CHAPTER 1 Mathematical Logic

LOGIC is the study of the method and principles used to distinguish good (correct) from bad (incorrect) reasoning.

Proposition is a statement which can be determined to be either true (T) or false (F), but not both. Usually, p, q and r are used to denote propositions.

Example

Which of the following statements are propositions?

(a) Beijing is the capital of China.

(b) $x + 1 = 2$.

(c) What time is it?

(d) $1 + 1 = 3$.

(e) Stop writing!

A Compound Proposition is made up of simple propositions with various logical connectives like:

		Read as	Notation	Example
1.	and	and	$\wedge$	$p \wedge q$
2.	or	or	$\vee$	$p \vee q$
3.	not	negation	$\sim$	$\sim p$
4.	implies (if…then…)	implies (if…then…)	$\rightarrow$	$p \rightarrow q$
5.	equivalent	if and only if	$\leftrightarrow$	$p \leftrightarrow q$

Ex. Let p = "It is hot"
and q = "It is sunny".

Write each of the following sentences symbolically,

(a) It is not hot but it is sunny.

(a) It is neither hot nor sunny.

TRUTH TABLES

p	q	$p \wedge q$	$p \vee q$	$\sim p$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Tautology is a proposition, which is always true.

Contradiction is a proposition, which is always false.

Why do we define “$p \rightarrow q$ is true” when “p is false”?

Consider the fact that human beings exist on this earth.

Some believe that God created human beings while others believe that we come from evolution.

Let p denote “God created human beings”, q denote “We come from evolution” and r denote “human beings exist on this earth”.

You may say that “$p \rightarrow r$ is true” if you believe in God.
You may say that “$q \rightarrow r$ is true” if you believe in evolution.

If you are not sure whether there is a God or whether evolution is true, you may think that both propositions are “reasonable”.

Therefore, the proposition is considered as true even though p is false.

Definition of **"Only if"**

p only if q means "if not q then not p" or , equivalently,

"if p then q".

Ex. [Necessary & sufficient conditions]

(a) Converting the following necessary condition to *if – then* form

"*George's attaining age 15 is a necessary condition for his being chairman of our mathematics club.*"

(b) Converting the following sufficient condition to *if – then* form

"*John's birth in Hong Kong is a sufficient condition for him to be a SAR citizen.*"

Biconditional of p and q

p if and only if *q*, denoted by

$$p \leftrightarrow q \quad \text{or} \quad p \text{ iff } q,$$

means

"p if q and p only if q" ≡ "$(q \rightarrow p) \wedge (p \rightarrow q)$".

Logical Equivalence

Propositions p and q are called *logically equivalent* if p and q have the same truth values. We usually write $p \Leftrightarrow q$ or $p \equiv q$.

Ex. 3.1 Show that the propositions $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

Solution : Using truth table,

p	q	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F T	F F F
T	F	T F	F T T
F	T	T F	T T F
F	F	T F	T T T

Ex. 3.2 Which of the following propositions is/are logically equivalent to

$$p \to q$$

(i) $\sim p \to \sim q$

(ii) $q \to p$

(iii) $\sim q \to \sim p$.

ALGEBRA OF PROPOSITIONS

Equivalence	Name
$p \wedge T \Leftrightarrow p, \quad p \vee F \Leftrightarrow p$	Identity Laws
$p \vee T \Leftrightarrow T, \quad p \wedge F \Leftrightarrow F$	Domination Laws
$p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$	Idempotent Laws
$\sim(\sim p) \Leftrightarrow p$	Double negation Law
$p \vee q \Leftrightarrow q \vee p, \quad p \wedge q \Leftrightarrow q \wedge p$	Commutative Laws
$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative Laws
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$	De Morgan's Laws

Ex. Show the Distributive Law by using truth table.

Solution :

p	$\vee$	$(q$	$\wedge$	$r)$	$\Leftrightarrow$	$(p$	$\vee$	$q)$	$\wedge$	$(p$	$\vee$	$r)$
T		T		T								
T		T		F								
T		F		T								
T		F		F								
F		T		T								
F		T		F								
F		F		T								
F		F		F								

Ex. 3.3 Rewrite the following statement in *if –then* form.

“*Either I have your money or I have your life*”.

Some Useful Identities

$$P \to Q \iff \sim P \vee Q \qquad \text{(implication)}$$

$$P \to Q \iff \sim Q \to \sim P \qquad \text{(contrapositive)}$$

$$P \leftrightarrow Q \iff (P \to Q) \wedge (Q \to P) \qquad \text{(equivalence)}$$

Ex. 3.5 Simplify the proposition

$$[(P \to Q) \vee (P \to R)] \to (Q \vee R).$$

Ex. 3.6 Show that $(p \to q) \vee p$ is a tautology.

PREDICATE & QUANTIFIERS

BASIC IDEA:

Consider the following statement:-
" *x/2 is an integer*."

This statement surely has true or false value. However, before we determine the truth value, we have to consider

(i) What is the "range" of x?
(ii) Are we talking about all values of x or a specific value of x?

Say, if we consider all possible values of x to be even integer, then the above statement is true.

On the other hand, if x can be any integer, then the above statement is **NOT** true for all possible values of x.

Predicate --- Statement with variables and it becomes a proposition when the variables are specified.

E.g. "*x/2 is an integer*" is denoted by P(x).
" x *equals* y" is denoted by Q(x, y).
" $x + y = z$" is denoted by R(x, y, z).

Ex. 4.1 What are the truth values of R(1, 2, 3) and R(0, 0, 1)?

Universe of discourse:

The set of values where x, y, z, … can be drawn. Usually denoted by U.

Quantifier: Change of predicate to proposition.

$\forall x P(x)$: means "for all (or every) x, $P(x)$ is true". It is called universal quantifier.

If $U = \{x_1, x_2, \ldots, x_n\}$, then

$$\forall x P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n).$$

$\exists x P(x)$: means "there exists x such that $P(x)$ is true". It is called <u>existential quantifier</u>

If $U = \{x_1, x_2, \ldots, x_n\}$, then

$$\exists x P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n).$$

Ex. 4.5 Let $P(x) = \{x^2 > 10\}$ and $U = \{1, 2, 3, 4\}$.

What is the truth value of $\exists x P(x)$?

Ex. 4.6 Let $Q(y) = \{y^2 - 9y + 25 > 0\}$ and U is the set of real numbers. What is the truth value of $\forall y Q(y)$?

The truth values of these quantifiers $\forall x P(x)$ and $\exists x P(x)$ depend on the universe of discourse.

Let $P(x)$ = "*x/2 is an integer*." Then,

universe of discourse	All integers	All even integers	All odd integers
$\forall x P(x)$	F	T	F
$\exists x P(x)$	T	T	F

Ex. 4.7 Rewrite each of the statements using quantifiers and variables.

(a) All triangles have three sides.
U = “All triangles”, P(x) = “x has 3 sides”.

(b) No dogs have wings.
U = “All dogs”, Q(y) = “y has wings”.

(c) Some programs are structured.
U = “All programs”, R(z) = “z is structured”.

Ex. 4.8 Rewrite each of the following statements using quantifiers and variables where U = "All positive numbers".

(a) Given any positive number, there is another positive number that is smaller than the given number.

(b) There is a positive number with the property that all positive numbers are smaller than this number.

Ex. 4.9 Let $P(x, y, z)$ stand for "$x + y = z$". Find the truth value of

$$\forall x \forall z \exists y P(x, y, z)$$

when the universe of discourse is

(a) the set of all real numbers,
(b) the set of all natural numbers.

Ex. 4.10 Let $U = \{x_1, x_2, \ldots, x_n\}$. Show that

$$\sim\forall x P(x) \Leftrightarrow \exists x[\sim P(x)].$$

Some identities

$$\sim \forall x P(x) \equiv \exists x \sim P(x) \qquad \sim \exists x P(x) \equiv \forall x \sim P(x)$$

$$\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y) \qquad \exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$$

$$\forall x\left[P(x) \wedge Q(x)\right] \equiv \forall x P(x) \wedge \forall y Q(y) \qquad \exists x\left[P(x) \vee Q(x)\right] \equiv \exists x P(x) \vee \exists y Q(y)$$

$$\dagger \ \forall x\left[P(x) \wedge Q\right] \equiv \forall x P(x) \wedge Q \qquad \exists x\left[P(x) \wedge Q\right] \equiv \exists x P(x) \wedge Q$$

$$\forall x\left[P(x) \vee Q\right] \equiv \forall x P(x) \vee Q \qquad \exists x\left[P(x) \vee Q\right] \equiv \exists x P(x) \vee Q$$

$$\forall x\left[P(x) \to Q\right] \equiv \exists x P(x) \to Q \qquad \exists x\left[P(x) \to Q\right] \equiv \forall x P(x) \to Q$$

$$\forall x\left[P \to Q(x)\right] \equiv P \to \forall x Q(x) \qquad \exists x\left[P \to Q(x)\right] \equiv P \to \exists x Q(x)$$

Ex. 4.11 Show that the identity

$$\exists x \exists y \left[P(x) \rightarrow Q(y) \right] \equiv \left(\forall x P(x) \rightarrow \exists y Q(y) \right)$$

is valid.

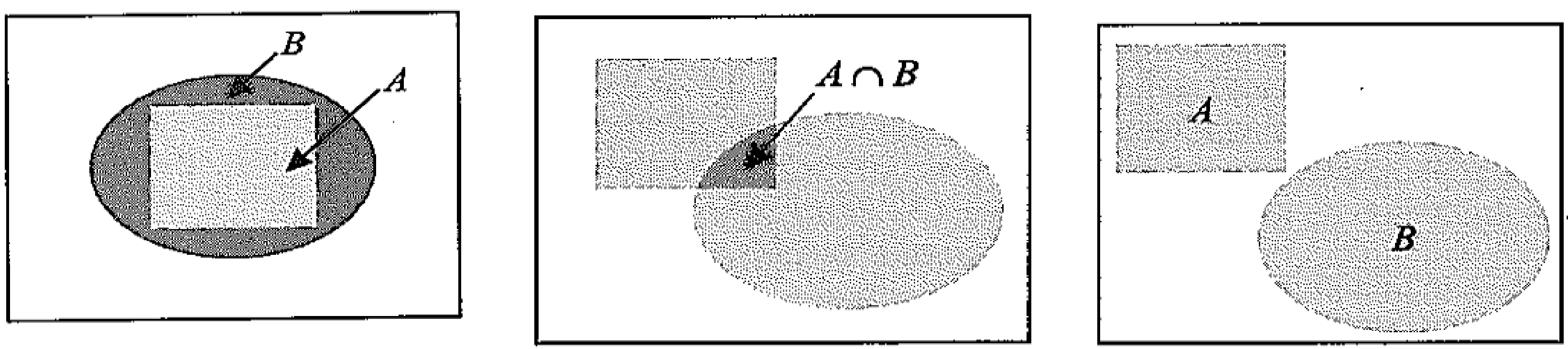
Let U be the collection of all objects with either property A or property B. Observe that there may be three different relations between A and B.

(i) Every member of A is a member of B. Then, $\forall x\left[A(x) \rightarrow B(x)\right]$ is a true proposition.

(ii) A and B have some member in common. Then, $\exists x\left[A(x) \wedge B(x)\right]$ is true.

(iii) A and B have no members in common. Then, $\forall x\left[A(x) \rightarrow \sim B(x)\right]$ is true.

U



Logical Inference

Based on facts or opinions that we assume to be true, we quite often would like to do some analysis, draw a conclusion so as to make a wise decision. Sometimes we may make mistake in the analysis and the conclusion does not follow from the assumptions. How can we make sure that the conclusion follows correctly from the assumptions? In other words, how can we make sure that our argument is valid.

Logical Inference

Suppose we know that
"*Samson is a boxing champion*."

and

"*If Samson is a boxing champion, then Samson is strong*."

We can conclude that
"*Samson is strong*."

This can be symbolically represented in the following way.

Let $p =$ "*Samson is a boxing champion.*"
$q =$ "*Samson is strong.*"

Then, we have

$$\left.\begin{array}{l} p \\ p \rightarrow q \end{array}\right\} \text{ Premises}$$

$$\overline{\therefore q} \quad \} \text{ Conclusion}$$

Premises: Propositions $p_1, p_2, \ldots, p_n$ which are assumed to hold.

Conclusion: Proposition q which is to follow the premises.

Argument: An argument with premises $p_1, p_2, \ldots, p_n$ and conclusion q is said to be *valid* if

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q$$

is a tautology.

Logical Inference by considering truth values

Method I:

The argument can be shown to be *valid* by using the truth table.

Ex. Use a truth table to verify

$$\begin{array}{r} p \wedge q \\ p \to r \\ \hline \therefore q \wedge r \end{array}.$$

Solution:

We have to show that $\left[(p \wedge q) \wedge (p \to r)\right] \Rightarrow (q \wedge r)$.

Logical Inference by considering truth values

Method II:

All the premises are assumed to be true. We try to establish the truth value of individual proposition by means of the definition of logical connectives and then check whether the conclusion is also true.

A number of valid arguments that are very common:

(1) addition

$$\frac{A}{\therefore A \vee B}$$

(2) simplification

$$\frac{A \wedge B}{\therefore A}$$

(3) modus ponens

$$\frac{\begin{array}{c} A \\ A \to B \end{array}}{\therefore B}$$

(4) modus tollens

$$\frac{\begin{array}{c} \sim B \\ A \to B \end{array}}{\therefore \sim A}$$

(5) hypothetical syllogism

$$\frac{\begin{array}{c} A \to B \\ B \to C \end{array}}{\therefore A \to C}$$

(6) disjunctive syllogism

$$\frac{\begin{array}{c} A \vee B \\ \sim A \end{array}}{\therefore B}$$

(7) conjunction

$$\frac{\begin{array}{c} A \\ B \end{array}}{\therefore A \wedge B}$$

(8) constructive dilemma

$$\frac{\begin{array}{c} (A \to B) \wedge (C \to D) \\ A \vee C \end{array}}{\therefore B \vee D}$$

(9) destructive dilemma

$$\frac{\begin{array}{c} (A \to B) \wedge (C \to D) \\ \sim B \vee \sim D \end{array}}{\therefore \sim A \vee \sim C}$$

Ex. 5.2 Show that the following is a valid argument:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}.$$

Ex. 5.4 Determine whether the following argument is valid:

"If I have a lot of friends, then I will be happy. If I study the BSCM Major, then I will be clever and have a lot of friends. If I am clever and happy, then I will not have any worries in life. However, I will have worries in life. Therefore, I do not study the BSCM Major."

Solution: Let

LF = "*I have a lot of friends*".

H = "*I will be happy.*"

CM = "*I study the BSCM Major.*"

C = "*I will be clever.*"

W = "*I will have worries in life.*"

Ex. 5.5 Determine whether the following argument is valid:

> "*Taxes will increase or government spending decrease. Government spending increases or more people have jobs. If people are poor or the economy is bad, then fewer people will have jobs. Therefore, if taxes decrease, people are rich.*"

Solution: Let

I = "*Taxes will increase*".

G = "*Government spending decreases.*"

J = "*More people have jobs.*"

P = "*People are rich.*"

E = "*Economy is bad.*"

Method III:

An argument is valid if and only if it is a tautology, i.e.

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$$

T

Always true in the truth table

We first assume that, in the truth table, there is a "*F*" under "$\rightarrow$" and then chase back the truth value of individual proposition to see whether such a case may exist.

NOT exist $\Rightarrow$ Valid

Exist $\Rightarrow$ Invalid.

Ex.5.7 Check if the following argument is valid.

$$\begin{array}{l} p \to q \\ r \to s \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

Ex. 5.8 Determine whether the following argument is valid:

"There are many job vacancies and the unemployment rate is low. If there are many job vacancies, then either the economy is thriving or wages are too low. If wages are not too low, then the unemployment rate is low. Therefore, the economy is thriving."

Solution: Let

J = "*There are many job vacancies*".

U = "*Unemployment rate is low.*"

W = "*Wages are too low.*"

E = "*Economy is thriving.*"

Assertions involving predicates and quantifiers:

(1) Universal Instantiation, abbreviated to *ui*

$$\frac{\forall x P(x)}{\therefore P(c)} \quad \text{or} \quad \frac{\forall x P(x)}{\therefore P(y)}$$

(2) Universal Generalization, *ug*

$$\frac{P(y)}{\therefore \forall x P(x)}$$

(3) Existential Instantiation, *ei*

$$\frac{\exists x P(x)}{\therefore P(c)}$$

(4) Existential Generalization, *eg*

$$\frac{P(c)}{\therefore \exists x P(x)}$$

Ex. 5.10 Let U = the set of creatures, $Q(x)$ = "x is a quadruped (a creature with four legs)", $H(x)$ = "x is a human being"; $W(x)$ = "x is a woman".

Express the following argument using predicates and quantifiers, and determine whether it is valid.

"*No human beings are quadrupeds. All women are human beings. Therefore, no women are quadrupeds.*"

Ex. Let $P(x)$ and $Q(x)$ be predicates in the variable x with a given universe of discourse. Give an example such that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x[\mathrm{P}(\mathrm{x}) \vee Q(x)]$ have different truth values.

Ex. 5.13

Determine whether the following argument is valid:

Some trigonometric functions are periodic. All periodic functions are bounded. Therefore, some trigonometric functions are bounded.

Solution: Let $T(x)$ —— x *is a trigonometric function*.

$P(x)$ —— x *is periodic*.

$B(x)$ —— x *is bounded*.

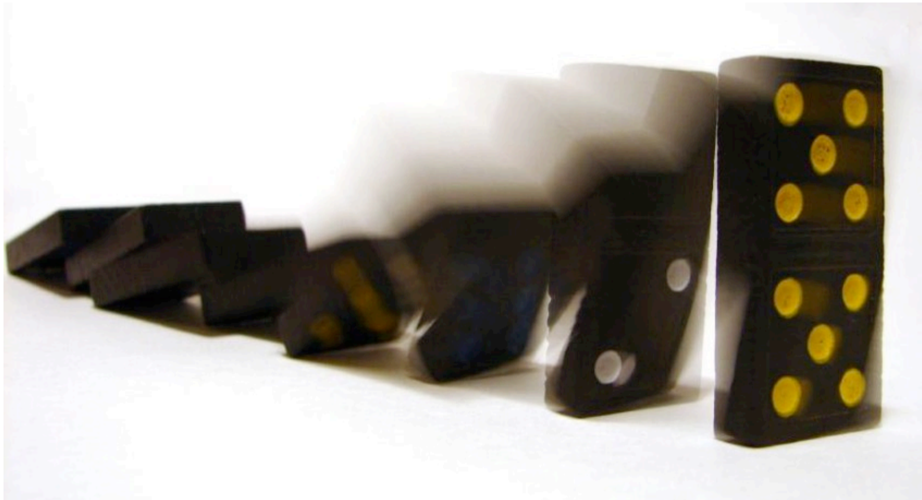
***Mathematical Induction* (M.I.)**

—— To prove that $P(n)$ is true for all positive integer n.

Two steps:

(i) $P(1)$ is true,

(ii) $P(n) \rightarrow P(n+1)$ is true for every positive integer n.



Ex. 6.1 Prove that $n < 2^n$ for all positive integer n.

Ex. 6.3 Use M.I. to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.