

Chapter 5

Transient Heat Conduction: Analytical Methods

1 Introduction

Many heat conduction problems encountered in engineering applications involve time as an independent variable. The goal of analysis is to determine the variation of the temperature as a function of time and position $T(\mathbf{x}, t)$ within the heat conducting body. In general, we deal with conducting bodies in a three dimensional Euclidean space in a suitable set of coordinates ($\mathbf{x} \in \mathbf{R}^3$) and the goal is to predict the evolution of the temperature field for future times ($t > 0$).

Here we investigate specifically solutions to selected special cases of the following form of the heat equation

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + g(\mathbf{r})$$

Solutions to the above equation must be obtained that also satisfy suitable initial and boundary conditions.

2 Fundamental Solutions

Transient problems resulting from the effect of instantaneous point, line and planar sources of heat lead to useful fundamental solutions of the heat equation. By considering media of infinite or semi-infinite extent one can conveniently ignore the effect of boundary conditions on the resulting solutions.

Let a fixed amount of energy H_0 (J) be instantaneously released (thermal explosion) at time $t = 0$ at the origin of a three dimensional system of coordinates inside a solid body of infinite extent initially at $T(\mathbf{x}, 0) = T(r, 0) = 0$ everywhere, where $\mathbf{x} = r = \sqrt{x^2 + y^2 + z^2}$. No other thermal energy input exists subsequent to the initial instantaneous release. Assuming constant thermal properties k, ρ and C_p , the heat equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right)$$

where $\alpha = k/\rho C_p$. This must be solved subject to the initial condition

$$T(r, 0) = 0$$

for all $r > 0$ plus the statement expressing the instantaneous release of energy at $t = 0$ at the origin. Since the body is infinitely large, the far field temperature never changes and all the released energy must dissipate within the body itself.

The fundamental solution of this problem is given by

$$T(r, t) = \frac{H_0}{(4\pi\alpha t)^{3/2}\rho C_p} \exp\left(-\frac{r^2}{4\alpha t}\right)$$

where $H_0/\rho C_p$ is the amount of energy released per unit energy required to raise the temperature of a unit volume of material by one degree.

This solution may be useful in the study of thermal explosions where a buried explosive load located at $r = 0$ is suddenly released at $t = 0$ and the subsequent distribution of temperature at various distances from the explosion is measured as a function of time. A slight modification of the solution produced by the method of reflexion constitutes an approximation to the problem of surface heating of bulk samples by short duration pulses of finely focused high energy beams.

Consider now the situation where heat is released instantaneously at $t = 0$ at $x = y = 0$ but along the entire z -axis in a circular cylindrical system of coordinates in the amount (per unit length) of H_0/L (J/m). The temperature is expected to be independent of z and the corresponding fundamental solution is

$$T(r, t) = \frac{(H_0/L)}{4\pi\alpha t\rho C_p} \exp\left(-\frac{r^2}{4\alpha t}\right)$$

where here $r = \sqrt{x^2 + y^2}$. This could be a model of heat conduction inside a massive body heated by a short pulse of current passing through a thin wire embedded within it.

Finally, if heat is instantaneously released at $t = 0$ at $x = 0$ but on the entire the $x - y$ plane, the corresponding fundamental solution is independent of x and y and is given as

$$T(r, t) = \frac{(H_0/A)}{(4\pi\alpha t)^{1/2}\rho C_p} \exp\left(-\frac{r^2}{4\alpha t}\right)$$

where $r = z$ and H_0/A is now the amount of thermal energy released per unit area (J/m^2). If the reflexion method is used here, the resulting expression may be a representation of the temperature distribution resulting from a short pulse of a uniformly distributed energy beam applied at the surface of a semi-infinite medium.

Another useful fundamental solution is obtained for the case of a semi-infinite solid ($x \geq 0$) initially at $T = T_0$ everywhere and suddenly exposed to a fixed temperature $T = 0$ at $x = 0$. The statement of the problem is

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

subject to

$$T(x, 0) = T_0$$

and

$$T(0, t) = 0$$

The solution is easily obtained introducing the Laplace transform as follows. Multiply the heat equation by $\exp(-st)$, where s is the parameter of the transform, and then integrate with respect to t from 0 to ∞ , i.e.

$$\frac{1}{\alpha} \int_0^\infty \exp(-st) \frac{\partial T}{\partial t} dt = \int_0^\infty \exp(-st) \frac{\partial^2 T}{\partial x^2} dt$$

introducing the notation $T^* = \int_0^\infty \exp(-st) T dt$ and allowing for the exchange of the integral and differential operators, the heat equation transforms into

$$\frac{d^2 T^*}{dx^2} = \frac{s}{\alpha} T^*$$

This is a simple ordinary differential equation which is readily solved for $T^*(x)$. The desired result $T(x, t)$ is finally obtained from T^* by an inversion process yielding

$$T(x, t) = T_0 \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where the error function, erf is defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\xi^2) d\xi$$

The above solution is an appropriate mathematical approximation of the problem of quenching hot bulk metal samples.

3 Separation of Variables and Integral Transforms in Transient Heat Conduction Problems

Analytical solutions to boundary value problems in linear ordinary differential equations are usually obtained by first producing a general solution. The general solution is obtained by simple integration and contains as many independent arbitrary constants as there are derivatives in the original differential equation. Incorporation of boundary conditions subsequently define the values of the integration constants.

In contrast, general solutions of linear partial differential equations involve arbitrary functions of specific functions. Incorporation of boundary conditions involves the determination

of functional relationships and is rarely feasible or practical. An alternative approach to finding solutions is based instead on first determining a set of particular solutions directly and then combining these so as to satisfy the prescribed boundary conditions. A specific, useful and simple implementation of the above idea is known as the method of separation of variables. This section illustrates the application of the separation of variables method for the determination of analytical solutions of steady and transient one dimensional linear heat conduction problems.

In essence, the method is based on the assumption that if one is looking for a solution to a transient, one-dimensional heat conduction problem of the form $T(x, t)$ it is possible to express it as a product

$$T(x, t) = X(x)\Gamma(t)$$

where the functions $X(x)$ and $\Gamma(t)$ are each functions of a single independent variable satisfying specific ordinary differential equations. One proceeds by first solving the associated ordinary differential equations (ODEs) which are then combined in the product form given above.

Regardless of the dimensionality of a problem, the equation for $\Gamma(t)$ is always of the first order and readily solvable by elementary methods.

The equation for $X(x)$ is always of second order and together with the boundary conditions leads to an eigenvalue problem (proper Sturm-Liouville system). The required solution for $X(x)$ can be formally and very generally expressed by the inversion formula

$$X(x) = \sum_{m=1}^{\infty} K(\beta_m, x) \bar{X}(\beta_m)$$

where the kernel functions $K(\beta_m, x)$ are the normalized eigenfunctions of the associated Sturm-Liouville system and the sum runs over all the eigenvalues of the system. Moreover, the integral transform $\bar{X}(\beta_m)$ is given by the formula

$$\bar{X}(\beta_m) = \int_{x'=0}^L K(\beta_m, x') X(x') dx'$$

Therefore, once the boundary conditions are translated from $T(x, t)$ to $X(x)$, the resulting Sturm-Liouville system is solved yielding the appropriate eigenfunctions and eigenvalues. From these one then determines the kernel $K(\beta_m, x)$ and the integral transform $\bar{X}(\beta_m)$. The inversion formula is next used to obtain the function $X(x)$. Finally, the desired solution $T(x, t)$ is obtained substituting in the assumed product form.

The transform method is also applicable to problems in semi-infinite domains. In this case the inversion formula is given instead by

$$X(x) = \int_{\beta=0}^{\infty} K(\beta, x) \bar{X}(\beta) d\beta$$

while the integral transform is

$$\bar{X}(\beta_m) = \int_{x'=0}^{\infty} K(\beta_m, x')X(x')dx'$$

Note that the sum over the discrete set of eigenvalues in the inversion formula has been replaced by an integral over the continuous spectrum of eigenvalues obtained in the semi-infinite case.

4 Separation of Variables in One-Dimensional Systems

In transient, one-dimensional heat conduction problems, the required temperature is a function of distance and time.

4.1 Imposed Boundary Temperature in Cartesian Coordinates

A simple but important conduction heat transfer problem consists of determining the temperature history inside a solid flat wall which is quenched from a high temperature. More specifically, consider the homogeneous problem of finding the one-dimensional temperature distribution inside a slab of thickness L and thermal diffusivity α , initially at some specified temperature $T(x, 0) = f(x)$ and exposed to heat extraction at its boundaries $x = 0$ and $x = L$ such that $T(0, t) = T(L, t) = 0$ (Dirichlet homogeneous conditions), for $t > 0$. The thermal properties are assumed constant.

The mathematical statement of the heat equation for this problem is:

$$\frac{\partial T(x, t)}{\partial t} = \alpha \frac{\partial^2 T(x, t)}{\partial x^2}$$

subject to

$$T(0, t) = T(L, t) = 0$$

and

$$T(x, 0) = f(x)$$

for all x when $t = 0$.

As indicated before, the method of separation of variables starts by assuming the solution to this problem has the following particular form

$$T(x, t) = X(x)\Gamma(t)$$

If the assumption is wrong, one discovers soon enough, but if it is correct then one may just find the desired solution to the problem! Fortunately, the latter turns out to be the case for this and for many other similar problems.

Introducing the above assumption into the heat equation and rearranging yields

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt}$$

However since $X(x)$ and $\Gamma(t)$, the left hand side of this equation is only a function of x while the right hand side is a function only of t . For this to avoid being a contradiction (for arbitrary values of x and t) both sides must be equal to a constant. For physical reasons (in this case we obviously are after a temperature function that either increases or decreases monotonically depending on the initial conditions and the imposed boundary conditions), the required constant must be negative; let us call it $-\omega^2$.

Therefore, the original heat equation, (a partial differential equation) is transformed into the following equivalent system of ordinary differential equations

$$\frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt} = -\omega^2$$

and

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\omega^2$$

General solutions to these equations are readily obtained by direct integration and are

$$\Gamma(t) = C \exp(-\omega^2 \alpha t)$$

and

$$X(x) = A' \cos(\omega x) + B' \sin(\omega x)$$

Substituting back into our original assumption yields

$$T(x, t) = X(x)\Gamma(t) = [A \cos(\omega x) + B \sin(\omega x)] \exp(-\omega^2 \alpha t)$$

where the constant C has been combined with A' and B' to give A and B without loosing any generality.

Now we introduce the boundary conditions. Since $T(0, t) = 0$, necessarily $A = 0$. Furthermore, since also $T(L, t) = 0$, then $\sin(\omega L) = 0$ (since $B = 0$ is an uninteresting trivial solution.) There is an infinite number of values of ω which satisfy this conditions, i.e.

$$\omega_n = \frac{n\pi}{L}$$

with $n = 1, 2, 3, \dots$. The ω_n 's are the eigenvalues and the functions $\sin(\omega_n x)$ are the eigenfunctions of the Sturm-Liouville problem satisfied by the function $X(x)$. These eigenvalues and eigenfunctions play a role in heat conduction analogous to that of the deflection modes

in structural dynamics, the vibration modes in vibration theory and the quantum states in wave mechanics.

Note that each value of ω_n yields an independent solution satisfying the heat equation as well as the two boundary conditions. Therefore we have now an infinite number of independent solutions $T_n(x, t)$ for $n = 1, 2, 3, \dots$ given by

$$T_n(x, t) = [B_n \sin(\omega_n x)] \exp(-\omega_n^2 \alpha t)$$

The principle of superposition allows the creation of a more general solution from the particular solutions above by simple linear combination to give

$$T(x, t) = \sum_{n=1}^{\infty} T_n(x, t) = \sum_{n=1}^{\infty} [B_n \sin(\omega_n x)] \exp(-\omega_n^2 \alpha t) = \sum_{n=1}^{\infty} [B_n \sin(\frac{n\pi x}{L})] \exp(-(\frac{n\pi}{L})^2 \alpha t)$$

The final step is to ensure the values of the constants B_n are such that they satisfy the initial condition, i.e.

$$T(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$$

But this is just the Fourier sine series representation of the function $f(x)$.

Recall that a key property of the eigenfunctions is that of orthonormality property. This is expressed here as

$$\int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) dx = \begin{cases} 0 & , n \neq m \\ L/2 & , n = m \end{cases}$$

Using the orthonormality property one can multiply the Fourier sine series representation of $f(x)$ by $\sin(\frac{m\pi x}{L})$ and integrate from $x = 0$ to $x = L$ to produce the result

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx$$

for $n = 1, 2, 3, \dots$

Finally, the resulting B_n 's can be substituted into the general solution above to give

$$T(x, t) = \sum_{n=1}^{\infty} T_n(x, t) = \sum_{n=1}^{\infty} [\frac{2}{L} \int_0^L f(x') \sin(\frac{n\pi x'}{L}) dx'] \sin(\frac{n\pi x}{L}) \exp(-\frac{n^2 \pi^2 \alpha t}{L^2})$$

Explicit expressions for the B_n 's can be readily obtained for simple $f(x)$'s, for instance if

$$f(x) = T_i = \text{constant}$$

then

$$B_n = -T_i \frac{2(-1 + (-1)^n)}{n\pi}$$

And if

$$f(x) = \begin{cases} x & , 0 \leq x \leq \frac{L}{2} \\ L - x & , \frac{L}{2} \leq x \leq L \end{cases}$$

then

$$B_n = \begin{cases} \frac{4L}{n^2\pi^2} & , n = 1, 5, 9, \dots \\ -\frac{4L}{n^2\pi^2} & , n = 3, 7, 11, \dots \\ 0 & , n = 2, 4, 6, \dots \end{cases}$$

For instance, for the second specific $f(x)$ given above, the result is

$$T(x, t) = \frac{4T_i}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left(\frac{(2n+1)\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)$$

And for the second case, the result is

$$T(x, t) = \frac{4L}{\pi^2} \left[\exp\left(-\frac{\pi^2\alpha t}{L^2}\right) \sin\left(\frac{\pi x}{L}\right) - \frac{1}{9} \exp\left(-\frac{9\pi^2\alpha t}{L^2}\right) \sin\left(\frac{3\pi x}{L}\right) + \dots \right]$$

4.2 Convection at the Boundary in Cartesian Coordinates

The separation of variables method can also be used when the boundary conditions specify values of the normal derivative of the temperature (Newmann conditions) or when linear combinations of the normal derivative and the temperature itself are used (Convective conditions; Mixed conditions; Robin conditions). Consider the homogeneous problem of transient heat conduction in a slab initially at a temperature $T = f(x)$ and subject to convection losses into a medium at $T = 0$ at $x = 0$ and $x = L$. Convection heat transfer coefficients at $x = 0$ and $x = L$ are, respectively h_1 and h_2 . Assume the thermal conductivity of the slab k is constant.

The mathematical formulation of the problem is to find $T(x, t)$ such that

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x, t)}{\partial t}$$

$$-k \frac{\partial T}{\partial x} + h_1 T = 0$$

at $x = 0$ and

$$k \frac{\partial T}{\partial x} + h_2 T = 0$$

at $x = L$, with

$$T(x, 0) = f(x)$$

for all x when $t = 0$.

Assume the solution is of the form $T(x, t) = X(x)\Gamma(t)$ and substitute to obtain

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt} = -\beta^2$$

The solution for $\Gamma(t)$ is

$$\Gamma(t) = \exp(-\alpha\beta^2 t)$$

while $X(x)$ is the solution of the following eigenvalue (proper Sturm-Liouville) problem

$$\frac{d^2 X}{dx^2} + \beta^2 X = 0$$

with

$$-k \frac{dX}{dx} + h_1 X = 0$$

at $x = 0$ and

$$k \frac{dX}{dx} + h_2 X = 0$$

at $x = L$.

Let the eigenvalues of this problem be β_m and the eigenfunctions $X(\beta_m, x)$. Since the eigenfunctions are orthogonal

$$\int_0^L X(\beta_m, x) X(\beta_n, x) dx = \begin{cases} 0 & , n \neq m \\ N(\beta_m) & , n = m \end{cases}$$

where

$$N(\beta_m) = \int_0^L X(\beta_m, x)^2 dx$$

is the norm of the problem.

It can be shown that for the above problem the eigenfunctions are

$$X(\beta_m, x) = \beta_m \cos \beta_m x + \frac{h_1}{k_1} \sin \beta_m x$$

while the eigenvalues are the roots of the transcendental equation

$$\tan \beta_m L = \frac{\beta_m \left(\frac{h_1}{k_1} + \frac{h_2}{k_2} \right)}{\beta_m^2 - \frac{h_1}{k_1} \frac{h_2}{k_2}}$$

Therefore, the complete solution is of the form

$$T(x, t) = \sum_{m=1}^{\infty} c_m X(\beta_m, x) \exp(-\alpha \beta_m^2 t)$$

The specific eigenfunctions are obtained by incorporating the initial condition

$$f(x) = \sum_{m=1}^{\infty} c_m X(\beta_m, x)$$

which expresses the representation of $f(x)$ in terms of eigenfunctions and requires that

$$c_m = \frac{1}{N(\beta_m)} \int_0^L X(\beta_m, x) f(x) dx$$

As an example, the case when $h_1 = 0$ and $h_2 = h$ yields the eigenfunctions

$$X(\beta_m, x) = \cos(\beta_m x)$$

and the eigenvalues are the roots of $\beta_m \tan(\beta_m L) = h/k$. Note that the eigenvalues in this case cannot be given explicitly but must be determined by numerical solution of the given transcendental equation. For this purpose, it is common to rewrite the transcendental equation as

$$\cot(\beta_m L) = \frac{\beta_m L}{Bi}$$

where $Bi = hL/k$. This can be easily solved either graphically or numerically by bisection, Newton's or secant methods. Finally, the norm in this case is

$$\begin{aligned} N(\beta_m) &= \int_0^L X(\beta_m, x)^2 dx = \int_0^L \cos(\beta_m x)^2 dx = \\ &= \frac{1}{2} \frac{\cos(\beta_m L) \sin(\beta_m L) + \beta_m L}{\beta_m} = \frac{L(\beta_m^2 + (h/k)^2) + (h/k)}{2(\beta_m^2 + (h/k)^2)} \end{aligned}$$

4.3 Imposed Boundary Temperature in Cylindrical Coordinates

Consider the quenching problem where a long cylinder (radius $r = b$) initially at $T = f(r)$ whose surface temperature is made equal to zero for $t > 0$.

The heat equation in this case is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T = 0$$

at $r = b$ and

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

According to the separation of variables method we assume a solution of the form $T(r, t) = R(r)\Gamma(t)$. Substituting this into the heat equation yields

$$\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = \frac{1}{\alpha\Gamma} \frac{d\Gamma}{dt}$$

This equation only makes sense when both terms are equal to a negative constant $-\lambda^2$. The original problem has now been transformed into two boundary value problems, namely

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma = 0$$

with general solution

$$\Gamma(t) = C \exp(-\alpha\lambda^2 t)$$

where C is a constant and

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + r^2 \lambda^2 R = 0$$

subject to $R(b) = 0$ and necessarily bounded at $r = 0$.

The last equation is a special case of Bessel's equation. The only physically meaningful solution of which has the form

$$R(r) = A' J_0(\lambda r)$$

where A' is another constant and $J_0(z)$ is the Bessel function of first kind of order zero of the argument given by

$$J_0(z) = 1 - \frac{z^2}{(1!)^2 2^2} + \frac{z^4}{(2!)^2 2^4} - \frac{z^6}{(3!)^2 2^6} + \dots$$

Since $R(b) = 0$ this requires that $J_0(\lambda b) = 0$ which defines the eigenvalues and eigenfunctions for this problem. The eigenvalues are thus given as the roots of

$$J_0(\lambda_n b) = 0$$

Specifically, the first four roots are: $\lambda_1 b = 2.405$, $\lambda_2 b = 5.520$, $\lambda_3 b = 8.654$, and $\lambda_4 b = 11.79$.

A particular solution is then

$$T_n(r, t) = R_n(r)\Gamma(t) = A_n J_0(\lambda_n r) \exp(-\alpha\lambda_n^2 t)$$

where $A_n = A'_n C$ and $n = 1, 2, 3, \dots$

Superposition of particular solutions yields the more general solution

$$T(r, t) = \sum_{n=1}^{\infty} T_n(r, t) = \sum_{n=1}^{\infty} R_n(r) \Gamma(t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

All is left is to determine the A_n 's. For this we use the given initial condition, i.e.

$$T(r, 0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

This is the Fourier-Bessel series representation of $f(r)$ and one can use the orthogonality property of the eigenfunctions to write

$$\begin{aligned} \int_0^b r J_0(\lambda_m r) f(r) dr &= \sum_{n=1}^{\infty} A_n \int_0^b r J_0(\lambda_m r) J_0(\lambda_n r) dr = A_n \int_0^b r J_0^2(\lambda_m r) dr = \\ &= \frac{b^2 A_m}{2} [J_0^2(\lambda_m b) + J_1^2(\lambda_m b)] = \frac{b^2 A_m}{2} J_1^2(\lambda_m b) \end{aligned}$$

where $J_1(z) = -dJ_0(z)/dz$ is the Bessel function of first kind of order one of the argument. Therefore,

$$A_n = \frac{2}{b^2 J_1^2(\lambda_n b)} \int_0^b r J_0(\lambda_n r) f(r) dr$$

so that the required solution is

$$T(r, t) = \frac{2}{b^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n b)} \exp(-\alpha \lambda_n^2 t) \int_0^b r' J_0(\lambda_n r') f(r') dr'$$

An important special case is obtained when $f(r) = T_i = \text{constant}$. The required solution then becomes

$$T(r, t) = \frac{2T_i}{b} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n b)} \exp(-\alpha \lambda_n^2 t)$$

4.4 Convection at the Boundary in Cylindrical Coordinates

Many problems involving more complex boundary conditions can also be solved using the separation of variables method. As an example consider the quenching problem where a long cylinder (radius $r = b$) initially at $T = f(r)$ is exposed to a cooling medium at zero temperature which extracts heat uniformly from its surface.

The heat equation in this case is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$-k \frac{\partial T}{\partial r} = hT$$

at $r = b$, and subject to

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

Separation of variables $T(r, t) = R(r)\Gamma(t)$ gives in this case

$$T(r, t) = \frac{2}{b^2} \sum_{m=1}^{\infty} \exp(-\alpha \beta_m t) \frac{\beta_m^2 J_0(\beta_m r)}{(\beta_m^2 + (\frac{h}{k})^2) J_0^2(\beta_m b)} \int_0^b r' J_0(\beta_m r') f(r') dr'$$

A special case of interest is when the initial temperature is constant $= T_i$ and the surrounding environment is at a non-zero temperature $= T_{\infty}$. In this case the above equation reduces to

$$T(r, t) = T_{\infty} + \frac{2}{b} (T_i - T_{\infty}) \sum_{m=1}^{\infty} \frac{1}{\beta_m} \frac{J_1(\beta_m b) J_0(\beta_m r)}{J_0^2(\beta_m b) + J_1^2(\beta_m b)} e^{-\beta_m^2 \alpha t}$$

where the eigenvalues β_m are obtained from the roots of the following transcendental equation

$$\beta b J_1(\beta b) - Bi J_0(\beta b) = 0$$

with $Bi = hb/k$.

4.5 Imposed Boundary Temperature in Spherical Coordinates

Consider the quenching problem where a sphere (radius $r = b$) initially at $T = f(r)$ whose surface temperature is made equal to zero for $t > 0$.

The heat equation in this case is

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial T}{\partial r}) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T = 0$$

at $r = b$ and

$$\frac{\partial T}{\partial r} = 0$$

at $r = 0$.

According to the separation of variables method we assume a solution of the form $T(r, t) = R(r)\Gamma(t)$. Substituting this into the heat equation yields

$$\frac{1}{r^2 R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \frac{1}{\alpha \Gamma} \frac{d\Gamma}{dt}$$

Again, this equation only makes sense when both terms are equal to a negative constant $-\lambda^2$. The original problem has now been transformed into two boundary value problems, namely

$$\frac{d\Gamma}{dt} + \lambda^2 \alpha \Gamma = 0$$

with general solution

$$\Gamma(t) = C \exp(-\alpha \lambda^2 t)$$

where C is a constant and

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \lambda^2 R = 0$$

subject to $R(b) = 0$ and necessarily bounded at $r = 0$. The general solution of the last problem is

$$R(r) = A' \frac{\sin(\lambda r)}{r} + B' \frac{\cos(\lambda r)}{r} = A' \frac{\sin(\lambda r)}{r}$$

where A' and B' are constants and $B' = 0$ since the temperature must be bounded at $r = 0$. Moreover, the boundary condition at $r = b$ yields the eigenvalues

$$\lambda_n = \frac{n\pi}{b}$$

and the eigenfunctions

$$R_n(r) = \frac{A'_n}{r} \sin(\lambda_n r) = \frac{1}{r} \sin\left(\frac{n\pi r}{b}\right)$$

for $n = 1, 2, 3, \dots$

A particular solution is then

$$T_n(r, t) = R_n(r)\Gamma(t) = \frac{A_n}{r} \sin(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

where $A_n = A'_n C$ and $n = 1, 2, 3, \dots$

Superposition of particular solutions yields the more general solution

$$T(r, t) = \sum_{n=1}^{\infty} T_n(r, t) = \sum_{n=1}^{\infty} R_n(r) \Gamma(t) = \sum_{n=1}^{\infty} \frac{A_n}{r} \sin(\lambda_n r) \exp(-\alpha \lambda_n^2 t)$$

To determine the A_n 's we use the given initial condition, i.e.

$$T(r, 0) = f(r) = \sum_{n=1}^{\infty} \frac{A_n}{r} \sin(\lambda_n r)$$

this is again a Fourier series representation with the Fourier coefficients given by

$$A_n = \frac{2}{b} \int_0^b f(r') \sin\left(\frac{n\pi r'}{b}\right) r' dr'$$

An important special case results when the initial temperature $f(r) = T_i = \text{constant}$. In this case the Fourier coefficients become

$$A_n = -\frac{T_i b}{n\pi} \cos(n\pi) = -\frac{T_i b}{n\pi} (-1)^n$$

4.6 Convection at the Boundary in Spherical Coordinates

Consider a sphere with initial temperature $T(r, 0) = F(r)$ and dissipating heat by convection into a medium at zero temperature at its surface $r = b$. The heat conduction equation in 1D spherical coordinates is

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

In terms of the new variable $U(r, t) = rT(r, t)$ the mathematical formulation of the problem is

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha} \frac{\partial U}{\partial t}$$

subject to

$$U(0, t) = 0, \quad r = 0$$

$$\frac{\partial U}{\partial r} + \left(\frac{h}{k} - \frac{1}{b}\right) U = 0, \quad r = b$$

and

$$U(r, 0) = rF(r), \quad t = 0$$

This is just like heat 1D conduction from a slab so the solution is

$$T(r, t) = \frac{2}{r} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{\beta_m^2 + (h/k - 1/b)^2}{b(\beta_m^2 + (h/k - 1/b)^2) + (h/k - 1/b)} \sin(\beta_m r) \int_{r'=0}^b r' F(r') \sin(\beta_m r') dr'$$

and the eigenvalues β_m are the positive roots of

$$\beta_m b \cot(\beta_m b) + b(h/k - 1/b) = 0$$

Consider now as another example a hollow sphere $a \leq r \leq b$ initially at $F(r)$ and dissipating heat by convection at both its surfaces via heat transfer coefficients h_1 and h_2 . Introducing the transformation $x = r - a$, the problem becomes identical to 1D heat conduction from a slab and the solution is

$$T(r, t) = \frac{1}{r} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{1}{N(\beta_m)} R(\beta_m, r) \int_{r'=a}^b r' F(r') R(\beta_m, r') dr'$$

where

$$R(\beta_m, r) = \beta_m \cos(\beta_m[r - a]) + (h_1/k + 1/a) \sin(\beta_m[r - a])$$

and the eigenvalues β_m are the positive roots of

$$\tan(\beta_m[b - a]) = \frac{\beta_m([h_1/k + 1/a] + [h_2/k - 1/b])}{\beta_m^2 - [h_1/k + 1/a][h_2/k - 1/b]}$$

4.7 Non-homogeneous Problems in One-Dimensional Systems

Consider the problem of finding the temperature field $T(x, t)$ resulting from transient heat conduction in a slab (1D) initially at $T(x, 0) = f(x)$ whose surfaces at $x = 0$ and $x = l$ are maintained respectively at constant non-zero temperatures T_0 and T_l for $t > 0$. The mathematical formulation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T(0, t) = T_0$$

$$T(l, t) = T_l$$

and

$$T(x, 0) = f(x)$$

This problem can be transformed into two equivalent but simpler problems. A steady state non homogeneous problem and a transient homogeneous problem.

Introduce new functions $v(x)$ and $w(x, t)$ such that

$$T(x, t) = v(x) + w(x, t)$$

Substituting into the original PDE one gets

$$\frac{d^2v}{dx^2} = 0$$

and

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha} \frac{\partial w}{\partial t}$$

subject to

$$w(0, t) = T_0 - v(0)$$

$$w(l, t) = T_l - v(l)$$

and

$$w(x, 0) = f(x) - v(x)$$

Now we select $v(x)$ such that the boundary conditions for w become homogeneous, i.e.

$$v(x) = T_0 + \frac{x}{l}[T_l - T_0]$$

With this, the solution for w with $F(x) = f(x) - v(x)$ is readily obtained by separation of variables. Finally

$$T(x, t) = T_0 + \left(\frac{x}{l}\right)(T_l - T_0) + \sum_{n=1}^{\infty} c_n e^{-n^2 \pi^2 \alpha t / l^2} \sin\left(\frac{n \pi x}{l}\right)$$

with

$$\begin{aligned} c_n &= \frac{2}{l} \int_0^l [f(x) - v(x)] \sin\left(\frac{n \pi x}{l}\right) dx \\ &= \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n \pi x}{l}\right) dx + \frac{2}{n \pi} [(-1)^n T_l - T_0] \end{aligned}$$

Consider a now long cylinder initially at $T = F(r)$ inside which heat is generated at a constant rate g_0 and whose boundary $r = b$ is subjected to $T = 0$. The mathematical statement of the problem consists of the heat equation *

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{k} g_0 = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

with $T = 0$ at $r = b$ and $t > 0$ and $T = F(r)$ at $t = 0$. The problem can be split into a steady state problem giving $T_s(r) = (g_0/4k)(b^2 - r^2)$ and a homogeneous transient problem giving $T_h(r, t)$. The solution to the original problem becomes

$$T(r, t) = T_s(r) + T_h(r, t).$$

5 Separation of Variables for the Multidimensional Heat Equation

The governing equation for three dimensional heat conduction is

$$\nabla^2 T(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t}$$

This must be solved subject to

$$k_i \frac{\partial T}{\partial n_i} + h_i T = 0$$

on boundary S_i and

$$T(\mathbf{r}, 0) = f(\mathbf{r})$$

initially.

Separation of variables in this case requires a solution of the form

$$T(\mathbf{r}, t) = \psi(\mathbf{r})\Gamma(t)$$

which leads to

$$\frac{1}{\psi(\mathbf{r})} \nabla^2 \psi(\mathbf{r}) = \frac{1}{\alpha \Gamma(t)} \frac{d\Gamma}{dt} = -\lambda^2$$

The solution for Γ is the same as before while ψ must be found to satisfy

$$\nabla^2 \psi(\mathbf{r}) + \lambda^2 \psi(\mathbf{r}) = 0$$

subject to

$$k_i \frac{\partial \psi}{\partial n_i} + h_i \psi = 0.$$

This is called the Helmholtz equation and can in turn be solved by separating variables as well.

5.1 Separation of Variables for the Helmholtz Equation

Separation of variables of the Helmholtz equation requires assuming the solution has the form

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

substituting this produces

$$X'' + \beta^2 X = 0$$

$$Y'' + \gamma^2 Y = 0$$

$$Z'' + \eta^2 Z = 0$$

where $\beta^2 + \gamma^2 + \eta^2 = \lambda^2$. Further, the solution for the time dependent part is

$$\Gamma = \exp(-\alpha\lambda^2 t)$$

Finally, the complete solution is obtained by superposition of the individual solutions.

5.2 Multidimensional Homogeneous Problems

Consider the problem of transient 2D conduction in a plate of dimensions $a \times b$, initially at $T(x, y, 0) = f(x, y)$ and subject to an insulating condition at the boundary $x = 0$, a zero temperature condition at the boundary $y = 0$ and to convective heat exchange with a surrounding environment at zero temperature at boundaries $x = a$ and $y = b$ with heat transfer coefficients h_2 and h_4 , respectively.

Separation of variables starts by assuming a solution of the form

$$T(x, y, t) = \Gamma(t)X(x)Y(y)$$

substituting one gets

$$X'' + \beta^2 X = 0$$

subject to

$$X' = 0$$

at $x = 0$ and

$$X' + \frac{h_2}{k} X = 0$$

at $x = a$, and

$$Y'' + \gamma^2 Y = 0$$

subject to

$$Y = 0$$

at $y = 0$ and

$$Y' + \frac{h_4}{k} Y = 0$$

at $y = b$. The complete solution is then

$$T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-\alpha(\beta_m^2 + \gamma_n^2)t} \cos(\beta_m x) \sin(\gamma_n y)$$

where $\cos(\beta_m x)$ and $\sin(\gamma_n x)$ are the eigenfunctions of this problem and the Fourier coefficients c_{mn} must be obtained by substituting the initial condition $T(x, y, 0) = f(x, y)$ and taking advantage of the orthogonality of the eigenfunctions to yield

$$c_{mn} = \frac{1}{N(\beta_m)N(\gamma_n)} \int_0^a \int_0^b \cos(\beta_m x) \sin(\gamma_n x) f(x, y) dx dy$$

where the norms $N(\beta_m)$ and $N(\gamma_n)$ are

$$N(\beta_m) = \int_0^a \cos^2(\beta_m x) dx = \frac{1}{2} \frac{a(\beta_m^2 + (h_2/k)^2) + h_2/k}{\beta_m^2 + (h_2/k)^2}$$

and

$$N(\gamma_n) = \int_0^b \sin^2(\gamma_n y) dy = \frac{1}{2} \frac{b(\gamma_n^2 + (h_4/k)^2) + h_4/k}{\gamma_n^2 + (h_4/k)^2}$$

While the eigenvalues β_m are obtained as the roots of

$$\beta_m \tan \beta_m a = \frac{h_2}{k}$$

and the eigenvalues γ_n are the roots of

$$\gamma_n \cot \gamma_n b = -\frac{h_4}{k}$$

5.3 Quenching of a Billet

Consider a long billet of rectangular cross section $2a \times 2b$ initially at a uniform temperature T_i which is quenched by exposing it to convective exchange with an environment at T_∞ by means of a heat transfer coefficient h . Since the billet is long it is appropriate to neglect conduction along its axis. Because of symmetry it is enough to analyze one quarter of the cross section using a rectangular Cartesian system of coordinates with the origin at the center of the billet.

Define first the **shifted temperature** $\theta(x, y, t) = T(x, y, t) - T_\infty$. In terms of the shifted temperature the mathematical formulation of the problem is as follows.

$$\frac{\partial \theta}{\partial t} = \alpha \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right)$$

subject to

$$\theta(x, y, 0) = T_i - T_\infty = \theta_i$$

$$\frac{\partial \theta}{\partial x} = 0$$

at $x = 0$,

$$\frac{\partial \theta}{\partial y} = 0$$

at $y = 0$,

$$\frac{\partial \theta}{\partial x} = -\frac{h}{k} \theta(a, y, t)$$

at $x = a$, and

$$\frac{\partial \theta}{\partial y} = -\frac{h}{k} \theta(x, b, t)$$

at $y = b$.

The solution to this problem obtained by separation of variables is

$$\frac{\theta(x, y, t)}{\theta_i} = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-\alpha(\lambda_n^2 + \beta_m^2)t} \times \frac{\sin(\lambda_n a) \cos(\lambda_n x) \sin(\beta_m b) \cos(\beta_m y)}{[\lambda_n a + \sin(\lambda_n a) \cos(\lambda_n a)][\beta_m b + \sin(\beta_m b) \cos(\beta_m b)]}$$

where the eigenvalues λ_n and β_m are the roots of the following transcendental equations

$$\lambda_n \tan(\lambda_n a) = \frac{h}{k}$$

and

$$\beta_m \tan(\beta_m b) = \frac{h}{k}$$

5.4 Product Solutions

The solution above can be rewritten as

$$\frac{\theta(x, y, t)}{\theta_i} = 2 \sum_{n=1}^{\infty} \frac{\sin(\lambda_n a) \cos(\lambda_n x)}{\lambda_n a + \sin(\lambda_n a) \cos(\lambda_n a)} e^{-\alpha \lambda_n^2 t} \\ \times 2 \sum_{m=1}^{\infty} \frac{\sin(\beta_m b) \cos(\beta_m y)}{\beta_m b + \sin(\beta_m b) \cos(\beta_m b)} e^{-\alpha \beta_m^2 t}$$

Note that the first sum is precisely the solution to the 1D problem of quenching of a slab of thickness $2a$ while the second sum is the corresponding solution for a slab of thickness $2b$. Therefore

$$\frac{\theta(x, y, t)}{\theta_i} \Big|_{billet} = \frac{\theta(x, t)}{\theta_i} \Big|_{2a-slab} \times \frac{\theta(y, t)}{\theta_i} \Big|_{2b-slab}$$

The method of obtaining solutions to multidimensional heat conduction problems by multiplication of the solutions of subsidiary problems of lower dimensionality is called the **method of product solutions**.

In general, if the initial temperature of the medium can be expressed as a product of single space variable functions one can produce solutions to multidimensional problems as products of single dimensional solutions.

Consider transient homogeneous 2D conduction in a plate with convective exchange on all four sides. Initially, $T(x, y, 0) = F(x)F(y)$. This problem can be regarded as a composite of two one-dimensional ones with solutions $T_1(x, t)$ and $T_2(y, t)$. The product solution for the 2D problem is then

$$T(x, y, t) = T_1(x, t)T_2(y, t)$$

5.5 Non-homogeneous Problems

Consider the non homogeneous problem of transient multidimensional heat conduction with internal heat generation

$$\nabla^2 T(\mathbf{r}, t) + \frac{1}{k} g(\mathbf{r}) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t}$$

subject to non homogeneous time independent conditions on at least part of the boundary.

This problem can be decomposed into a set of steady state non homogeneous problems in each of which a single non homogenous boundary condition occurs and a transient homogeneous problem. If the solutions of the steady state problems are $T_{0j}(\mathbf{r})$ and that for the transient problem is $T_h(\mathbf{r}, t)$, the solution of the original problem is

$$T(\mathbf{r}, t) = T_h(\mathbf{r}, t) + \sum_{j=0}^N T_{0j}(\mathbf{r})$$

6 Transient Temperature Nomographs: Heisler Charts

The solutions obtained for 1D non homogeneous problems with Neumann boundary conditions in Cartesian coordinate systems using the method of separation of variables have been collected and assembled in the form of transient temperature nomographs by Heisler. The given charts are a very useful baseline against which to validate one's own analytical or numerical computations. The Heisler charts summarize the solutions to the following three important problems.

The first problem is the 1D transient homogeneous heat conduction in a plate of span L from an initial temperature T_i and with one boundary insulated and the other subjected to a convective heat flux condition into a surrounding environment at T_∞ . (This problem is equivalent to the quenching of a slab of span $2L$ with identical heat convection at the external boundaries $x = -L$ and $x = L$).

Introduction of the following non dimensional parameters simplifies the mathematical formulation of the problem. First is the dimensionless distance

$$X = \frac{x}{L}$$

next, the dimensionless time

$$\tau = \frac{\alpha t}{L^2}$$

then the dimensionless temperature

$$\theta(X, \tau) = \frac{T(x, t) - T_\infty}{T_i - T_\infty}$$

and finally, the **Biot number**

$$Bi = \frac{hL}{k}$$

With the new variables, the mathematical formulation of the heat conduction problem becomes

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

subject to

$$\frac{\partial \theta}{\partial X} = 0$$

at $X = 0$,

$$\frac{\partial \theta}{\partial X} + Bi\theta = 0$$

at $X = 1$, and

$$\theta = 1$$

in $0 \leq X \leq 1$ for $\tau = 0$.

The solutions obtained for one-dimensional nonhomogeneous problems with Neumann boundary conditions in cylindrical and spherical coordinate systems using the method of separation of variables have also been collected and assembled in the form of transient temperature nomographs by Heisler. As in the Cartesian case, the charts are a useful baseline against which to validate one's own analytical or numerical computations. The Heisler charts summarize the solutions to the following three important problems.

The first problem is the 1D transient homogeneous heat conduction in a solid cylinder of radius b from an initial temperature T_i and with one boundary insulated and the other subjected to a convective heat flux condition into a surrounding environment at T_∞ .

Introduction of the following nondimensional parameters simplifies the mathematical formulation of the problem. First is the dimensionless distance

$$R = \frac{r}{b}$$

next, the dimensionless time

$$\tau = \frac{\alpha t}{b^2}$$

then the dimensionless temperature

$$\theta(X, \tau) = \frac{T(r, t) - T_\infty}{T_i - T_\infty}$$

and finally, the **Biot number**

$$Bi = \frac{hb}{k}$$

With the new variables, the mathematical formulation of the heat conduction problem becomes

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \theta}{\partial R} \right) = \frac{\partial \theta}{\partial \tau}$$

subject to

$$\frac{\partial \theta}{\partial R} = 0$$

at $R = 0$,

$$\frac{\partial \theta}{\partial R} + Bi\theta = 0$$

at $R = 1$, and

$$\theta = 1$$

in $0 \leq R \leq 1$ for $\tau = 0$.

As the second problem consider the cooling of a sphere ($0 \leq r \leq b$) initially at a uniform temperature T_i and subjected to a uniform convective heat flux at its surface into a medium at T_∞ with heat transfer coefficient h . In terms of the dimensionless quantities $Bi = hb/k$, $\tau = \alpha t/b^2$, $\theta = (T(r, t) - T_\infty)/(T_i - T_\infty)$ and $R = r/b$, the mathematical statement of the problem is

$$\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 \frac{\partial \theta}{\partial R}) = \frac{\partial \theta}{\partial \tau}$$

in $0 < R < 1$, subject to

$$\frac{\partial \theta}{\partial R} = 0$$

at $R = 0$, and

$$\frac{\partial \theta}{\partial R} + Bi\theta = 0$$

at $R = 1$, and

$$\theta = 1$$

in $0 \leq R \leq 1$, for $\tau = 0$.

As before, the solutions to the above problems are well known and are readily available in graphical form (Heisler charts).

7 Other Analytical Methods

There is a variety of heat conduction problems which although linear, are not readily solved using the separation of variables method. Fortunately, a number of methods not based on separation of variables are available for the solution of these heat conduction problems.

Four such methods are briefly described below, namely: the Laplace transform method, the Duhamel method, the Green's function method and Goodman's approximate integral method. In each case, examples are used to illustrate the application of the method.

7.1 Laplace Transform Method

The Laplace transform method makes use of the Laplace transform to convert the original heat equation into an ordinary differential equation which is more readily solved. The solution of the transformed problem must then be inverted in order to obtain the solution to the original problem.

7.1.1 Theory

Consider a real function $F(t)$. The Laplace transform of $F(t)$, $\mathcal{L}[F(t)] = \bar{F}(s)$ is defined as

$$\mathcal{L}[F(t)] = \bar{F}(s) = \int_{t'=0}^{\infty} e^{-st'} F(t') dt'$$

The inverse transform (to recover $F(t)$ from $\mathcal{L}[F(t)]$) is

$$F(t) = \frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{F}(s) ds$$

where γ is large enough that all the singularities of $\bar{F}(s)$ lie to the left of the imaginary axis.

For the transform and inverse to exist, $F(t)$ must be at least piecewise continuous, of exponential order and $t^n F(t)$ must be bounded as $t \rightarrow 0^+$.

Laplace transforms have the following properties:

Linearity.

$$\mathcal{L}[c_1 F(t) + c_2 G(t)] = c_1 \bar{F}(s) + c_2 \bar{G}(s)$$

Transforms of Derivatives.

$$\mathcal{L}[F'(t)] = s\bar{F}(s) - F(0)$$

$$\mathcal{L}[F''(t)] = s^2 \bar{F}(s) - sF(0) - F'(0)$$

$$\mathcal{L}[F'''(t)] = s^3 \bar{F}(s) - s^2 F(0) - sF'(0) - F''(0)$$

$$\mathcal{L}[F^{(n)}(t)] = s^n \bar{F}(s) - s^{n-1} F(0) - s^{n-2} F^{(1)}(0) - s^{n-3} F^{(2)}(0) - \dots - F^{(n-1)}(0)$$

Transforms of Integrals.

Let $g(t) = \int_0^t F(\tau) d\tau$ (i.e. $g'(t) = F(t)$).

$$\mathcal{L}\left[\int_0^t F(\tau) d\tau\right] = \frac{1}{s} \bar{F}(s)$$

$$\mathcal{L}\left[\int_0^t \int_0^{\tau_2} F(\tau_1) d\tau_1 d\tau_2\right] = \frac{1}{s^2} \bar{F}(s)$$

$$\mathcal{L}\left[\int_0^t \dots \int_0^{\tau_n} F(\tau_1) d\tau_1 \dots d\tau_n\right] = \frac{1}{s^n} \bar{F}(s)$$

Scale change.

Let $a > 0$ be a real number.

$$\mathcal{L}[F(at)] = \frac{1}{a} \bar{F}\left(\frac{s}{a}\right)$$

$$\mathcal{L}\left[F\left(\frac{1}{a}t\right)\right] = a\bar{F}(as)$$

Shift.

$$\mathcal{L}[e^{\pm at}F(t)] = \bar{F}(s \mp a)$$

Transform of translated functions.

The translated function is

$$U(t-a)F(t-a) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

where

$$U(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

The Laplace transform of the translated function is

$$\mathcal{L}[U(t-a)F(t-a)] = e^{-as}\bar{F}(s)$$

Also

$$\mathcal{L}[U(t-a)] = e^{-as}\frac{1}{s}$$

Transform of the δ function.

$$\mathcal{L}[\delta(t)] = 1$$

Transform of convolution.

The convolution of two functions $f(t)$ and $g(t)$ is

$$f * g = \int_0^t f(t-\tau)g(\tau)d\tau = g * f$$

The Laplace transform is

$$\mathcal{L}[f * g] = \bar{f}(s)\bar{g}(s)$$

Derivatives.

If $\bar{F}(s)$ is the Laplace transform of $F(t)$ then

$$\frac{d\bar{F}}{ds} = \bar{F}'(s) = \mathcal{L}[(-t)F(t)]$$

$$\frac{d^n \bar{F}}{ds^n} = \bar{F}^{(n)}(s) = \mathcal{L}[(-t)^n F(t)]$$

Integrals.

$$\int_s^\infty \bar{F}(s') ds' = \mathcal{L}\left[\frac{F(t)}{t}\right]$$

Functions of More than One Variable.

Let $f(x, t)$ be a function of the two independent variables x and t . The Laplace transform of $f(x, t)$, $\mathcal{L}[f(x, t)] = \bar{f}(x, s)$ is defined as

$$\bar{f}(x, s) = \int_0^\infty e^{-st} f(x, t) dt$$

With this, the following useful formulae are obtained

$$\mathcal{L}\left[\frac{\partial f(x, t)}{\partial x}\right] = \frac{d\bar{f}(x, s)}{dx}$$

$$\mathcal{L}\left[\frac{\partial^2 f(x, t)}{\partial x^2}\right] = \frac{d^2 \bar{f}(x, s)}{dx^2}$$

$$\mathcal{L}\left[\frac{\partial f(x, t)}{\partial t}\right] = s\bar{f}(x, s) - f(x, 0)$$

$$\mathcal{L}\left[\frac{\partial^2 f(x, t)}{\partial t^2}\right] = s^2 \bar{f}(x, s) - sf(x, 0) - \frac{\partial f(x, 0)}{\partial t}$$

The inverse Laplace transform of $\mathcal{L}[F(t)]$ is

$$F(t) = \frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{\gamma+i\infty} e^{st} \bar{F}(s) ds$$

The integration must be carried out in the complex domain and requires a basic understanding of analytic functions of a complex variable and residue calculus. Fortunately, inversion formulae for many transforms have been computed and are readily available in tabular form. A short table of useful transforms is included below.

$F(t)$	$\bar{F}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
\sqrt{t}	$\frac{\sqrt{\pi}}{2\sqrt{s^3}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$F(t-a); t \geq a; \text{otherwise}$	$e^{-as} \bar{F}(s)$
$\frac{k}{2\sqrt{\pi t^3}} e^{-\frac{k^2}{4t}}$	$e^{-k\sqrt{s}}$
$erfc(\frac{k}{2\sqrt{t}})$	$\frac{1}{s} e^{-k\sqrt{s}}$
$-e^{ak} e^{a^2 t} erfc(a\sqrt{t} + \frac{k}{s\sqrt{t}}) + erfc(\frac{k}{2\sqrt{t}})$	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}$
$1 - 2 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t/a^2} J_0(\lambda_n x/a)}{\lambda_n J_1(\lambda_n)}$; λ_n roots of $J_0(\lambda_n) = 0$	$\frac{J_0(ix\sqrt{s})}{sJ_0(ia\sqrt{s})}$

7.1.2 Examples

Consider the problem of finding $T(x, t)$ in $0 \leq x < \infty$ satisfying

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

and subject to

$$T(0, t) = f(t)$$

$$T(x \rightarrow \infty, t) = 0$$

and

$$T(x, 0) = 0$$

Taking Laplace transforms one gets

$$\frac{d^2 \bar{T}(x, s)}{dx^2} - \frac{s}{\alpha} \bar{T}(x, s) = 0$$

$$\bar{T}(0, s) = \bar{f}(s)$$

$$\bar{T}(x \rightarrow \infty, s) = 0$$

The solution of this problem is

$$\bar{T}(x, s) = \bar{f}(s)e^{-x\sqrt{s/\alpha}} = \bar{f}(s)\bar{g}(x, s) = \mathcal{L}[f(t) * g(x, t)]$$

Inversion then produces

$$T(x, t) = f(t) * g(x, t) = \int_0^t f(\tau)g(x, t - \tau)d\tau$$

Inversion of $\bar{g}(x, s)$ to get $g(x, t)$ finally gives

$$T(x, t) = \frac{x}{\sqrt{4\pi\alpha}} \int_{\tau=0}^t \frac{f(\tau)}{(t - \tau)^{3/2}} \exp\left[-\frac{x^2}{4\alpha(t - \tau)}\right] d\tau$$

If $f(t) = T_0 = \text{constant}$, the solution is

$$T(x, t) = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Consider now the transient problem of quenching a solid sphere (radius b) initially at T_0 by maintaining its surface at zero. The Laplace transform method can be used to find an expression for $T(r, t)$. The transformed problem is

$$\frac{1}{r} \frac{d^2(r\bar{T})}{dr^2} - \frac{s}{\alpha} \bar{T} = -\frac{T_0}{\alpha}$$

the solution of which is

$$\bar{T}(r, s) = \frac{T_0}{s} - \frac{T_0 b \sinh(r\sqrt{s\alpha})}{rs \sinh(b\sqrt{s\alpha})}$$

To invert this the trigonometric functions are expressed as asymptotic series and then inverted term by term. The final result is

$$T(r, t) = T_0 - \frac{bT_0}{r} \sum_{n=0}^{\infty} \left\{ \operatorname{erfc}\left(\frac{b(2n+1) - r}{2\sqrt{\alpha t}}\right) - \operatorname{erfc}\left(\frac{b(2n+1) + r}{2\sqrt{\alpha t}}\right) \right\}$$

7.2 Duhamel's Method

Duhamel's method is based on Duhamel's theorem which allows expressing the solution of a problem subjected to time dependent boundary conditions in terms of an integral of the solution corresponding to time independent boundary conditions.

7.2.1 Theory

Many practical transient heat conduction problems involve time dependent boundary conditions. Duhamel's theorem allows the analysis of transient heat conduction problems involving time varying thermal conditions at boundaries.

Consider the following problem

$$\nabla^2 T(\mathbf{r}, t) + \frac{1}{k} g(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t}$$

subject to

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i(\mathbf{r}, t)$$

on boundaries S_i for $i = 1, 2, \dots, N$ and for $t > 0$, as well as

$$T(\mathbf{r}, 0) = F(\mathbf{r})$$

Solution by separation of variables is not possible because of the time dependent nonhomogeneous terms $g(\mathbf{r}, t)$ and $f_i(\mathbf{r}, t)$.

Consider instead an auxiliary problem obtained by introducing a new parameter τ , unrelated to t . Find $\Phi(\mathbf{r}, t, \tau)$ such that

$$\nabla^2 \Phi(\mathbf{r}, t, \tau) + \frac{1}{k} g(\mathbf{r}, \tau) = \frac{1}{\alpha} \frac{\partial \Phi(\mathbf{r}, t, \tau)}{\partial t}$$

subject to

$$k_i \frac{\partial \Phi}{\partial n_i} + h_i \Phi = f_i(\mathbf{r}, \tau)$$

on boundaries S_i for $i = 1, 2, \dots, N$ and for $t > 0$, as well as

$$\Phi(\mathbf{r}, 0, \tau) = F(\mathbf{r})$$

Since here $g(\mathbf{r}, \tau)$ and $f_i(\mathbf{r}, \tau)$ do not depend on time, the problem can be solved by separation of variables. Duhamel's theorem states that once $\Phi(\mathbf{r}, t, \tau)$ has been determined, $T(\mathbf{r}, t)$ can be found by

$$T(\mathbf{r}, t) = F(\mathbf{r}) + \int_{\tau=0}^t \frac{\partial}{\partial t} \Phi(\mathbf{r}, t - \tau, \tau) d\tau$$

If the nonhomogeneity occurs only in the boundary condition at one boundary and the initial temperature is $F(\mathbf{r}) = 0$, the appropriate form of Duhamel's theorem is

$$T(\mathbf{r}, t) = \int_{\tau=0}^t f(\tau) \frac{\partial}{\partial t} \Phi(\mathbf{r}, t - \tau) d\tau$$

Sometimes the nonhomogeneous term at a boundary $f(t)$ changes discontinuously with time. The integral in Duhamel's theorem must then be done separately integrating by parts. For a total of N discontinuities at times over the time range $0 < t < \tau_N$, the temperature $T(\mathbf{r}, t)$ for $\tau_{N-1} < t < \tau_N$ is given by

$$T(\mathbf{r}, t) = \int_{\tau=0}^t \Phi(\mathbf{r}, t - \tau) \frac{df(\tau)}{d\tau} d\tau + \sum_{j=0}^{N-1} \Phi(\mathbf{r}, t - \tau_j) \Delta f_j$$

where $\Delta f_j = f^+(\tau_j) - f^-(\tau_j)$ is the step change in the boundary condition at $t = \tau_j$. If the values of f are constant between the steps changes, Duhamel's theorem becomes

$$T(\mathbf{r}, t) = \sum_{j=0}^{N-1} \Phi(\mathbf{r}, t - j\Delta t) \Delta f_j$$

for the time interval $(N - 1)\Delta t < t < N\Delta t$.

7.2.2 Examples

Consider the following 1D transient problem in a slab of thickness L . Find $T(x, t)$ satisfying

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T(0, t) = 0$$

$$T(L, t) = f(t) = \begin{cases} bt; & 0 < t < \tau_1 \\ 0; & t > \tau_1 \end{cases}$$

and

$$T(x, 0) = 0$$

The appropriate auxiliary problem here is, find $\Phi(x, t)$

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \Phi}{\partial t}$$

subject to

$$\Phi(0, t) = 0$$

$$\Phi(L, t) = 1$$

and

$$\Phi(x, 0) = 0$$

The desired function $\Phi(x, t - \tau)$ is obtained from the solution to the above problem by replacing t by $t - \tau$, i.e.

$$\Phi(x, t) = \frac{x}{L} + \frac{2}{L} \sum_{m=1}^{\infty} e^{-\alpha\beta_m^2 t} \frac{1}{\beta_m} \sin \beta_m x$$

where $\beta_m = m\pi/L$. So, for $t < \tau_1$

$$\begin{aligned} T(x, t) &= \int_{\tau=0}^t \Phi(x, t - \tau) \frac{df(t)}{d\tau} d\tau = \\ &= b \frac{x}{L} t + b \frac{2}{L} \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha\beta_m^3} (1 - e^{-\alpha\beta_m^2 t}) \sin \beta_m x \end{aligned}$$

And for $t > \tau_1$

$$T(x, t) = \int_{\tau=0}^{\tau_1} \Phi(x, t - \tau) \frac{df}{d\tau} d\tau + \int_{\tau_1}^t \Phi(x, t - \tau) \frac{df}{d\tau} d\tau + \Phi(x, t - \tau_1) \Delta f_1$$

since $df/d\tau = b$ when $t < \tau_1$, $df/d\tau = 0$ for $t > \tau_1$ and $\Delta f_1 = -b\tau_1$, so

$$\begin{aligned} T(x, t) &= \int_{\tau=0}^{\tau_1} \left\{ \frac{x}{L} + \frac{2}{L} \sum_{m=1}^{\infty} e^{-\alpha\beta_m^2(t-\tau)} \frac{(-1)^m}{\beta_m} \sin \beta_m x \right\} b d\tau \\ &\quad - b\tau_1 \left[\frac{x}{L} + \frac{2}{L} \sum_{m=1}^{\infty} e^{-\alpha\beta_m^2(t-\tau_1)} \frac{(-1)^m}{\beta_m} \sin \beta_m x \right] \end{aligned}$$

Consider now the transient 1D problem in a solid cylinder (radius b) with internal heat generation $g(t)$. Initially and at the boundary the temperature is zero. The auxiliary problem is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{k} = \frac{1}{\alpha} \frac{\partial \Phi}{\partial t}$$

subject to

$$\Phi(b, t) = 0$$

$$\Phi(r, 0) = 0$$

The solution to this problem is

$$\Phi(r, t) = \frac{b^2 - r^2}{4k} - \frac{2}{bk} \sum_{m=1}^{\infty} e^{-\alpha\beta_m^2 t} \frac{J_0(\beta_m r)}{\beta_m^3 J_1(\beta_m b)}$$

where β_m are the roots of $J_0(\beta_m b) = 0$. From Duhamel's theorem, the desired solution is

$$\begin{aligned} T(r, t) &= \int_{\tau=0}^t g(\tau) \frac{\partial \Phi}{\partial t} d\tau = \\ &= \frac{2\alpha}{bk} \sum_{m=1}^{\infty} e^{-\alpha\beta_m^2 t} \frac{J_0(\beta_m r)}{\beta_m J_1(\beta_m b)} \int_{\tau=0}^t g(\tau) e^{\alpha\beta_m^2 \tau} d\tau \end{aligned}$$

7.3 Green's Function Method

The method of Green's functions is based on the notion of fundamental solutions of the heat equation described before. Specifically, Green's functions are solutions of heat conduction problems involving instantaneous point, line or planar sources of heat. Solutions to other heat conduction problems can then be expressed as special integrals of the Green's functions.

7.3.1 Theory

Consider the problem of a 3D body (volume V) initially at zero temperature. Suddenly, a thermal explosion at a point \mathbf{r}' instantaneously releases a unit of energy at time τ while the entire outer surface S of the body undergoes convective heat exchange with the surrounding environment at zero temperature. The thermal explosion is an instantaneous point source of thermal energy which then diffuses throughout the body. The mathematical formulation of this problem consists of finding the function $G(\mathbf{r}, t | \mathbf{r}', \tau)$ satisfying the nonhomogeneous transient heat equation in V ,

$$\nabla^2 G + \frac{1}{k} \delta(\mathbf{r} - \mathbf{r}') \delta(t - \tau) = \frac{1}{\alpha} \frac{\partial G}{\partial t}$$

subject to

$$k \frac{\partial G}{\partial n} + hG = 0$$

on the boundary S . Here, $\delta(x)$ is Dirac's delta function (which is zero everywhere except at the origin, where it is infinite). The function $G(\mathbf{r}, t | \mathbf{r}', \tau)$ represents the temperature at location \mathbf{r} and time t resulting from an instantaneous point source of heat releasing a unit of thermal energy at location \mathbf{r}' and time τ and it is called a Green's function.

Consider now the related problem of a 3D body of volume V , initially at $T(\mathbf{r}, 0) = F(\mathbf{r})$ inside which thermal energy is generated at a volumetric rate $g(\mathbf{r}, t)$ while its outer surface(s) S_i exchange heat by convection with a medium at T_∞ . The solution to this problem can be expressed in terms of the Green's function above as

$$\begin{aligned} T(\mathbf{r}, t) &= \int \int \int_V G(\mathbf{r}, t | \mathbf{r}', \tau) |_{\tau=0} F(\mathbf{r}') dV' + \\ &\quad \frac{\alpha}{k} \int_{\tau=0}^t d\tau \int \int \int_V G(\mathbf{r}, t | \mathbf{r}', \tau) g(\mathbf{r}', \tau) dV' + \\ &\quad \alpha \int_{\tau=0}^t d\tau \sum_{i=1}^N \int \int_{S_i} G(\mathbf{r}, t | \mathbf{r}', \tau) |_{\mathbf{r}'=\mathbf{r}_i} \frac{1}{k_i} h_i T_\infty dS'_i \end{aligned}$$

The terms on the RHS above are: i) the contribution due to the initial condition, (ii) the contribution due to the internal energy generation, and (iii) the contribution due to the nonhomogeneity in the boundary conditions.

Line sources and plane sources can be similarly defined respectively for two- and one-dimensional systems and nonhomogeneous transient problems in 2D and 1D are the solvable in terms of appropriate Green's functions. For instance, for a 1D system of extent L ,

$$\begin{aligned} T(x, t) = & \int_L G(x, t|x', \tau)|_{\tau=0} F(x') dx' + \\ & \frac{\alpha}{k} \int_{\tau=0}^t d\tau \int_L G(x, t|x', \tau) g(x', \tau) dx' + \\ & \alpha \int_{\tau=0}^t d\tau \sum_{i=1}^2 G(x, t|x', \tau)|_{x'=x_i} \frac{1}{k_i} h_i T_\infty \end{aligned}$$

Here, the Green's function $G(x, t|x', \tau)$ represents the temperature at x, t resulting from the instantaneous release of a unit of energy at time τ , by a planar source located at x' .

Point, line and surface sources can be instantaneous or continuous. Subscript and superscript signs are used to make the distinction explicit. For instance g_p^i denotes an instantaneous point source of heat. In Cartesian coordinates, the relationship between g_p^i and $g(x, y, z, t)$ is

$$g_p^i \delta(x - x') \delta(y - y') \delta(z - z') \delta(t - \tau) = g(x, y, z, t)$$

Once the Green function associated with a particular problem is known, the actual temperature can be calculated simply by integration. So, one needs techniques for Green's function determination.

Consider the following homogeneous problem in a region R . Find $T(\mathbf{r}, t)$ satisfying

$$\nabla^2 T(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$\frac{\partial T}{\partial n_i} + H_i T = 0$$

on all surfaces S_i and

$$T(\mathbf{r}, t) = F(\mathbf{r})$$

Many special cases of this problem have already been solved by separation of variables. In all instances one can symbolically write the solution as

$$T(\mathbf{r}, t) = \int_R K(\mathbf{r}, \mathbf{r}', t) F(\mathbf{r}') dv'$$

where $K(\mathbf{r}, \mathbf{r}', t)$ is called the kernel of the integration. Now, applying the Green's function approach to this problem yields

$$K(\mathbf{r}, \mathbf{r}', t) = G(\mathbf{r}, t | \mathbf{r}', 0)$$

so the kernel is the Green's function in this case. The associated nonhomogeneous problem can also be solved once $G(\mathbf{r}, t | \mathbf{r}', 0)$ is known by simply replacing t by $t - \tau$ to obtain $G(\mathbf{r}, t | \mathbf{r}', \tau)$.

Consider as an example the 1D transient heat conduction in a cylinder.

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) + \frac{1}{k} g(r, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T(b, t) = f(t)$$

and

$$T(r, 0) = F(r)$$

The associated homogeneous problem with homogeneous boundary condition is readily solved by separation of variables and the solution is

$$\begin{aligned} \psi(r, t) = \int_{r'=0}^b \left[\frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{1}{J_1^2(\beta_m b)} r' J_0^2(\beta_m r) J_0^2(\beta_m r') \right] F(r') dr' = \\ \int_{r'=0}^b r' G(r, t | r', 0) F(r') dr' \end{aligned}$$

where β_m are the roots of $J_0(\beta_m b) = 0$. Therefore, the Green's function with $\tau = 0$ is

$$G(r, t | r', \tau) |_{\tau=0} = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{1}{J_1^2(\beta_m b)} r' J_0^2(\beta_m r) J_0^2(\beta_m r')$$

and replacing t by $t - \tau$, the desired Green's function is

$$G(r, t | r', \tau) = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 (t-\tau)} \frac{1}{J_1^2(\beta_m b)} r' J_0^2(\beta_m r) J_0^2(\beta_m r')$$

For homogeneous 1D transient conduction in an infinite medium, the Green's function is

$$G(x, t | x', \tau) |_{\tau=0} = \frac{1}{(4\pi\alpha t)^{1/2}} \exp\left(-\frac{(x-x')^2}{4\alpha t}\right)$$

For homogeneous 1D transient conduction in a semiinfinite medium, the Green's function is

$$G(x, t | x', \tau) |_{\tau=0} = \frac{1}{(4\pi\alpha t)^{1/2}} \left[\exp\left(-\frac{(x-x')^2}{4\alpha t}\right) - \exp\left(-\frac{(x+x')^2}{4\alpha t}\right) \right]$$

For homogeneous 1D transient conduction in a slab of thickness L ,

$$G(x, t | x', \tau) |_{\tau=0} = 2 \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{\beta_m^2 + H^2}{L(\beta_m^2 + H^2) + H} \cos(\beta_m x) \cos(\beta_m x')$$

7.3.2 Examples

The problem of finding $\psi(x, t)$ for $-\infty < x < \infty$ satisfying

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}$$

and subject to

$$\psi(x, 0) = F(x)$$

has the solution

$$\begin{aligned} \psi(x, t) &= \int_{x'=-\infty}^{\infty} [(4\pi\alpha t)^{-1/2} \exp(-\frac{(x-x')^2}{4\alpha t})] F(x') dx' = \\ &= \int_{x'=-\infty}^{\infty} G(x, t|x', \tau)|_{\tau=0} F(x') dx' \end{aligned}$$

Therefore, the Green's function of the problem of finding $T(x, t)$ for $-\infty < x < \infty$ satisfying

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{k} g(x, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

and subject to

$$T(x, 0) = F(x)$$

is simply

$$G(x, t|x', \tau) = (4\pi\alpha[t - \tau])^{-1/2} \exp(-\frac{(x-x')^2}{4\alpha[t - \tau]})$$

and the desired solution is

$$\begin{aligned} T(x, t) &= (4\pi\alpha t)^{-1/2} \int_{x'=-\infty}^{\infty} \exp(-\frac{(x-x')^2}{4\alpha t}) F(x') dx' \\ &+ \frac{\alpha}{k} \int_{\tau=0}^t d\tau \int_{x'=-\infty}^{\infty} (4\pi\alpha[t - \tau])^{-1/2} \exp(-\frac{(x-x')^2}{4\alpha[t - \tau]}) g(x', \tau) dx' \end{aligned}$$

The problem of finding $\psi(r, t)$ for $0 \leq r \leq b$ satisfying

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}$$

and subject to

$$\psi(0, t) = 0$$

and

$$\psi(r, 0) = F(r)$$

has the solution

$$\begin{aligned}\psi(r, t) &= \int_{r'=0}^b r' \left[\frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 t} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} J_0(\beta_m r') \right] F(r') dr' \\ &= \int_{r'=0}^b r' G(r, t | r', \tau) |_{\tau=0} F(r') dr'\end{aligned}$$

Therefore, the Green's function of the problem of finding $T(r, t)$ for $0 \leq r \leq b$ satisfying

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial T}{\partial r} \frac{1}{k} g(x, t) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

and subject to

$$T(b, t) = f(t)$$

and

$$T(r, 0) = F(r)$$

is simply

$$G(r, t | r', \tau) = \frac{2}{b^2} \sum_{m=1}^{\infty} e^{-\alpha \beta_m^2 [t-\tau]} \frac{J_0(\beta_m r)}{J_1^2(\beta_m b)} J_0(\beta_m r')$$

and the desired solution is

$$\begin{aligned}T(r, t) &= \int_{r'=0}^b r' G(r, t | r', \tau) |_{\tau=0} F(r') dr' + \frac{\alpha}{k} \int_{\tau=0}^t d\tau \int_{r'=0}^b r' G(r, t | r', \tau) g(r', \tau) dr' \\ &\quad - \alpha \int_{\tau=0}^t \left[r' \frac{\partial G}{\partial r'} \right] f(\tau) d\tau\end{aligned}$$

7.4 Goodman Approximate Integral Method

The integral method provides approximate solutions to heat conduction problems in which the energy balance is satisfied only in an average sense.

The integral method is usually implemented in four steps:

- The heat equation is first integrated over a distance $\delta(t)$ called the thermal layer to obtain the heat balance integral. The thermal layer is selected so that the effect of boundary conditions is negligible for $x > \delta(t)$.

- A low order polynomial is then selected to approximate the temperature distribution over the thermal layer. The coefficients in the polynomial are in general functions of time and must be determined from the conditions of the problem.
- The approximating polynomial is substituted in the heat balance integral to obtain a differential equation for $\delta(t)$.
- The resulting $\delta(t)$ is substituted in the polynomial formula to obtain the required temperature distribution.

Consider the case of a semiinfinite medium initially at T_i where at the boundary $x = 0$ the temperature is maintained at $T_0 > T_i$. The formulation is, find $T(x, t)$ satisfying

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T(x, 0) = T_i$$

and to

$$T(0, t) = T_0$$

First integrate the heat equation from $x = 0$ to $x = \delta(t)$ to obtain

$$\begin{aligned} \int_{x=0}^{x=\delta(t)} d\left(\frac{\partial T}{\partial x}\right) &= \frac{\partial T}{\partial x}\bigg|_{x=\delta(t)} - \frac{\partial T}{\partial x}\bigg|_{x=0} = \\ &= \frac{1}{\alpha} \int_{x=0}^{x=\delta(t)} \frac{\partial T}{\partial t} dx = \frac{1}{\alpha} \left[\frac{d}{dt} \left(\int_{x=0}^{x=\delta(t)} T dx \right) - T|_{x=\delta(t)} \frac{d\delta}{dt} \right] \end{aligned}$$

where Leibnitz rule has be used on the right hand side. Since $(dT/dx)|_{x=\delta(t)} = 0$ and $T|_{x=\delta(t)} = T_i$

$$-\alpha \frac{\partial T}{\partial x}\bigg|_0 = \frac{d}{dt} \left(\int_{x=0}^{x=\delta(t)} T dx - T_i \delta \right)$$

this is the heat balance integral equation.

Now assume that the temperature distribution in the thermal layer can be represented as a low order (fourth order) polynomial of x , i.e.

$$T(x, t) = a + bx + cx^2 + dx^3 + ex^4$$

The coefficients a , b , c , d and e are determined by introducing the conditions

$$T(0, t) = T_0$$

$$T(\delta, t) = T_i$$

$$\frac{\partial T}{\partial x}_{x=\delta} = 0$$

$$\frac{\partial^2 T}{\partial x^2}_{x=0} = 0$$

and

$$\frac{\partial^2 T}{\partial x^2}_{x=\delta} = 0$$

Substituting into the polynomial, the result is

$$\frac{T(x, t) - T_i}{T_0 - T_i} = 1 - 2\frac{x}{\delta} + 2\left(\frac{x}{\delta}\right)^3 - \left(\frac{x}{\delta}\right)^4$$

Substituting this last expression for $T(x, t)$ into the heat balance integral results in

$$\frac{20}{3}\alpha = \delta \frac{d\delta}{dt}$$

which must be solved subject to $\delta(0) = 0$ to give

$$\delta(t) = \sqrt{\frac{40}{3}\alpha t}$$

This expression for $\delta(t)$ can now be substituted into the expression for $T(x, t)$ or in $q(0, t) = -k(\partial T/\partial x)|_{x=0}$ to give

$$q(0, t) = \frac{2k}{\delta}(T_0 - T_i)$$

Consider now instead the case of a semiinfinite medium initially at $T_i = 0$ where the boundary $x = 0$ is subjected to a prescribed, time dependent heat flux $f(t)$ and in which the thermal properties are all functions of temperature.

The formulation is, find $T(x, t)$ satisfying

$$\frac{\partial}{\partial x}(k(T) \frac{\partial T}{\partial x}) = \rho(T)C_p(T) \frac{\partial T}{\partial t}$$

subject to

$$T(x, 0) = 0$$

and to

$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = f(t)$$

Introducing now a new variable U defined as $U = \int_0^T \rho C_p dT$ the problem transforms into

$$\frac{\partial}{\partial x} (\alpha(U) \frac{\partial U}{\partial x}) = \frac{\partial U}{\partial t}$$

subject to

$$U(x, 0) = 0$$

and to

$$-(\alpha(U) \frac{\partial U}{\partial x}) \Big|_{x=0} = f(t)$$

Integrating over the thermal layer $\delta(t)$ and since $(\partial U / \partial x)|_{x=\delta} = U|_{x=\delta} = 0$, leads to the heat balance integral equation

$$\frac{d}{dt} \int_0^\delta U dx = f(t)$$

Now, assuming that the distribution of U is cubic in x and using the conditions of the problem to define the coefficients leads to

$$U(x, t) = \frac{\delta f(t)}{3\alpha_{x=0}} \left(1 - \frac{x}{\delta}\right)^3$$

Substituting this into the heat balance integral gives

$$\frac{d}{dt} \left[\frac{\delta^2 f(t)}{12\alpha_{x=0}} \right] = f(t)$$

In this case, however, the solution depends on $\alpha_{x=0}$ which is unknown but can be determined from the value of U at $x = 0$. The final result is

$$(U \sqrt{\alpha})|_{x=0} = \left[\frac{4}{3} f(t) \int_0^t f(t') dt' \right]^{1/2}$$

This one can be used to find $\alpha_{x=0}(t)$. From that, $\delta(t)$ can be calculated, which in turn gives U , which in turn gives T .