MA1300 Assignment 2

1. (a). $\lim_{x\to 1} f_1(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$

$= \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 2$

at $x=1$, $f_1(x)$ is not well defined $f_1(1) \neq \lim_{x\to 1} f_1(x)$

so $f_1(x)$ is not continuous at $x=a$

(b). $\because \lim_{x\to 0} f_2(x)$ DNE $\therefore f_2(x)$ is not continuous at $x=0$

proof: suppose $\lim_{x\to 0^+} f_2(x)$ exists. $\lim_{x\to 0^+} \sin(\frac{\pi}{x}) = L$

$\forall \varepsilon>0$, $\exists \delta>0$. such that if $0<x<\delta$, $|f(x)-L|<\varepsilon$

Take $x_1 = \frac{1}{2n+\frac{1}{2}}$ where $n > \frac{1}{2\delta}$ and n is an integer

$\Rightarrow 0<x_1<\delta$. then $|\sin(2n\pi+\frac{\pi}{2})-L|<\varepsilon$

$\therefore |1-L|<\varepsilon$

Take $x_2 = \frac{1}{2n}$, where $n>\frac{1}{2\delta}$ and n is an integer

$\Rightarrow \begin{cases} |L|<\varepsilon \\ |1-L|<\varepsilon \end{cases}$ Take $\varepsilon = 0.1$. There is no L satisfy both.

$\lim_{x\to 0} f_2(x) \neq f_2(0)$

$\therefore \lim_{x\to 0} f_2(x)$ DNE $\therefore f_2(x)$ is not continuous at $x=0$.

(c). $f_3(x) = \begin{cases} \frac{|x|}{x} = \frac{x}{x} = 1, & x>0 \\ 0, & x=0 \\ \frac{|x|}{x} = -1, & x<0 \end{cases}$

$\lim_{x\to 0^+} f_3(x) = \lim_{x\to 0^+} 1 = 1$ $\lim_{x\to 0^-} f_3(x) = \lim_{x\to 0^-} -1 = -1$

$\therefore \lim_{x\to 0^+} f_3(x) \neq \lim_{x\to 0^-} f_3(x)$

$\therefore \lim_{x\to 0} f_3(x)$ DNE $\therefore \lim_{x\to 0} f_3(x) \neq f_3(0)$

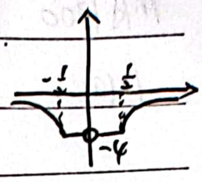
$\therefore f_3(x)$ is not continuous at $x=0$

2.

Yes and $a=-4$

$$f(x)=\begin{cases} \frac{2}{x}, & x\le -\frac{1}{2} \\ -4, & x\in(-\frac{1}{2},0)\cup(0,\frac{1}{2}) \\ a, & x=0 \\ -\frac{2}{x}, & x\ge \frac{1}{2}\end{cases}$$

It's easy to see that $f(x)$ is continuous except $x=0$



To make $f(x)$ continuous at $x=0$. $\lim_{x\to 0} f(x) = f(0) = a$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(-4) = -4$ $\quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-4) = -4$

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = -4$

$\therefore f(x) = a = \lim_{x\to 0} f(x) = -4$

3. $a=\frac{5}{6}$, $b=\frac{8}{3}$

To make f continuous for $x>0$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

and $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{2x^2-4x}{x^2-4} = \lim_{x\to 2^-} \frac{(x-2)2x}{(x-2)(x+2)} = \lim_{x\to 2^-} \frac{2x}{x+2} = 1$

$\therefore \begin{cases} 4a-2b+3=1 \\ 9a-3b+3=6-a-b \end{cases} \Rightarrow \begin{cases} a=\frac{5}{6} \\ b=\frac{8}{3}\end{cases}$

4. $a=4$. $b=4$

To make f continuous at $x=0$, $\lim_{x\to 0} f(x) = f(0)$

$\therefore \lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = f(0) = 1$

$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = \lim_{x\to 0} \frac{ax+b-4}{x(\sqrt{ax+b}+2)}$

when $a=4$, $b=4$

$\lim_{x\to 0} \frac{4x+4-4}{x(\sqrt{4x+4}+2)} = \lim_{x\to 0} \frac{4x}{x(\sqrt{4x+4}+2)} = \lim_{x\to 0} \frac{4}{\sqrt{4x+4}+2} = \frac{4}{\sqrt{4}+2} = 1$

5. $a=\frac{1\pm\sqrt{5}}{2}$

To make f continuous everywhere $\lim_{x\to a} f(x) = f(a)$

$f(a) = a+1 = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} x^2$

$\therefore a^2 = a+1 \Rightarrow a = \frac{1\pm\sqrt{5}}{2}$

6. (a). is correct

proof: suppose $|f(x)|$ is not continuous at $x=a$

and that $\lim_{x\to a^-} |f(x)| = A$; $\lim_{x\to a^+} |f(x)| = B$ $A \neq B$, $A\geq 0$, $B \geq 0$

$\therefore \lim_{x\to a^-} f(x) = \pm A$ $\lim_{x\to a^+} f(x) = \pm B$

$\because A \neq B$ $\therefore \pm A \neq \pm B$ $\therefore \lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ ①

① is contradictory to $f(x)$ is continuous

$\therefore |f(x)|$ is continuous

(b) is incorrect

Take $f(x) = \begin{cases} 1, & x>0 \\ -1, & x\leq 0 \end{cases}$ $|f(x)| = 1$

$|f(x)|$ is continuous, but $f(x)$ is not continuous.

7. let $g(x)$ be the continuous function and $f(x) = x$

$\begin{cases} 0 \leq g(0) \leq 1 \\ 0 \leq g(1) \leq 1 \end{cases}$ while $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$

let $h(x) = f(x) - g(x)$

$h(0) = f(0) - g(0) \leq 0$

$h(1) = f(1) - g(1) \geq 0$

By Intermediate Value Theorem.

there must be $x_0 \in [0,1]$

such that $h(x_0) = 0$

$h(x_0) = f(x_0) - g(x_0) = 0$ $g(x_0) = f(x_0) = x_0$

$\therefore$ there exists a fixed point (x_0, x_0)