MA1301 Assignment 1

1(a). $\int e^{1+x^{-2}}(x^{-3}dx) = \int e^{1+x^{-2}} d(-\frac{1}{2}x^{-2})$

let $u = -\frac{1}{2}x^{-2}$, $1+x^{-2} = 1-2u$

原式 $= \int e^{1-2u} du = -\frac{1}{2}e^{1-2u} + C$

$\therefore \int \frac{e^{1+\frac{1}{x^2}}}{x^3} dx = -\frac{1}{2}e^{1+x^{-2}} + C$

1(b). $\int x^{11}\sqrt{1+x^4}\,dx = \int \sqrt{1+x^4}\,x^{11}dx = \int \sqrt{1+x^4}\,\frac{1}{12}\,dx^{12}$

$= \frac{1}{12}\int \sqrt{1+x^4}\,dx^{12}$ let $u = 1+x^4$

原式 $= \frac{1}{12}\int \sqrt{u}\,d(u-1)^3 = \frac{1}{4}\int \sqrt{u}\,(u-1)^2\,du$

$= \frac{1}{4}\int u^{\frac{1}{2}}(u^2-2u+1)\,du = \frac{1}{4}\int u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}\cdot u^{\frac{3}{2}}\right) + C$

$= \frac{1}{4}\left(\frac{2}{7}(1+x^4)^{\frac{7}{2}} - \frac{4}{5}(1+x^4)^{\frac{5}{2}} + \frac{2}{3}(1+x^4)^{\frac{3}{2}}\right) + C$

1(c). $\int \sin 2x\sqrt{\cos x}\,dx = 2\int \sin x\cos^{\frac{3}{2}}x\,dx$

$= -2\int \cos^{\frac{3}{2}}x\,(-\sin x)\,dx$

$= -2\int \cos^{\frac{3}{2}}x\,d\cos x$

let $u = \cos x$ 原式 $= -2\int u^{\frac{3}{2}}du$

$= -2\left(\frac{2}{5}u^{\frac{5}{2}}\right) + C$

$= -\frac{4}{5}u^{\frac{5}{2}} + C$

$\therefore \int \sin 2x\sqrt{\cos x}\,dx = -\frac{4}{5}(\cos x)^{\frac{5}{2}} + C$

11(d). $\int_1^2 x e^{x^2-1} dx = \int_1^2 e^{x^2-1} x\, dx$

$= \frac{1}{2}\int_1^2 e^{x^2-1}\, dx^2 = \frac{1}{2}\int_1^2 e^{x^2-1}\, d(x^2-1)$

let $u = x^2-1$, 原式 $= \frac{1}{2}\int_0^3 e^u du = \frac{1}{2} e^u \Big|_0^3$

$\therefore \int_1^2 x e^{x^2-1} dx = \frac{1}{2}(e^3-1)$

11(e). $\int_1^5 \frac{\sin^2(\ln x)}{x} dx = \int_1^5 \sin^2(\ln x)\frac{1}{x} dx$

$= \int_1^5 \sin^2(\ln x)\, d\ln x$

let $\ln x = u$ 原式 $= \int_0^{\ln 5} \sin^2 u\, du$

~~let $\sin u = v$ 原式 $= \int_0^{\sin(\ln 5)} v^2 \cos u\, du$~~

原式 $= \int_0^{\ln 5} \frac{1-\cos 2u}{2} du$

$= \frac{1}{2}\left(\int_0^{\ln 5} 1\, du - \int_0^{\ln 5} \cos 2u\, du\right)$

$= \frac{1}{2}\left( u\Big|_0^{\ln 5} - \frac{\sin 2u}{2}\Big|_0^{\ln 5}\right)$

$= \frac{1}{2}\left(\ln 5 - \frac{\sin 2\ln 5}{2}\right)$

$= \frac{\ln 5}{2} - \frac{\sin(2\ln 5)}{4}$

1(f). $\int \sin^7 x\, dx = \int \sin^6 x \sin x\, dx = -\int \sin^6 x\, d\cos x$

原式 $= -\int (1-\cos^2 x)^3\, d\cos x$

let $u = \cos x$. 原式 $= -\int (1-u^2)^3 du$

$= \int (u^2-1)^3 du = \int u^6 - 3u^4 + 3u^2 - 1\, du$

$= \frac{1}{7}u^7 - \frac{3}{5}u^5 + u^3 - u + C$

$= \frac{1}{7}(\cos x)^7 - \frac{3}{5}(\cos x)^5 + \overbrace{(\cos x)^3 - \cos x}^{\cos x(\cos^2 x - 1)} + C$

$= \frac{1}{7}(\cos x)^7 - \frac{3}{5}(\cos x)^5 - \cos x(\sin x)^2 + C$

1(g). $\int \sqrt{9-16x^2}\,dx = \int \sqrt{16\left(\frac{9}{16}-x^2\right)}\,dx$

$= 4\int \sqrt{\left(\frac{9}{16}-x^2\right)}\,dx$

let $x = \frac{3}{4}\sin\theta$. 原式 $= 4\int \frac{3}{4}\cos\theta \, d\left(\frac{3}{4}\sin\theta\right)$

$= 3\int \cos\theta \, d\frac{3}{4}\sin\theta = 3\int \cos\theta \frac{3}{4}\cos\theta \, d\theta$

$= \frac{9}{4}\int \cos^2\theta \, d\theta$

原式 $= \frac{9}{4}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{9}{4}\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right) + C$

$= \frac{9}{8}\arcsin\frac{4}{3}x + \frac{x\sqrt{9-16x^2}}{2} + C$

1(h). $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

let $x = \sin\theta$. $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin^2\theta\cos\theta}\,d\sin\theta$

原式 $= \int \frac{1}{\sin^2\theta\cos\theta}\cos\theta\,d\theta = \int \frac{1}{\sin^2\theta}\,d\theta = \int \csc^2\theta\,d\theta$

$= -\cot\theta + C = -\frac{\sqrt{1-x^2}}{x} + C$

2(a). $\int x e^{-3x}\,dx = \int -\frac{x}{3}\,de^{-3x} = -\frac{1}{3}\int x\,de^{-3x}$

$= -\frac{1}{3}\left(xe^{-3x} - \int e^{-3x}\,dx\right)$

$= -\frac{1}{3}\left(xe^{-3x} - \left(-\frac{e^{-3x}}{3} + C\right)\right)$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C = C - \frac{3x+1}{9}e^{-3x}$

2(b). $\int_1^e \sqrt{x}\ln x\,dx = \int_1^e \ln x\sqrt{x}\,dx = \frac{2}{3}\int_1^e \ln x\,dx^{\frac{3}{2}}$

$= \frac{2}{3}\left(\ln x\cdot x^{\frac{3}{2}}\Big|_1^e - \int_1^e x^{\frac{3}{2}}\,d(\ln x)\right)$

$= \frac{2}{3}\left(e^{\frac{3}{2}} - \int_1^e \sqrt{x}\,dx\right) = \frac{2}{3}\left(e^{\frac{3}{2}} - \frac{2}{3}\left(e^{\frac{3}{2}}-1\right)\right)$

$= \frac{2}{3}\left(\frac{1}{3}e^{\frac{3}{2}} + \frac{2}{3}\right) = \frac{2}{9}e^{\frac{3}{2}} + \frac{4}{9}$

2(c). $\int x^2\sin x\,dx = -\int x^2\,d\cos x$

$= -\left(x^2\cos x - \int \cos x\,dx^2\right)$

$= \int \cos x\,dx^2 - x^2\cos x = 2\int x\cos x\,dx - x^2\cos x$

$= 2\int x\,d\sin x - x^2\cos x = 2\left(x\sin x - \int \sin x\,dx\right) - x^2\cos x$

$\therefore \int x^2\sin x\,dx = 2x\sin x + 2\cos x - x^2\cos x + C$

2(d). $\int x\sin^2 x\,dx = \int \frac{x(1-\cos 2x)}{2}\,dx$

$= \frac{1}{2}\left(\int x\,dx - \int x\cos 2x\,dx\right) = \frac{1}{4}x^2 + C - \frac{1}{2}\int x\cos 2x\,dx$

$\int x\cos 2x\,dx = \frac{1}{2}\int x\,d\sin 2x = \frac{1}{2}\left(x\sin 2x - \int \sin 2x\,dx\right)$

$= \frac{1}{2}\left(x\sin 2x + \frac{\cos 2x}{2} + C\right)$

$\therefore \int x\sin^2 x\,dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

2(e). $\int_1^e \left(\frac{\ln x}{x}\right)^2 dx = \int_1^e x^{-1}\,d\ln^2 x - \frac{\ln^2 x}{x}\Big|_1^e$

$= \frac{-2\ln x}{x}\Big|_1^e - \frac{\ln^2 x}{x}\Big|_1^e + 2\int_1^e \frac{1}{x}\,d\ln x$

$= \frac{-2\ln x}{x}\Big|_1^e - \frac{\ln^2 x}{x}\Big|_1^e - \frac{2}{x}\Big|_1^e$

$= \frac{-2}{e} - \frac{1}{e} - \frac{2}{e} + 2 = 2 - \frac{5}{e}$

2(f). $\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$

$= \int \cos^2 x\, d\sin x = \int 1-\sin^2 x\, d\sin x$

let $u = \sin x$, 原式 $= \int 1-u^2\, du$

$= u - \frac{1}{3}u^3 + C$ $\quad\therefore \int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$

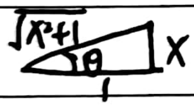
2(g). $\int e^x \sin 3x\, dx = -\frac{1}{3}(e^x \cos 3x - \int \cos 3x\, de^x)$

$= -\frac{1}{3} e^x \cos 3x + \frac{1}{3}\int \cos 3x\, de^x = -\frac{1}{3} e^x \cos 3x + \frac{1}{9}\int e^x\, d\sin 3x$

$\therefore \int e^x \sin 3x\, dx = -\frac{1}{3} e^x \cos 3x + \frac{1}{9} e^x \sin 3x - \frac{1}{9} e^x \sin 3x\, dx$

$\frac{10}{9}\int e^x \sin 3x\, dx = \frac{e^x \sin 3x - 3e^x \cos 3x}{9} + C$

$\therefore \int e^x \sin 3x\, dx = \frac{e^x \sin 3x - 3e^x \cos 3x}{10} + C$

2(h). 

$\tan^{-1} x = \theta \quad \tan\theta = x$

$\therefore \int \tan^{-1} x\, dx = \int \theta\, d\tan\theta = \int \theta \frac{1}{\cos^2\theta}\, d\theta$

$\int \theta\, d\tan\theta = \theta\tan\theta - \int \tan\theta\, d\theta = \theta\tan\theta - \int \frac{1}{\cos\theta}\sin\theta\, d\theta$

$\int \theta\, d\tan\theta = \theta\tan\theta + \int \frac{1}{\cos\theta}\, d\cos\theta = \theta\tan\theta + \ln|\cos\theta| + C$

$\cos\theta = \frac{1}{\sqrt{x^2+1}} \quad \therefore \int \tan^{-1} x\, dx = x\tan^{-1} x + \ln\frac{1}{\sqrt{x^2+1}} + C$

$\int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C$

3(a). $\int e^{2x}\sin(2e^x+1)\,dx = -\int e^{2x}\frac{d\cos(2e^x+1)}{2e^x} = -\frac{1}{2}\int e^x\,d\cos(2e^x+1)$

$= -\frac{1}{2}\left(e^x\cos(2e^x+1) - \int\cos(2e^x+1)e^x\,dx\right)$

$= -\frac{1}{2}e^x\cos(2e^x+1) + \frac{1}{2}\int\cos(2e^x+1)e^x\,dx$

$= -\frac{1}{2}e^x\cos(2e^x+1) + \frac{1}{4}\int 1\,d\sin(2e^x+1)$

$\therefore \int e^{2x}\sin(2e^x+1) = -\frac{1}{2}e^x\cos(2e^x+1) + \frac{1}{4}\sin(2e^x+1)+C$

3(b). $\int_0^1 \sin(2\sqrt{x})\,dx$ let $u=2\sqrt{x}$

$\int_0^1 \sin(2\sqrt{x})\,dx = \frac{1}{2}\int_0^2 u\sin u\,du = -\frac{1}{2}\int_0^2 u\,d\cos u = -\frac{1}{2}\left(u\cos u\Big|_0^2 - \int_0^2\cos u\,du\right)$

$= -\frac{1}{2}u\cos u\Big|_0^2 + \frac{1}{2}\int_0^2\cos u\,du = -\frac{1}{2}(2\cos 2-0) + \frac{1}{2}\sin 2$

$\therefore \int_0^1 \sin(2\sqrt{x})\,dx = \frac{1}{2}\sin 2 - \cos 2$

3(c). $\int_0^1 \ln(1+x^{\frac{1}{3}})\,dx$ let $u = 1+x^{\frac{1}{3}}$

原式 $= 3\int_1^2 \ln u\,(u-1)^2\,du = \int_1^2 \ln u\,d(u-1)^3$

$= \ln u\,(u-1)^3\Big|_1^2 - \int_1^2 (u-1)^3\,d\ln u$

$= \ln u\,(u-1)^3\Big|_1^2 - \int_1^2 \frac{(u-1)^3}{u}\,du$

$= \ln 2 - \left(\frac{8}{3} - 6 + 6 - \ln 2 - \frac{1}{3} + \frac{3}{2} - 3\right)$

$= 2\ln 2 - \frac{7}{3} + \frac{3}{2} = 2\ln 2 - \frac{5}{6} = \frac{12\ln 2 - 5}{6}$

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3(d). $\int \cos(\ln x)\,dx$ let $\ln x = u$

$\int \cos u\, de^u = e^u \cos u - \int e^u d\cos u = e^u \cos u + \int e^u \sin u\, du$

$= e^u \cos u + \int \sin u\, de^u = e^u \cos u + e^u \sin u - \int e^u d\sin u$

$= e^u(\cos u + \sin u) - \int e^u \cos u\, du = \int \cos u\, de^u$

$\therefore \int \cos u\, de^u = \frac{1}{2} e^u (\sin u + \cos u)$

$\int \cos(\ln x)\,dx = \frac{x}{2}(\sin(\ln x) + \cos(\ln x)) + C$

3(e). $\int \sin(2x)\ln(\sin x)\,dx$ $\int \ln(\sin x)\sin(2x)\,dx = -\frac{1}{2}\int \ln(\sin x)\, d\cos 2x$

$= -\frac{1}{2}\ln(\sin x)\cos 2x + \underline{\frac{1}{2}\int \cos 2x\, d\ln(\sin x)}$

$= -\frac{1}{2}\ln(\sin x)\cos 2x + \frac{1}{2}\int \cos 2x \frac{\cos x}{\sin x}\,dx$

$= -\frac{1}{2}\ln(\sin x)\cos 2x + \frac{1}{2}\int \frac{1}{\sin x} - 2\sin x\, d\sin x$

$\therefore \int \sin(2x)\ln(\sin x)\,dx = -\frac{1}{2}\ln(\sin x)\cos 2x + \frac{1}{2}\ln(\sin x) - \frac{1}{2}\sin^2 x + C$

$= \ln(\sin x)(\sin x)^2 - \frac{(\sin x)^2}{2} + C$

3(f). $\int (x+1)\ln(x+3)\,dx = \frac{1}{2}\int \ln(x+3)\, d(x+1)^2$

$= \frac{1}{2}(x+1)^2 \ln(x+3) - \frac{1}{2}\int (x+1)^2\, d\ln(x+3)$

$\cancel{= \frac{1}{2}(x+1)^2\ln(x+3) - \frac{1}{6}\int \frac{1}{x+3}\, d(x+1)^3}$

$\cancel{= \frac{1}{2}(x+1)^2\ln(x+3) - \frac{1}{6}\frac{(x+1)^3}{x+3} + \frac{1}{6}\int (x+1)^3\, d\frac{1}{x+3}}$

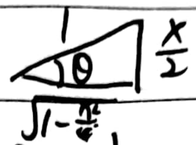
$= \frac{1}{2}(x+1)^3\ln(x+3) - \frac{1}{2}\int \frac{(x+3)^2 - 6x - 12 + 4}{x+3}\, d(x+3)$

$= \frac{1}{2}(x+1)^3\ln(x+3) - \frac{1}{2}\int (x+3) + \frac{4}{x+3} - 4\; d(x+3)$

$= \frac{1}{2}(x+1)^3\ln(x+3) - \frac{1}{4}(x+3)^2 - 2\ln(x+3) + 2(x+3) + C$

$= \frac{2(x-1)(x+3)\ln(x+3) - x^2 + 2x}{4} + C$

3(g). $\int x^2\sqrt{4-x^2}\,dx$ let $x=2\sin\theta$




原式 $= \int 4\sin^2\theta\cdot 2\cos\theta\,dx = 8\int \sin^2\theta\cos\theta\,dx$

$= 16\int \sin^2\theta\cos^2\theta\,d\theta = 4\int (1-\cos^2 2\theta)\,d\theta = 2\int \sin^2 2\theta\,d2\theta$

$= \frac{1}{2}\int (1-\cos 4\theta)\,d4\theta = \frac{1}{2}(4\theta-\sin 4\theta)+C = 2\theta-\frac{1}{2}\sin 4\theta+C$

$\theta = \arcsin\frac{x}{2}$

$\therefore \int x^2\sqrt{4-x^2}\,dx = 2\arcsin\frac{x}{2} - \frac{1}{2}\sin(4\arcsin\frac{x}{2})+C$

$\cancel{3(h).\ \int x^3\sin(4+x^2)\,dx = \int \sin(4+x^2)x^3\,dx = \frac{1}{4}\int \sin(4+x^2)\,dx^4}$

$\cancel{= \frac{1}{4}x^4\sin(4+x^2) - \frac{1}{4}\int x^4\,d\sin(4+x^2)}$

$\cancel{= \frac{1}{4}x^4\sin(4+x^2) - \frac{1}{12}\int \cos(4+x^2)\,dx^6}$

$\cancel{= \frac{1}{4}x^4\sin(4+x^2) - \frac{1}{12}x^6\cos(4+x^2) + \frac{1}{12}\int x^6\,d\cos(4+x^2)}$

3(h). $\int x^3\sin(4+x^2)\,dx = -\frac{1}{2}\int x^2\,d\cos(4+x^2)$

$= -\frac{1}{2}\cos(4+x^2)x^2 + \frac{1}{2}\int \cos(4+x^2)\,dx^2$

$= -\frac{x^2}{2}\cos(4+x^2) + \frac{1}{2}\int \cos(4+x^2)\,d(x^2+4)$

$= -\frac{x^2}{2}\cos(4+x^2) + \frac{1}{2}\sin(4+x^2)+C$

4. Easy to see $f(x)=\sqrt{1+x^2}$ is an even function

so the condition $\Rightarrow 1 \le \int_0^1 \sqrt{1+x^2}\,dx \le \sqrt{2}$

when $x \in (0,1)$. $1 \le \sqrt{1+x^2} \le \sqrt{2}$

$\therefore \int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{2}\,dx$

$\therefore 1 \le \int_0^1 \sqrt{1+x^2}\,dx \le \sqrt{2}$ so $2 \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le 2\sqrt{2}$

5. $\int_a^b x f'(x)\,dx = \int_a^b x\,df(x)$

$= x f(x) \Big|_{x=a}^{x=b} - \int_a^b f(x)\,dx$

$= b - a - 0 = b - a$

So the value of $\int_a^b x f'(x)\,dx$ is $b-a$

6. $\int_0^a x^3 f(x^2)\,dx = \int_0^a x^3 f(x^2) \frac{1}{2x}\,dx^2$

$= \frac{1}{2}\int_0^a x^2 f(x^2)\,dx^2$ let $u=x^2$. Since $a>0$,

原式 $= \frac{1}{2}\int_0^{a^2} u f(u)\,du = \frac{1}{2}\int_0^{a^2} x f(x)\,dx$

$\therefore \int_0^a x^3 f(x^2)\,dx = \frac{1}{2}\int_0^{a^2} x f(x)\,dx$

so $\int_0^a x^3 f(x^2)\,dx - \frac{1}{2}\int_0^{a^2} x f(x)\,dx = 0$

7. (a). easy to see $f(x) = \cos^n x$ is even

$I_n = \frac{n-1}{n} I_{n-2} \Longrightarrow \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx$

$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\,d\sin x$

$= \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,d\cos^{n-1} x$

$= \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x\,dx$

$= \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} (1-\cos^2 x) \cos^{n-2} x\,dx$

$= (0-0) + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$\therefore (n-1+1) \int_0^{\frac{\pi}{2}} \cos^n x\,dx = (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx$

$\therefore \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \cos^{n-2} dx$

即 $I_n = \frac{n-1}{n} I_{n-2}$

7(b) $I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} I_1 = \frac{8}{15} I_1$

$I_1 = 2 \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= 2 \sin x \Big|_0^{\frac{\pi}{2}} = 2$

$\therefore I_5 = 2 \times \frac{8}{15} = \frac{16}{15}$