MA1300 Hand in Assignment #4

1① $f(x) = \sin 2x - 2\cos x$, for $0 \le x \le 2\pi$

$f'(x) = 2\cos 2x + 2\sin x = 2(1-2\sin^2 x) + 2\sin x = 2(-2\sin^2 x + \sin x + 1) = 2(-\sin x + 1)(2\sin x + 1)$

(a) when $0 \le x \le 2\pi$, $\sin x \in [-1, 1]$

when f is increasing $f'(x) > 0$, $(0, \frac{7}{6}\pi) \cup (\frac{11}{6}\pi, 2\pi)$

when f is decreasing $f'(x) < 0$, $(\frac{7}{6}\pi, \frac{11}{6}\pi)$

(b) local maximum $= f(\frac{7}{6}\pi) = \sin\frac{\pi}{3} + 2\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \sqrt{3} = \frac{3}{2}\sqrt{3}$

local minimum $= f(\frac{11}{6}\pi) = \sin\frac{11}{3}\pi - 2\cos\frac{11}{6}\pi = -\frac{\sqrt{3}}{2} - \sqrt{3} = -\frac{3}{2}\sqrt{3}$

(c). $f''(x) = -4\sin 2x + 2\cos x = -8\sin x\cos x + 2\cos x = 2\cos x(1 - 4\sin x)$

let $\sin\alpha = \frac{1}{4}$, $\alpha \in (0, \frac{\pi}{2})$ $\sin\beta = \frac{1}{4}$, $\beta \in (\frac{\pi}{2}, \pi)$

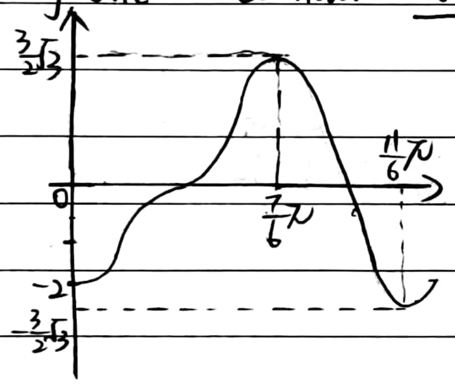
x	$(0,\alpha)$	α	$(\alpha, \frac{\pi}{2})$	$\frac{\pi}{2}$	$(\frac{\pi}{2}, \beta)$	β	$(\beta, \frac{3}{2}\pi)$	$\frac{3}{2}\pi$	$(\frac{3}{2}\pi, 2\pi)$
$f''(x)$	+	0	−	0	+	0	−	0	+

∴ Concave upward : $(0, \alpha) \cup (\frac{\pi}{2}, \beta) \cup (\frac{3}{2}\pi, 2\pi)$

Concave downward : $(\alpha, \frac{\pi}{2}) \cup (\beta, \frac{3}{2}\pi)$ inflection points

$x = \alpha, \frac{\pi}{2}, \beta, \frac{3}{2}\pi$

(d).



1.② $f(x) = \frac{x^2-4}{x^2-2x} = \frac{(x-2)(x+2)}{(x-2)x}$ 又 ∵ $x \ne 2$ ∴ $f(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$ $(x \ne 0, 2)$

(a). $f'(x) = -\frac{2}{x^2} < 0$ ∴ $f(x)$ is decreasing on $(-\infty, 0) \cup (0, 2) \cup (2, +\infty)$

(b). f has no local maximum nor local minimum

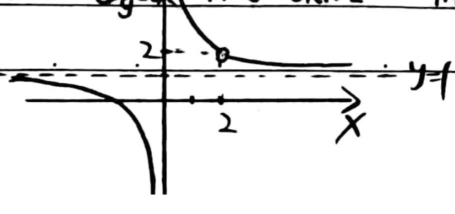
(c). $f''(x) = 4x^{-3}$

∴ Concave downward : $(-\infty, 0)$

Concave upward : $(0, 2) \cup (2, +\infty)$

There does not exist inflection points

(d).



2. proof: let $f(x) = 2\sqrt{x} + \frac{1}{x} - 3$

$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{x^2-\sqrt{x}}{x^{\frac{5}{2}}}$

let $g(x) = x^2 - \sqrt{x}$ $\quad g'(x) = 2x - \frac{1}{2\sqrt{x}} = \frac{4x^{\frac{3}{2}}-1}{2\sqrt{x}}$

又 $\because$ $4x^{\frac{3}{2}}$ is increasing on $(1, +\infty)$ $\therefore$ $4x^{\frac{3}{2}} > 4(1)^{\frac{3}{2}}$

$\therefore \begin{cases} 4x^{\frac{3}{2}} - 1 > 0 \\ 2\sqrt{x} > 0 \end{cases} \Rightarrow g'(x) > 0$ for $\forall x > 1$

$\because g'(x) > 0$ $\quad\therefore g(x)$ is increasing on $(1, +\infty)$

$g(x) > (1)^2 - \sqrt{1} = 0$ $\quad\therefore x^2 - \sqrt{x} > 0$ for $\forall x > 1$

$\therefore f'(x) = \frac{x^2-\sqrt{x}}{x^{\frac{5}{2}}} > 0$ $\quad\therefore f(x)$ is increasing on $(1, +\infty)$

$\therefore f(x) > f(1) = 2+1-3 = 0$

$\therefore$ for $\forall x > 1$, $f(x) > 0$ that's to say $2\sqrt{x} > 3 - \frac{1}{x}$

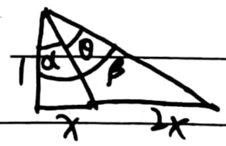
3. suppose the surface area is x cm^2: and length $a = \sqrt{\frac{x}{6}}$

$V = \left(\frac{x}{6}\right)^{\frac{3}{2}}$ take the derivative of time on both sides.

$\frac{dv}{dt} = \frac{3}{2} \times \frac{1}{6} \times \left(\frac{x}{6}\right)^{\frac{1}{2}} \frac{dx}{dt}$ 又 $\because$ $\frac{dv}{dt} = 10\ cm^3/min$ $\quad x = 30cm$

$\Rightarrow \frac{dx}{dt} = \frac{4}{3}\ cm^2/min$

$\therefore$ the surface area is increasing at $\frac{4}{3}\ cm^2/min$

4. 

we can get $\tan\alpha = \frac{x}{1} = x$ $\quad \tan\beta = \frac{3x}{1} = 3x$

$\tan(\theta) = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \frac{2x}{3x^2+1} = \frac{2}{3x + \frac{1}{x}}$

when $\left(\frac{2}{3x+\frac{1}{x}}\right)' = 0$ $\quad x = \frac{\sqrt{3}}{3}$ so $\tan\theta \le \frac{\sqrt{3}}{3}$

$\because \tan\theta < \frac{\sqrt{3}}{3}$ $\left(\theta \in (0, \frac{\pi}{2})\right)$ $\quad\therefore \theta_{max} = \frac{\pi}{6}$

$\therefore$ the maximum value of the observer's angle of sight is $\frac{\pi}{6}$ (30°)

5. By MVT, $\exists\, p \in (0, a)$, $f'(p) = \frac{f(a)-f(0)}{a-0}$

$\exists\, q \in (a, 2a)$, $f'(q) = \frac{f(2a)-f(a)}{2a-a}$

又 $\because p \in (0,a)$, $q \in (a, 2a)$

$\therefore q > p$ 且 $f'(x)$ is increasing $\therefore f'(q) > f'(p)$

$\therefore \frac{f(a)-f(0)}{a-0} \leq \frac{f(2a)-f(a)}{2a-a}$ $\quad \frac{f(a)}{a} \leq \frac{f(2a)-f(a)}{a}$

$\because a > 0$ $\therefore f(a) \leq f(2a) - f(a)$ that's to say $f(2a) \geqslant 2f(a)$

6. By MVT, $\exists\, c \in (0, \frac{a+b}{2})$ $\quad f'(c) = \frac{f(\frac{a+b}{2}) - f(a)}{\frac{a+b}{2} - a}$

$d \in (\frac{a+b}{2}, b)$ $\quad f'(d) = \frac{f(b) - f(\frac{a+b}{2})}{b - \frac{a+b}{2}}$

① when $f(x) = C$ (C is a constant)

$f'(x) = 0$ $\quad f''(x) = 0$ $\therefore$ there exists some $\xi \in (a,b)$ s.t. $f''(\xi) = 0$

② when $f(x)$ is not a constant

$\because f(a) = f(b) = f(\frac{a+b}{2})$ $\quad \therefore f'(c) = f'(d) = 0$

By MVT, $\exists\, \xi \in (c, d)$ s.t. $f''(\xi) = \frac{f'(c) - f'(d)}{c-d} = 0$

$\therefore$ there exists some $\xi \in (a,b)$ s.t. $f''(\xi) = 0$

7. let $\log_b(a) = C$

so we have $b^c = a$

$\because b^c = a$ $\quad \therefore \ln(b^c) = \ln(a)$

$\therefore c \ln(b) = \ln(a)$

$\therefore c = \frac{\ln a}{\ln b} = \log_b(a)$