# Midterm 7 December Autumn 2020, answers

Discrete Mathematics (City University of Hong Kong)

1. (5 points) Find the truth table of the compound proposition

$$(p \lor \neg q) \to (p \land r).$$

| $p$ | $q$ | $r$ | $\neg q$ | $p \lor \neg q$ | $p \land r$ | $(p \lor \neg q) \to (p \land r)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T |
| T | F | T | T | T | T | T |
| F | T | T | F | F | F | T |
| F | F | T | T | T | F | F |
| T | T | F | F | T | F | F |
| T | F | F | T | T | F | F |
| F | T | F | F | F | F | T |
| F | F | F | T | T | F | F |

2. Determine the truth value of each of these statements if the domain consists of all real numbers.

(a) (3 points) $\exists x(x^3 = -1)$

T

(b) (3 points) $\neg\forall x(x^2 > 0)$

T

(c) (3 points) $\forall x(x^2 \neq x)$

F

3. If $p$ stands for "it rains in Pari" and $q$ stands for "we bring umbrellas", then express each of the following in symbolic form.

   (a) (5 points) If it rains in Paris, then we bring our umbrellas.

   $p \to q$

   (b) (5 points) It is not the case that either it rains in Paris or we do not bring umbrellas.

   $\neg(p \vee \neg q)$

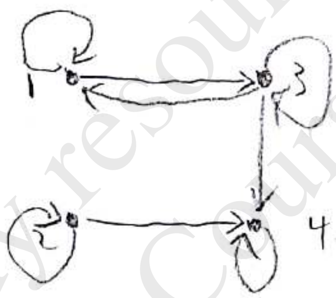
   (c) (5 points) Raining in Paris is a necessary condition for us to bring umbrellas.

   $q \to p$

4. Let $A = \{1, 2, 3, 4\}$. $R$ is a relation on $A$ with

$$R = \{(1,1), (1,3), (3,1), (2,2), (2,4), (3,3), (3,4), (4,4)\}.$$

   (a) (3 points) Draw the directed diagram for $R$.




   (b) (5 points) Determine whether $R$ is reflexive, symmetric, antisymmetric, or transitive. Then explain why.

R is reflexive b/c every number is relateal to itself.

R is not symmetric, b/c 2R4 but $4 \not R 2$

R is not antisymmetric, b/c 1R3 and 3R1

R is not transitive, b/c 1R3, 3R4 but $1 \not R 4$

5. (10 points) Show the validity of

$$\begin{array}{rl} & p \to q \quad ① \\ & r \to s \quad ② \\ & p \vee r \quad ③ \\ \hline \therefore & q \vee s \end{array}$$

④ $\neg p \to r$ from ③

⑤ $\neg p \to s$ from ② ④

⑥ $\neg q \to \neg p$ from ①

⑦ $\neg q \to s$ from ⑤ ⑥

⑧ $\neg(\neg q) \vee s$ from ⑦

⑨ $q \vee s$ from ⑧

6. Let $P(x)$ be the statement "$x$ has visited Europe," where the domain consists of the students in your school. Express each of these quantifications in English.

(a) (3 points) $\forall x P(x)$

Every student in the university has visited Europe.

(b) (3 points) $\neg \exists x\ P(x)$

It's not the case that there is a student who has visited Europe.

(c) (3 points) $\forall x\ (\neg P(x))$

Every student hasn't visited Europe.

7. (10 points) Show that $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

$\neg p \to (q \to r) \equiv p \vee (q \to r)$

$\equiv p \vee (\neg q \vee r) \equiv \neg q \vee (p \vee r)$

$\equiv q \to (p \vee r)$

8. (10 points) Determine whether the following argument is correct or incorrect and explain why. "Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university."

Let $P(x)$ be "x enroll in the university".

let $Q(x)$ be "x has lived in a dormitory".

Let c represent Mia.

Then. $\forall x\, (P(x) \to Q(x))$ ①

$\neg Q(c)$ ②

$\therefore \neg P(c)$

③ $P(c) \to Q(c)$ from ①

④ $\neg Q(c) \to \neg P(c)$ from ③.

⑤ $\neg P(c)$ from ② ④.

9. Let $A$ be the set $\{a, b\}$.

(a) (5 points) Find the power set $\mathcal{P}(A)$ of $A$.

$$\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

(b) (5 points) Consider the inclusion relation $R$ on $\mathcal{P}(A)$, i.e. $R = \{(X, Y) \mid X, Y \in \mathcal{P}(A),\ X \subseteq Y\}$. Show that $R$ is a partial order.

① reflexive : every set is a subset of itself.

② antisymmetric : If $A \subseteq B$, $B \subseteq A$, then $A = B$.

③ transitive : If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

10. (5 points) Let $A = \{a, b, c\}$, $B = \{x, y\}$ and $C = \{0, 1\}$. Find $A \times C \times B$ and $B \times B \times B$.

$$A \times C \times B = \{ (a, x, 0)\ (a, x, 1),\ (a, y, 0),\ (a, y, 1)$$
$$(b, x, 0)\ (b, x, 1)\ (b, y, 0),\ (b, y, 1)$$
$$(c, x, 0)\ (c, x, 1)\ (c, y, 0),\ (c, y, 1) \}$$

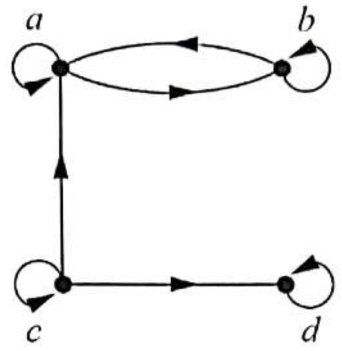
11. (5 points) Let $A_i = \{i, i+1, \ldots, i+10\}$. Find

$$\bigcap_{i=1}^{\infty} A_i \quad \text{and} \quad \bigcup_{i=1}^{\infty} A_i.$$

$$\bigcap_{i=1}^{\infty} A_i = \phi.$$

$$\bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, \ldots\} = \mathbb{Z}_{>0}.$$

12. (5 points) List the ordered pairs in the relation represented by the following directed graphs.




$R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,a), (c,a), (c,d)\}$

13. Answer the following questions for the poset $\{\{1\},\{3\},\{1,2\},\{1,2,4\},\{1,3\},\subseteq\}$.

(a) (3 points) Find the minimal(s) in the poset.

$\{1\}, \{3\}$

(b) (3 points) Find the maximal(s) in the poset.

$\{1,2,4\}$ $\{1,3\}$

14. (6 points) Determine if the following relation $R$ on the set of all people is an equivalence relation? Explain why.

$$R = \{(a, b) | a \text{ and } b \text{ speak a common language}\}.$$

R is not an equivalence relation

b/c R is not transitive.



15. (10 points) Show that the following compound proposition is a tautology without using truth table.

$$(\neg q \wedge (p \to q)) \to \neg p.$$

$$\equiv (\neg q \wedge (\neg p \vee q)) \to \neg p$$

$$\equiv ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \to \neg p$$

$$\equiv (\neg q \wedge \neg p) \to \neg p$$

$$\equiv \neg(\neg q \wedge \neg p) \vee \neg p$$

$$\equiv q \vee p \vee \neg p$$

$$\equiv q \vee \mathbf{T}$$

$$\equiv \mathbf{T}$$

16. (10 points) Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Let s, c, g, t, h be "It's sunny this afternoon" "it's colder than yesterday" "We will go swimming" "we take a canoe trip". "we will be home by sunset".

$\neg s \wedge c$ ①
$g \to s$ ②
$\neg g \to t$ ③
$t \to h$ ④

$\therefore h$

⑤ $\neg s \to \neg g$ from ②
⑥ $\neg s$ from ①
⑦ $\neg g$ from ⑤⑥
⑧ $t$ from ③⑦
⑨ $h$ from ⑧④



**Optional Question.**(5 points) The logical operator NAND, written as | is defined by

$$p \mid q = \neg(p \vee q).$$

Using the NAND operator only, rewrite

(a) $\neg p$

$\neg p \equiv \neg(p \vee p) \equiv p \mid p$

(b) $p \wedge q$

$\neg(\neg p \vee \neg q) = \neg p \mid \neg q \equiv (p \mid p) \mid (q \mid q)$

(c) $p \to q$

$\neg p \vee q \equiv \neg(p \wedge \neg q) \equiv p \wedge \neg q \mid p \wedge \neg q$

$\equiv (p \mid p) \mid (\neg q \mid \neg q) \mid (p \mid p) \mid (\neg q \mid \neg q)$

$\equiv [(p \mid p) \mid (q \mid q) \mid (q \mid q)] \mid [(p \mid p) \mid (q \mid q) \mid (q \mid q)]$