1. f is not differentiable at $x=0$

proof: suppose f is differentiable, then $\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=L$ exists.

$\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{h\cos(\frac{1}{h})-0}{h}=\lim_{h\to0}\cos(\frac{1}{h})=L$ exists

then $\forall\varepsilon>0$, $\exists\delta>0$, s.t. if $|h-0|<\delta$, $|\cos(\frac{1}{h})-L|<\varepsilon$.

Take $h_1=\frac{1}{2n\pi}$. where $n>\frac{1}{2\pi\delta}$ and n is an integer

$|\cos(\frac{1}{2n\pi})-L|=|1-L|<\delta$ ①

Take $h_2=\frac{1}{2n\pi+\frac{\pi}{2}}$, where $n>\frac{1}{2\pi\delta}$ and n is an integer

$|0-L|<\delta$ ②

Take $\delta=0.1$ $\begin{cases}|1-L|<0.1 ① & 0.9<L<1.1\\ |0-L|<0.1 ② & -0.1<L<0.1\end{cases}$

$\therefore$ There doesn't exist a L satisfying both ① and ②

$\therefore \lim_{h\to0}\frac{f(h)-f(0)}{h}$ DNE $\therefore$ f is not differentiable.

2(a). $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{(x+h)^3+5(x+h)+4-x^3-5x-4}{h}$

$\frac{f(x+h)-f(x)}{h}=\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{+5x}+5h\cancel{-x^3}\cancel{-5x}}{h}$

$=3x^2+3xh+h^2+5$ $\because h\to0$ $\therefore f'(x)=3x^2+5$

2(b). $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\frac{4-x-h}{3+x+h}-\frac{4-x}{3+x}}{h}$

$\frac{f(x+h)-f(x)}{h}=\frac{(4-x-h)(3+x)-(4-x)(3+x+h)}{(3+x)(3+x+h)h}=\frac{-7h}{(3+x)(3+x+h)h}=\frac{-7}{(3+x)(3+x+h)}$ $\because h\to0$ $\therefore f'(x)=-\frac{7}{(3+x)^2}$

$(x\neq-3)$

3. $y'=3(x^4-3x^2+5)^2\cdot(4x^3-6x)$

let $f(x)=x^3$ $g(x)=x^4-3x^2+5$

$f'(x)=3x^2$ $g'(x)=4x^3-6x$ $y=f(g(x))=f\circ g(x)$

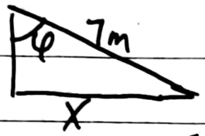
$\therefore y'=(f\circ g(x))'=3(x^4-3x^2+5)^2(4x^3-6x)$

4. $y' = \frac{1}{2\sqrt{4+4\sin x}} \cdot 4\cos x = \frac{4\cos x}{2\cdot 2\sqrt{1+\sin x}} = \frac{\cos x}{\sqrt{1+\sin x}}$

when $x=0$, $y' = \frac{1}{\sqrt{1+0}} = 1$

tangent line: $y = x+2$

nomal line: $y = -x+2$

5. 

It's easy to get that $x = 7\sin\varphi$

$\therefore \dot{x} = 7\cos\varphi$

when $\varphi = \frac{\pi}{3}$, $\dot{x} = 7\cos\frac{\pi}{3} = \frac{7}{2}$

6. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$(f(x)g(x))'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$

$(f(x)g(x))''' = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)$

......

$(f(x)g(x))^{(n)} = C_0^n f^{(n)}(x)g^{(0)}(x) + C_1^n f^{(n-1)}(x)g^{(1)}(x) + \dots + C_r^n f^{(n-r)}(x)g^{(r)}(x) + \dots + C_n^n f^{(0)}(x)g^{(n)}(x)$

$\therefore (f(x)g(x))^{(n)} = \sum_{k=0}^{n} C_k^n f^{(n-k)}(x)g^{(k)}x$

7(a) $\frac{d^{30}}{dx^{30}}(\sin 2x)$ let $f(x) = \sin 2x$ $\frac{d^{30}}{dx^{30}}(\sin 2x) = f^{(30)}(x)$

$f'(x) = 2\cos 2x$ $f''(x) = -4\sin 2x$ $f'''(x) = -8\cos 2x$ $f''''(x) = 16\sin 2x$

easy to see. $f^{(30)}(x) = -2^{30}\sin 2x$

7(b). let $u(x) = x$ $v(x) = \sin x$

$\frac{d}{dx^{30}}(x\sin x) = (u(x)v(x))^{(30)}$

$= \sum_{k=0}^{30} C_k^{30} u^{(30-k)}(x) v^{(k)}(x)$

$= C_0^n u^{(30)}(x)v^{(0)}(x) + \dots + C_{29}^{30} u^{(1)}(x)v^{(29)}(x) + C_{30}^{30} u^{(0)}(x)v^{(30)}(x)$

又∵ $\begin{cases} u^{(30)}(x) = u^{(29)}(x) = \dots = 0 \\ u^{(1)}(x) = 1 \quad u^{(0)}(x) = x \end{cases}$ $v^{(29)}(x) = \cos x$ $v^{(30)}(x) = -\sin x$

$\therefore \frac{d^{30}}{dx^{30}}(x\sin x) = 30v^{(29)}(x) + xv^{(30)}(x)$

$= 30\cos x - x\sin x$

8(a). It is always true.

proof: $\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$, s.t. if $|x-c| < \delta$,

$|h(x) - h(c)| \leq 3|x-c| < 3\delta = \varepsilon$ $(c \in \mathbb{R})$

$\therefore$ $h(x)$ is continuous

$\because \lim_{x\to 0}(f(x)-g(x))=0$ $\therefore \lim_{x\to 0} f(x) - \lim_{x\to 0} g(x) = 0$

let $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = L$

$\because$ $h(x)$ is continuous $\therefore \lim_{x\to L} h(x) = h(L)$

$\therefore \begin{cases} \lim_{x\to 0} h(f(x)) = \lim_{x\to L} h(x) = h(L) \\ \lim_{x\to 0} h(g(x)) = \lim_{x\to L} h(x) = h(L) \end{cases} \Rightarrow \lim_{x\to 0} h(f(x)) = \lim_{x\to 0} h(g(x))$

$\therefore \lim_{x\to 0} h(f(x)) - \lim_{x\to 0} h(g(x)) = 0$

$\lim_{x\to 0}(h(f(x)) - h(g(x))) = 0$

8(b). It is always true.

proof: $\because \lim_{x\to 0}(f(x) - g(x)) = 0$

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$

let $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = L$

$\because$ $h(x)$ is continuous $\therefore \lim_{x\to L} h(x) = h(L)$

$\begin{cases} \lim_{x\to 0} h(f(x)) = \lim_{x\to L} h(x) = h(L) \\ \lim_{x\to 0} h(g(x)) = \lim_{x\to L} h(x) = h(L) \end{cases} \Rightarrow \lim_{x\to 0} h(f(x)) = \lim_{x\to 0} h(g(x))$

$\therefore \lim_{x\to 0}(h(f(x)) - h(g(x))) = 0$