CITY UNIVERSITY OF HONG KONG

Course code and title : MA2504 Discrete Mathematics

Session : Semester B, 2007-2008

Time allowed : Three Hours

This paper has FIVE pages (including this page).

Instructions to candidates:

1. This paper has SIX questions.
2. Attempt ALL questions.
3. The paper has 110 marks in total.
4. The maximum obtainable mark is 100 marks.
5. Start each question on a new page.
6. Show <u>ALL</u> workings.

Materials, aids & instruments which students are permitted to use during examination: Approved calculators

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Do not remove this from exam

Answer ALL Questions

Question 1

(a) Use *Proof by Contradiction* to show the validity of $\dfrac{\forall x(P(x) \to \sim Q(x))}{\therefore\ \sim \exists x(P(x) \land Q(x))}$.

(7 marks)

(b) The following derivation is to show the validity of $\dfrac{\forall x(P(x) \to Q(x))}{\therefore \exists x P(x) \to \exists x Q(x)}$. However, there is a mistake in the derivation, find it and provide a correct proof using *cp rule*.

1	$\forall x(P(x) \to Q(x))$	p
2	$P(c) \to Q(c)$	$1, ui$
3	$\exists x P(x)$	add p
4	$P(c)$	$3, ei$
5	$Q(c)$	$2, 4$
6	$\exists x Q(x)$	$5, eg$
7	$\exists x P(x) \to \exists x Q(x)$	3,6 cp rule

(5 marks)

(c) Does there exist a simple graph with five vertices of the following degrees? If yes, please draw the graph. If no, state your reason.

(i) $(0, 1, 2, 2, 3)$

(2 marks)

(ii) $(1, 1, 2, 2, 3)$

(2 marks)

Question 2

(a) Simplify $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)) \cap \mathcal{P}(\emptyset))$. ($\mathcal{P}(A)$ is the power set of A)

(4 marks)

(b) Let A be a non-empty set, R and S are binary relations on $A \times A$. State whether the following statements true or not. If yes, give a proof, if no, give a counter example.

(i) If R and S are reflexive, then $R \cup S$ is reflexive.

(2 marks)

(ii) If R and S are transitive, then $R \cup S$ is transitive.

(2 marks)

(iii) If R and S are antisymmetric, then $R \cap S$ is antisymmetric.

(2 marks)

(c) Let $f : A \to B$ and $g : B \to C$ be functions.

(i) Prove that if $g \circ f$ is surjective, then g is surjective.

(3 marks)

(ii) Prove that if $g \circ f$ is injective, then f is injective.

(3 marks)

Question 3

(a) It is given 13 integers $c_1, c_2, \ldots, c_{13}$ (some of them may be the same). Use pigeonhole principle to prove that there exist i and j with $0 \leq i < j \leq 13$ such that $c_{i+1} + c_{i+2} + \cdots + c_j$ is divisible by 13, for example, $c_4 + c_5 + c_6 + c_7$ is divisible by 13. (Hint: consider the following 13 integers

$$\begin{aligned} n_1 &= c_1 \\ n_2 &= c_1 + c_2 \\ &\vdots \\ n_{13} &= c_1 + c_2 + \cdots + c_{13} \end{aligned}$$

and their remainder when divided by 13)

(10 marks)

(b) How many solutions are there to the equation $x_1 + x_2 + x_3 = 30$, where x_1, x_2 and x_3 are integers such that $x_1 \geq 3, x_2 \geq 5$ and $3 \leq x_3 \leq 14$?

(6 marks)

Question 4

(a) Find a recurrence relation for the number of ways to climb n stairs if stairs can be climbed two or three at a time. Also, determine the initial condition(s) to solve the recurrence relation. (You are not required to solve the recurrence relation)

(4 marks)

(b) Find the solution for the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ with initial conditions $a_0 = 0$ and $a_1 = 3$.

(8 marks)

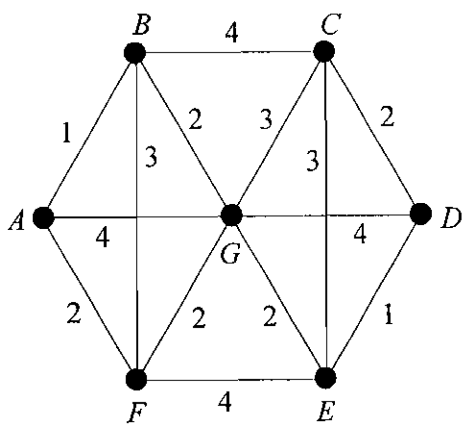
(c) A graph is called *self-complementary* if it is isomorphic to its complement. State, with reason, whether a graph with 22 vertices is self-complementary or not.

(4 marks)

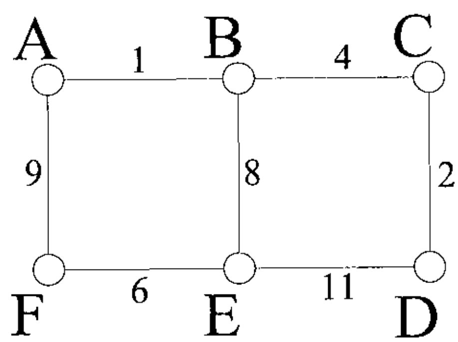
Question 5

(a) Use Dijkstra's algorithm to find the shortest path from A to D in the following weighted graph.

(7 marks)



(b) Consider the weighted graph G



(i) Write down the adjacency matrix of G.

(2 marks)

(ii) Use Prim's algorithm with starting at D to find a minimal spanning tree.

(7 marks)

Question 6

(a) Let A be a set of nonzero integers and let R be the relation on $A \times A$ defined by $((a,b),(c,d)) \in R$ if $ad = bc$, show that R is an equivalence relation.

(9 marks)

(b) Given $R = \{(x,y) | x - y \text{ is an integer}\}$ is an equivalence relation on the set of rational number. What are the equivalence class of 0 and $\frac{1}{2}$?

(6 marks)

(c) A complete graph K_n is a simple graph with n vertices and each vertex is adjacent to all other vertices.

(i) For which value of n with $n \geq 2$, K_n has an Euler circuit?

(6 marks)

(ii) For which value of n with $n \geq 2$, K_n has a Hamilton circuit?

(3 marks)

(d) Give an example to show that there is a simple graph $G = (V, E)$ with n vertices and $d(v) \geq \frac{n-1}{2}$ for all $v \in V$ but G does not contain a Hamilton circuit.

(6 marks)

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