Apr. 24 9:30 – 12:30

Review (Midterm)

Friday, 3 March 2023 12:08 PM

$\lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \int_a^b f(x)\,dx$

Chapter 1-3

1. (Chapter 1) Definition of Riemann sum and definite integral.
2. (Chapter 1) FTC and Comparison properties of the integral .
3. (Chapter 2) Using integration to get Area, Volume, Average value, MVT.
4. (Chapter 3) Learn how to solve the following types of integrals:

a) $\int x\sin(2x)\cos(2x)dx$;

b) $\int \frac{1+x^2}{(x-1)^2(x^2+x+3)}dx$;

c) $\int_0^9 \frac{1}{2\sqrt{x}+1}dx$;

d) $\frac{d}{dx}\int_x^{x^2}\sin(y^2)dy$;

e) $\int_{-\pi/3}^{\pi/3}\frac{x^3\tan(x)\sin(x)}{x^2+\cos(x)}dx$. $=0$

f) $\int_0^1 |2x-1|dx$

$\int_a^b f(x)\,dx = F(b) - F(a)$ when $F'(x) = f(x)$

5. (Chapter 3) Definition of improper integral.
6. (Chapter 3) Comparison Test for improper integrals

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_0^t f(x)\,dx + \lim_{t\to-\infty}\int_t^0 f(x)\,dx$

7. Approximation

$\lim_{a\to0^+}\int_a^1 \frac{1}{x}dx = \int_0^1 \frac{1}{x}dx$

Left endpoint approximation:

$$\int_a^b f(x)dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

Right endpoint approximation:

$$\int_a^b f(x)dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

Midpoint Rule:

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

where $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$.

Trapezoidal Rule:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

where $\Delta x = \frac{b-a}{n}$.

Error bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Simpson's Rule

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)],$$

where n is even and $\Delta x = \frac{b-a}{n}$.

Error Bounds for Simpson's Rule

Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_s is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Chapter 4

1. Arc length of a curve $y = f(x)$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

2. Area of surface generated by rotating $y = f(x)$ about $y = k$ $(f(x) > k)$:

$$A = 2\pi \int_a^b (f(x) - k)\sqrt{1 + [f'(x)]^2}dx.$$

Important questions: all suggested questions in MA1301

3. Application in Phys

Hydrostatic Force and Pressure

Physical laws:

Suppose that a thin horizonal plate with A m^2 is submerged in a fluid of density ρ kg/m^3 at a depth d m below the surface of the fluid. The force F exerted by the fluid on the plate is

$$F = mg = \rho g A d.$$

The pressure P on the plate is

$$P = \frac{F}{A} = \rho g d.$$

An important principal of fluid pressure is that *at any point in a liquid the pressure is the same in all directions.*

Consider a flat plate with uniform density ρ that occupies a region $\mathfrak{R}$ of the plane. Assume that $\mathfrak{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis and beneath the graph of f, where f is a continuous function.

Then the **moment of $\mathfrak{R}$ about the y-axis** is

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i)\Delta x = \rho \int_a^b x f(x) dx.$$

The **moment of $\mathfrak{R}$ about the x-axis** is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{1}{2}[f(\bar{x}_i)]^2 \Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx.$$

The **center of mass of the plate** (or **the centroid of $\mathfrak{R}$**) is located at the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx, \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx,$$

where $A = \int_a^b f(x)\, dx$ is the area of $\mathfrak{R}$.

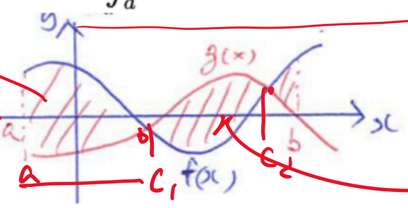
Note that the location of the centroid is independent of the density ρ.

Area and Volunms

1. (p. 4-17) Area of the region bounded by the curves $y = f(x)$ and $y = g(x)$:

$$Area = \int_a^b |f(x) - g(x)| dx.$$

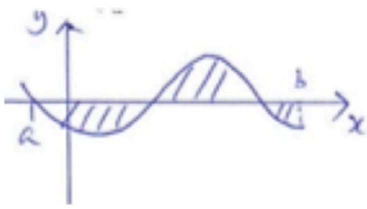
$\int_a^{c_1} f(x) - g(x)\, dx$



$\int_{c_1}^{c_2} g(x) - f(x)\, dx$

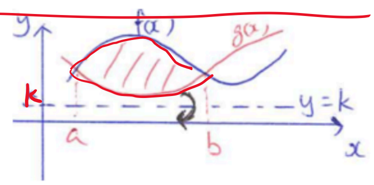
If the area enclosed by the curves $y = f(x)$ and x-axis (g(x)=0):

$$Area = \int_a^b |f(x)| dx.$$



2. (p. 18-27, 30-31) Volume of the solid formed by rotating an area between $y = f(x)$ and $y = g(x)$ about $y = k$ ($f(x) > g(x)$ and $y = k$ not cut the region):

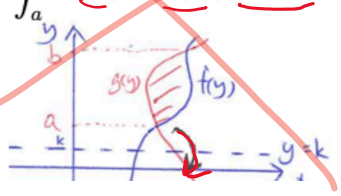
$$V_x = \pi \int_a^b (f(x) - k)^2 - (g(x) - k)^2 dx.$$



$f(x) \geq g(x)$ in $[a, b]$

(shell method, p.32) Volume of the solid formed by rotating an area between $x = f(y)$ and $x = g(y)$ about $y = k$ ($f(y) > g(y)$ and $y = k$ not cut the region):

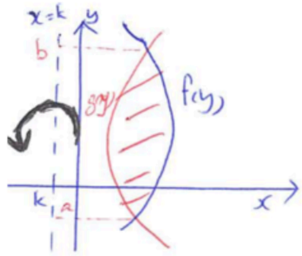
$$V_x = 2\pi \int_a^b (f(y) - g(y))|y - k| dy.$$



$f(y) \geq g(y)$ in $[a, b]$

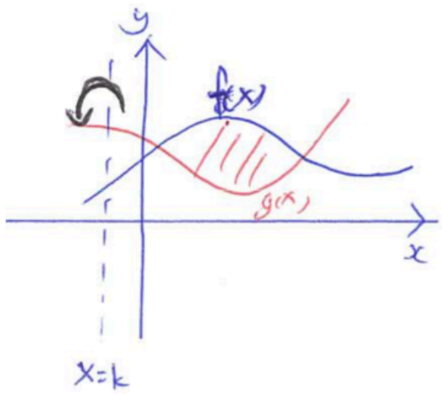
2.5. (p. 28-29) Volume of the solid formed by rotating an area between $x = f(y)$ and $x = g(y)$ about $x = k$ ($f(y) > g(y)$ and $x = k$ not cut the region):

$$V_y = \pi \int_a^b (f(y) - k)^2 - (g(y) - k)^2 dy.$$



(shell method) Volume of the solid formed by rotating an area between $y = f(x)$ and $y = g(x)$ about $y = k$ ($f(x) > g(x)$ and $x = k$ not cut the region):

$$V_y = 2\pi \int_a^b (f(x) - g(x))|x - k| dx.$$



3. (p.39-46) Arc length of a curve $y = f(x)$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

4. (p. 47-64) Area of surface generated by rotating $y = f(x)$ about $y = k$ ($f(x) > k$):

$$A = 2\pi \int_a^b (f(x) - k)\sqrt{1 + [f'(x)]^2} dx.$$

Chapter 5

1. (p. 4, 5, 6, 10, 17) Magnitude of vector $\vec{a}$:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

2. (p. 9, 10) Change a vector $\vec{a}$ to a unit vector $\vec{n}$ with same direction:

$$\vec{n} = \frac{\vec{a}}{|\vec{a}|}$$

with $|\vec{a}| \neq 0$.

3. (p.21-38) Scalar Product:

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

where $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where $0 \leq \theta \leq \pi$ is the angle between two vectors (see figure on p.23).
If $\vec{a} \perp \vec{b}$, then $\theta = \pi/2$ and

$$\vec{a} \cdot \vec{b} = 0.$$

If $\vec{a} = \vec{b}$, then

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{a} = |\vec{a}|^2.$$

4. (p. 39-62) Vector Product:

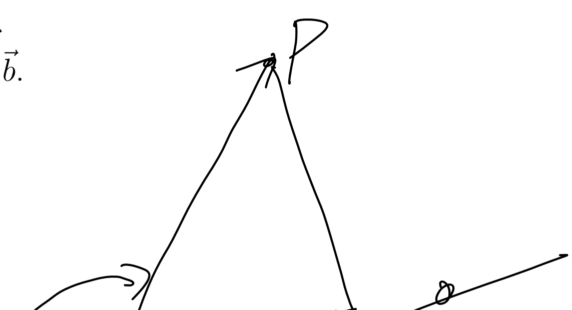
$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n},$$

where $0 \leq \theta \leq \pi$ is the angle between two vectors (see figure on p.23), and $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (see figure on p.40). **Read the list on p.41.**
If $\vec{a}$ is parallel to $\vec{b}$, then $\theta = 0$ and

$$\vec{a} \times \vec{b} = 0.$$

5. (p.31- 38) Projection vector of $\vec{a}$ onto $\vec{b}$:

$$Proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}.$$



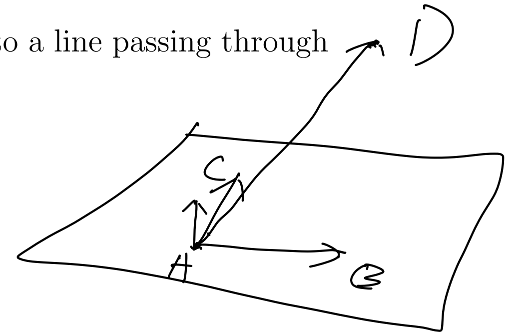
6. (p.35-36) Distance from a point P to a line passing through A and B:

$$d = \sqrt{|\vec{AP}|^2 - |proj_{\vec{AB}}\vec{AP}|^2}.$$

7. (p.48-49) Distance from a point D to a plane containing three points A, B and C:

$$d = |proj_{\vec{n}}\vec{AD}|,$$

where $\vec{n} = \vec{AB} \times \vec{AC}$.



8. (p.52-53) Distance from a line passing through A and B to a line passing through C and D:

$$d = |proj_{\vec{n}}\vec{AD}|,$$

where $\vec{n} = \vec{AB} \times \vec{CD}$.

9 (p.45-46) Area of Triangle ABC:

$$Area = |\vec{AC} \times \vec{AB}|/2.$$

10 (p.45-46) Area of Parallelogram formed by $\vec{AB}$ and $\vec{AC}$ (see figure on p.45):

$$Area = |\vec{AC} \times \vec{AB}|.$$

(p. 47) If A, B and C are collinear, then

$$Area = |\vec{AC} \times \vec{AB}| = 0.$$

11 (p.57-59) Volume of Parallelepiped formed by A, B, C and D:

$$Volume = |\vec{AB} \cdot (\vec{AC} \times \vec{AD})|. = \left|\det\begin{pmatrix}\vec{AB}\\ \vec{AC}\\ \vec{AD}\end{pmatrix}\right|$$

(p.60) If A, B, C and D are coplanar, then

$$Volume = |\vec{AB} \cdot (\vec{AC} \times \vec{AD})| = 0.$$

12 (p.63-73) Definition of Linearly Independent (Linearly dependent). How to check it in R^2 and R^3.

Important questions: 1, 5, 11, 16, 18, 19, 20, 21, 23 in MA1201 problem set, all questions in MA1301 problem set

Chapter 6

1. (p. 7) Division between complex numbers

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}.$$

2. (p. 13-16) Polar form

$$z = a+bi = r(\cos\phi + i\sin\phi)$$

with the modulus $r = \sqrt{a^2+b^2} \geq 0$ and principle value $(-\pi < \phi \leq \pi)$ of argument can be calculated by following method:

3. (p. 24-26) Multiplication and division of complex numbers in polar form

$$z_1 z_2 = r_1(\cos\phi_1 + i\sin\phi_1)r_2(\cos\phi_2 + i\sin\phi_2) = r_1 r_2(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)),$$

$$z_1/z_2 = \frac{r_1(\cos\phi_1 + i\sin\phi_1)}{r_2(\cos\phi_2 + i\sin\phi_2)} = \frac{r_1}{r_2}(\cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2)).$$

Remark: $\phi_1+\phi_2$ and $\phi_1-\phi_2$ may not be principle values.

4. (p.31) Euler Form

$$z = r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

5. (p. 33) Multiplication and division of complex numbers in Euler form

$$z_1 z_2 = r_1(e^{i\phi_1})r_2(e^{i\phi_2}) = r_1 r_2 e^{i(\phi_1+\phi_2)},$$

$$z_1/z_2 = \frac{r_1(e^{i\phi_1})}{r_2(e^{i\phi_2})} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}.$$

6 (p.35-36) Key examples

$$i = e^{i\pi/2}, \ -1 = e^{i\pi}$$

$$e^{ia} \pm e^{ib} = e^{i(a+b)/2} \left(e^{ia - i(a+b)/2} \pm e^{ib - i(a+b)/2}\right) = e^{i(a+b)/2} \left(e^{i(a-b)/2} \pm e^{-i(a-b)/2}\right)$$

$$2\cos\phi = e^{i\phi} + e^{-i\phi}, \ 2i\sin\phi = e^{i\phi} - e^{-i\phi},$$

7. Relations among three different forms, see p. 42.

8. (p.43-65) DeMoivre's Theorem (n, m are integers):

$$z^{n/m} = (r(\cos\phi + i\sin\phi)^{n/m} = (r^n(\cos(n\phi) + i\sin(n\phi)))^{1/m}$$

$$= r^{n/m}\left(\cos\frac{2k\pi + n\phi}{m} + i\sin\frac{2k\pi + n\phi}{m}\right) \text{ for } k = 0, 1, ..., m-1.$$

9. (p.57) Definition of nth root of unity $w^n = 1$. **10**. (p.66-70) Application of complex numbers:

identities of trigonometric functions: Binomial Theorem (p.67) vs. DeMoivre's Theorem

11. (p.81-83) Using complex conjugate to obtain roots of polynomials:

If $z = a + bi$ is a root of a polynomial function, the complex conjugate $\bar{z} = a - bi$ is also a root of the function.

Important questions: 3, 5, 8, 11, 12, 13 in MA1201 problem set, all questions in MA1301 problem set

Chapter 7

1. (p.5-6) Multiplication of matrices A, $m \times p$ matrix, and B, $p \times n$ matrix:

$$AB = \begin{pmatrix} a_{11} & ... & a_{1p} \\ ... & ... & ... \\ a_{i1} & ... & a_{ip} \\ ... & ... & ... \\ a_{m1} & ... & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & ... & b_{1j} & ... & b_{1n} \\ ... & ... & ... & ... & ... \\ b_{p1} & ... & b_{pj} & ... & b_{pn} \end{pmatrix} = C,\ m \times n \text{ matrix}$$

$$= \begin{pmatrix} ... & ... & ... \\ ... & C_{ij} & ... \\ ... & ... & ... \end{pmatrix}$$

where $C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj}$.

2. (p. 14). Transpose of matrix:

$$\begin{pmatrix} a_{11} & ... & a_{1p} \\ ... & ... & ... \\ a_{i1} & ... & a_{ip} \\ ... & ... & ... \\ a_{m1} & ... & a_{mp} \end{pmatrix}^T = \begin{pmatrix} a_{11} & ... & a_{i1} & ... & a_{m1} \\ ... & ... & ... & ... & .... \\ a_{1p} & ... & a_{ip} & ... & a_{mp} \end{pmatrix}$$

$$\begin{pmatrix} a_{ji} & \Leftrightarrow & a_{ij} \end{pmatrix}$$

$A = A^T$

3. (p. 15-19) Definitions of upper (lower) triangular matrix, diagonal matrix, symmetric matrix, anti-symmetric matrix and identity matrix.

$\Rightarrow -A = A^T$

~~**4**. Determinant~~ of matrix:

(2 × 2 matrix):

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

(3 × 3 matrx):

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

($n \times n$ matrix) see p.31.

Properties of determinant: $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$, $\det(A^{-1}) = 1/\det(A)$, $\det(cA) = c^n \det(A)$, where A is $n \times n$ matrix.

5. Cofactor matrix of A, see p. 30.

$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$

6. (p.37) Definition of inverse matrix.

Inverse of square matrix:

If $\det A \neq 0$, then inverse of A exists (A is non-singular, A is invertible).

$\Downarrow$

$\det A^{-1} = \dfrac{1}{\det A}$

If $\det A = 0$, then inverse of A does not exist (A is singular, A is not invertible).
Inverse of $A =$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & ... & A_{1m} \\ ... & ... & .... \\ A_{m1} & ... & A_{mm} \end{pmatrix}^{T}$$

where A_{ij} is the cofactor of the matrix A.
7. (p. 60) Definition of non-homogeneous system and homogeneous system.
8. (p. 63). A system of linear equations is consistent if the system has at least one solution (one or infinitely many).
A system of linear equations is inconsistent if the system has no solution.
9. (p. 64-66) Matrix representation of the system:

$$\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ ... \\ b_m \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & ... & a_{1n} & b_1 \\ ... & ... & ... & ... \\ a_{m1} & ... & a_{mn} & b_m \end{array}\right)$$

10. (p.71-73) Gaussian Elimination and reduced row echelon form: Definitions of pivot and elementary row operations.
11. (p.81-83) In the reduced row echelon form,

- **Case 1**: No solution (inconsistent)
  There is a row $(0\ 0...\ 0|b)$ where $b \neq 0$.
- **Case 2**: Infinitely many solutions (consistent)
  Not **Case 1** and there is a column with no pivot (corresponding to free variable).
- **Case 3**: Only one solution (consistent)
  Not **Case 1** and there is no column with no pivot.

A^{-1}

12. Three methods to solve a system of linear equations:

$A\vec{x} = b$

$\Downarrow$

$\vec{x} = A^{-1}b$

- **Method 1**: (p.91-92) By the inverse of Matrix
  Only for square coefficient matrix and **Case 3**.
- **Method 2**: (p.81-90) Gaussian Elimination
  For any case. (including general solution for Case 2)
- **Method 3**: (p.93-94) Cramer's Rule
  Only for square coefficient matrix and **Case 3**.

12. Applications of Gaussian Elimination:
a. Finding inverse (Gauss Jordan Method) see p. 96-101.
b. Checking the linear independency of vectors see p.108-114.

Important questions: 1, 4, 5, 9, 14, 15, 18, 21, 22, 23 in MA1201 problem set, all questions in MA1301 problem set