MA1301 Assignment 3 刘恒澈 57854329

1. suppose $z_1 = a+bi \quad z_2 = x+yi$

$|z_1+z_2|^2 = (a+x)^2+(b+y)^2$

$|z_1-z_2|^2 = (a-x)^2+(b-y)^2$

$|z_1+z_2|^2 - |z_1-z_2|^2 = (a+x+a-x)(a+x-a+x)+(b+y+b-y)(b+y-b+y)$

$4Re(\bar{z_1}z_2) = 4Re(ax+ayi-bxi+by) = 4(ax+by)$

$|z_1+z_2|^2-|z_1-z_2|^2 = 4ax+4by = 4Re(\bar{z_1}z_2)$

$\therefore |z_1+z_2|^2 - |z_1-z_2|^2 = 4Re(\bar{z_1}z_2)$

2(a). $z^6 = 2\sqrt{3}(-\frac{\sqrt{3}}{2}+\frac{1}{2}i) = 2\sqrt{3}(\cos(150^\circ)+i\sin(150^\circ))$

$\therefore z = (2\sqrt{3})^{\frac{1}{6}}(\cos(\frac{150^\circ}{6}+\frac{2k\pi}{6})+i\sin(\frac{150^\circ}{6}+\frac{2k\pi}{6}))$ $k=0,1,2,3,4,5$

$\therefore z_1 = (2\sqrt{3})^{\frac{1}{6}}(\cos(25^\circ)+i\sin(25^\circ)) \quad z_2 = (2\sqrt{3})^{\frac{1}{6}}(\cos(85^\circ)+i\sin(85^\circ))$

$z_3 = (2\sqrt{3})^{\frac{1}{6}}(\cos(145^\circ)+i\sin(145^\circ)) \quad z_4 = (2\sqrt{3})^{\frac{1}{6}}(\cos(205^\circ)+i\sin(205^\circ))$

$z_5 = (2\sqrt{3})^{\frac{1}{6}}(\cos(265^\circ)+i\sin(265^\circ)) \quad z_6 = (2\sqrt{3})^{\frac{1}{6}}(\cos(325^\circ)+i\sin(325^\circ))$

2(b). $(1+z)^7 = (z-1)^7 \quad \therefore (\frac{1+z}{z-1})^7 = 1$

$(\frac{1+z}{z-1})^7 = \cos 0 + i\sin 0$

$\therefore \frac{1+z}{z-1} = \cos(0+\frac{2n\pi}{7}) + i\sin(0+\frac{2n\pi}{7}) = e^{\frac{2n\pi}{7}i}$ $n=0,1,2\dots6$

$\therefore z_1 = \frac{2}{\cos 0+i\sin 0-1}$ DNE $\quad z_2 = \frac{1+e^{\frac{2}{7}\pi i}}{e^{\frac{2}{7}\pi i}-1}$

$z_3 = \frac{1+e^{\frac{4}{7}\pi i}}{e^{\frac{4}{7}\pi i}-1} \quad z_4 = \frac{1+e^{\frac{6}{7}\pi i}}{e^{\frac{6}{7}\pi i}-1}$

$z_5 = \frac{1+e^{\frac{8}{7}\pi i}}{e^{\frac{8}{7}\pi i}-1} \quad z_6 = \frac{1+e^{\frac{10}{7}\pi i}}{e^{\frac{10}{7}\pi i}-1}$

$z_7 = \frac{1+e^{\frac{12}{7}\pi i}}{e^{\frac{12}{7}\pi i}-1}$

2(c). suppose $u = z^5$ we have $u^2 - 5u - 6 = 0$

$(u-6)(u+1) = 0$ so $u = 6$ or $u = -1$

$\therefore$ ① $z^5 = 6(\cos(0) + i\sin(0))$ or ② $z^5 = \cos(\pi) + i\sin(\pi)$

solve ①: $z_1 = \sqrt[5]{6}(\cos 0 + i\sin 0)$ $z_2 = \sqrt[5]{6}\left(\cos\left(\frac{2}{5}\pi\right) + i\sin\left(\frac{2}{5}\pi\right)\right)$

$z_3 = \sqrt[5]{6}\left(\cos\left(\frac{4}{5}\pi\right) + i\sin\left(\frac{4}{5}\pi\right)\right)$ $z_4 = \sqrt[5]{6}\left(\cos\left(\frac{6}{5}\pi\right) + i\sin\left(\frac{6}{5}\pi\right)\right)$

$z_5 = \sqrt[5]{6}\left(\cos\left(\frac{8}{5}\pi\right) + i\sin\left(\frac{8}{5}\pi\right)\right)$

solve ②: $z_6 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ $z_7 = \cos\frac{3}{5}\pi + i\sin\frac{3}{5}\pi$

$z_8 = \cos\pi + i\sin\pi$ $z_9 = \cos\frac{7}{5}\pi + i\sin\frac{7}{5}\pi$ $z_{10} = \cos\frac{9}{5}\pi + i\sin\frac{9}{5}\pi$

3(a). $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\sin^3\theta\cos^2\theta$

$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$

Take the real parts: $\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

After simplification we have: $\cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

So the real parts must be the same.

we have $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

3(b). By De Moive's formula, we have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta$$

Take the imaginary parts:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 3\sin^3\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

So we have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

4(a).

$$\begin{array}{cc}\left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ -3 & 2 & 1 & 4 \\ 2 & -3 & 2 & 9 \end{array}\right) & \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}\end{array} \sim \begin{array}{cc}\left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & -1 & -4 & -21 \end{array}\right) & \begin{array}{l} r_1 \\ r_2+3r_1 \\ r_3-2r_1 \end{array}\end{array} \sim$$

$$\begin{array}{cc}\left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & 0 & -14 & -70 \end{array}\right) & \begin{array}{l} r_1 \\ r_2 \\ r_3-r_2 \end{array}\end{array} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 5 \end{cases}$$

4(b).

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 4 & 0 & 1 & 5 \\ 3 & -1 & 2 & 1 \end{array}\right) \sim \begin{array}{cc}\left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 3 & -1 & 2 & 1 \\ 7 & -1 & 3 & 6 \end{array}\right) & \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}\end{array} \sim$$

$$\begin{array}{cc}\left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 0 & -\frac{5}{2} & \frac{13}{2} & -17 \\ 0 & -\frac{9}{2} & \frac{27}{2} & -36 \end{array}\right) & \begin{array}{l} r_1 \\ r_2-\frac{3}{2}r_1 \\ r_3-\frac{7}{2}r_1 \end{array}\end{array} \sim \begin{array}{cc}\left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 0 & -\frac{5}{2} & \frac{13}{2} & -17 \\ 0 & 0 & \frac{9}{5} & -\frac{27}{5} \end{array}\right) & \begin{array}{l} r_1 \\ r_2 \\ r_3-\frac{9}{5}r_2 \end{array}\end{array}$$

$$\Rightarrow \begin{cases} x = 2 \\ y = -1 \\ z = -3 \end{cases}$$

5(a). $\begin{pmatrix} 2 & 1 & -b & | & 3 \\ 0 & a & -1 & | & 2 \\ -2 & 5 & 0 & | & 1 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \sim \begin{pmatrix} 2 & 1 & -b & | & 3 \\ 0 & a & -1 & | & 2 \\ 0 & 0 & \frac{6}{a}-b & | & 4-\frac{12}{a} \end{pmatrix}$

If the system has unique solution, then rank A = rank B = n

① so $\begin{cases} a \neq 0 \\ \frac{6}{a} - b \neq 0 \end{cases}$ ② $\begin{pmatrix} 2 & 1 & -b & | & 3 \\ 0 & 6 & -b & | & 4 \\ 0 & 0 & -1 & | & 2 \end{pmatrix}$ when $a=0$ also unique solution.

then if it has unique solution, then $\begin{cases} a \neq 0 \\ ab \neq 6 \end{cases}$ or $a=0$

so when $\boxed{ab \neq 6}$, the system has unique solution.

(b). rank A = rank B < n, so $\frac{6}{a} - b = 0$, $4 - \frac{12}{a} = 0$

we have $a=3$ and $b=2$

when $\boxed{\begin{cases} a=3 \\ b=2 \end{cases}}$, the system has infinitely many solutions

(c). rank A < rank B so $\begin{cases} \frac{6}{a} - b = 0 \\ 4 - \frac{12}{a} \neq 0 \end{cases}$

when $\boxed{\begin{cases} ab=6 \\ a \neq 3 \end{cases}}$ then we have no solution.

6(a). Suppose $x\begin{bmatrix}0\\1\\1\end{bmatrix} + y\begin{bmatrix}1\\2\\0\end{bmatrix} + z\begin{bmatrix}-1\\1\\1\end{bmatrix} = 0$

then we have $\begin{cases} y - z = 0 \\ x + 2y + z = 0 \\ x + z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

so $\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\begin{pmatrix}1\\2\\0\end{pmatrix}$ $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ are linearly independent

6(b). $\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right)$

$\Rightarrow$ the unique solution is trivial, $x_1 = x_2 = x_3 = 0$

so $\begin{pmatrix}1\\2\\1\end{pmatrix}$ $\begin{pmatrix}-2\\1\\0\end{pmatrix}$ $\begin{pmatrix}2\\1\\1\end{pmatrix}$ are linearly independent

6(c). $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right) \sim \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right) \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_1 + r_2 - r_3 \end{array}$

rank A = rank B = $3 < 4$, so we have infinite solutions

then $\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\begin{pmatrix}1\\2\\3\end{pmatrix}$ are linearly dependent

7 (a). $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$ the rank is 2

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{matrix} r_1 \\ r_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{matrix} r_1 \\ r_2-2r_1 \end{matrix} \quad \therefore \text{rank} = 2$$

(b) $$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{matrix} r_1 \\ r_2-3r_1 \\ r_3+r_1 \end{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 3 & 4 \end{pmatrix}\begin{matrix} r_1 \\ r_2 \\ r_3+\frac{3}{4}r_2 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & -2 \end{pmatrix}\begin{matrix} r_1 \\ r_2 \\ r_3 + \end{matrix}$$

So the rank is 3

(c). $$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So the rank is 1