MA1301 Assignment 2.

1(a). $y=\frac{1}{3}x^{\frac{3}{2}}\ (0\le x\le 5)$

$[f'(x)]^2=\frac{1}{4}x \qquad L=\int_0^5\sqrt{1+\frac{1}{4}x}\,dx$

$L=\frac{1}{2}\int_0^5\sqrt{4+x}\,dx$ let $4+x=u$. $L=\frac{1}{2}\int_4^9\sqrt{u}\,du$

$L=\frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=4}^{u=9}=\frac{19}{3}$

$\therefore$ the legth is $\frac{19}{3}$

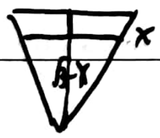
1(b). $(y-1)^{\frac{3}{2}}=\frac{3}{2}x \qquad \therefore x=\frac{2}{3}(y-1)^{\frac{3}{2}} \qquad \because 0\le x\le\left(\frac{2}{3}\right)(3)^{\frac{3}{2}}$

$\therefore 1\le y\le 4 \qquad \text{length}=\int_1^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$=\int_1^4\sqrt{1+y-1}\,dy=\int_1^4\sqrt{y}\,dy=\frac{2}{3}y^{\frac{3}{2}}\Big|_1^4=\frac{16}{3}-\frac{2}{3}=\frac{14}{3}$

$\therefore$ the length is $\frac{14}{3}$.

2(a)



For the i-th strip, $\Delta S=\left(\frac{\sqrt{3}-x_i}{\sqrt{3}}\right)\times 2\cdot(x_{i+1}-x_i)$

$\therefore S=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{\sqrt{3}-x_i}{\sqrt{3}}\right)\times 2\cdot(\Delta x)$

By $F=PS$ we have $F=\rho g\int_0^{\sqrt{3}}\left(\frac{\sqrt{3}-x}{\sqrt{3}}\right)\times 2\times x\,dx$

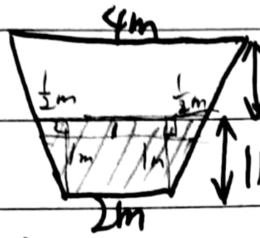
$\therefore F=1000\times 9.8\times\frac{2\sqrt{3}}{3}\int_0^{\sqrt{3}}(\sqrt{3}-x)x\,dx$

$\int_0^{\sqrt{3}}(\sqrt{3}-x)x\,dx=\int_0^{\sqrt{3}}(\sqrt{3}x-x^2)\,dx=\frac{\sqrt{3}}{2}x^2-\frac{1}{3}x^3\Big|_0^{\sqrt{3}}$

$=\frac{3}{2}\sqrt{3}-\sqrt{3}=\frac{\sqrt{3}}{2}$

$\therefore F=1000\times 9.8\times\frac{2\sqrt{3}}{3}\times\frac{\sqrt{3}}{2}=9800\ N$

$\therefore$ The force is 9800 N

2(b). 

i-th strip

$S=(3-x_i)(x_{i+1}-x_i)$

By $F=PS$ we have $F=\rho g\int_0^1 (3-x)x\,dx$

$F=9800\int_0^1 (3x-x^2)dx = 9800\times\frac{7}{6} = \frac{34300}{3}$ N

∴ The force is $\frac{34300}{3}$ N

2(c). The width of the i-th strip would be $2\sqrt{4-(10-x)^2}$

∴ $S=2\sqrt{4-(10-x)^2}\,dx$

By $F=PS$ we have $F=\int_8^{10} 9800x\cdot 2\sqrt{4-(10-x)^2}\,dx$

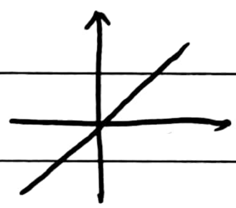
let $u=10-x$, $du=-dx$ then we have

$F=\int_{-2}^{0} 9800(10-u)\,2\sqrt{4-u^2}(-du) = 2\cdot 9800\int_0^2 (10-u)\sqrt{4-u^2}\,du$

$=196000\int_0^2 \sqrt{4-u^2}\,du - 9800\int_0^2 2u\sqrt{4-u^2}\,du$

$=196000\pi - \frac{16}{3}\times 9800 = 196000\pi - \frac{156800}{3}$

3(a). $M_x=\frac{1}{2}\int_{-2}^{1} (2-x^2)^2-x^2\,dx = 1.8$



$M_y=\int_{-2}^{1} x(2-x^2-x)\,dx = -2.25$

$\int_{-2}^{1}(2-x^2-x)\,dx=\frac{9}{2}$

$\bar{x}=\frac{1}{A}M_y=\frac{2}{9}\times(-2.25)=-\frac{1}{2}$

$\bar{y}=\frac{2}{9}\times 1.8=\frac{2}{5}$ ∴ centroid $(-\frac{1}{2},\frac{2}{5})$

3(b). $M_x = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos x)^2 - (\sin x)^2 \, dx = \frac{1}{4}$

$M_y = \int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\, dx = \frac{\sqrt{2}\pi}{4} - 1$

$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = \sqrt{2} - 1$

$\bar{x} = \frac{1}{1-\sqrt{2}}\left(\frac{4-\sqrt{2}\pi}{4}\right) = \frac{(\sqrt{2}\pi - 4)(\sqrt{2}+1)}{4}$

$\bar{y} = \frac{1}{4} \times \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{4}$ $\therefore$ Centroid $\left(\frac{(\sqrt{2}\pi-4)(\sqrt{2}+1)}{4}, \frac{\sqrt{2}+1}{4}\right)$

3(c). $M_x = \frac{1}{2}\int_{-1}^{1} (x^3 - x)^2 - (x^2-1)^2 \, dx = -\frac{16}{35}$

$M_y = \int_{-1}^{1} x(x^3 - x - x^2 + 1)\, dx = -\frac{4}{15}$

$A = \int_{-1}^{1} (x^3 - x - x^2 + 1)\, dx = \frac{4}{3}$

$\bar{y} = \frac{3}{4} \times \left(-\frac{16}{35}\right) = -\frac{12}{35}$ $\bar{x} = \frac{3}{4} \times \left(-\frac{4}{15}\right) = -\frac{1}{5}$

$\therefore$ Centroid $\left(-\frac{1}{5}, -\frac{12}{35}\right)$

4.(a). $\overrightarrow{AB} = (1, 1, 1)$ $\overrightarrow{AX} = \frac{2}{3}\overrightarrow{AB} = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

(b). $X = \left(\frac{2}{3}, \frac{5}{3}, -\frac{1}{3}\right)$ $\therefore \overrightarrow{OX} = \left(\frac{2}{3}, \frac{5}{3}, -\frac{1}{3}\right)$

5. $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}| \cdot |\overrightarrow{AC}| \cdot \cos\theta = 4 \cdot 4 \cdot \cos\theta = 2$ $\therefore \cos\theta = \frac{1}{8}$

let $A = (0, 0)$ $B = \left(\frac{1}{2}, \frac{\sqrt{63}}{2}\right)$ $C(4, 0)$

$\therefore BC^2 = \left(\frac{7}{2}\right)^2 + \left(\frac{\sqrt{63}}{2}\right)^2 = \frac{63}{4} + \frac{49}{4} = \frac{112}{4}$

$\therefore BC = \sqrt{\frac{112}{4}} = \frac{4\sqrt{7}}{2} = 2\sqrt{7}$

or $BC^2 = |\overrightarrow{BC}|^2 = (\overrightarrow{BA} + \overrightarrow{AC})^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2 = AC^2 + AB^2 - 2\overrightarrow{AC}\cdot\overrightarrow{AB}$

$= 4^2 + 4^2 - 2 \times 2 = 32 - 4 = 28$

$\therefore |\overrightarrow{BC}| = 2\sqrt{7}$

6 (a). $\vec{a}\times\vec{b} = 15\vec{i} - 5\vec{j} - 2\vec{k}$

(b). $\vec{a}\times\vec{b} = (1\times5-(-2)\times2)\vec{i} + ((-3)\times(-1)-5\times1)\vec{j} + (1\times2-(-3)\times1)\vec{k}$

$= 9\vec{i} + \vec{j} + 5\vec{k}$

(c). $\vec{a}\times\vec{b} = (1\times1-3\times6)\vec{i} + (-3\times1)\vec{j} + (-3\times6)\vec{k}$

$= -17\vec{i} + 3\vec{j} - 18\vec{k}$

7 (a). $\vec{AB} = (2,-3,-2)$ $\vec{AC} = (-3,-2,1)$

设 $\vec{n} = (x,y,z)$ we have $\begin{cases} 2x-3y-2z=0 \\ -3x-2y+z=0 \end{cases}$

$\therefore \vec{n} = (7,-4,13)$

$\therefore (7,-4,13)$ is perpendicular to $\vec{AB}$ and $\vec{AC}$

(b). set $\vec{a} = (7a,-4a,13a)$ $\vec{BC} = (-5,1,3)$

$|\vec{a}| = \sqrt{7^2+4^2+13^2}\,a = \sqrt{234}\,a$

$|\vec{BC}| = \sqrt{5^2+1^2+3^2} = \sqrt{35}$

$\therefore a = \frac{\sqrt{35}}{\sqrt{234}} = \frac{\sqrt{910}}{78}$

$\therefore \vec{a} = \left(\frac{7}{78}\sqrt{910}, -\frac{2}{39}\sqrt{910}, \frac{1}{6}\sqrt{910}\right)$

or $\vec{a}$ is in the opposite direction $\left(-\frac{7}{78}\sqrt{910}, \frac{2\sqrt{910}}{39}, -\frac{\sqrt{910}}{6}\right)$

so $\vec{a}$ is $\left(-\frac{7}{78}\sqrt{910}, \frac{2}{39}\sqrt{910}, -\frac{1}{6}\sqrt{910}\right)$

or $\left(\frac{7}{78}\sqrt{910}, -\frac{2}{39}\sqrt{910}, \frac{1}{6}\sqrt{910}\right)$

8(a). suppose $\vec{a} = n\vec{b}$ n DNE

So $\vec{a}$ and $\vec{b}$ are linearly independent

(b). suppose $\vec{c} = x\vec{a} + y\vec{b}$

$$\begin{cases} x+2y=3 \\ -2x+5y=2 \\ 3x+y=-3 \end{cases} \Rightarrow \begin{cases} x= \\ y= \end{cases} \text{DNE}$$

So $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent

(c). suppose $\vec{c} = x\vec{a} + y\vec{b}$

$$\begin{cases} x-y=3 \\ 2x+2y=-2 \\ -5x-3y=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=-2 \end{cases}$$

So $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent.