

CITY UNIVERSITY OF HONG KONG

Course code & title: MA2504/2509 Discrete Mathematics
Session: Semester A, 2022/2023
Time allowed: 3 hours, 18:30 – 21:30, December 15, 2022

This exam has THREE pages (including this cover page).

1. This exam contains **6** questions. The total number of points is **70**.
 2. Please answer all questions and justify all your steps.
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The exam is **closed book**. You may use a calculator.

No materials or aids are allowed in the examination. If any unauthorized materials or aids are found on a student during the examination, the student will be subject to disciplinary action.

1. (a) (5 points) Is

$$[(p \vee q \vee r \vee s \vee t) \wedge [u \vee v \vee w \rightarrow \sim (s \vee t)]] \rightarrow p \vee q \vee r \vee \sim v$$

a tautology? Justify your answer **without** using the truth table.

- (b) (7 points) Let Ω = the set of human beings; $S(x)$: x is a student; $P(x)$: x is a professor; $L(x)$: x is a Liar; $T(x, y)$: x trusts y .

Express the following argument using predicates and quantifiers, and determine whether it is valid.

“Some students trust every professor, and no student trusts any liar. Therefore, no professor is a liar.”

2. Let x_1, x_2, \dots be a sequence of positive real numbers.

- (a) (5 points) Let k be a positive integer, and $n := 2^k$. Show that for all k we have

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}. \quad (1)$$

- (b) (5 points) Show that (1) holds for any positive integer n . (Hint: Suppose (1) is true for $n + 1$, can you show that (1) is true for n ?)

3. (a) (4 points) Let A, B , and C be sets. Show that

$$(A - B) - C = (A - C) - (B - C)$$

without using Venn diagram.

- (b) (6 points) Is there a bijective function $f : (0, 1) \rightarrow [0, 1]$? Justify your answer.

4. (a) (4 points) How many ways to sit 5 men and 3 women in a row, where all men must sit next to one another?
- (b) (4 points) How many integers between 1000 and 9999 have sum of digits exactly 8? e.g. 1025, 2222, 7001.
- (c) (4 points) Find the number of positive integers less than 1,000,000 that are divisible neither by 4 nor by 6.

5. (a) (6 points) Solve the recurrent relation

$$a_n = 2a_{n-1} - a_{n-2} + 2^n + 3^n$$

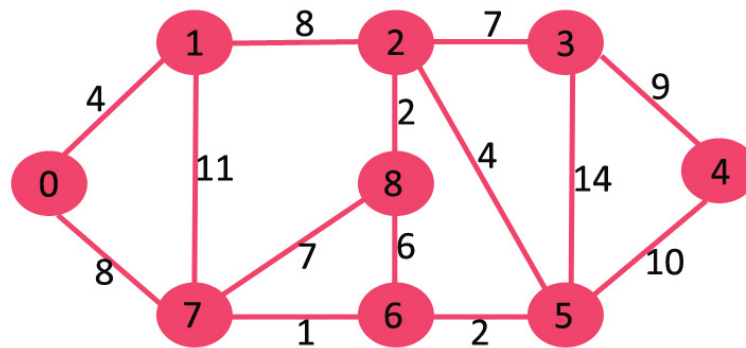
where $a_0 = 0$, $a_1 = 1$.

- (b) (6 points) Is

$$\sqrt{3}(1 + \sqrt{3})^{10000} + (1 - \sqrt{3})^{10000}$$

a rational number? Justify your answer.

6. (a) (4 points) Consider the following weighted graph of 9 vertices.



Draw the subgraph containing vertices 0,1,2,6,7,8 with all the induced edges (i.e. all edges have both end points in $\{0,1,2,6,7,8\}$). Does this subgraph has an Euler path or Euler circuit?

- (b) (5 points) For the original graph in (a), use Dijkstra's Algorithm to find the shortest path between 0 and 4.
- (c) (5 points) What is the number of binary trees with five vertices labelled from v_1 to v_5 that have height 2?

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