Problem Set 2 - Refer to title . Solution also available, click on my page and find out

Discrete Mathematics (City University of Hong Kong)

Problem Set Chapter 2

1. Let $M = \{r, s, t\}$. State whether each of the four statements is correct or incorrect. If a statement is incorrect, explain why.

   (a) $r \in M$, (b) $r \subset M$, (c) $\{r\} \in M$, (d) $\{r\} \subset M$.

2. (a) Write a negation for the following statement: $\forall$ sets S, $\exists$ a set T such that $S \cap T = \varnothing$. Which is true, the statement or its negation? Explain.

   (b) Write a negation for the following statement: $\exists$ a set S such that $\forall$ sets T, $S \cup T = \varnothing$. Which is true, the statement or its negation? Explain.

3. (a) Let A, B, C be sets. If $A \in B$ and $B \in C$, is it possible that $A \in C$? Is it always true that $A \in C$? Give examples to support your assertions.

   (b) List all subsets of A where $A = \{\{\phi, 2\}, \{2\}\}$.

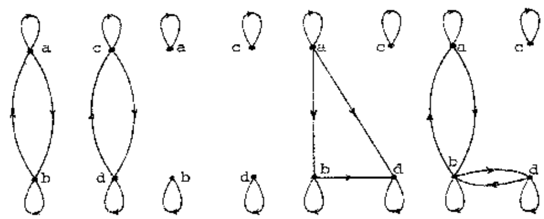
4. Given $A - B = A \cap B^c$. we define the operation Set Difference, denoted by "$-$".

Determine which of the following statements is true. If a statement is true, prove it using logical inference. Find a counter example for each statement that is false. Assume all sets are subsets of a universal set U.

(a) For all sets A and B, $(A - B) \cap (A \cap B) = \varnothing$.
(b) For all sets A and B, if $A \subset B$ then $A \cap B^c = \varnothing$.
(c) For all sets A and B, if $B \subset A^c$ then $A \cap B = \varnothing$.
(d) For all sets A, B and C, if $B \cap C \subset A$, then $(A - B) \cap (A - C) = \varnothing$.
(e) For all sets A, B and C, if $C \subset B - A$, then $A \cap C = \varnothing$.
(f) For all sets A, B and C, if $B \cap C \subset A$, then $(C - A) \cap (B - A) = \varnothing$.

5. Each of the following digraphs depicts a binary relation on $A=\{a,b,c,d\}$.

(a) Which of the following are reflexive?
(b) Which of the following are symmetric?
(c) Which of the following are antisymmetric?
(d) Which of the following are transitive?



(i) (ii) (iii) (iv)

6. On the set N, define a binary relation R such that $(a,b)\in R$ iff a divides b^2.

(a) Is R reflexive?
(b) Is R symmetric?
(c) Is R antisymmetric?
(d) Is R transitive?

7. Let $f(x)=\dfrac{2x+1}{x-2}$ and $g(x)=\dfrac{x-5}{3x+1}$ where x is real.

(a) Find $g\circ f$ and $f\circ g$.

(b) Find $(g\circ f)^{-1}$ and $f^{-1}\circ g^{-1}$. Are they equal?

8. Determine whether each of the following functions is a bijection from R to R.

(a) $f(x)=x^2+1$,
(b) $f(x)=x^3$,
(c) $f(x)=(x^2+1)/(x^2+2)$.

9. Let R be the binary relation on I, the set of integers, defined by $(x,y)\in R$ iff x^2-y^2 is divisible by 5. Is R an equivalence relation? If so, find all distinct equivalence classes.

10. Let f be a function from the set A to the set B. Let S and T be subsets of A. We define the set $f(S)$ in B as

$$f(s)=\{b\in B\,|\,b=f(a)\text{ and }a\in S\}.$$

Now, given that $A = \{-3, -2, -1, 0, 1, 2, 3\}$, $B = \{0, 1, 4, 9\}$, $S = \{0, 1\}$, $T = \{-1\}$ and $f(x) = x^2$.

(a) Find the sets $f(S \cup T)$ and $f(S) \cup f(T)$. Are these two sets equal?

(b) Find the sets $f(S \cap T)$ and $f(S) \cap f(T)$. Are these two sets equal?

11. (a) In how many ways can five people be seated at a round table, so that two of them are never to be separated?

(b) Eight people are to be seated at a round table. Suppose two persons refuse to sit next to each other. How many arrangements are possible?

12. How many positive integers less than 1000

(a) have exactly three decimal digits?

(b) have an odd number of decimal digits?

(c) have at least one decimal digit equal to 9?

(d) have no odd decimal digits?

(e) have two consecutive decimal digits equal to 5?

(f) are palindromes (that is, read the same forward and backward)?

13. In how many arrangements of the letters in the word *luscious* does the *l* immediately precede the *c*? In how many arrangements does a *s* immediately precede the *c*?

14. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

where $x_i, i = 1, 2, 3, 4, 5$ is a nonnegative integer such that

(a) $x_1 \geq 1$?

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

(c) $0 \leq x_1 \leq 10$?

15. The number 42 has the prime factorization $2 \cdot 3 \cdot 7$. Thus 42 can be written in four ways as a product of two positive integer factors: $1 \cdot 42, 6 \cdot 7, 14 \cdot 3$, and $2 \cdot 21$.

(a) How many distinct ways can the number 60 be written as a product of two positive integer factors?

(b) If $n = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5$, where p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?

16. (a) How many reflexive relations are there on a set with n elements?

(b) How many symmetric relations are there on a set with n elements?

(c) How many antisymmetric relations are there on a set with n elements?

17. (a) For $n \geq 1$, prove that

$$\sum_{i=2}^{n+1} \binom{i}{2} = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}.$$

(b) Let m be any nonnegative integer. For $n \geq 0$, prove that

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}.$$

18. (a) For $n > 1$ such that

$$(a + x)^n = 3b + 6bx + 5bx^2 + \ldots,$$

find the values of a, b and n.

(b) Find the term independent of x in the expansion of $(x - \frac{2}{x^3})^8$.

19. Prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n},$$

for all integers $n \geq 0$.

20. How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2 and x_3 are nonnegative integers less than 6?

21. How many integers from 1 through 999,999 contain each of the digits 1, 2, and 3 at least once?

22. (a) How many permutations of *abcde* are there in which the first character is *a*, *b*, or *c* and the last character is *c*, *d*, or *e*?

(b) In how many ways can the 5 digits 1, 2, …, 5 be arranged as strings of 5 digits so that none of the patterns ‘12’, ‘34’ or ‘45’ occur in the strings?

23. (a) With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking any combination of one- and two-stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let C_n be the number of different ways to climb the staircase. Find a recurrence relation for $C_1, C_2, C_3, \ldots$.

(b) What are the initial conditions?

(c) How many ways can you climb a staircase of eight stairs?

24. (a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0’s.

(b) What are the initial conditions?

(c) How many bit strings of length seven contain two consecutive 0’s?