CITY UNIVERSITY OF HONG KONG

Department of Mathematics

Course Code & Title	:	MA1300 Enhanced Calculus and Linear Algebra I
Session	:	Semester A, 2020-2021
Time Allowed	:	Three Hours

This paper has **Three** pages. (including this cover page)

Instructions to candidates:

1. Answer **all** questions.
2. Start each main question on a new page.
3. Show all step.
4. If technical issues happen, then inform immediately (to invigilators, course leader, etc.) via "chat" in Zoom, email (wingclo@cityu.edu.hk), or call the departmental hotline 3442 8646.

*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable portable battery operated calculator.

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

NOT to be taken away
NOT to share the questions out

1. Solve the following problems:

   (a) [6 marks] Compute $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

   (b) [7 marks] Compute $\lim_{x \to \infty} \left(1 + \frac{4}{\ln x} \right)^{\ln(x^2+x)}$.

   (c) [7 marks] Let $f(x) = (x^3 + \cos x)^{\sin x}$, find $f'(x)$.

2. [20 marks] For the following function:

$$f(x) = \sqrt[3]{x^3 - x^2 - x + 1}, \text{ for } -2 \le x \le 2,$$

   (a) Find the local maximum and minimum values of f.
   (b) Find the intervals of concavity and the inflection points.
   (c) Sketch the curve with the information above.

3. [15 marks] Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1} - \sqrt{n-1}}{2^n} x^{3n}.$$

4. [15 marks] Given two positive numbers a and b such that $a < b$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences which satisfy

$$x_1 = a, \ y_1 = b,$$

$$x_{n+1} = \sqrt{x_n y_n}, \text{ and } \ y_{n+1} = \frac{x_n + y_n}{2}$$

   for all integers $n \ge 1$.

   (a) Apply the Monotonic Sequence Theorem to show that $\lim_{n \to \infty} x_n$ and $\lim_{n \to \infty} y_n$ exist.

   (b) If the two limits exist, show that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$.

5. [10 marks] Suppose that f is a function such that $f'(x)$ and $f''(x)$ exist for any real number x. Define that g is a function with $g(0) = f'(0)$ and $g(x) = f(x)/x$ for $x \ne 0$. Given that $g(x)$ is continuous for any real number x. Prove that $g'(x)$ exists for any real number x.

6. [10 marks] Let f be a differentiable function on (a, b) and continuous function on $[a, b]$, with $f(a) < 0$ and $f(b) > 0$. Given that $f(x)$ have **exactly two** solutions in $[a, b]$ for $f(x) = 0$. Prove that there exists $x_0 \in (a, b)$ satisfying $f(x_0) = 0$ and $f'(x_0) = 0$.

7. [10 marks] If functions f and g are continuous on the interval $[-1, 1]$ and differentiable on $(-1, 1)$, and they satisfy $f(-1) = f(1)$ and $g(-1) = g(1)$, prove that there exists some $\xi \in (-1, 1)$ such that

$$f'(\xi) = g'(\xi)f(\xi).$$

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