w$, accelerates the vehicle (or retards it when $F_w > F$). Letting $a = \frac{F - F_w}{k_m \cdot m}$, where a is the acceleration and k_m is a coefficient that compensates for the apparent increase in vehicle mass due to rotating masses (wheels, flywheel, crankshaft, etc.):

- Show that car acceleration,¹⁶ a , may be determined from the equation:

$$F = fmg \cos \alpha + mg \sin \alpha + 0.5\rho C_w A(v + v_{hw})^2 + k_m ma$$

¹⁶ Other quantities, such as top speed, climbing ability, etc., may also be calculated by manipulation from that equation.

- b.** Assuming constant acceleration and using the average value for speed, find the average motive force, F_{av} (in N), and power, P_{av} (in kW) the car needs to accelerate from 40 to 60 km/h in 4 seconds on a level road, ($\alpha = 0^\circ$), under windless conditions, where $v_{hw} = 0$. You are given the following parameters: $m = 1590 \text{ kg}$, $A = 2 \text{ m}^2$, $f = 0.011$, $\rho = 1.2 \text{ kg/m}^3$, $C_w = 0.3$, $\eta_{tot} = 0.9$, $k_m = 1.2$. Furthermore, calculate the additional power, P_{add} , the car needs after reaching 60 km/h to maintain its speed while climbing a hill with a gradient $\alpha = 5^\circ$.
- c.** The equation derived in Part **a** describes the non-linear car motion dynamics where $F(t)$ is the input to the system, and $v(t)$ the resulting output. Given that the aerodynamic drag is proportional to v^2 under

windless conditions, linearize the resulting equation of motion around an average speed, $v_o = 50 \text{ km/h}$, when the car travels on a level road,¹⁷ where $\alpha = 0^\circ$. (Hint: Expand $v^2 - v_0^2$ in a truncated Taylor series). Write that equation of motion and represent it with a block diagram in which the block G_v represents the vehicle dynamics. The output of that block is the car speed, $v(t)$, and the input is the excess motive force, $F_e(t)$, defined as: $F_e = F - F_{St} - F_{Ro} + F_o$, where F_o the constant component of the linearized aerodynamic drag.

- d.** Use the equation in Part **c** to find the vehicle transfer function: $G_v(s) = V(s)/F_e(s)$.

¹⁷ Note that on a level road the climbing resistance is $F_{St} = 0$, since $\sin \alpha = \sin 0^\circ = 0$.

Cyber Exploration Laboratory

Experiment 2.1

Objectives To learn to use MATLAB to (1) generate polynomials, (2) manipulate polynomials, (3) generate transfer functions, (4) manipulate transfer functions, and (5) perform partial-fraction expansions.

Minimum Required Software Packages MATLAB and the Control System Toolbox

Prelab

- Calculate the following by hand or with a calculator:

- The roots of $P_1 = s^6 + 7s^5 + 2s^4 + 9s^3 + 10s^2 + 12s + 15$
- The roots of $P_2 = s^6 + 9s^5 + 8s^4 + 9s^3 + 12s^2 + 15s + 20$
- $P_3 = P_1 + P_2$; $P_4 = P_1 - P_2$; $P_5 = P_1P_2$

- Calculate by hand or with a calculator the polynomial

$$P_6 = (s + 7)(s + 8)(s + 3)(s + 5)(s + 9)(s + 10)$$

- Calculate by hand or with a calculator the following transfer functions:

- $G_1(s) = \frac{20(s+2)(s+3)(s+6)(s+8)}{s(s+7)(s+9)(s+10)(s+15)}$,

represented as a numerator polynomial divided by a denominator polynomial.

- $G_2(s) = \frac{s^4 + 17s^3 + 99s^2 + 223s + 140}{s^5 + 32s^4 + 363s^3 + 2092s^2 + 5052s + 4320}$,

expressed as factors in the numerator divided by factors in the denominator, similar to the form of $G_1(s)$ in Prelab 3a.

- $G_3(s) = G_1(s) + G_2(s)$; $G_4(s) = G_1(s) - G_2(s)$; $G_5(s) = G_1(s)G_2(s)$ expressed as factors divided by factors and expressed as polynomials divided by polynomials.

4. Calculate by hand or with a calculator the partial-fraction expansion of the following transfer functions:

a. $G_6 = \frac{5(s+2)}{s(s^2 + 8s + 15)}$

b. $G_7 = \frac{5(s+2)}{s(s^2 + 6s + 9)}$

c. $G_8 = \frac{5(s+2)}{s(s^2 + 6s + 34)}$

Lab

1. Use MATLAB to find P_3 , P_4 , and P_5 in Prelab 1.
2. Use only one MATLAB command to find P_6 in Prelab 2.
3. Use only two MATLAB commands to find $G_1(s)$ in Prelab 3a represented as a polynomial divided by a polynomial.
4. Use only two MATLAB commands to find $G_2(s)$ expressed as factors in the numerator divided by factors in the denominator.
5. Using various combinations of $G_1(s)$ and $G_2(s)$, find $G_3(s)$, $G_4(s)$, and $G_5(s)$. Various combinations implies mixing and matching $G_1(s)$ and $G_2(s)$ expressed as factors and polynomials. For example, in finding $G_3(s)$, $G_1(s)$ can be expressed in factored form and $G_2(s)$ can be expressed in polynomial form. Another combination is $G_1(s)$ and $G_2(s)$ both expressed as polynomials. Still another combination is $G_1(s)$ and $G_2(s)$ both expressed in factored form.
6. Use MATLAB to evaluate the partial fraction expansions shown in Prelab 4.

Postlab

1. Discuss your findings for Lab 5. What can you conclude?
2. Discuss the use of MATLAB to manipulate transfer functions and polynomials. Discuss any shortcomings in using MATLAB to evaluate partial fraction expansions.

Experiment 2.2

Objectives To learn to use MATLAB and the Symbolic Math Toolbox to (1) find Laplace transforms for time functions, (2) find time functions from Laplace transforms, (3) create LTI transfer functions from symbolic transfer functions, and (4) perform solutions of symbolic simultaneous equations.

Minimum Required Software Packages MATLAB, the Symbolic Math Toolbox, and the Control System Toolbox

Prelab

1. Using a hand calculation, find the Laplace transform of:

$$f(t) = 0.0075 - 0.00034e^{-2.5t} \cos(22t) + 0.087e^{-2.5t} \sin(22t) - 0.0072e^{-8t}$$

2. Using a hand calculation, find the inverse Laplace transform of

$$F(s) = \frac{2(s+3)(s+5)(s+7)}{s(s+8)(s^2 + 10s + 100)}$$

3. Use a hand calculation to solve the circuit for the loop currents shown in Figure P2.42.

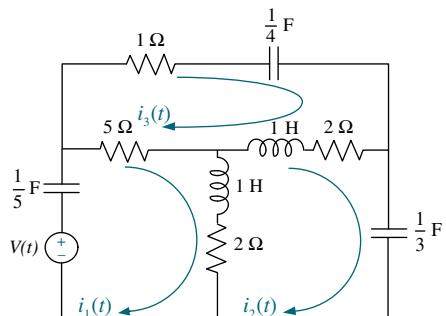


FIGURE P2.42

Lab

1. Use MATLAB and the Symbolic Math Toolbox to
 - a. Generate symbolically the time function $f(t)$ shown in Prelab 1.
 - b. Generate symbolically $F(s)$ shown in Prelab 2. Obtain your result symbolically in both factored and polynomial forms.
 - c. Find the Laplace transform of $f(t)$ shown in Prelab 1.
 - d. Find the inverse Laplace transform of $F(s)$ shown in Prelab 2.
 - e. Generate an LTI transfer function for your symbolic representation of $F(s)$ in Prelab 2 in both polynomial form and factored form. Start with the $F(s)$ you generated symbolically.
 - f. Solve for the loop currents in Prelab 3.

Postlab

1. Discuss the advantages and disadvantages between the Symbolic Math Toolbox and MATLAB alone to convert a transfer function from factored form to polynomial form and vice versa.
2. Discuss the advantages and disadvantages of using the Symbolic Math Toolbox to generate LTI transfer functions.
3. Discuss the advantages of using the Symbolic Math Toolbox to solve simultaneous equations of the type generated by the electrical network in Prelab 3. Is it possible to solve the equations via MATLAB alone? Explain.
4. Discuss any other observations you had using the Symbolic Math Toolbox.

Experiment 2.3

Objective To learn to use LabVIEW to generate and manipulate polynomials and transfer functions.

Minimum Required Software Packages LabVIEW and the LabVIEW Control Design and Simulation Module.

Prelab

1. Study Appendix D, Sections D.1 through Section D.4, Example D.1.
2. Perform by hand the calculations stated in Prelab 1 of Experiment 2.1.
3. Find by a hand calculation the polynomial whose roots are: $-7, -8, -3, -5, -9$, and -10 .
4. Perform by hand a partial-fraction expansion of $G(s) = \frac{5s + 10}{s^3 + 8s^2 + 15s}$.
5. Find by a hand calculation $G_1(s) + G_2(s)$, $G_1(s) - G_2(s)$, and $G_1(s)G_2(s)$, where $G_1(s) = \frac{1}{s^2 + s + 2}$ and $G_2(s) = \frac{s + 1}{s^2 + 4s + 3}$.

Lab

1. Open the LabVIEW functions palette and select the **Mathematics/Polynomial** palette.
2. Generate the polynomials enumerated in Prelab 1a and 1b of Experiment 2.1.
3. Generate the polynomial operations stated in Prelab 1c of Experiment 2.1.
4. Generate a polynomial whose roots are those stated in Prelab 3 of this experiment.

5. Generate the partial fraction expansion of the transfer function given in Prelab 4 of this experiment.
6. Using the **Control Design and Simulation/Control Design/Model Construction** palette, construct the two transfer functions enumerated in Prelab 5.
7. Using the **Control Design and Simulation/Control Design/Model Interconnection** palette, display the results of the mathematical operations enumerated in Prelab 5 of this experiment.

Postlab

1. Compare the polynomial operations obtained in Lab 3 to those obtained in Prelab 2.
2. Compare the polynomial displayed in Lab 4 with that calculated in Prelab 3.
3. Compare the partial-fraction expansion obtained in Lab 5 with that calculated in Prelab 4.
4. Compare the results of the mathematical operations found in Lab 7 to those calculated in Prelab 5.

Bibliography

- Aggarwal, J. K. *Notes on Nonlinear Systems*. Van Nostrand Reinhold, New York, 1972.
- Bosch, R. GmbH, *Bosch Automotive Handbook*, 7th ed. John Wiley & Sons Ltd., UK, 2007.
- Boutayeb, A., Twizell, E. H., Achouayb, K., and Chetouani, A. Mathematical Model for the Burden of Diabetes and Its Complications. *BioMedical Engineering OnLine*, 2004, Retrieved from <http://www.biomedical-engineering-online.com/content/3/1/20>, pp. 1–19.
- Cannon, R. H., Jr., *Dynamics of Physical Systems*. McGraw-Hill, New York, 1967.
- Carlson, L. E., and Griggs, G. E. *Aluminum Catenary System Quarterly Report*. Technical Report Contract Number DOT-FR-9154, U.S. Department of Transportation, 1980.
- Chignola, R., and Foroni, R. I. Estimating the Growth Kinetics of Experimental Tumors from as Few as Two Determinations of Tumor Size: Implications for Clinical Oncology. *IEEE Transactions on Biomedical Engineering*, vol. 52, no. 5, May 2005, pp. 808–815.
- Cochin, I. *Analysis and Design of Dynamic Systems*. Harper and Row, New York, 1980.
- Cook, P. A. *Nonlinear Dynamical Systems*. Prentice Hall, United Kingdom, 1986.
- Craig, I. K., Xia, X., and Venter, J. W. Introducing HIV/AIDS Education into the Electrical Engineering Curriculum at the University of Pretoria. *IEEE Transactions on Education*, vol. 47, no. 1, February 2004, pp. 65–73.
- Davis, S. A., and Ledgerwood, B. K. *Electromechanical Components for Servomechanisms*. McGraw-Hill, New York, 1961.
- Doebelin, E. O. *Measurement Systems Application and Design*. McGraw-Hill, New York, 1983.
- Dorf, R. *Introduction to Electric Circuits*, 2d ed. Wiley, New York, 1993.
- D'Souza, A. *Design of Control Systems*. Prentice Hall, Upper Saddle River, NJ, 1988.
- Edelstein-Keshet, L. *Mathematical Models in Biology*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2005.
- Elkins, J. A. *A Method for Predicting the Dynamic Response of a Pantograph Running at Constant Speed under a Finite Length of Overhead Equipment*. Technical Report TN DA36, British Railways, 1976.
- Franklin, G. F., Powell, J. D., and Emami-Naeini, A. *Feedback Control of Dynamic Systems*. Addison-Wesley, Reading, MA, 1986.

- Galvão, R. K. H., Yoneyama, T., and de Araújo, F. M. U. A Simple Technique for Identifying a Linearized Model for a Didactic Magnetic Levitation System. *IEEE Transactions on Education*, vol. 46, no. 1, February 2003, pp. 22–25.
- Hsu, J. C., and Meyer, A. U. *Modern Control Principles and Applications*. McGraw-Hill, New York, 1968.
- Johansson, R., Magnusson, M., and Åkesson, M. Identification of Human Postural Dynamics. *IEEE Transactions on Biomedical Engineering*, vol. 35, no. 10, October 1988, pp. 858–869.
- Kailath, T. *Linear Systems*. Prentice Hall, Upper Saddle River, NJ, 1980.
- Kermurjian, A. From the Moon Rover to the Mars Rover. *The Planetary Report*, July/August 1990, pp. 4–11.
- Krieg, M., and Mohseni, K. Developing a Transient Model for Squid Inspired Thrusters, and Incorporation into Underwater Robot Control Design. *2008 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, France*, September 2008.
- Kuo, F. F. *Network Analysis and Synthesis*. Wiley, New York, 1966.
- Lago, G., and Benningfield, L. M. *Control System Theory*. Ronald Press, New York, 1962.
- Lin, J.-S., and Kanellakopoulos, I. Nonlinear Design of Active Suspensions. *IEEE Control Systems Magazine*, vol. 17, issue 3, June 1997, pp. 45–59.
- Mablekos, V E. *Electric Machine Theory for Power Engineers*. Harper & Row, Cambridge, MA, 1980.
- Marttinen, A., Virkkunen, J., and Salminen, R. T. Control Study with Pilot Crane. *IEEE Transactions on Education*, vol. 33, no. 3, August 1990, pp. 298–305.
- Milsum, J. H. *Biological Control Systems Analysis*. McGraw-Hill, New York, 1966.
- Minorsky, N. *Theory of Nonlinear Control Systems*. McGraw-Hill, New York, 1969.
- Nilsson, J. W., and Riedel, S. A. *Electric Circuits*, 5th ed. Addison-Wesley, Reading, MA, 1996.
- O'Connor, D. N., Eppinger, S. D., Seering, W. P., and Wormly, D. N. Active Control of a High-Speed Pantograph. *Journal of Dynamic Systems, Measurements, and Control*, vol. 119, March 1997, pp. 1–4.
- Ogata, K. *Modern Control Engineering*, 2d ed. Prentice Hall, Upper Saddle River, NJ, 1990.
- Raven, F. H. *Automatic Control Engineering*, 5th ed. McGraw-Hill, New York, 1995.
- Schnell, S., and Mendoza, C. The Condition for Pseudo-First-Order Kinetics in Enzymatic Reactions Is Independent of the Initial Enzyme Concentration. *Biophysical Chemistry* (107), 2004, pp. 165–174.
- Van Valkenburg, M. E. *Network Analysis*. Prentice Hall, Upper Saddle River, NJ, 1974.
- Vidyasagar, M. *Nonlinear Systems Analysis*. Prentice Hall, Upper Saddle River, NJ, 1978.
- Wang, J. Z., Tie, B., Welkowitz, W., Semmlow, J. L., and Kostis, J. B. Modeling Sound Generation in Stenosed Coronary Arteries. *IEEE Transactions on Biomedical Engineering*, vol. 37, no. 11, November 1990, pp. 1087–1094.