CHAPTER 3 Combinatorics

- Basic counting techniques
- Elementary Probability

2. Basic Counting Techniques

2.1 Pigeonhole Principle

The pigeons are placed into pigeonholes.

If the number of pigeons is more than k times the number of pigeonholes, then some pigeonholes must contain at least $k + 1$ pigeons.

Example 2.2

Choose any five points from the interior of an equilateral triangle having sides of length 1. Show that the distance between some pair of these points does not exceed 1/2.

Example 2.3

(a) If five integers are selected from the first eight positive integers, there must be a pair of these integers with sum equal to 9.

(b) Is the conclusion in part (a) still true if four integers are selected rather than five?

2.4 Permutation ${}_nP_r$

— the total arrangements of r objects from n distinct objects:

$$*\ {}_nP_r = \frac{n!}{(n-r)!}$$

where $n! = n\,(n-1)\,\ldots\,2 \times 1$

$$0! = 1.$$

What an incredible answer:

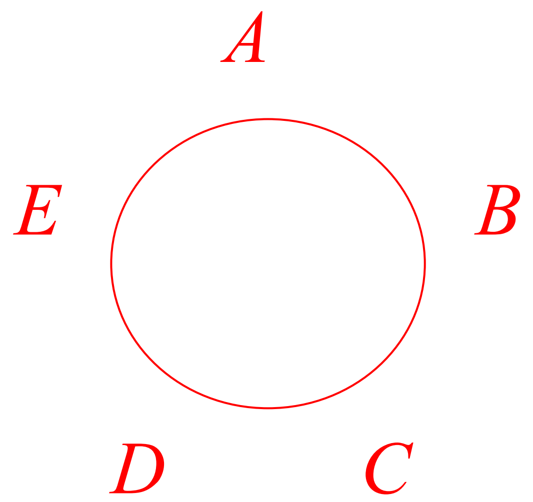
$$\mathbf{1 + 2 + 3 + \ldots + 14 + 15 = 5!}$$

Example 2.10

Of the permutations of the letters of the word *MEANS* taken all together, how many do not begin with *AS*?

Example 2.11

In how many different ways can 5 persons be seated at a round table?



2.5 Combination ${}_nC_r$

How many selections of r objects are there from n distinct objects?

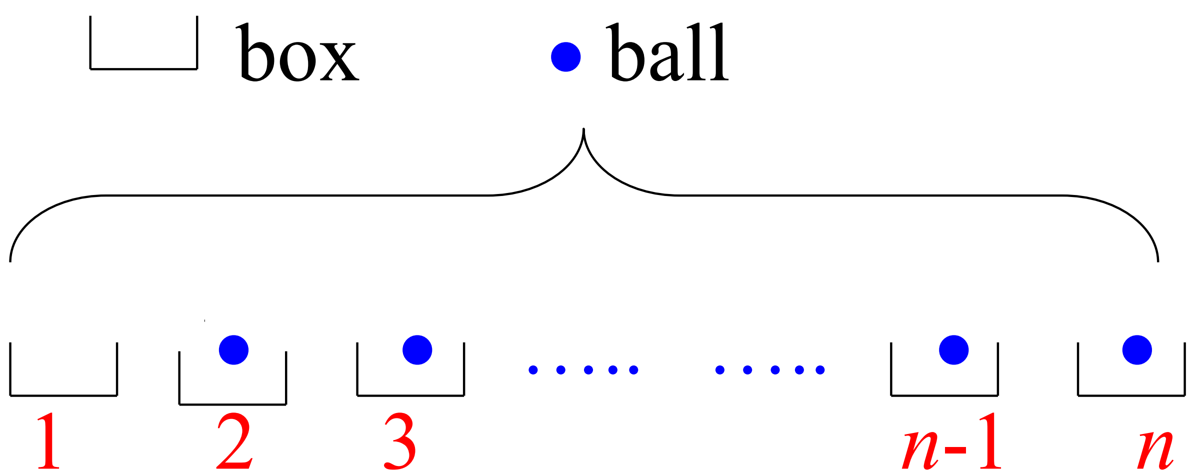
1. If we choose r objects from n distinct objects, number of ways = ${}_nC_r$.
2. If we permute the r chosen objects among themselves, number of ways = $r!$.

Steps 1 & 2 are equivalent to the arrangement of r objects from n distinct objects. Therefore,

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Equivalent model

How many ways are there to put r white balls into n distinct boxes?



No. of ways $= \binom{n}{r}$.

Example 2.14

If an alphabet consists of 5 vowels and 15 consonants, how many words of 3 different vowels and 2 different consonants can be made?

Example 2.15

For $q \leq p + 1$, find the number of ways to arrange p 1's and q 0's in a row so that no two 0's are adjacent.

Example 2.16

How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 11$$

where x_i (i = 1, 2, 3) is a non-negative integer such that

(a) $x_i \geq 0$ for i =1, 2, 3?

(b) $x_1 \geq 5$ and $x_2, x_3 \geq 0$?

(c) $7 \geq x_1 \geq 5$ and $x_2, x_3 \geq 0$?

Example 2.17 (Repeated case)

Suppose the n objects are

n_1 of the first kind
n_2 of the second kind
$\vdots$
n_r of the r^{th} kind

where $n_1 + n_2 + \ldots + n_r = n$, then

Total arrangement of these n objects $= \dfrac{n!}{n_1!\, n_2! \ldots n_r!}$

Example 2.18

How many strings of length 13 can be formed from the word *INFINITESIMAL*?

Example 2.19

There are 20 persons and they are 10 couples.

(a) How many ways are there to select 6 persons among the 20 persons if there is no couple among them?

(b) How many ways are there to select 6 persons among the 20 persons if there is exactly one couple among them?

(c) How many ways are there to select 6 persons among the 20 persons if there are exactly two couples among them?

3. Elementary Probability

Reason to study probability: understand just how likely or unlikely some events happen.

Examples: cards, grade distribution, stock market, insurance, quantum mechanics, forest fire…

Key terminologies:

Sample space (Ω): the set of possible outcomes.

E.g. if I toss a coin 3 times, the sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Element (x): elements of the sample space.

Event (E): subsets of the sample space.

E.g. The event that the first two tosses are heads is {HHT, HHH}

- We assign each element x a weight $p(x) \geq 0$.

- For each event E, we have

$$P(E) = \sum_{x : x \in E} p(x)$$

- We also require $P(\Omega) = 1$, and thus

$$0 \leq p(x) \leq 1$$

for all x.

Exercise 3.1.1 Try flipping a coin five times.

- What is the probability of getting at least one head?
- What is the probability of no heads?

Theorem 3.1

If two events E and F are complementary, that is they have nothing in common ($E \cap F = \emptyset$) and their union is the whole sample space ($E \cup F = \Omega$), then

$$P(E) = 1 - P(F).$$

We often write $F = E^c$, as F is the complement of E.

Exercise 3.1.2

Find a good sample space for rolling two dice.

- What weights are appropriate for the members of your sample space?
- What is the probability of getting a 6 or 7 total on the two dice?
- Assume the dice are of different colors. What is the probability of getting less than 3 on the red one and more than 3 on the green one?

Theorem 3.2

Suppose P is the uniform probability measure defined on a sample space Ω. Then for any event E,

$$P(E) = |E|/|\Omega|,$$

the size of E divided by the size of Ω.

Exercise 3.2.1

What is the probability of an odd number of heads in three tosses of a coin?

Exercise 3.2.2

A sample space consists of the numbers 0, 1, 2 and 3. We assign weight 1/8 to 0, 3/8 to 1, 3/8 to 2, and 1/8 to 3.
What is the probability that an element of the sample space is positive?

Exercise 3.2.3

Use the set $\{0, 1, 2, 3\}$ as a sample space for the process of flipping a coin three times and counting the number of heads. Determine the appropriate probability weights $P(0)$, $P(1)$, $P(2)$, and $P(3)$.

Conditional Probability

Let E and F be two events, with $P(F)>0$. The probability that E holds true, conditioned on F holds is defined as

$$P(E|F) = P(E \cap F)/P(F).$$

We say that E and F are independent if $P(E|F) = P(E)$.

Exercise 3.3.1

When we roll two ordinary dice, what is the probability that the sum of the tops comes out even, given that the sum is greater than or equal to 10?

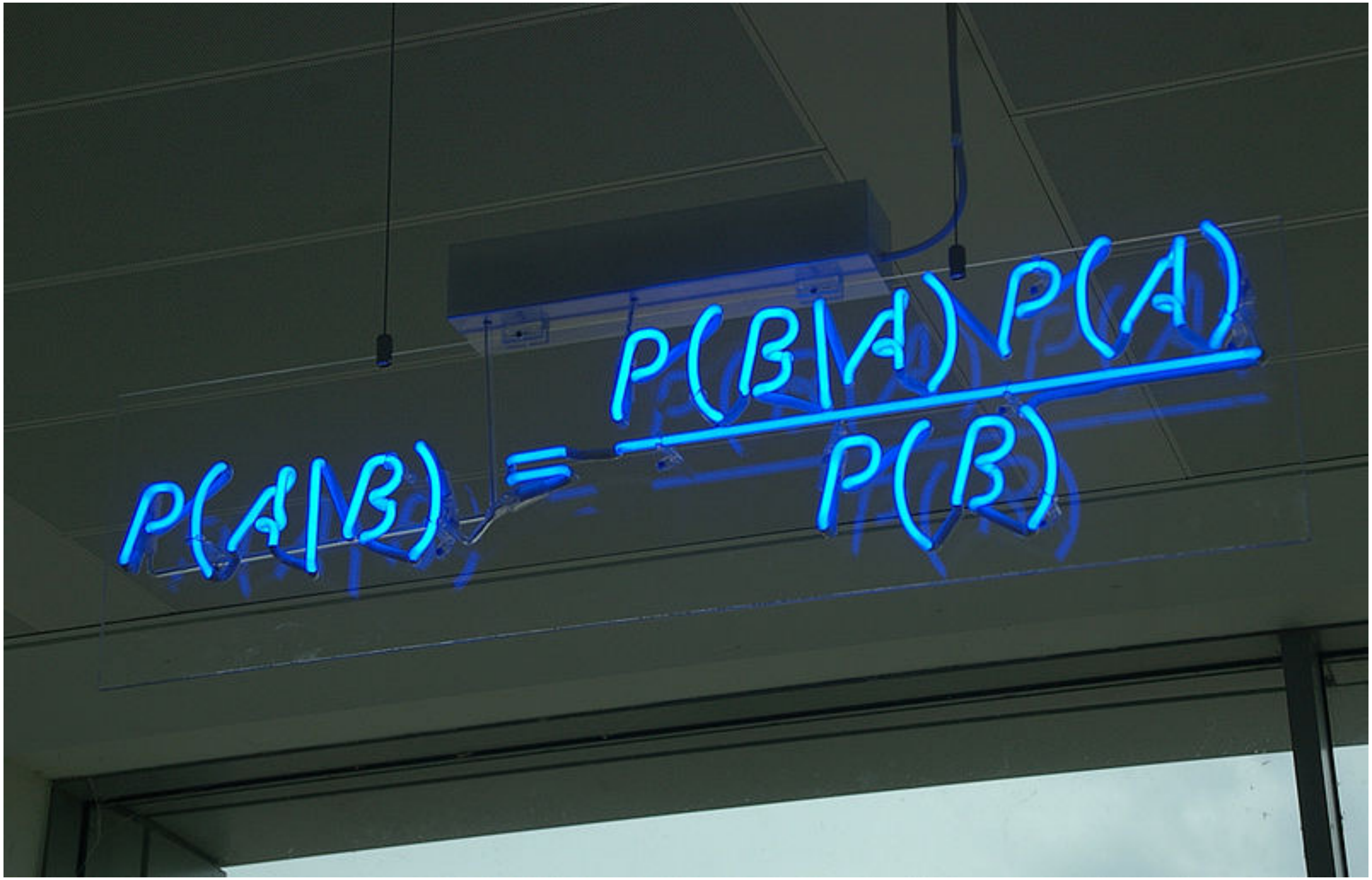
Exercise 3.3.2

Show that when we roll two dice, one red and one green, the event "The total number of dots on top is odd" is independent of the event "The red die has an odd number of dots on top."

Exercise 3.3.3

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test has 1% rate of false negatives (that is, about 1% of people who do have the disease will test negative); the false positive rate is 2% (that is, about 2% of people who do not have the disease will test). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Bayes rule



Random Variables:

Variable whose values depend on outcomes of a random phenomenon.

Let Ω be the sample space. A random variable X is a function that assigns each element ω of Ω to a real value $X(\omega)$.

Example

Let Ω be the outcomes of flipping a coin 5 times, and let X be the number of times that the the flip shows a tail.

Expected value

For a random X, its expected value is defined as

$$E(X)= \Sigma_a a \times P(X=a),$$

where the sum ranges from all possible values of X.

Exercise 3.3.5

A student takes a ten-question objective test. Suppose that a student has probability .8 of success an any question, independently of how the student did on any other problem. What is the probability that this student earns a grade of 80 or better (out of 100)? What grade would you expect the student to get?

Exercise 3.3.6

You pay \$5 to roll a die. If you roll a one or a two you win \$13. Should you play the game?

Theorem 3.4

Suppose X and Y are random variables on the (finite) sample space. Then $E(X + Y) = E(X) + E(Y)$.

In addition, $E(cX)=cE(X)$ for any deterministic c.

Exercise 3.3.6

We perform n trials, and each has a success probability p. What is the expected number of successful trials?

Exercise 3.3.4

If you flip a coin six times, how many heads do you expect?

Variance

For a random variable X, its variance is defined as

$$Var(X)=E(X^2)-(E(X))^2=E((X-E(X))^2)$$

Variance is the measure of dispersion.

Theorem 3.5

Suppose X and Y are independent random variables.

Then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

In addition, $\mathrm{Var}(cX)=c^2\mathrm{Var}(X)$ for any deterministic c.

Moreover, the variance is always nonnegative, and it is 0 iff the random variable is constant.

Exercise 3.4.1

What is the variance for the number of heads in one flip of a coin?

What is the variance for the number of heads in 5 flips of a coin?