MA1300 Hand-in Assignment 5

1. (1). $\lim_{x\to\infty}(1+\frac{4}{x})^x = e^{\lim_{x\to\infty}\ln(1+\frac{4}{x})^x} = e^{\lim_{x\to\infty}x\ln(1+\frac{4}{x})} = e^{\lim_{x\to\infty}\frac{\ln(1+\frac{4}{x})}{\frac{1}{x}}}$

$= e^{\lim_{x\to\infty}\frac{\frac{1}{1+\frac{4}{x}}(-\frac{4}{x^2})}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\frac{4}{1+\frac{4}{x}}} = e^4$

(2). $\lim_{x\to 0}\frac{e^x-1}{\tan x} = \lim_{x\to 0}\frac{e^x}{\sec^2 x} = \lim_{x\to 0}\frac{e^x}{\frac{1}{\cos^2 x}} = 1$

(3). $\lim_{x\to 0^+}x^2\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+}-\frac{\ln\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+}-\frac{\frac{1}{\frac{1}{x}}(-\frac{1}{x^2})}{-2\frac{1}{x^3}} = \lim_{x\to 0^+}-\frac{\frac{1}{x}}{\frac{2}{x^3}} = \lim_{x\to 0^+}-\frac{x^2}{2} = 0$

$\therefore \lim_{x\to 0^+}x^2\ln x = 0$

2 (1). $\because y = 3^{x\ln x}$ $\therefore \ln y = x\ln x\cdot\ln 3$ Take $\frac{d}{dx}$ on both sides

$\frac{1}{y}\frac{dy}{dx} = \ln 3\,(x\cdot\frac{1}{x}+\ln x)$

$\therefore \frac{dy}{dx} = y\ln 3\,(1+\ln x)$

(2). $\because xe^y = y-1$ $\therefore \ln x + y = \ln(y-1)$ Take $\frac{d}{dx}$ on both sides

$\frac{1}{x}+\frac{dy}{dx} = \frac{1}{y-1}\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -\frac{y-1}{x(y-2)}$

(3). $\because y = x^{2x}$ $\therefore \ln y = 2x\ln x$ Take $\frac{d}{dx}$ on both sides

$\frac{1}{y}\frac{dy}{dx} = 2(x\cdot\frac{1}{x}+\ln x)$ $\frac{1}{y}\frac{dy}{dx} = 2(1+\ln x)$

$\therefore \frac{dy}{dx} = 2y(1+\ln x)$

3 (1). $\because \sum_{n=1}^{\infty}\frac{n}{n^3+1}$, $\frac{n}{n^3+1} = \frac{1}{n^2+\frac{1}{n}} < \frac{1}{n^2}$

By p-series test, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent

By comparison test, we have $\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ convergent

$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$ $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$

(2). $\because 3^n > n^3$ for $n>1$ $\therefore \frac{n^3}{5^n} < \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n = \frac{3}{5}\times\left(\frac{3}{5}\right)^{n-1}$

By Geometric Series we have $\sum_{n=1}^{\infty}\frac{3}{5}\times\left(\frac{3}{5}\right)^{n-1}$ convergent

By comparison test we have $\sum_{n=1}^{\infty}\frac{n^3}{5^n}$ convergent

(3). $\because |\cos 3n| < 1$, $1+(1.2)^n > 1$

$\therefore \left|\frac{\cos 3n}{1+(1.2)^n}\right| < \frac{1}{(1.2)^n} = \left(\frac{5}{6}\right)^n = \frac{5}{6}\left(\frac{5}{6}\right)^{n-1}$

By Geometric series we have $\sum_{n=1}^{\infty}\frac{5}{6}\left(\frac{5}{6}\right)^{n-1}$ convergent

By comparison test we have $\sum_{n=1}^{\infty}\left|\frac{\cos 3n}{1+(1.2)^n}\right|$ convergent

since $\sum_{n=1}^{\infty}\left|\frac{\cos 3n}{1+(1.2)^n}\right|$ is convergent. then $\sum_{n=1}^{\infty}\frac{\cos 3n}{1+(1.2)^n}$ is convergent

(4). when $n>1$ 且 $n\in N^+$

$$\frac{1}{n^{\frac{5}{4}}} - \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{1}{n^{\frac{5}{4}}} - \frac{(\sqrt{n+1}-\sqrt{n-1})n^{\frac{1}{4}}}{n\cdot n^{\frac{1}{4}}} = \frac{1-n^{\frac{1}{4}}(\sqrt{n+1}-\sqrt{n-1})}{n^{\frac{5}{4}}} = \frac{n^{\frac{1}{4}}\left((1)^{k} - (\sqrt{n+1}-\sqrt{n-1})\right)}{n^{\frac{5}{4}}}$$

$n^{\frac{1}{4}} > \sqrt{n+1}-\sqrt{n-1}$ $\therefore \frac{1}{n^{\frac{5}{4}}} > \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

By P-series we have $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{5}{4}}$ convergene

By comparison test we have $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ convergent

(5) $\lim_{n\to\infty}\frac{n}{3n^2+1} = \lim_{n\to\infty}\frac{1}{6n} = 0$

$$b_n - b_{n+1} = \frac{n}{3n^2+1} - \frac{n+1}{3n^2+6n+4} = \frac{\overset{3n^2+3n-1}{\cancel{3n^3+6n^2+4n-3n^3-n-3n^2-1}}}{(3n^2+1)(3n^2+6n+4)} > 0$$

for $n\geq 1$

so ① $b_{n+1}\leq b_n$ for all n

② $\lim_{n\to\infty} b_n = 0$ Then $\sum_{n=1}^{\infty}(-1)^n\frac{n}{3n^2+1}$ converge.

4(1). By root test, $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{x^2}{2} < 1$ $\therefore x^2 < 2$

$-\sqrt{2} < x < \sqrt{2}$ $\therefore$ the radius of convergence $R=\sqrt{2}$.

Interval of convergence of the series is $(-\sqrt{2}, \sqrt{2})$

when $x=\pm\sqrt{2}$ $a_n = \frac{((\pm\sqrt{2})^2)^n}{2^n} = 1$ $\sum_{n=1}^{\infty} 1 = \infty$ diverge

overall the interval is $(-\sqrt{2}, \sqrt{2})$

4 (2) By alternating series test

we have $\begin{cases} \frac{x^{n+1}}{3(n+1)^2+1} \le \frac{x^n}{3n^2+1} \\ \lim_{n\to\infty} \frac{x^n}{3n^2+1} = 0 \end{cases} \Rightarrow \begin{cases} -1 \le x \le 1 \\ \frac{x^{n+1}}{6} = 0 \end{cases} \Rightarrow -1 \le x \le 1$

when $x=1$ $a_n = (-1)^n \frac{1}{3n^2+1}$

$|a_n| < \frac{1}{3n^2}$ By comparison and p-series test,

we have a_n covergent

when $x=-1$, $a_n = \frac{1}{3n^2+1}$, we also have a_n convergent

so the radius of convergence is 1

the interval of convergence is $[-1, 1]$

5. let $\lim_{n\to\infty} a_n = L$

$\forall \varepsilon > 0$, $\exists N > 0$, s.t. if $n > N$, then $|a_n - L| < \varepsilon$

let $N_1 = N$. so that if $n > N_1$, $n+1 > N_1 = N$

$\therefore \forall \varepsilon > 0$, $\exists N > 0$, s.t. if $n+1 > N$, then $|a_{n+1} - L| < \varepsilon$

so it shows that $\lim_{n\to\infty} a_{n+1} = L$

$\therefore \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$

6. If n is even: we have $n = 2m$.

by $\lim_{n\to\infty} a_{2n} = L$, we have $\lim_{n\to\infty} a_{2m} = L$

$\forall \varepsilon > 0$. $\exists N_0$. s.t. if $2m > N_0$. $|a_{2m} - L| < \varepsilon$

$\forall \varepsilon > 0$, $\exists N_0$, s.t. if $n > N_0$, $|a_n - L| < \varepsilon$

If n is odd, then we have $n = 2m+1$

by $\lim_{n\to\infty} a_{2n+1} = L$. we have $\lim_{n\to\infty} a_{2m+1} = L$

$\forall \varepsilon > 0$, $\exists N_1$, s.t. if $n = 2m+1 > N_1$. $|a_n - L| < \varepsilon$

overall, $|a_n - L| < \varepsilon$ is true for $n > \max\{N_0, N_1\}$

so $\lim_{n\to\infty} a_n = L$

7. 1° suppose the limit exists and $\lim\limits_{n\to\infty} a_n = L$

then $\lim\limits_{n\to\infty} a_{n+1} = 3 - \frac{1}{\lim\limits_{n\to\infty} a_n}$

$L = 3 - \frac{1}{L} \Rightarrow L = \frac{3\pm\sqrt{5}}{2}$

2° suppose $1 < a_k < 3$ for some $k \in N_+$

then $a_{k+1} = 3 - \frac{1}{a_k} < 3 - \frac{1}{3} = \frac{8}{3} < 3$

$a_{k+1} = 3 - \frac{1}{a_k} > 3 - 1 = 2 > 1 \quad \therefore 1 < a_{k+1} < 3$

the By M.I., we have $1 < a_k < 3$

$\frac{3-\sqrt{5}}{2} < 1 < \frac{3+\sqrt{5}}{2} < 3 \quad \therefore L = \frac{3+\sqrt{5}}{2}$

3° $a_{k+1} - a_k = 3 - \frac{1}{a_k} - a_k = \frac{-a_k^2 + 3a_k - 1}{a_k}$

又 $\because 1 < a_k < \frac{3+\sqrt{5}}{2} \quad \therefore a_{k+1} - a_k > 0$. a_k increases

We have a_n increasing and upper bounded then by MST,

we proved that $\{a_n\}$ is convergent