Suggested Solution of 2023 Mathematics Competition

(1) Use the definition of derivative to determine if the following function $f(x)$ is differentiable at $x = 0$.

$$f(x) = \tan|x|.$$

Solution

(1)

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(0+x) - f(0)}{x} = \lim_{x \to 0^+} \frac{\tan x - \tan 0}{x} = \lim_{x \to 0^+} \frac{\tan x}{x} = 1,$$

$$f'_-(0) = \lim_{x \to 0^+} \frac{f(0-x) - f(0)}{x} = \lim_{x \to 0^+} \frac{-\tan x - \tan 0}{x} = \lim_{x \to 0^+} \frac{-\tan x}{x} = -1.$$

From $f'_+(0) = 1 \neq -1 = f'_-(0)$, $f(x)$ is not differentiable at $x = 0$.

(2) Calculate the following quantities

$$\lim_{n \to +\infty} \sqrt{\tfrac{1}{n^2} + \tfrac{1}{n^3}} + \sqrt{\tfrac{1}{n^2} + \tfrac{2}{n^3}} + \cdots + \sqrt{\tfrac{1}{n^2} + \tfrac{n}{n^3}}$$

Solution

(2) $I = \lim_{n \to +\infty} \sqrt{\tfrac{1}{n^2} + \tfrac{1}{n^3}} + \sqrt{\tfrac{1}{n^2} + \tfrac{2}{n^3}} + \cdots + \sqrt{\tfrac{1}{n^2} + \tfrac{n}{n^3}}$

$$I = \lim_{n \to +\infty} \frac{1}{n}\left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \cdots + \sqrt{1 + \frac{n}{n}}\right) = \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(2\sqrt{2} - 1)$$

(3) $\lim_{x \to +\infty} \frac{\int_0^x (arctant)^2 dt}{\sqrt{x^2+1}}$

Solution

(3) $I = \lim_{x \to +\infty} \frac{\int_0^x (arctant)^2 dt}{\sqrt{x^2+1}}$, from Hospital's law, we can get

$$I = \lim_{x \to +\infty} \frac{(\int_0^x (arctant)^2 dt)'}{(\sqrt{x^2+1})'} = \lim_{x \to +\infty} \frac{(arctanx)^2}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x \to +\infty} (arctanx)^2 \frac{\sqrt{x^2+1}}{x} = (\frac{\pi}{2})^2 \cdot 1 = \frac{\pi^2}{4}$$

(4) Find the area of the surface obtained by rotating the following curve about the x-axis,

$$y = \sqrt{4 - x^2}, -1 \le x \le 1.$$

Solution

(4)

$$S = \int_\Omega 2\pi y dl = \int_{-1}^{1} 2\pi y\sqrt{1+(y')^2}dx = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\sqrt{1+(\frac{-x}{\sqrt{4-x^2}})^2}dx$$

$$= \int_{-1}^{1} 2\pi\sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}dx = \int_{-1}^{1} 4\pi dx = 4\pi x|_{-1}^{1} = 8\pi.$$

(5) Let L_1 be a line passing through $A(1,0,1)$ and $B(2,3,0)$, and L_2 be a line passing through $C(1,1,0)$ and $D(2,1,1)$. Use the concept of vectors to find the shortest distance between L_1 and L_2.

Solution

(5) Let L_1 be a line passing through $A(1,0,1)$ and $B(2,3,0)$, and L_2 be a line passing through $C(1,1,0)$ and $D(2,1,1)$. Use the concept of vectors to find the shortest distance between L_1 and L_2.

We know that $\vec{AB} = (1,3,-1), \vec{CD} = (1,0,1)$.

Let $\vec{n} = (x,y,z)$ satisfies $\vec{n} \perp \vec{AB}, \vec{n} \perp \vec{CD}$. Then we have

$$\begin{cases} \vec{n}\cdot\vec{AB} = 0 \\ \vec{n}\cdot\vec{CD} = 0 \end{cases} \Rightarrow \begin{cases} x+3y-z=0 \\ x+z=0 \end{cases} \Rightarrow 2x = -3y = -2z \tag{1}$$

Let $y = 2$, then $\vec{n} = (-3,2,3)$. Meanwhile, $\vec{AD} = (1,1,0)$.

$$d = |\vec{AD}|\cdot cos\langle\vec{AD},\vec{n}\rangle = \frac{|\vec{AD}\cdot\vec{n}|}{|\vec{n}|} = \frac{|-3\times1+2\times1+3\times0|}{\sqrt{(-3)^2+2^2+3^2}} = \frac{1}{\sqrt{22}}.$$

(6)Find the modulus and argument of $(\frac{1-i}{1+i}+2i)^3$.

Solution

(6)

$$(\frac{1-i}{1+i}+2i)^3 = (\frac{1-i+2i(1+i)}{1+i})^3 = [\frac{(-1+i)(1-i)}{(1+i)(1-i)}]^3 = (\frac{2i}{2})^3 = -i$$

Therefore, the modulus is 1, the argument is π (principal value).

(7) Consider the following linear system

$$\begin{cases} x_1+2x_2+3x_3 = \lambda, \\ x_1+3x_2+(5-\lambda)x_3 = \alpha+\lambda, \\ 2x_1+3x_2+5x_3 = 2\lambda. \end{cases} \tag{2}$$

Find all possible values of α and λ such that the system has infinitely many solutions and find the general form of the solutions.

Solution

(7)

$$\begin{cases} x_1 + 2x_2 + 3x_3 = \lambda, \\ x_1 + 3x_2 + (5-\lambda)x_3 = \alpha + \lambda, \\ 2x_1 + 3x_2 + 5x_3 = 2\lambda. \end{cases} \tag{3}$$

$$\begin{pmatrix} 1 & 2 & 3 & \lambda \\ 1 & 3 & 5-\lambda & \alpha+\lambda \\ 2 & 3 & 5 & 2\lambda \end{pmatrix} \xrightarrow[r_3-2r_1]{r_2-r_1} \begin{pmatrix} 1 & 2 & 3 & \lambda \\ 0 & 1 & 2-\lambda & \alpha \\ 0 & -1 & -1 & 0 \end{pmatrix} \xrightarrow{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & \lambda \\ 0 & 1 & 2-\lambda & \alpha \\ 0 & 0 & 1-\lambda & \alpha \end{pmatrix} \tag{4}$$

Therefore, $1-\lambda = \alpha = 0$. Let $x_3 = k$, we have

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1, \\ x_2 + x_3 = 0. \end{cases} \Rightarrow \begin{cases} x_1 = 1 - k, \\ x_2 = -k. \end{cases} \tag{5}$$

The general solution is $x = (x_1, x_2, x_3)^T = (1, 0, 0)^T + k(-1, -1, 1)^T, k \in \mathbb{R}$.

(8) (a) Find the convergence radius and convergence domain of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$.

(b) Find the sum function $S(x)$ of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$.

Solution

(8) (a) Find the convergence radius and convergence domain of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$.

From $\lim\limits_{n\to\infty} \frac{2^{n+1}\cdot n}{(n+1)\cdot 2^n} = 2$, we know the convergence radius of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$ is $\frac{1}{\sqrt{2}}$.

When $x = -\frac{1}{\sqrt{2}}$, $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1} = \sum\limits_{n=1}^{\infty} \frac{2^n}{n} (-\frac{1}{\sqrt{2}})^{2n+1} = -\frac{1}{\sqrt{2}} \sum\limits_{n=1}^{\infty} \frac{1}{n}$ divergence.

When $x = \frac{1}{\sqrt{2}}$, $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1} = \sum\limits_{n=1}^{\infty} \frac{2^n}{n} (\frac{1}{\sqrt{2}})^{2n+1} = \frac{1}{\sqrt{2}} \sum\limits_{n=1}^{\infty} \frac{1}{n}$ divergence.

Therefore, the convergence domain of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$ is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(b) Find the sum function $S(x)$ of series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1}$.

$$S(x) = \sum_{n=1}^{\infty} \frac{2^n}{n} x^{2n+1} = x \sum_{n=1}^{\infty} \frac{(2x^2)^n}{n} = x \cdot (-\ln(1-2x^2)) = -x\ln(1-2x^2).$$

(9) (a) Let the positive term series $\sum\limits_{n=1}^{\infty} a_n$ converge. Prove the convergence of $\sum\limits_{n=1}^{\infty} a_n^2$.

(b) Prove the convergence of $\sum\limits_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$.

Solution

(9) (a) Let the positive term series $\sum\limits_{n=1}^{\infty} a_n$ converge. Prove the convergence of $\sum\limits_{n=1}^{\infty} a_n^2$.

From the convergence of $\sum\limits_{n=1}^{\infty} a_n$, we know $\lim\limits_{n\to\infty} a_n = 0$. In other words, for any $\epsilon > 0$, there exists $N > 0$ such that $|a_n - 0| < \epsilon$ when $n > N$.

Let $\epsilon = 1$, then $0 \leq a_n < 1$ when $n > N \Rightarrow 0 \leq a_n^2 \leq a_n < 1$ when $n > N$.

From the comparison test, $\sum\limits_{n=N+1}^{\infty} a_n^2$ converge, then $\sum\limits_{n=1}^{\infty} a_n^2$ also converge.

(b) Prove the convergence of $\sum\limits_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$.

We know that $\ln(1+x) < x$ when $x > 0$, so $\sum\limits_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$ is a positive term series.

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{1 - \frac{1}{1+x}}{2x} = \frac{1}{2} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n} - \ln(1+\frac{1}{n})}{\frac{1}{n^2}} = \frac{1}{2}$$

Therefore, from the comparison test, $\sum\limits_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$ converge when $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converge.

(10) Assume that the set of vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is linearly independent, then a sufficient condition for the vector group $a_1\alpha_1 + b_1\alpha_2, a_2\alpha_2 + b_2\alpha_3, a_3\alpha_3 + b_3\alpha_4, a_4\alpha_4 + b_4\alpha_1$ to be linearly independent is $a_1a_2a_3a_4 \neq b_1b_2b_3b_4$.

Solution

(10) Let $\beta_1 = a_1\alpha_1 + b_1\alpha_2, \beta_2 = a_2\alpha_2 + b_2\alpha_3, \beta_3 = a_3\alpha_3 + b_3\alpha_4, \beta_4 = a_4\alpha_4 + b_4\alpha_1$.

Then

$$\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\end{pmatrix} = \begin{pmatrix}a_1 & b_2 & 0 & 0\\ 0 & a_2 & b_3 & 0\\ 0 & 0 & a_3 & b_4\\ b_1 & 0 & 0 & a_4\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix} \tag{6}$$

$$\begin{vmatrix}a_1 & b_2 & 0 & 0\\ 0 & a_2 & b_3 & 0\\ 0 & 0 & a_3 & b_4\\ b_1 & 0 & 0 & a_4\end{vmatrix} = a_1a_2a_3a_4 - b_1b_2b_3b_4 \neq 0. \tag{7}$$

Therefore, a sufficient condition for the vector group $a_1\alpha_1+b_1\alpha_2, a_2\alpha_2+b_2\alpha_3, a_3\alpha_3+b_3\alpha_4, a_4\alpha_4+b_4\alpha_1$ to be linearly independent is the set of vectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is linearly independent.

(11) Let $f(x)$ be a continuous function on the interval $[0,\infty)$. For any $x_0 > 0$, we can define a sequence $a_n = f(n+x_0)$ for $n = 1, 2, \cdots$ and $\lim\limits_{n\to\infty} a_n = 0$. Is $\lim\limits_{x\to\infty} f(x) = 0$ true. If yes, provide a proof; if not, provide a counterexample.

Solution

(11) False. Counterexample:

$$f(x) = \begin{cases} 2n(x-n), \text{if } x \in [n, n+\frac{1}{2n}], n = 1, 2, \cdots \\ -2n(x-n-\frac{1}{n}), \text{if } x \in (n+\frac{1}{2n}, n+\frac{1}{n}], n = 1, 2, \cdots \\ 0, \text{ otherwise.} \end{cases} \tag{8}$$

(12)

$$A = \begin{pmatrix}a & b & c\\ d & e & f\\ x & y & z\end{pmatrix}, A^{-1} = \begin{pmatrix}-1 & -2 & -1\\ -2 & 1 & 0\\ 0 & -3 & -1\end{pmatrix}, B = \begin{pmatrix}a-2b & b-3 & -c\\ d-2e & e-3f & -f\\ x-2y & y-3z & -z\end{pmatrix} \tag{9}$$

Find X such that $X + (B(A^TB^2)^{-1}A^T)^{-1} = X(A^2(B^TA)^{-1}B^T)^{-1}(A+B)$

Solution

(12)

$$|B| = \begin{vmatrix}a-2b & b-3 & -c\\ d-2e & e-3f & -f\\ x-2y & y-3z & -z\end{vmatrix} = \begin{vmatrix}a-2b & b-3+3c & -c\\ d-2e & e & -f\\ x-2y & y & -z\end{vmatrix} \tag{10}$$

$$= \begin{vmatrix} a+6c-6 & b-3+3c & -c \\ d & e & -f \\ x & y & -z \end{vmatrix} = \begin{vmatrix} a & b & -c \\ d & e & -f \\ x & y & -z \end{vmatrix} + \begin{vmatrix} 6(c-1) & 3(c-1) & 0 \\ d & e & -f \\ x & y & -z \end{vmatrix} \tag{11}$$

$$= -|A| + 3(c-1) \begin{vmatrix} 2 & 1 & 0 \\ d & e & -f \\ x & y & -z \end{vmatrix} = -|A| - 3(c-1) \begin{vmatrix} 2 & 1 & 0 \\ d & e & f \\ x & y & z \end{vmatrix} \tag{12}$$

Let A_{11}, A_{12} are the algebraic remainders of the first two elements of the first row of the matrix. $|A| = |A^{-1}|^{-1} = (-1)^{-1} = -1$. Expand the determinant by the first row, by definition of the cofactor formula we have $|B| = 1 - 3(c-1)(2A_{11} - A_{12})$.

$$A^* = |A|A^{-1} = -A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \tag{13}$$

Thus, $A_{11} = 1, A_{12} = 2$. Then $|B| = 1 \neq 0$. So B is an invertible matrix.

$$(B(A^T B^2)^{-1} A^T)^{-1} = (A^T)^{-1} A^T B^2 B^{-1} = B$$
$$(A^2 (B^T A)^{-1} B^T)^{-1} = (B^T)^{-1} B^T A (A^2)^{-1} = A^{-1}$$

So, $X + B = XA^{-1}(A+B) \Rightarrow B = XA^{-1}B \Rightarrow X = A$.

$$X = A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ -6 & 3 & 5 \end{pmatrix} \tag{14}$$