1. Let $S(x, y, z)$ denote the predicate "$x+y=z$', $P(x, y, z)$ denote "$x*y=z$", and "$L(x, y)$ denote $x<y$. Let the universe of discourse be the set of all natural numbers. Express the following assertions using the predicates above.
   (a) For every x and y, there exists a z such that $x+y=z$.
   (b) No x is less than 0.
   (c) For all x, $x+0=x$.
   (d) There exists an x such that $x*y=y$, unless $x<y$.

2. Of 1000 applicants for a mountain climbing trip in the Himalayas, 450 get altitude sickness, 622 are not in good enough shape, and 30 have allergies. An applicant qualifies if and only if this applicant does not get altitude sickness, is in good shape, and does not have allergies. If there are 111 applicants who get altitude sickness and are not in good enough shape, 14 who get altitude sickness and have allergies, 18 who are not in good enough shape and have allergies, and 9 who get altitude sickness, are not in good enough shape, and have allergies, how many applicants qualify?

3. Let $P(x)$ be a predicate with universe of discourse $\{a, b, c\}$, The quantifier $\exists!$ is used to assert that there is a unique element of the universe of discourse which makes a predicate true. Now express $\exists! x P(x)$ using only the operators $\wedge$, $\sim$ and $\vee$.

4. List all subsets of the following sets:
   (a) $\{\{1,\{2,3\}\}\}$.

5. Suppose that A is a nonempty set. Let R be a relation on A. Show that the relation $S=\{(a, b): \exists c \in A$ such that $(a, c),(c, b) \in R\}$ is an equivalence relation on A if R is an equivalence relation on A.

6.
   (a) Define the relation R on the set I by $(a, b) \in R$ if $a-b=2n$ where $n \geq 0$. Verify that R defines a partial order for I.
   (b) Let S be the set of all points (x, y) in R^2 with $y \leq 0$. Define an ordering by $(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1=x_2$ and $y_1 \leq y_2$. Show that this is a partial ordering of S.

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8. Let A, B, C be sets. Show that
   (a) if $C \subset A$ and $C \subset B$, then $C \subset(A \cap B)$.

(b) if $A \subset C$ and $B \subset C$, then $(A \cup B) \subset C$.

9. How many strings of four decimal digits
   (a) do not contain the same digit twice?
   (b) end with an even digit?
   (c) have exactly three digits that are 9s?

Prove that for all integers $n \geq 1$,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

(a) $f(S \cup T) = f(S) \cup f(T)$,

(b) $f(S \cap T) \subseteq f(S) \cap f(T)$; give an example to show that the inclusion may be proper,

(c) $f(A) \setminus f(S) \subseteq f(S^c)$; give an example to show that the inclusion may be proper.

(d) Give counterexamples to show that neither $f(S^c) \subseteq \big(f(S)\big)^c$ nor $\big(f(S)\big)^c \subseteq f(S^c)$.

2. Let f be a function (not necessarily bijective) from the set A to the set B. Let S be a subset B. We define the inverse image of S to be the subset of A containing all preimages of all elements of S. We denote the inverse image of S by $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$. Show that

   (0) this introduced notation is consistent with the old notation: if f is bijective and the inverse function is denoted by $g = f^{-1}$, then the inverse image $f^{-1}(S)$ as defined above coincides with the image of the inverse function $g(S)$,

   (a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$,

   (b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$,

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_1, x_2, x_3, x_4, x_5 are natural numbers such that

(a) $x_1 \geq 1$?

(b) $x_2, x_3, x_4, x_5 \geq 2$?

(c) $3 \leq x_1 \leq 10$?

How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

How many strings of six lowercase letters from the English alphabet contain

(a) the letter a?

(b) the letters a or b?

(c) the letters a and b in consecutive positions with a preceding b, and all the letters distinct?

(d) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

The logical operator NOR, written as $\downarrow$, is defined by $p \downarrow q \equiv \sim (p \vee q)$.

Using the NOR operator only, rewrite

(a) $\sim p$,

(b) $p \vee q$,

(c) $p \wedge q$,

(d) $p \to q$,

(e) $p \leftrightarrow q$.

For the following assertions, establish those which are true and find interpretations for P and Q which provide counterexamples for those which are false.

(a) $[\exists x P(x) \to \forall x Q(x)] \to \forall x [P(x) \to Q(x)]$

(b) $\forall x [P(x) \to Q(x)] \to [\exists x P(x) \to \forall x Q(x)]$.

(a) Find a recurrence relation with initial conditions for the number of bit strings of length n that contain at least two consecutive 0s.

(b) Find the number of bit strings of length five with at least two consecutive 0s.

(c) How many bit strings of length five contain two consecutive 0s <u>and</u> two consecutive 1s?

(d) Using the principle of inclusion-exclusion or otherwise, find the number of bit strings of length five with at least two consecutive 0s <u>or</u> at least two consecutive 1s.

Give inductive definitions for the following sets:

(a) The set of unsigned integers in binary form without leading zeros.

(b) The set of all real numbers with terminating fractional parts in decimal form. (Leading zeros are permitted here.)

(c) The set of all even natural numbers in binary form without leading zeros.