CHAPTER 2 Set Theory

Set — A collection of objects.

The objects comprising a set are called its elements. We write

$x \in A$ if x is an element in the set A.

Example 2.1

(a) What is the set V of all vowels of the English alphabet E?

(b) What are the elements in the set A where

$$A = \{x : x \in Z \text{ and } x < 4 \text{ and } x \geq 0\}.$$

Subset — A is a subset of B

$$\Leftrightarrow \forall x[x \in A \to x \in B]$$

$$\Leftrightarrow A \subset B.$$

Example 2.4

Which of the following statements is/are correct?

(i) $a \in V$, (ii) $a \subset V$, (iii) $\{a\} \in V$, (iv) $\{a\} \subset V$.

Example 2.5

Let A, B and C be sets such that

$$A \subset B \text{ and } B \subset C.$$

Show that

$$A \subset C.$$

Equality of two sets

$$A = B \quad \Leftrightarrow \quad \forall x[x \in A \rightarrow x \in B] \wedge \forall x[x \in B \rightarrow x \in A]$$

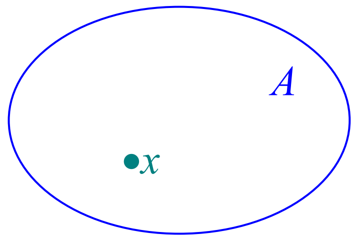
$$\Leftrightarrow \quad A \subset B \text{ and } B \subset A.$$

That is, A and B have exactly the same elements.

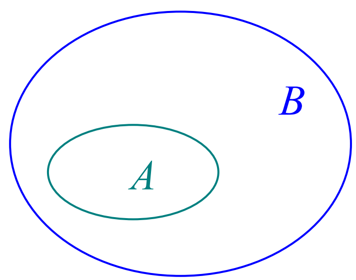
Venn Diagram

Sets can be represented by simple plane diagrams called *Venn Diagrams*.

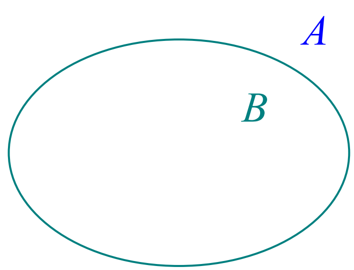
$x \in A \equiv$



$A \subset B \equiv$



$A = B \equiv$



The negations of $x \in A$, $A \subset B$ and $A = B$ are written as

$x \notin A$, $A \not\subset B$ and $A \neq B$, respectively.

<u>Universal Set</u> (denoted by U) — The largest possible set under consideration.

<u>Empty Set</u> (denoted by ϕ) — The set contains no element and is expressed as $\boldsymbol{\sim \exists x \, (x \in \phi)}$.

For any set A, we have

$$\phi \subset A \subset \boldsymbol{U}.$$

Example 2.8

In human population studies, the universal set consists of all the people in the world.

Example 2.9 Determine which of the following sets are equal:

$$\phi, \quad \{0\}, \quad \{\phi\}.$$

Finite Set — Set S has exactly n distinct elements.

We usually write: $|S| = n$.

Example 2.10

(a) Let S be the set of letters in the English alphabet. Then,

$$|S| = 26.$$

(b) Since ϕ has no elements,

$$|\phi| = 0.$$

A set is said to be *infinite* if it is not finite.

For instance, the set of positive integers is infinite.

Algebra of Set

The *union* of two sets A and B, denoted by $A \cup B$, is the set of elements which belong to either A or B:

$$A \cup B = \{x: x \in A \vee x \in B\}.$$

Example 2.11 Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

Find $A \cup B$.

The *intersection* of A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

$$A \cap B = \{x\colon\ x \in A \wedge x \in \mathrm{B}\}.$$

Example 2.12 Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4,5, 6\}$.

Find $A \cap B$.

If $A \cap B = \phi$, then A and B are said to be *disjoint*.

For instance, the set of even numbers and the set of odd numbers are disjoint.

The *difference* of A from B, denoted by $A - B$, is the set of elements which belong to A but not to B:

$$A - B = \{x: x \in A \wedge x \notin B\}.$$

Observe that $(A - B) \cap B = \phi$.

Example 2.13 Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

Find $A - B$.

The *complement* of A, denoted by A^c, is the set of elements which do not belong to A:

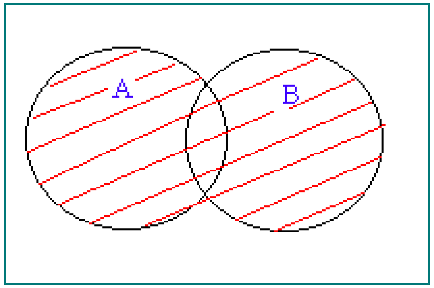
$$A^c = \{x : x \in U \text{ and } x \notin A\}.$$

From definition, $A^c = U - A$.

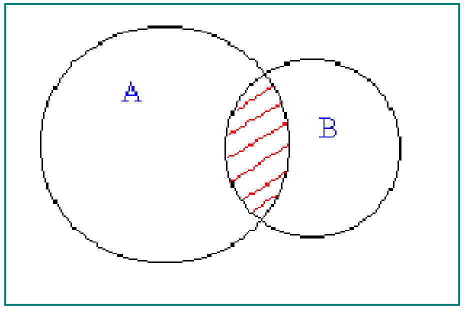
Example 2.14 Let $A = \{1, 2, 3, 4\}$ and $U = \{1, 2, 3, \ldots\}$.

Find A^c.

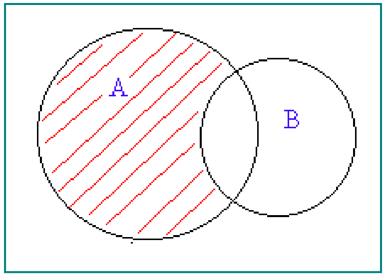
Venn Diagram of the set operations (U is represented by the area in the entire rectangle):



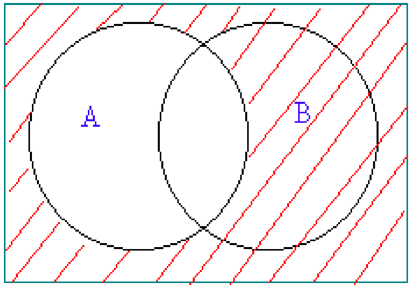
$A \cup B$ is shaded



$A \cap B$ is shaded



$A - B$ is shaded



A^c is shaded

Laws of the algebra of Sets

Idempotent Laws

(1a) $A \cup A = A$ (1b) $A \cap A = A$

Associative Laws

(2a) $(A \cup B) \cup C = A \cup (B \cup C)$

(2b) $(A \cap B) \cap C = A \cap (B \cap C)$

Commutative Laws

(3a) $A \cup B = B \cup A$ (3b) $A \cap B = B \cap A$

Distributive Laws

(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity Laws

(5a) $A \cup \phi = A$ (5b) $A \cap U = A$

(6a) $A \cup U = U$ (6b) $A \cap \phi = \phi$

Complement Laws

(7a) $A \cup A^c = U$ (7b) $A \cap A^c = \phi$

(8a) $(A^c)^c = A$ (8b) $U^c = \phi$, $\phi^c = U$

De Morgan's Laws

(9a) $(A \cup B)^c = A^c \cap B^c$ (9b) $(A \cap B)^c = A^c \cup B^c$

Ex. 2.17 Show by using Venn Diagram that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Ex. 2.18 Prove the above identity from the definitions.

Ex. 2.22 Let A, B be two sets. Show that

$$(A \cup B) \cap (A \cap B) = A \cap B.$$

Correspondence of logic connectives with set operators

Logic	Set
$\wedge$	$\cap$
$\vee$	$\cup$
$\sim$	^{c}
T	U
F	ϕ
$\equiv$, $\Leftrightarrow$	$=$

* For a logical identity, there corresponds a set identity.

Ex. 2.23 Determine which of the following statements is/are true. Find a counter example for each statement that is false. Assume all sets are subsets of a universal set U.

(a) For all sets A, B and C, $(A - C) \cap (B - C) \cap (A - B) = \varnothing$.

(b) For all sets A, B and C, if $\mathrm{A} \subseteq \mathrm{B}$ then $\mathrm{A} \cap (\mathrm{B} \cap \mathrm{C})^{c} = \varnothing$.

(c) For all sets A and B, if $\mathrm{A}^{c} \subseteq \mathrm{B}$ then $\mathrm{A} \cup \mathrm{B} = \mathrm{U}$.

(d) For all sets A, B and C,

$$\text{if } \mathrm{A} \subseteq \mathrm{B} \text{ and } \mathrm{B} \cap \mathrm{C} = \varnothing \text{ then } \mathrm{A} \cap \mathrm{C} = \varnothing.$$

(e) For all sets A, B, C and D,

$$(A-C) \cap (B-C) \cap (A-D) \cap (B-D) = \varnothing.$$

Cartesian Products

The *ordered n–tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

In particular, *2–tuples* are called *ordered pairs*.

Equality of two ordered pairs:

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

The *Cartesian product* of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b)\colon\ a \in A \ \wedge\ b \in B\}.$$

Ex. 2.24 What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Ex. 2.25 If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

Ex. 2.26 Is $A \times B$ equal to $B \times A$?

Ex. 2.28 Prove that

$$(A \cap B) \times C = (A \times C) \cap (B \times C).$$

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n–tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ $(i = 1, 2, \ldots, n)$, i.e.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Ex. 2.29 What is the Cartesian product $A \times B \times C$ where $A = \{0, 1\}$, $B = \{\text{a}, \text{b}\}$ and $C = \{X, Y, Z\}$?

Relations

For two sets A and B, $A \times B$ may be regarded as all the possible combinations of elements between A and B. A subset of $A \times B$ gives a specific relation between the elements of A and B. A *binary relation* between A and B is a subset of $A \times B$. If a relation is a subset of $A \times A$, it is called a *binary relation on A*.

Ex. 3.1 Let A = {David, John, Ray} and B = {Lily, Mary, Tracy}. Consider the relation of marriage. $A \times B$ gives all the possible marriages. Assume that Lily, Mary, Tracy are married to David, John, Ray, respectively. Then, the relation R is expressed as

$$R = \{(\text{David, Lily}), (\text{John, Mary}), (\text{Ray, Tracy})\}.$$

If Mary is married to Ray and there is no other marriage, then

$$R = \{(\text{Ray, Mary})\}.$$

Ex. 3.2 The following are relations:

(a) $M = \{(x,y) \mid x \textit{ is married to } y\}$ is a relation on the set of people in the human race.

(b) $R = \{(x,y) \mid \textit{Language } x \textit{ is available on computer } y\}$ is a relation between the set of languages and the set of computers.

(c) $F = \{(x,y) \mid \textit{Airline } x \textit{ flies to city } y\}$ is a relation between the set of airlines and the set of cities with airports.

Ex. 3.3 Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(x,y) \mid x \textit{ divides } y\}$?

If $(x, y) \in R$, we write

$$x \, \boldsymbol{R} \, y.$$

Otherwise,

$$x \, \not{\boldsymbol{R}} \, y.$$

An *n–ary relation* is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

Domain and Range

Given two sets A and B,

$$dom\ R = \{a \in A : \exists b \in B \text{ such that } (a,b) \in R\},$$

$$range\ R = \{b \in B : \exists a \in A \text{ such that } R(a) = b\}.$$

Note that $dom\ R \subset A$ and $range\ R \subset B$.

Ex. 3.4 Let

$$f = \{(1,a),(1,b),(2,b),(3,c)\} \subset \{1,2,3,4\} \times \{a,b,c,d\}.$$

Find $dom\ f$ and $range\ f$.

Ex. 3.5 Express the following set of data on 3 students using a 4-ary relation.

Name	Student No.	Tutorial group	Tutor
A E Leung	93101245	1	Dr. Cheung
C C Chan	93050036	2	Mr. Lee
B B Wong	93011236	5	Mr. Berstein

Solution: R = {(A E Leung, 54101245, 1, Dr. Cheung),
(C C Chan, 52050036, 2, Mr. Lee),
(B B Wong, 59011236, 5, Mr. Berstein)}.

The *inverse relation* of R is the subset of $B \times A$ which is defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}.$$

Ex. 3.6 Let $R = \{(1,1), (1,2), (1,3)\}$. Find R^{-1}.

Ex. 3.7 Let $R = \{(x, y) \mid x < 2y\}$. Find R^{-1}.

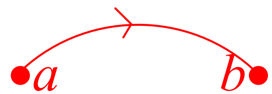
Ex. 3.8 Let $A = \{a\}$ and $B = \{1, 2\}$. How many possible relations are there from A to B?

Question

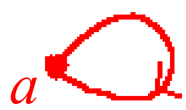
Let $|A| = m$ and $|B| = n$. How many different relations are there from A to B?

Digraph (Pictorial representation of relations)

A directed graph, or *digraph*, consists of a set V of *vertices* together with a set E of ordered pairs of elements of V called *edges*. The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge:

$(a, b) \in E \iff$ 

Loop: Edge having the same initial and terminal vertex.



Ex. 3.9 Let $A = \{1, 2, 3, 6, 12\}$ and the relation R on A be defined as

$$(x, y) \in R \quad \text{iff} \quad x \text{ divides } y\,.$$

Find the digraph representation of R.

Special properties of binary relations

Let R be a binary relation on A. Then, R is

(1) *reflexive* iff $(x,x)\in R$ for all x in A, that is,

$$\forall x\left[(x,x)\in R\right] \text{ is true.}$$

(2) *symmetric* iff $\forall x \forall y\left[(x,y)\in R \Rightarrow (y,x)\in R\right]$.

(3) *antisymmetric* iff $\forall x \forall y\left[\left((x,y)\in R \wedge (y,x)\in R\right) \Rightarrow (x=y)\right]$.

(4) *transitive* iff $\forall x \forall y \forall z\left[\left((x,y)\in R \wedge (y,z)\in R\right) \Rightarrow (x,z)\in R\right]$.

Ex. 3.10 Let $A = \{1, 2, 3, 4\}$. Let R_1, R_2 and R_3 be defined as

$R_1 = \{(1, 1), (1, 2), (2, 1)\}$,
$R_2 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Here, R_1, R_2, R_3 and ϕ are relations on A.

Which of these relations are reflexive?

Ex. 3.11 Let A be the set of triangles and R be the binary relation on A defined by

$$(x, y) \in R \quad \text{iff} \quad x \text{ is similar to } y.$$

Determine whether R is

(a) reflexive,

(b) symmetric,

(c) antisymmetric,

(d) transitive.

Ex. 3.12 Let Q be the binary relation defined on N by

$$(x, y) \in Q \quad \text{iff} \quad x - y \text{ is divisible by } 3.$$

Determine whether Q is

(a) reflexive,

(b) symmetric,

(c) antisymmetric,

(d) transitive.

3.1 Equivalence Relations

The purpose is to group the elements of a set into classes according to a certain relation instead of individual elements.

R is an equivalence relation on A. $\Leftrightarrow$ R is (i) reflexive,
(ii) symmetric,
(iii) transitive.

Two related elements are called *equivalent*.

Ex. 3.14 Let $A = \{1, 2, 3, 4\}$. Let R_1 and R_2 be relation on A defined as

$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$.

Are R_1 and R_2 equivalent relations on A?

Exercise

(a) Let A be the set of mankind and

$$R = \{(x, y) : x \text{ and } y \text{ are of the same age}\}.$$

Show that R is an equivalence relation on A.

(b) Let A be the set of English letters and R is a relation such that

$$aRb \text{ for } a, b \in A \quad \Leftrightarrow \quad L(a) = L(b),$$

where $L(x)$ is the length of the string x. Show that R is an equivalence relation on A.

Ex. 3.15 Define that

$$a \equiv b \pmod{m} \iff m \text{ divides } a - b.$$

Let A be the set of integers and R be a relation on A such that

$$aRb \iff a \equiv b \pmod{3}.$$

Is R an equivalent relations on A?

Functions

— assignment of each element of a set a particular element of a second set.

— dependence of one varying quantity on another.

A *function f from a set A to a set B* is a relation between elements of A and elements of B with the property that each element of A is related to a unique element of B, i.e.

(i) For $x \in A$ and $y \in B$ such that

$$\forall x\, \exists y\ \ (x, y) \in f;$$

(ii) If $(a, b) \in f$ and $(a, b_1) \in f$, then $b = b_1$.

Ex. 4.2 Given sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Are the following relations functions?

(a) $R_1 = \{(a, 1), (b, 3)\}$

(b) $R_2 = \{(a, 1), (a, 2), (b, 3)\}$

(c) $R_1^{-1} = \{(1, a), (3, b)\}$.

For a function f from A to B, we write

$$f : A \to B.$$

For any $a \in A$, we use $f(a)$ to denote the unique element b in B related to a. We write

$$f(a) = b.$$

— A is the **domain** of f.

— B is the **codomain** of f.

— b is the **image** of a.

— a is a **pre-image** of b.

— the **range** of f is the set of all images of elements of A.

— f **maps** A to B.

A function can be expressed explicitly by a *digraph*.

Ex. 4.3 Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

$f : A \to B$ is defined as

$$f(1) = a,$$

$$f(2) = c,$$

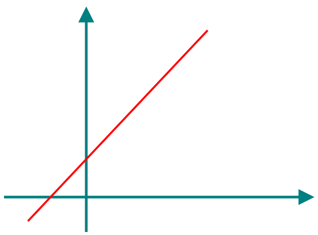
$$f(3) = c.$$

$f: A \to B$ can be expressed by means of a digraph as

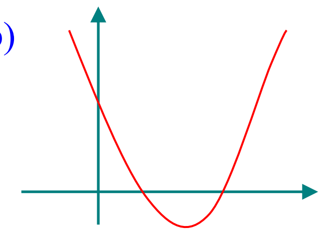
$$\begin{tikzcd} 1 \arrow[r, "f"] & a \\ 2 \arrow[dr] & b \\ 3 \arrow[r] & c \end{tikzcd}$$

Ex. 4.4 Which of the following relations on the set of real numbers are functions?

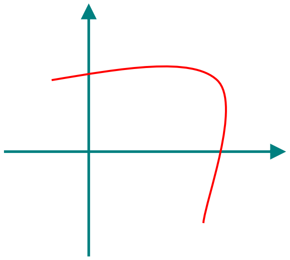
(a)



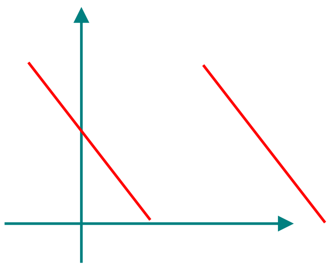
(b)



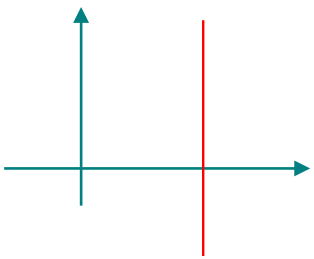
(c)



(d)



(e)



Composition of Functions

Let $g : A \to B$ and $f : B \to C$ be functions with the property that the range of g is a subset of the domain of f. Define a new function $f \circ g : A \to C$ as follows:

$$(f \circ g)(x) = f(g(x)) \text{ for all } x \in A.$$

The function $f \circ g$ is called the composition of f and g.

Ex. 4.8,4.9 Find $g \circ f$ and $f \circ g$ and determine whether $f \circ g$ equals $g \circ f$.

(a) $f(x) = 2x + 1$ and $g(x) = x^2$, for all real numbers.

(b) $f(x) = \log_2 x$ and $g(x) = 8^{4x}$, for all positive numbers.

A function f is **one–to–one** or **injective**

$\Leftrightarrow f(x) = f(y) \Rightarrow x = y \quad \forall\, x, y \in A.$

$\Leftrightarrow x \neq y \Rightarrow f(x) \neq f(y) \quad \forall\, x, y \in A$ (contra-positive identity).

Ex. Determine whether the following functions are one–to–one?

(a) $f(x) = x^2 \ (x \in \boldsymbol{R})$,

(b) $f(x) = x + 1$.

Ex. Consider the function $f : \Re \times \Re \to \Re \times \Re$ defined by

$$f(x, y) = (x + y, x - y).$$

Prove that f is one-to-one on $\Re \times \Re$.

A function f from A to B is **onto** or **surjective**

$\Leftrightarrow$ for every element $y \in B$, there is an element $x \in A$ with $f(x) = y$.

$\Leftrightarrow$ $\forall y\, \exists x\, (f(x) = y)$

Ex. Determine whether the following functions from the set of integers to itself are surjective?

(a) $f(x) = x + 1$,

(b) $f(x) = x^2$.

A function f is **bijective**

$\Leftrightarrow$ f is both one–to–one and onto.

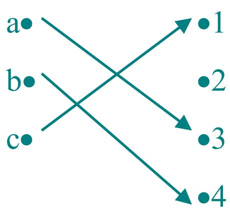
Ex. 4.15 Construct a bijection

$$f : (0, 1) \to (0, 2)$$

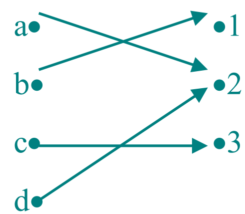
From the open interval (0, 1) to the open interval (0, 2).

Ex. What types of functions are the following?

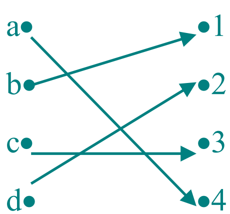
(a)



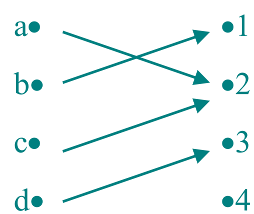
(b)



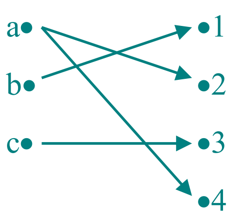
(c)



(d)



(e)



Ex. Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} x+7, & x \le 0 \\ -2x+5, & 0 < x \le 3 \\ x-1, & 3 < x. \end{cases}$$

(i) Find $f^{-1}(0)$ and $f^{-1}(4)$.

(ii) Let S be a subset of R and $f^{-1}(S)$ contains the pre-images of all the elements of S, i.e.

$$f^{-1}(S) = \{a \in R : f(a) \in S\}.$$

Determine $f^{-1}([0, 4])$.

Ex. 4.17 Let $f: A \to B$ be a function from set A to set B. Let S be a subset of B and $f^{-1}(S)$ contains the pre-images of all the elements of S, i.e.

$$f^{-1}(S) = \{a \in A: f(a) \in S\}.$$

(a) Let T be a subset of B and $S \subset T$. Show that $f^{-1}(S) \subset f^{-1}(T)$.
(b) Show that $f(f^{-1}(S)) \subset S$.
(c) Let T be a subset of A. Show that $T \subset f^{-1}(f(T))$.