Proofs – Artificial Noisy MIMO Systems under Correlated Scattering Rayleigh Fading – A Physical Layer Security Approach

It provides the proofs of Theorem 1, 2, 3, and 4 of the paper:

[S] Yiliang Liu, Hsiao-Hwa Chen, Liangmin Wang, and Weixiao Meng, Artificial Noisy MIMO Systems under Correlated Scattering Rayleigh Fading – A Physical Layer Security Approach, IEEE Systems Journal, 2019.

I. APPENDICES

A. Proof of Theorem 1

Lemma 1 (Proved in [1, Th. 2.1]): For $r \times r$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, if $\mathbf{A}$ and $\mathbf{B}$ are Hermitian positive (semi-)definite, then

$$\sigma_1(\mathbf{A} \circ \mathbf{B}) \leq \max_{1 \leq i \leq r} a_{ii} \sigma_1(\mathbf{B}), \tag{1}$$

where a_{ii} is a diagonal element of $\mathbf{A}$, and "$\circ$" denotes the Schur product defined as $\mathbf{A} \circ \mathbf{B} = [a_{ij} b_{ij}]$.

Lemma 2 (Proved in [2, Th. 3]): For two $r \times r$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, if $\mathbf{A}$ and $\mathbf{B}$ are Hermitian positive (semi-)definite, then

$$\prod_{i=1}^{r} \sigma_i(\mathbf{A} \circ \mathbf{B}) \geq \prod_{i=1}^{r} \sigma_i(\mathbf{B}) a_{ii}. \tag{2}$$

We begin to prove Theorem 1 as follows. For $d_1 > d_2$, we can build up $\mathbf{R}_a(d_1)$ via $\mathbf{R}_a(d_2)$ as

$$\mathbf{R}_a(d_1) = \mathbf{M} \circ \mathbf{R}_a(d_2), \tag{3}$$

where $\mathbf{M}$ is a Hermitian matrix whose diagonal elements are all one. If $d_1 > d_2$ and $i \neq j$, based on [S, Eq. (6)], we can find that the modulus value of $[\mathbf{R}_a(d_1)]_{i,j}$ is smaller, i.e.,

$$|[\mathbf{R}_a(d_1)]_{i,j}| < |[\mathbf{R}_a(d_2)]_{i,j}|. \tag{4}$$

Thus, $\mathbf{M}$ is positive (semi-)definite because all diagonal elements of $\mathbf{M}$ are one, and the modulus of non-diagonal elements is smaller than one. Based on Lemma 1 and Eqn. (3), we can get

$$\sigma_1[\mathbf{R}_a(d_1)] \leq \max_{1 \leq i \leq n} m_{ii} \sigma_1[\mathbf{R}_a(d_2)], \tag{5}$$

and m_{ii} is a diagonal element of $\mathbf{M}$ such that $m_{ii} = 1$. Hence, $\sigma_1[\mathbf{R}_a(d_1)] < \sigma_1[\mathbf{R}_a(d_2)]$. With the same argument, we can show that $\sigma_1[\mathbf{R}_a(\bar{\theta})]$ and $\sigma_1[\mathbf{R}_a(\delta)]$ have the same property.

Based on Lemma 2 and $a_{ii} = 1$, we get

$$\begin{aligned} \det[\mathbf{R}_a(d_1)] &= \prod_{i=1}^{r} \sigma_i \big[\mathbf{M} \circ \mathbf{R}_a(d_2)\big] \\ &\geq \prod_{i=1}^{r} \sigma_i \big[\mathbf{R}_a(d_2)\big] m_{ii} = \det[\mathbf{R}_a(d_2)]. \end{aligned} \tag{6}$$

Note $\det[\mathbf{R}_a(d_1)] \neq \det[\mathbf{R}_a(d_2)]$, and thus "$>$" is held. Similarly, $\det[\mathbf{R}_a(\bar{\theta})]$ and $\det[(\mathbf{R}_a(\delta)]$ have the same property.

∎

B. Proof of Theorem 2

Lemma 3 (Proved in [3, Th. 2.3.2]): If $\mathbf{H}_e \sim \mathcal{CN}_{e,t}(\mathbf{0}, \mathbf{R}_e \otimes \mathbf{I}_t)$, the characteristic function of $\mathbf{H}_e$ is

$$\phi_{\mathbf{H}_e}(\mathbf{X}) = \mathrm{E}\big\{\mathrm{etr}[i\mathbf{H}_e\mathbf{X}^\dagger)]\big\} = \mathrm{etr}\big(-\frac{1}{2}\mathbf{X}^\dagger\mathbf{R}_e\mathbf{X}\mathbf{I}_t\big), \tag{7}$$

where $i = \sqrt{-1}$.

Next, we can prove Theorem 2 based on Lemma 3. For a given $(t \times s)$ unitary matrix $\mathbf{B}$, the characteristic function of $\mathbf{H}_e\mathbf{B}$ is

$$\phi_{\mathbf{H}_e\mathbf{B}}(\mathbf{X}) = \mathrm{E}[\mathrm{etr}(i\mathbf{H}_e\mathbf{B}\mathbf{X}^\dagger)] = \mathrm{E}[\mathrm{etr}(i\mathbf{H}_e\mathbf{Y}^\dagger)], \tag{8}$$

where $\mathbf{Y}^\dagger = \mathbf{B}\mathbf{X}^\dagger$. Viewing $\mathbf{Y}$ as a variable, from Lemma 1, we get

$$\begin{aligned} \mathrm{E}[\mathrm{etr}(i\mathbf{H}_e\mathbf{Y}^\dagger)] &= \mathrm{etr}\big(-\frac{1}{2}\mathbf{Y}^\dagger\mathbf{R}_e\mathbf{Y}\big) \\ &= \mathrm{etr}\big(-\frac{1}{2}\mathbf{X}^\dagger\mathbf{R}_e\mathbf{X}\mathbf{B}^\dagger\mathbf{B}\big). \end{aligned} \tag{9}$$

Since $\mathbf{B}$ is a $(t \times s)$ unitary matrix, we have $\mathbf{B}^\dagger\mathbf{B} = \mathbf{I}_s$. Then, Eqn. (8) can be written as

$$\begin{aligned} \phi_{\mathbf{H}_e\mathbf{B}}(\mathbf{X}) &= \mathrm{etr}\big(-\frac{1}{2}\mathbf{X}^\dagger\mathbf{R}_e\mathbf{X}\mathbf{B}^\dagger\mathbf{B}\big) \\ &= \mathrm{etr}\big(-\frac{1}{2}\mathbf{X}^\dagger\mathbf{R}_e\mathbf{X}\mathbf{I}_s\big). \end{aligned} \tag{10}$$

As Eqn. (10) is the characteristic function of a complex Gaussian matrix with its covariance matrix $\mathbf{R}_e \otimes \mathbf{I}_s$, the proof is completed. ∎

C. Proof of Theorem 3

Let us define the cdf $F_{\lambda_k}(x)$ as

$$F_{\lambda_k}(x) = P(\lambda_k \leq x) \tag{11}$$
$$= P(\lambda_{k-1} \leq x) + p,$$

where $p = P(\lambda_n < \cdots < \lambda_k < x < \lambda_{k-1} < \cdots < \lambda_1)$. Let the domain be $D_1 = \{0 < \lambda_1 < \cdots < \lambda_n < x\}$, $D_2 = \{x < \lambda_1 < \cdots < \lambda_n < \infty\}$, and $D_3 = \{\lambda_n < \cdots < \lambda_k < x < \lambda_{k-1} < \cdots < \lambda_1\}$.

Lemma 6 (Proved in [4]): The joint pdf of the ordered eigenvalues $\lambda_1 > \cdots > \lambda_n > 0$ of a receiver-side correlated central Wishart matrix $\mathbf{W} \sim W_n(m, \mathbf{0}_n, \mathbf{R}_a)$ is

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = K_0^{-1} \det\left[\mathbf{G}, \mathbf{E}(\boldsymbol{\lambda})\right] \prod_{i<j}^{n} (\lambda_i - \lambda_j) \prod_{i=1}^{n} \lambda_i^{b-n}, \tag{12}$$

where

$$K_0 = \begin{cases} \prod_{i=1}^{a} \sigma_i^{b-n} (b-i)! \prod_{i<j}^{a} \sigma_i - \sigma_j, & b \geq a, \\ \prod_{i=1}^{b} (b-i)! \prod_{i<j}^{a} \sigma_i - \sigma_j, & b < a, \end{cases} \tag{13}$$

and $\mathbf{G}$ is a $a \times (a-n)$ matrix, whose (i,j)th element is σ_i^{j-1}. $\boldsymbol{\sigma} = (\sigma_1, ...\sigma_a)$ are the eigenvalues of $\mathbf{R}_a$, such that $\sigma_1 > ... > \sigma_a > 0$. $\mathbf{E}(\boldsymbol{\lambda})$ is a $a \times n$ matrix, whose (i,j)th element is $[\sigma_i^{a-n-1} \exp(-\lambda_{j-a+n}/\sigma_i)]$.

Integrating Eqn. (12) over D_3, we can get the probability p as

$$p = K_0^{-1} \int_{D_3} \det[\mathbf{G}, \mathbf{E}(\boldsymbol{\lambda})] \prod_{i<j}^{n} (\lambda_i - \lambda_j) \prod_{i=1}^{n} \lambda_i^{b-n} d\lambda_i. \tag{14}$$

Performing the Laplace expansion over the first $a-n$ columns of $[\mathbf{G}, \mathbf{E}(\boldsymbol{\lambda})]$, we gave

$$\det[\mathbf{G}, \mathbf{E}(\boldsymbol{\lambda})] = \sum_{\boldsymbol{\kappa} \in \mathcal{Q}(i)} (-1)^{\sum_{i=1}^{a-n}(\kappa_i + i)} \det[\mathbf{G}^{\boldsymbol{\kappa}}] \det[\mathbf{E}^{\boldsymbol{\kappa}}(\boldsymbol{\lambda})], \tag{15}$$

where $\mathcal{Q}(i)$ is a set of all permutations $(\kappa_1, ..., \kappa_a)$ of the integers $(1, ..., a)$, such that $(\kappa_1 < \kappa_2 < ... < \kappa_{a-n})$ and $(\kappa_{a-n+1} < \kappa_{a-n+2} < ... < \kappa_a)$. Hence, $\sum_{\boldsymbol{\kappa} \in \mathcal{Q}(i)}$ denotes the summation over two combinations $(\kappa_1 < \kappa_2 < ... < \kappa_{a-n})$ and $(\kappa_{a-n+1} < \kappa_{a-n+2} < ... < \kappa_a)$. $[\mathbf{E}^{\boldsymbol{\kappa}}(\boldsymbol{\lambda})]$ is a $n \times n$ matrix, i.e., $[\mathbf{E}^{\boldsymbol{\kappa}}(\boldsymbol{\lambda})]_{i,j} = \sigma_{\kappa_{a-n+i}}^{a-n-1} \exp(-\lambda_j/\sigma_{\kappa_{a-n+i}})$ for $i, j = 1, ..., n$. $[\mathbf{G}^{\boldsymbol{\kappa}}]$ is a $(a-n) \times (a-n)$ Vandermonde matrix, i.e., $[\mathbf{G}^{\boldsymbol{\kappa}}]_{i,j} = \sigma_{\kappa_i}^{j-1}$ for $i, j = 1, ..., a-n$. When $a = n$, we set $\det[\mathbf{G}^{\boldsymbol{\kappa}}] = 1$.

$$\det\left[\mathbf{E}^{\boldsymbol{\kappa}}(\boldsymbol{\lambda})\right]\prod_{i<j}^{n}(\lambda_i-\lambda_j)=\prod_{i=1}^{n}\sigma_{\kappa_{a-n+i}}^{a-n-1}\sum_{q}^{\sim}\sum_{\iota}^{\sim}(-1)^{\mathrm{per}(\iota_1,\ldots,\iota_n)}\prod_{i=1}^{n}\lambda_{q_i}^{\iota_i-1}\exp(-\frac{\lambda_{q_i}}{\sigma_{\kappa_{a-n+i}}}). \quad (16)$$

Next, we prove Eqn. (16) for simplifying Eqn. (14). In Eqn. (16), $\sum_q^{\sim}$ denotes the summation over all permutations $(q_1,\ldots,q_n)$ of $(1,\ldots,n)$, $\sum_\iota^{\sim}$ is the summation over all permutations $(\iota_1,\ldots,\iota_n)$ of $(1,\ldots,n)$, and $\mathrm{per}(\iota_1,\ldots,\iota_n)$ is either 0 or 1, corresponding to even or odd value of the permutation $(\iota_1,\ldots,\iota_n)$. Then, p can be written as

$$\begin{aligned} p &= K_0^{-1}\int_{D_3}\det[\mathbf{G},\mathbf{E}(\boldsymbol{\lambda})]\prod_{i<j}^{n}(\lambda_i-\lambda_j)\prod_{i=1}^{n}\lambda_i^{b-n}d\lambda_i \\ &= K_0^{-1}\sum_{\boldsymbol{\kappa}\in\mathcal{Q}(i)}(-1)^{\sum_{i=1}^{a-n}(\kappa_i+i)}\det[\mathbf{G}^{\boldsymbol{\kappa}}]\prod_{i=1}^{n}\sigma_{\kappa_{a-n+i}}^{a-n-1}\sum_{q}^{\sim}\sum_{\iota}^{\sim} \\ &\times(-1)^{\mathrm{per}(\iota_1,\ldots,\iota_n)}\int_{D_3}\prod_{i=1}^{n}\lambda_{q_i}^{\iota_i-1}\exp(-\frac{\lambda_{q_i}}{\sigma_{\kappa_{a-n+i}}})\prod_{i=1}^{n}\lambda_i^{b-n}d\lambda_{q_i} \\ &= K_0^{-1}\sum_{\boldsymbol{\mu}\in\mathcal{P}(k)}\sum_{\boldsymbol{\kappa}\in\mathcal{Q}(i)}(-1)^{\sum_{i=1}^{a-n}(\kappa_i+i)}\det[\mathbf{G}^{\boldsymbol{\kappa}}]\prod_{i=1}^{n}\sigma_{\kappa_{a-n+i}}^{a-n-1} \\ &\times\sum_{\iota}^{\sim}(-1)^{\mathrm{per}(\iota_1,\ldots,\iota_n)}I_1(\mu,\iota,\kappa)I_2(\mu,\iota,\kappa), \end{aligned} \quad (17)$$

where $\sum_q^{\sim}=\sum_{\boldsymbol{\mu}\in\mathcal{P}(k)}\sum_{q_{\mu_\psi}}^{\sim}\sum_{q_{\mu_\omega}}^{\sim}$, and $\sum_{q_{\mu_\psi}}^{\sim}$ denotes the summation over the permutations $(q_{\mu_1},\ldots,q_{\mu_{k-1}})$ of $(1,\ldots,k-1)$, $\sum_{q_{\mu_\omega}}^{\sim}$ calculates the summation over the permutations $(q_{\mu_k},\ldots,q_{\mu_n})$ of $(k,\ldots,n)$, $\sum_{\boldsymbol{\mu}\in\mathcal{P}(k)}$ is the summation over the combination of sets $(\mu_1<\mu_2<\cdots<\mu_{k-1})$ and $(\mu_k<\mu_{k+1}<\cdots<\mu_n)$, and $(\mu_1,\ldots,\mu_n)$ is a permutation of

$(1,\ldots,n)$. From [5, Eqs. (4.20) and (4.21)], we obtain

$$\begin{aligned}
I_1(\mu,\iota,\kappa) &= \sum_{q_{\mu_\psi}}^{\sim}\int_{D_4}\prod_{i=1}^{k-1}\lambda_{q_{\mu_i}}^{b-n+\iota_i-1}\exp(-\frac{\lambda_{q_{\mu_i}}}{\sigma_{\kappa_{a-n+\mu_i}}})d\lambda_{q_{\mu_i}} \\
&= \prod_{i=1}^{k-1}\int_x^\infty \lambda_{\mu_i}^{b-n+\iota_i-1}\exp(-\frac{\lambda_{\mu_i}}{\sigma_{\kappa_{a-n+\mu_i}}})d\lambda_{\mu_i} \\
&= \prod_{i=1}^{k-1}\sigma_{\kappa_{a-n+\mu_i}}^{b-n+\iota_i}\Gamma(b-n+\iota_i,\frac{\lambda_{\mu_i}}{\sigma_{\kappa_{a-n+\mu_i}}}), \qquad (18) \\
I_2(\mu,\iota,\kappa) &= \sum_{q_{\mu_\omega}}^{\sim}\int_{D_5}\prod_{i=k}^{n}\lambda_{q_{\mu_i}}^{b-n+\iota_i-1}\exp(-\frac{\lambda_{q_{\mu_i}}}{\sigma_{\kappa_{a-n+\mu_i}}})d\lambda_{q_{\mu_i}} \\
&= \prod_{i=k}^{n}\int_0^x \lambda_{\mu_i}^{\iota_i-1}\exp(-\frac{\lambda_{\mu_i}}{\sigma_{\kappa_{a-n+\mu_i}}})d\lambda_{\mu_i} \\
&= \prod_{i=k}^{n}\sigma_{\kappa_{a-n+\mu_i}}^{b-n+\iota_i}\gamma(b-n+\iota_i,\frac{\lambda_{\mu_i}}{\sigma_{\kappa_{a-n+\mu_i}}}), \qquad (19)
\end{aligned}$$

where $D_4=\{x<\lambda_{k-1}<\cdots<\lambda_1<\infty\}$ and $D_5=\{0<\lambda_n<\cdots<\lambda_k<x\}$. ι_i is the ith position after re-ordering $(\iota_1,\ldots,\iota_n)$, which can be viewed as the column index of the determinant of an $(n\times n)$ matrix. μ_i is the row index of the determinant of the $(n\times n)$ matrix dependent on k. Hence, $\sum_\iota^\sim(-1)^{\mathrm{per}(\iota_1,\ldots,\iota_n)}I_1(\mu,\iota,\kappa)I_2(\mu,\iota,\kappa)$ denotes the determinant of a matrix, each element of which is expressed by $[\boldsymbol{\Theta}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\kappa},k;x)]_{\mu_i,i}$. We can re-define the order index numbers of rows and columns of the determinant as u and μ_v. Finally, we get

$$\begin{aligned}
p = K_0^{-1}&\sum_{\boldsymbol{\mu}\in\mathcal{P}(k)}\sum_{\boldsymbol{\kappa}\in\mathcal{Q}(i)}(-1)^{\sum_{i=1}^{a-n}(\kappa_i+i)}\det[\mathbf{G}^{\boldsymbol{\kappa}}] \qquad (20)\\
&\times\prod_{i=1}^{n}\sigma_{\kappa_{a-n+i}}^{a-n-1}\det\left[\boldsymbol{\Theta}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\kappa},k;x)\right],
\end{aligned}$$

where $(n\times n)$ real matrix $\boldsymbol{\Theta}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\kappa},k;x)$ is defined as

$$\begin{aligned}
&\left[\boldsymbol{\Theta}(\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\kappa},k;x)\right]_{u,\mu_v} \qquad (21)\\
&=\begin{cases}\sigma_{\kappa_{a-n+u}}^{b-n+\mu_v}\Gamma(b-n+\mu_v,\frac{x}{\sigma_{\kappa_{a-n+u}}}), & v=1,...,k-1,\\ \sigma_{\kappa_{a-n+u}}^{b-n+\mu_v}\gamma(b-n+\mu_v,\frac{x}{\sigma_{\kappa_{a-n+u}}}), & v=k,...,n,\end{cases}
\end{aligned}$$

for $u,v=1,...,n$, where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the upper and lower incomplete Gamma functions defined in [S, Eqs. (24) and (25)].

Since we have

$$\prod_{i=1}^{n} \sigma_{\kappa_{a-n+i}}^{a-n-1} \det\left[\boldsymbol{\Theta}(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\kappa}, k; x)\right] \tag{22}$$
$$= \prod_{i=1}^{n} \sigma_{\kappa_{a-n+i}}^{b-n} \det[\boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\kappa}, k; x)],$$

where $(n \times n)$ real matrix $\boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, k, \kappa; x)$ is defined as

$$[\boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\kappa}, k; x)]_{u,\mu_v} \tag{23}$$
$$= \begin{cases} \sigma_{\kappa_{a-n+u}}^{a-n+\mu_v-1} \Gamma(b-n+\mu_v, \frac{x}{\sigma_{\kappa_{a-n+u}}}), & v = 1, ..., k-1, \\ \sigma_{\kappa_{a-n+u}}^{a-n+\mu_v-1} \gamma(b-n+\mu_v, \frac{x}{\sigma_{\kappa_{a-n+u}}}), & v = k, ..., n, \end{cases}$$

for $u, v = 1, ..., n$. Substituting Eqn. (22) to Eqn. (20) and performing the inverse Laplace expansion of Eqn. (20), we obtain

$$p = K_0^{-1} \prod_{i=1}^{n} \sigma_i^{b-n} \sum_{\boldsymbol{\mu} \in \mathcal{P}(k)} \det[\mathbf{G}, \boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, k; x)] \tag{24}$$
$$= K^{-1} \sum_{\boldsymbol{\mu} \in \mathcal{P}(k)} \det[\mathbf{G}, \boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, k; x)],$$

where

$$K = \prod_{i<j}^{n} \sigma_i - \sigma_j \prod_{i=1}^{n} (b-i)!. \tag{25}$$

$F_{\lambda_k}(x)$ can be expressed by

$$F_{\lambda_k}(x) = K^{-1} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu} \in \mathcal{P}(i)} \det[\mathbf{G}, \boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, i; x)], \tag{26}$$

which is the marginal cdf of the kth largest eigenvalue λ_k of a receiver-side correlated central Wishart matrix $\mathbf{W} \sim W_n(m, \mathbf{0}_n, \mathbf{R}_a)$. The marginal pdf of the kth largest eigenvalue can be easily derived from the derivative of a determinant as shown in [6], which is

$$f_{\lambda_k}(x) = \frac{d}{dx} \left\{ K^{-1} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu} \in \mathcal{P}(i)} \det\left[\mathbf{G}, \boldsymbol{\Psi}(\boldsymbol{\mu}, \boldsymbol{\sigma}, i; x)\right] \right\}$$
$$= K^{-1} \sum_{i=1}^{k} \sum_{\boldsymbol{\mu} \in \mathcal{P}(i)} \sum_{j=1}^{n} \det\left[\mathbf{G}, \boldsymbol{\Omega}(\boldsymbol{\mu}, \boldsymbol{\sigma}, i, j; x)\right], \tag{27}$$

where $(n \times n)$ real matrix $\boldsymbol{\Omega}(\boldsymbol{\mu}, \boldsymbol{\sigma}, i, j; x)$ is defined in [S, Eq. (23)]. This completes the proof. ∎

D. Proof of Theorem 4

According to Jensen's inequality, we have

$$
\begin{aligned}
C_{\mathbf{A}}(\mathbf{R}_a,\rho,\eta) &= \sum_{i=1}^{\eta} \mathrm{E}\big\{ \log_2[1+(P/t)\lambda_i(\mathbf{A}\mathbf{A}^\dagger)]\big\} \\
&\le \sum_{i=1}^{\eta} \log_2\big\{1+(P/t)\mathrm{E}[\lambda_i(\mathbf{A}\mathbf{A}^\dagger)]\big\},
\end{aligned} \tag{28}
$$

where $\lambda_1(\mathbf{A}\mathbf{A}^\dagger) > \lambda_2(\mathbf{A}\mathbf{A}^\dagger) > \cdots > \lambda_n(\mathbf{A}\mathbf{A}^\dagger)$ are the ordered eigenvalues of $\mathbf{A}\mathbf{A}^\dagger$. Thus, $C_{\mathbf{H}}(\mathbf{R}_r,\rho,s_1)$ in [S, Eq. (19)] can be expressed as

$$
C_{\mathbf{H}}(\mathbf{R}_r,\rho,s_1) = \chi_1 = \sum_{i=1}^{s_1} \log_2\big\{1+\rho\mathrm{E}[\lambda_i(\mathbf{H}\mathbf{H}^\dagger)]\big\}. \tag{29}
$$

From [7, Eqn. (21)] or [8, Eqn. (27)], we get

$$
C_{\mathbf{H}_3}(\mathbf{R}_e,\rho,n_1) = \log_2\left[1+\sum_{k=1}^{e}\rho^k\prod_{i=0}^{k-1}(m_1-i)\varrho_k\right], \tag{30}
$$

and

$$
C_{\mathbf{H}_4}(\mathbf{R}_e,\rho,e) = \log_2\left[1+\sum_{k=1}^{e}\rho^k\prod_{i=0}^{k-1}(t-i)\varrho_k\right], \tag{31}
$$

respectively, where ϱ_k, n_1, and m_1 are defined in [S, Eqs. (29) and (31)]. We can simplify $C_{\mathbf{H}_3}(\mathbf{R}_e,\rho,n_1) - C_{\mathbf{H}_4}(\mathbf{R}_e,\rho,e)$ as

$$
\begin{aligned}
&C_{\mathbf{H}_3}(\mathbf{R}_e,\rho,n_1) - C_{\mathbf{H}_4}(\mathbf{R}_e,\rho,e) \\
&= \chi_2 = \log_2\left[\frac{1+\sum_{k=1}^{e}\rho^k\prod_{i=0}^{k-1}(m_1-i)\varrho_k}{1+\sum_{k=1}^{e}\rho^k\prod_{i=0}^{k-1}(t-i)\varrho_k}\right].
\end{aligned} \tag{32}
$$

Hence, [S, Eq. (13)] can be expressed approximately by

$$
R_s^{\mathrm{app}} = [\chi_1+\chi_2]^+. \tag{33}
$$

This completes the proof. ■

References

[1] R. A. Horn and R. Mathias, "Block-matrix generalizations of Schur's basic theorems on Hadamard products," Linear Algebra and its Applications, vol. 172, pp. 337–346, Jul. 1992.

[2] R. Bapat and V. Sunder, "On majorization and schur products," Linear Algebra and its Applications, vol. 72, pp. 107 – 117, Dec. 1985.

[3] A. K. Gupta and D. K. Nagar, Matrix variate distributions. Boca Raton, Florida: CRC Press, 2018.

[4] A. T. James et al., "Distributions of matrix variates and latent roots derived from normal samples," The Annals of Mathematical Statistics, vol. 35, no. 2, pp. 475–501, 1964.

[5] T. Ratnarajah, "Topics in complex random matrices and information theory," Master's thesis, Mathematics and Statistics of University of Ottawa, May 2003.

[6] J. E. H. John G. Christiano, "On the n-th derivative of a determinant of the j-th order," Math. Magazine, vol. 37, no. 4, pp. 215–217, Sep. 1964. [Online]. Available: http://www.jstor.org/stable/2688589

[7] Q. T. Zhang, X. W. Cui, and X. M. Li, "Very tight capacity bounds for MIMO-correlated Rayleigh-fading channels," IEEE Trans. Wireless Commun., vol. 4, no. 2, pp. 681–688, Mar. 2005.

[8] X. W. Cui, Q. T. Zhang, and Z. M. Feng, "Generic procedure for tightly bounding the capacity of MIMO correlated Rician fading channels," IEEE Trans. Commun., vol. 53, no. 5, pp. 890–898, May 2005.