MA 2509 Discrete Mathematics
Autumn 2022
20.10.2022
Time Limit: 100 Minutes

Name: ____________________

Student ID: ____________________

This exam contains 5 pages (including this cover page) and 4 questions. Total number of points is 24.

Question	Points	Score
1	6	
2	6	
3	6	
4	6	
Total:	24	

Justify all your steps.

Q1(c) cont. Suppose E, then $\sim I \wedge \sim M$ (by 5)

Then $\sim I$, then A (by 2)

Then $\sim M$, then W (by 3)

then $A \wedge W$, then P (by 1)

This contradicts 4

As a result, the argument is valid.

1. (a) (2 points) Determine whether $[(p \vee q) \wedge (p \to r)] \to (q \vee r)$ is a tautology.
   (b) (2 points) Determine whether $\forall x(P(x) \vee Q(x))$ is logically equivalent to $(\forall x\, P(x)) \vee (\forall x\, Q(x))$.
   (c) (2 points) Determine whether this argument is valid.
   If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

(a) $[(p \vee q) \wedge (p \to r)] \to (q \vee r)$

$\Leftrightarrow [(p \vee q) \wedge (\sim p \vee r)] \to (q \vee r)$

Suppose $[(p \vee q) \wedge (\sim p \vee r)]$ is true.

- If p is true, then $\sim p$ is false
  Since $\sim p \vee r$ is true, r is true $\Rightarrow$ $q \vee r$ is true.
- If p is false, then q is true, then $q \vee r$ is true.

Hence it is a tautology.

(b) U: real numbers, $P(x): x > 0$, $Q(x): x \leq 0$

$\forall x(P(x) \vee Q(x))$ is true

$(\forall x\, P(x)) \vee (\forall x\, Q(x))$ is false

(c) A: able, W: willing, P: prevents evil
I: Impotent, M: malevolent, E: exists

1. $A \wedge W \to P$
2. $\sim A \to I$
3. $\sim W \to M$
4. $\sim P$
5. $E \to \sim I \wedge \sim M$

$\to \sim E$

2. (a) (3 points) Show that $\sqrt{2}+\sqrt{3}$ is not a rational number, i.e. $\sqrt{2}+\sqrt{3} \neq \frac{m}{n}$ where m, n are integers and n is nonzero.

 (b) (3 points) Prove that if $x > 0$ is any fixed real number, then $(1+x)^n > 1 + nx$ for all $n \geq 2$.

(a) Suppose $\sqrt{2}+\sqrt{3}$ is rational, then

$\sqrt{2}+\sqrt{3} = \frac{m}{n}$, $(\sqrt{2}+\sqrt{3})^2 = \frac{m^2}{n^2}$

$5+2\sqrt{6} = \frac{m^2}{n^2}$ $\sqrt{6} = \frac{m^2-5n^2}{2n^2}$

Then $\sqrt{6}$ is rational. denote $\sqrt{6} = \frac{p}{q}$, $\gcd(p, q) = 1$

Then $6 = \frac{p^2}{q^2} \Rightarrow 6q^2 = p^2 \Rightarrow 2$ divides p

$\Rightarrow 4$ divides $p^2 \Rightarrow 4$ divides $6q^2$

$\Rightarrow 2$ divides $q^2 \Rightarrow 2$ divides q

Contradiction. Thus $\sqrt{2}+\sqrt{3}$ is irrational.

(b) For $n=2$ $(1+x)^2 = 1+2x+x^2 > 1+2x$.

Suppose $(1+x)^n > 1+nx$

Then $(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x)$

$= 1+(n+1)x+nx^2 > 1+(n+1)x$.

3. Let $\mathbf{R}, \mathbf{S}$ be relations on the set of integers, such that $(x, y) \in \mathbf{R}$ if and only if $x^3 + y^3$ is divisible by 5, and $(x, y) \in \mathbf{S}$ if and only if $x^3 - y^3$ is divisible by 5.
   (a) (2 points) Is $\mathbf{R}$ an equivalent relation?
   (b) (2 points) Is $\mathbf{S}$ an equivalent relation?
   (c) (2 points) If either $\mathbf{R}$ or $\mathbf{S}$ is a equivalent relation, find its equivalent classes that form a partition of the integers.

(a) $2^3+3^3 = 8+27 = 35$ $(2,3) \in R.$

$8^3+2^3 = 8+512 = 520$ $(8,2) \in R.$

$8^3+3^3 = 512+27 = 539$ $(8,3) \notin R$

Not transitive ⇒ Not equivalent

(b) Yes, reflexive ...

symmetric ...

transitive ...

(c) $\{5n, n \in \mathbb{Z}\}$

$\{5n+1, n \in \mathbb{Z}\}$

$\{5n+2, n \in \mathbb{Z}\}$

$\{5n+3, n \in \mathbb{Z}\}$

$\{5n+4, n \in \mathbb{Z}\}$

4. (a) (2 points) Let $\mathbb{R}$ denote the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \mathrm{e}^{|x|} - 1\,. \tag{1}$$

Is f bijective?

(b) (2 points) Let $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$. Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ as in (1). Is f bijective?

(c) (2 points) Find a function $g : \mathbb{R} \to \mathbb{R}$, which is surjective but not injective.

(a) $f(-1) = f(1)$ not injective $\Rightarrow$ not bijective

(b) yes. $f(x) = f(y) \Rightarrow e^{|x|} - 1 = e^{|y|} - 1$

$\Rightarrow e^{x} - 1 = e^{y} - 1 \Rightarrow x = y \Rightarrow$ injective

$y, x > 0$

$f(0) = 0$, $\lim_{x \to +\infty} f(x) = +\infty$ f is increasing

$\Rightarrow f$ surjective.

$\therefore f$ bijective

(c) $f(x) = \begin{cases} x+1 & x \le 0 \\ x-1 & x > 0 \end{cases}$