Characterization of truthful multi-dimensional mechanisms

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Overview

1. General results for truthful multi-dimensional mechanisms
2. Truthful pseudo multi-dimensional mechanisms
    - General one-parameter mechanisms
    - Multi-dimensional mechanisms with structure-related variables

Simple Myerson Lemma

Simple Myerson lemma

1. An allocation rule is implementable if and only if it is monotonic.
2. If π is monotonic, then there is a unique payment policy p for which (π, p) is IR and IC and the winner pays critical bid and the losers pay zero.

Critical bid

$v_i^*(\theta'_{-i}) = \inf\{v_i' \text{ s.t. } \pi_i(v_i', \theta'_{-i}) = 1\}$ for any given θ'_{-i}

Cyclical Monotonicity

Cyclical Monotonicity

Social choice function f satisfies cyclical monotonicity if for each agent i, fix the report type of other agents. For all finite type cycle $\Theta = (\theta_1, \theta_2, \cdots, \theta_k, \theta_{k+1} = \theta_1)$, we have: $\sum_{j=1}^{k}(v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1})) \leq 0$

Rochet Theorem

A social choice fuction f is implementable if and only if it satisfies cyclical monotonicity.

Jean-Charles Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. Journal of Mathematical Economics, 16(2):191– 200, 1987.

Proof.

If f can be implementable:
There is a payment rule $p_i(\theta')$ such that:
$v_i(\theta_{j+1}, f(\theta_{j+1})) - p_i(\theta_{j+1}) \geq v_i(\theta_{j+1}, f(\theta_j)) - p_i(\theta_j) \quad \forall i \in N, j \in \{1, \cdots, k\}$
$p_i(\theta_j) - p_i(\theta_{j+1}) \geq v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1}))$
Add above equations:
$\sum_{j=1}^{k}(p_i(\theta_j) - p_i(\theta_{j+1})) \geq \sum_{j=1}^{k}(v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1})))$
$p_i(\theta_1) - p_i(\theta_{k+1}) \geq \sum_{j=1}^{k}(v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1})))$
$\sum_{j=1}^{k}(v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1}))) \leq 0$ $\square$

Proof.

Given a θ_1'

For any type θ, let $p_{\theta_1'-\theta}$ be any type path from θ_1' to θ. If the length of a specific path is L, denote as $(\theta_1, \cdots, \theta_L)$. Then $\theta_1 = \theta_1'$, $\theta_L = \theta$

Define $g(\theta) = \sup\limits_{p_{\theta_1'-\theta}} \sum_{j=1}^{L-1} (v_i(\theta_{j+1}, f(\theta_j)) - v_i(\theta_{j+1}, f(\theta_{j+1})))$

If $\theta = \theta_1'$, the path becomes a cycle, $g(\theta_1') = 0$

From the definition of cyclical monotonicity:

$g(\theta) + v_i(\theta_1', f(\theta)) - v_i(\theta_1', f(\theta_1')) \leq 0$

$g(\theta) + v_i(\theta_1', f(\theta)) - v_i(\theta_1', f(\theta_1')) \leq g(\theta_1')$

$v_i(\theta_1', f(\theta_1')) + g(\theta_1') \geq v_i(\theta_1', f(\theta)) + g(\theta)$

Define $p(\theta) = -g(\theta)$

$v_i(\theta_1', f(\theta_1')) - p(\theta_1') \geq v_i(\theta_1', f(\theta)) - p(\theta)$

Since θ_1' and θ can be any type, we have done. □

Weakness of Rochet theorem

The value domain of each agent is unrestricted.(i.e. If there are m outcomes, the value vector of outcomes $V_i = R^m$)
But it is rare that value domain is unrestricted.

Combinatorial auctions

The domain is restricted, for each agent i:

1. If i gets the same bundle of items in two different outcomes a_1, a_2, then $v_i(a_1) = v_i(a_2)$
2. If i gets nothing in outcome a, $v_i(a) = 0$
3. If i gets bundle A in outcome a_A and bundle B in outcome a_B with $a_A \subseteq a_B$, then $v_i(a_A) \leq v_i(a_B)$

Towards a characterization of truthful combinatorial auctions. Proc. of the 44th Annual Symposium on Foundations of Computer Science (FOCS'03), 2003.

Weak Monotonicity

Weak Monotonicity

A social choice function f satisfies Weak Monotonicity if for any $(v_i, v_{-i}) \in V$, player i, and $u_i \in V_i : f(v_i, v_{-i}) = a, f(u_i, v_{-i}) = b$ implies that $u_i(b) - v_i(b) \geq u_i(a) - v_i(a)$.

Theorem

For any social choice function with convex domain, weak monotonicity is necessary and suffcient for implementation.

Convex value set is more common, such as combinatorial auctions.

Saks, M. E. and L. Yu (2005): Weak Monotonicity Suffces for Truthfulness on Convex Domains," in Proceedings of 7th ACM Conference on Electronic Commerce, ACM Press, 286-293.

Truthful pseudo multi-dimensional mechanisms

General one-parameter mechanisms

Single-minded case

Single-minded agent

If there exists a bundle of items S_i and a value v^* such that $v_i(S) = v^*$ for all $S \supseteq S_i$ and $v_i(S) = 0$ for the other S.
Agents report (v_i', S_i') in mechanism

Theorem

A deterministic mechanism for single-minded bidders in which losing bidders pay 0 is incentive compatible if and only if it satisfies the following two conditions:

1. **Monotonicity:** If i wins for a bid (S_i, v_i), then i still wins for any bid (S_i', v_i') in which $v_i' \geq v_i, S_i' \subseteq S_i$
2. **Critical payment:** $p_i(S_i, v_i) = \inf\{v_i' : \text{i wins for } (S_i, v_i')\}$

A concrete mechanism

(i) Reorder the bid such that $\frac{v_1}{\sqrt{|S_1|}} \geq \cdots \geq \frac{v_n}{\sqrt{|S_n|}}$

(ii)Allocate bundles from 1 to n.

For player i in above order, if S_i is compatible with the winnner bundle so far. Allocate S_i to i.

(iii)If i win S_i, then $p_i = \frac{v_j}{\sqrt{\frac{|S_j|}{|S_i|}}}$

(j is smallest index ($i < j$) of the agent whose bundle is incompatible with i's ($S_i \cap S_j \neq \emptyset$) and $S_k \cap S_j = \emptyset$ for all $k < j, k \neq i$. However,if no such j exists, $p_i = 0$.)

Observation

p_i is critical bid since $\frac{p_i}{\sqrt{|S_i|}} = \frac{v_j}{\sqrt{|S_j|}}$

D. Lehmann, L. O'Callaghan, and Y. Shoham. Truth revelation in rapid, approximately efficient combinatorial auctions. In ACM Conf. on Electronic Commerce, 96–102, 1999.

General one-parameter mechanism

Property of agents

For each agent i, the utility function can be defined as:
$u_i(b_i, b_{-i}) = v_i\pi_i(b_i, b_{-i}) - p_i(b_i, b_{-i})$
where v_i is true value, b_i is bid value

Find characterization of allocation and payment.

Fix b_{-i}, u_i is only function of b_i
$u_i(b_i) = v_i\pi_i(b_i) - p_i(b_i)$
Assume $\pi_i(b_i)$ and $p_i(b_i)$ are twice differentiable
Since truthful-telling maximizes utility:
$\frac{du_i(b_i)}{db_i}\Big|_{b_i=v_i} = 0$
$\frac{d^2u_i(b_i)}{db_i^2}\Big|_{b_i=v_i} \leq 0$ $\square$

Proof.

Using first order condition

$$[v_i \frac{d\pi_i(b_i)}{db_i} - \frac{dp_i(b_i)}{db_i}]\Big|_{b_i=v_i} = 0$$

$$b_i \frac{d\pi_i(b_i)}{db_i} - \frac{dp_i(b_i)}{db_i} = 0$$

$$p_i(b_i) = p_i(0) + b_i\pi_i(b_i) - \int_0^{b_i} \pi_i(u)du$$

$\square$

Proof.

Using second order condition

$$[v_i \frac{d^2\pi_i(b_i)}{db_i^2} - \frac{d^2 p_i(b_i)}{db_i^2}]\Big|_{b_i=v_i} \leq 0$$

$$b_i \frac{d^2\pi_i(b_i)}{db_i^2} - \frac{d^2 p_i(b_i)}{db_i^2} \leq 0$$

$$b_i \frac{d^2\pi_i(b_i)}{db_i^2} + \frac{d\pi_i(b_i)}{db_i} - \frac{d^2 p_i(b_i)}{db_i^2} = 0$$

$$\frac{d\pi_i(b_i)}{db_i} \geq 0$$

□

Generalize above result

Social choice function $\pi(b_i, b_{-i})$ is implementable if and only if $\pi_i(b_i, b_{-i})$ is increasing function of b_i for all i and b_{-i}. Such mechanism is truthful if and only if:
$p_i(b_i, b_{-i}) = h_i(b_{-i}) + b_i\pi_i(b_i, b_{-i}) - \int_0^{b_i} \pi_i(u, b_{-i})du$
Where h_i are arbitrary functions

A further result

Social choice function $\pi(b_i, b_{-i})$ is implementable through an IC and IR mechanism if and only if $\pi_i(b_i, b_{-i})$ is increasing function and of b_i and $\int_0^{\infty} \pi_i(u, b_{-i})du < \infty$ for all i and b_{-i}. The payment of such mechanism should be:
$p_i(b_i, b_{-i}) = b_i\pi_i(b_i, b_{-i}) + \int_{b_i}^{\infty} \pi_i(u, b_{-i})du$

Aaron Archer and Eva Tardos. Truthful mechanisms for one-parameter agents. Proc. of the 42nd Annual Symposium on Foundations of Computer Science (FOCS'01), 2001.

Truthful pseudo multi-dimensional mechanisms

Multi-dimensional mechanisms with structure-related variables

Online mechanism design

Settings

Consider discrete time periods $T = \{1, 2, \cdots\}$. Seller sells a single item in each time period.
Agents come to and leave a system dynamicly, type profile of each agent is $\theta_i = (a_i, d_i, v_i)$ where a_i is arrival time, d_i is departure time, v_i is value for the item.
Each agent only wants one item.

Main Problem

If for every agent a_i and d_i is fixed, then the structure of auction system is determined. And it is exactly one-dimensional mechanism design problem. However, agents may misreport a_i, d_i.

Parkes, David C. 2007. Online mechanisms. In Algorithmic Game Theory, ed. N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, 411-439. Cambridge: Cambridge University Press.

Online mechanism design

Monotonicity

For two types $\theta_1 = (a_1, d_1, v_1), \theta_2 = (a_2, d_2, v_2)$, $\theta_1 \succ \theta_2$ if and only if $v_1 \geq v_2$, $a_1 \leq a_2$, $d_1 \geq d_2$.
Allocation policy π is monotonic if $\pi_i(\theta_2, \theta_{-i}) = 1 \Rightarrow \pi_i(\theta_1, \theta_{-i}) = 1$ for all $\theta_1 \succ \theta_2$ and θ_{-i}

A structure-related critical bid:

$v^*_{(a_i,d_i)}(\theta'_{-i}) = \inf\{v'_i \text{ s.t. } \pi_i((a_i, d_i, v'_i), \theta'_{-i}) = 1\}$

Lemma

Fixed report type of other agents θ'_{-i}. If $a'_i \leq a_i$, $d'_i \geq d_i$, then:
$v^*_{(a_i,d_i)}(\theta'_{-i}) \geq v^*_{(a'_i,d'_i)}(\theta'_{-i})$

Online mechanism design

Theorem

1. Any monotonic deterministic allocation policy π is implementable with no early-arrival or late-departure misreports.
2. The payment for winner is critical bid $v^*_{(a_i,d_i)}(\theta'_{-i})$.

Intuitively, with no early-arrival or late-departure misreports, agents have no incentive to misreport to a "tighter" intervel or critical bid will increase

Online mechanism design

A concrete mechanism

A bid from an agent is a claim about its type, $\theta'_i = (a'_i, d'_i, v'_i)$,
(i) In each period t, allocate the item to the highest unassigned bid, breaking ties at random.
(ii) Every allocated agent pays its critical-value payment, collected upon its reported departure.

Mechanism design in social network

Settings

Each agent has neighbors r_i, type profile $\theta_i = (v_i, r_i)$

Monotonicity

For two types $\theta_1 = (v_1, r_1), \theta_2 = (v_2, r_2)$, $\theta_1 \succ \theta_2$ if and only if $v_1 \geq v_2$, $r_1 \subseteq r_2$
Allocation policy π is monotonic if $\pi_i(\theta_2, \theta_{-i}) = 1 \Rightarrow \pi_i(\theta_1, \theta_{-i}) = 1$ for all $\theta_1 \succ \theta_2$ and θ_{-i}

Theorem

In diffusion auctions, any monotonic allocation policy is implementable.

Mechanism design in social network

Critical bid

$v_i^*(r_i, \theta'_{-i}) = \inf\{v'_i \text{ s.t. } \pi_i((v'_i, r_i), \theta'_{-i}) = 1\}$ for any given θ'_{-i}

A sufficient and necessary condition for one-item auction

$x_i(\theta') = \pi(\theta')\tilde{x}_i(\theta') + (1 - \pi(\theta'))\bar{x}_i(\theta')$

1.Allocation policy π is value monotonicity

2. $\tilde{x}_i$ and $\bar{x}_i$ are bid-independent

3.$\tilde{x}_i(r_i) - \bar{x}_i(r_i) = v_i^*(r_i)$

4.$\tilde{x}_i(r_i)$ and $\bar{x}_i(r_i)$ are diffusion-monotonic

Diffusion-monotonic

For any $r'_i \subseteq r_i$, $x_i(r'_i) \geq x_i(r_i)$

Bin Li, Dong Hao, and Dengji Zhao. 2020. Incentive-Compatible Diffusion Auctions. In Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020, Christian Bessiere (Ed.). ijcai.org,231–237

Mechanism design in social network

A sufficient and necessary condition for single-minded auction

$\theta_i = (v_i, S_i, r_i)$

1. Allocation policy π is value monotonic and bundle monotonic
2. $\tilde{x}_i$ is bid-independent, denote as $\tilde{x}_i(S_i, r_i)$
3. $\bar{x}_i$ is bid-independent and bundle-independent, denote as $\bar{x}_i(r_i)$
4. $\tilde{x}_i(S_i, r_i) - \bar{x}_i(r_i) = v_i^*(S_i, r_i)$

($v_i^*(S_i, r_i)$ is critical bid of i for getting bundle S_i when diffusing information to r_i)

5. $\tilde{x}_i(S_i, r_i)$ is diffusion-monotonic and bundle-monotonic
6. $\bar{x}_i(r_i)$ is diffusion-monotonic