Midterm 7 December Autumn 2020, answers

Discrete Mathematics (City University of Hong Kong)

Solution of Midterm

Q1: (1) T (2) T (3) F

Q2:

Step 1: $p \to q \equiv \neg q \to \neg p$ Contrapositive

Step 2: $p \vee r \equiv \neg p \to r$

Step 3: $\neg q \to r$ Hypothetical syllogism

Step 4: $r \to s$

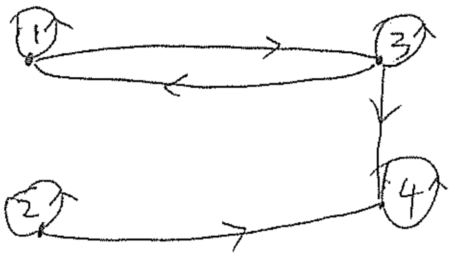
Step 5: $\neg q \to s$ Hypothetical syllogism

Step 6: $\neg q \to s \equiv \neg(\neg q) \vee s \equiv q \vee s$

Q3: $\neg p \to (q \to r) \equiv \neg(\neg p) \vee (q \to r) \equiv p \vee (q \to r)$

$\equiv p \vee (\neg q \vee r) \equiv \neg q \vee (p \vee r) \equiv q \to p \vee r$

Q4: ①



② R is reflexive, since $(1,1), (2,2), (3,3), (4,4) \in R$

R is not symmetric, since $(2,4) \in R$, but $(4,2) \notin R$

R is not antisymmetric, since $(1,3) \in R$ and $(3,1) \in R$, but $1 \neq 3$

R is not transitive, since $(1,3) \in R$ and $(3,4) \in R$, but $(1,4) \notin R$

Q5: ① $P(A) = \{\Phi, \{a\}, \{b\}, \{a,b\}\}$

② $A \times A \times A = \{(a,a,a), (a,a,b), (a,b,a), (a,b,b), (b,a,a)$
$(b,a,b), (b,b,a), (b,b,b)\}$

③ Since $f(B) = \overline{B}$

For any two subsets $B_1 \neq B_2$, then $\overline{B_1} \neq \overline{B_2}$. Hence f is injective.

On the other hand, for any subset B of A, that is $B \in P(A)$, there exists $\overline{B} \in P(A)$ such that $f(\overline{B}) = \overline{\overline{B}} = B$. thus f is surjective. Hence f is bijective.

Q6: We need to check reflexive, symmetric, and transitive.

① reflexive, since $a \equiv a \pmod 7$ and hence $a = \pm a \pmod 7$

② symmetric: If $a = \pm b \pmod 7$, then $a = \pm b + 7k$ for some integer k. Thus $a = b + 7k$ or $a = -b - 7k$

Therefore $b = a - 7k$ or $b = -a \mp 7k$.

This implies $b = \pm a \pmod 7$. Hence it is symmetric

③ transitive: If $a = \pm b \pmod 7$ and $b = \pm c \pmod 7$

Then $\begin{cases} a = \pm b + 7k \\ b = \pm c + 7l \end{cases} \Rightarrow \begin{cases} a = b + 7k & \text{or} \quad a = -b + 7k \\ b = c + 7l & \text{or} \quad b = -c + 7l \end{cases}$

This implies

$a = c + 7(k+l)$ or $a = -c + 7(k+l)$

or $a = -c + 7(k-l)$ or $a = c + 7(k-l)$

In both cases, $a = \pm c \pmod 7$

Thus it is transitive.

Q7:

① $\neg p \equiv \neg(p \vee p) \equiv p|p$

② $p \wedge q \equiv \neg(\neg p \vee \neg q) \equiv \neg p | \neg q \overset{①}{\equiv} (p|p)|(q|q)$

③ $p \to q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q) \overset{①}{\equiv} p \wedge \neg q \,|\, p \wedge \neg q$

$\overset{②}{\equiv} (p|p)|(\neg q)|(\neg q) \,|\, (p|p)|(\neg q)|(\neg q)$

$\overset{①}{\equiv} (p|p)|(q|q)|(q|q) \,|\, (p|p)|(q|q)|(q|q)$

Q8:

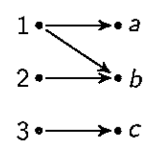
① If $A \subset B$, then for any $x \in A$, we know $x \in B$. Thus $A \subseteq A \cap B$. On the other hand, obviously, $A \cap B \subseteq A$, thus $A = A \cap B$

② If $A \cap B = A$, then for any $x \in A$, we know $x \in A \cap B$

Hence $x \in B$. therefore $A \subset B$

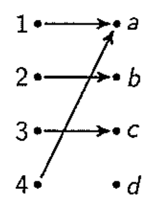
Example: Do the following diagrams define functions?

- $A = \{1, 2, 3\}$, $B = \{a, b, c\}$,



No, 1 has two images.

- $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$,



Yes, every element in A has assigned a unique element in B.