A Collection of Testing Problems

Testing Characteristic Set Algorithm, Zeros Decompostion etc.

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CTP_I: Qin-Heron Formula of Triangle's Area

The Geometry Figure

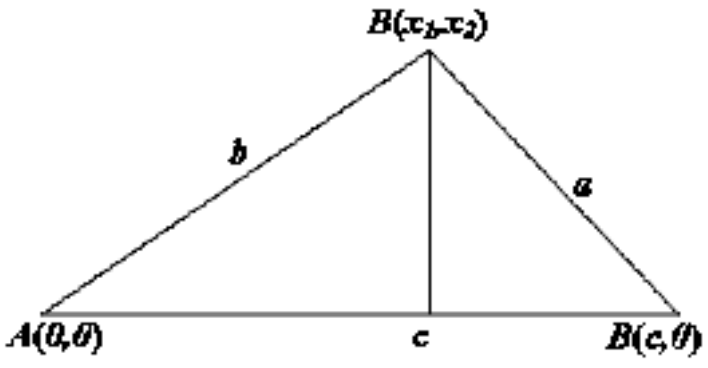

Figure 1

Polynomial System with ord and const

```
p1 = c x2 - 2 S;
p2 = x1^2 + x2^2 - b^2;
p3 = (c - x1)^2 + x2^2 - a^2;
pset = {p1, p2, p3};
ord = {S, x1, x2};
const = {a, b, c};
```

Characteristic Set and Characteristic For

```
defcs = CharacteristicSet[pset, ord, const, TracePrintOn → True]
```

{CS_STEP:1, $\{-2\,S + c\,x_2\}$}

$\{CS\_STEP:2, \{-b^2\,c^2 + 4\,S^2 + c^2\,x_1^2, -2\,S + c\,x_2\}\}$

$\{CS\_STEP:3, \{c^2\,(-a^2 + b^2 + c\,(c - 2\,x_1)), -2\,S + c\,x_2\}\}$

$\{A\ New\ Component:1, a^2 - b^2 - c^2 + 2\,c\,x_1\}$

$\{CS\_STEP:4, \{c^2\,(a^4 + b^4 - 2\,b^2\,c^2 + c^4 - 2\,a^2\,(b^2 + c^2) + 16\,S^2), c^2\,(-a^2 + b^2 + c\,(c - 2\,x_1)), -2\,S + c\,x_2\}\}$

$\{A\ New\ Component:1, a^4 - 2\,a^2\,b^2 + b^4 - 2\,a^2\,c^2 - 2\,b^2\,c^2 + c^4 + 16\,S^2\}$

$\{A\ New\ Component:1, a^2 - b^2 - c^2 + 2\,c\,x_1\}$

{Total 3 Branch(s) of New Component(s) Discovered}

$\{a^4\,c^2 - 2\,a^2\,b^2\,c^2 + b^4\,c^2 - 2\,a^2\,c^4 - 2\,b^2\,c^4 + c^6 + 16\,c^2\,S^2, -a^2\,c^2 + b^2\,c^2 + c^4 - 2\,c^3\,x_1, -2\,S + c\,x_2\}$

```
CharacteristicForm[defcs, ord, Padding → "0"]
```

$$\begin{pmatrix} a^4\,c^2 - 2\,a^2\,b^2\,c^2 + b^4\,c^2 - 2\,a^2\,c^4 - 2\,b^2\,c^4 + c^6 + 16\,c^2\,S^2 & \{a, b, c, S, 0, 0\} \\ -a^2\,c^2 + b^2\,c^2 + c^4 - 2\,c^3\,x_1 & \{a, b, c, 0, x_1, 0\} \\ -2\,S + c\,x_2 & \{0, 0, c, S, 0, x_2\} \end{pmatrix}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[defcs, ord] // FullSimplify
```

$$\left\{\left\{S \to -\frac{1}{4}\sqrt{(-(a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c))},\ x_1 \to \frac{-a^2+b^2+c^2}{2\,c},\ x_2 \to -\frac{1}{2\,c}\left(\sqrt{(-(a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c))}\right)\right\},\ \left\{S \to \frac{1}{4}\sqrt{(-(a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c))},\ x_1 \to \frac{-a^2+b^2+c^2}{2\,c},\ x_2 \to \frac{1}{2\,c}\left(\sqrt{(-(a-b-c)\,(a+b-c)\,(a-b+c)\,(a+b+c))}\right)\right\}\right\}$$

CTP_II: Deduction of Side Relation in Parallelogram

Polynomial System with ord and const

```
p1 = a12 - u12 v12;
p2 = a13 - u13 v13;
p3 = a14 - u14 v14;
p4 = a23 - (u13 - u12) (v13 - v12);
p5 = a24 - (u14 - u12) (v14 - v12);
p6 = a34 - (u14 - u13) (v14 - v13);
pset = {p1, p2, p3, p4, p5, p6};
ord = {u14, u13, u12, v12, v13, v14};
```

Characteristic Set and Characteristic For

```
defcs = CharacteristicSet[pset, ord, const, TracePrintOn → True]
```

{CS_STEP:1, $\{a_{12} - u_{12}\, v_{12},\ a_{13} - u_{13}\, v_{13},\ a_{14} - u_{14}\, v_{14}\}$}

{CS_STEP:2, $\{a_{13}\, u_{14}\, (-u_{13} + u_{14}) + u_{13}\, (a_{14}\, (u_{13} - u_{14}) + a_{34}\, u_{14}),$
$a_{12}\, u_{13}\, (-u_{12} + u_{13}) + u_{12}\, (a_{13}\, (u_{12} - u_{13}) + a_{23}\, u_{13}),\ a_{12} - u_{12}\, v_{12},\ a_{13} - u_{13}\, v_{13},\ a_{14} - u_{14}\, v_{14}\}$}

{CS_STEP:3, $\{a_{13}\, u_{14}\, (-u_{13} + u_{14}) + u_{13}\, (a_{14}\, (u_{13} - u_{14}) + a_{34}\, u_{14}),\ u_{12}\, (-a_{14}\, a_{23}\, u_{13} + a_{13}\, (a_{14}\, (u_{13} - u_{14}) + a_{24}\, u_{14})) +$
$a_{12}\, (a_{13}\, u_{14}\, (-u_{12} - u_{13} + 2\, u_{14}) + u_{13}\, (a_{14}\, (u_{12} - u_{14}) + a_{34}\, u_{14})),\ a_{12} - u_{12}\, v_{12},\ a_{13} - u_{13}\, v_{13},\ a_{14} - u_{14}\, v_{14}\}$}

{CS_STEP:4, $\Big\{-a_{12}\, a_{13}\, \left(a_{13}^2\, a_{24} + a_{12}^2\, a_{34} + a_{23}\, \left(a_{14}^2 + a_{14}\, (a_{23} - a_{24} - a_{34}) + a_{24}\, a_{34}\right) +$
$a_{12}\, (a_{13}\, (a_{23} - a_{24} - a_{34}) - (a_{23} + a_{24} - a_{34})\, a_{34} - a_{14}\, (a_{23} - a_{24} + a_{34})) - a_{13}\, (a_{14}\, (a_{23} + a_{24} - a_{34}) + a_{24}\, (a_{23} - a_{24} + a_{34}))\right)$
$u_{14}^3\, ((-a_{14} + a_{34})\, u_{13} + a_{13}\, (-u_{13} + u_{14})),\ u_{12}\, (-a_{14}\, a_{23}\, u_{13} + a_{13}\, (a_{14}\, (u_{13} - u_{14}) + a_{24}\, u_{14})) +$
$a_{12}\, (a_{13}\, u_{14}\, (-u_{12} - u_{13} + 2\, u_{14}) + u_{13}\, (a_{14}\, (u_{12} - u_{14}) + a_{34}\, u_{14})),\ a_{12} - u_{12}\, v_{12},\ a_{13} - u_{13}\, v_{13},\ a_{14} - u_{14}\, v_{14}\Big\}$}

{A New Component:1, a_{13}}

$\Big\{$A New Component:2,
$a_{12}\, a_{13}\, a_{23} - a_{12}\, a_{14}\, a_{23} - a_{13}\, a_{14}\, a_{23} + a_{14}^2\, a_{23} + a_{14}\, a_{23}^2 - a_{12}\, a_{13}\, a_{24} + a_{13}^2\, a_{24} + a_{12}\, a_{14}\, a_{24} - a_{13}\, a_{14}\, a_{24} - a_{13}\, a_{23}\, a_{24} - a_{14}\, a_{23}\, a_{24} +$
$a_{13}\, a_{24}^2 + a_{12}^2\, a_{34} - a_{12}\, a_{13}\, a_{34} - a_{12}\, a_{14}\, a_{34} + a_{13}\, a_{14}\, a_{34} - a_{12}\, a_{23}\, a_{34} - a_{14}\, a_{23}\, a_{34} - a_{12}\, a_{24}\, a_{34} - a_{13}\, a_{24}\, a_{34} + a_{23}\, a_{24}\, a_{34} + a_{12}\, a_{34}^2\Big\}$

$\Big\{$A New Component:3, $u_{14}^3\Big\}$

{A New Component:4, $a_{13}\, u_{13} + a_{14}\, u_{13} - a_{34}\, u_{13} - a_{13}\, u_{14}$}

$\Big\{$CS_STEP:5, $\Big\{a_{12}\, a_{13}^3\, a_{14}\, \left(a_{13}^2\, a_{24} + a_{12}^2\, a_{34} + a_{23}\, \left(a_{14}^2 + a_{14}\, (a_{23} - a_{24} - a_{34}) + a_{24}\, a_{34}\right) +$
$a_{12}\, (a_{13}\, (a_{23} - a_{24} - a_{34}) - (a_{23} + a_{24} - a_{34})\, a_{34} - a_{14}\, (a_{23} - a_{24} + a_{34})) - a_{13}\, (a_{14}\, (a_{23} + a_{24} - a_{34}) + a_{24}\, (a_{23} - a_{24} + a_{34}))\right)\, u_{14}^5,$
$-a_{12}\, a_{13}\, \left(a_{13}^2\, a_{24} + a_{12}^2\, a_{34} + a_{23}\, \left(a_{14}^2 + a_{14}\, (a_{23} - a_{24} - a_{34}) + a_{24}\, a_{34}\right) +$
$a_{12}\, (a_{13}\, (a_{23} - a_{24} - a_{34}) - (a_{23} + a_{24} - a_{34})\, a_{34} - a_{14}\, (a_{23} - a_{24} + a_{34})) - a_{13}\, (a_{14}\, (a_{23} + a_{24} - a_{34}) + a_{24}\, (a_{23} - a_{24} + a_{34}))\right)$
$u_{14}^3\, ((-a_{14} + a_{34})\, u_{13} + a_{13}\, (-u_{13} + u_{14})),\ u_{12}\, (-a_{14}\, a_{23}\, u_{13} + a_{13}\, (a_{14}\, (u_{13} - u_{14}) + a_{24}\, u_{14})) +$
$a_{12}\, (a_{13}\, u_{14}\, (-u_{12} - u_{13} + 2\, u_{14}) + u_{13}\, (a_{14}\, (u_{12} - u_{14}) + a_{34}\, u_{14})),\ a_{12} - u_{12}\, v_{12},\ a_{13} - u_{13}\, v_{13},\ a_{14} - u_{14}\, v_{14}\Big\}\Big\}$

$\Big\{$A New Component:1, $a_{13}^3\Big\}$

{A New Component:2, a_{14}}

$\Big\{$A New Component:3,
$a_{12}\, a_{13}\, a_{23} - a_{12}\, a_{14}\, a_{23} - a_{13}\, a_{14}\, a_{23} + a_{14}^2\, a_{23} + a_{14}\, a_{23}^2 - a_{12}\, a_{13}\, a_{24} + a_{13}^2\, a_{24} + a_{12}\, a_{14}\, a_{24} - a_{13}\, a_{14}\, a_{24} - a_{13}\, a_{23}\, a_{24} - a_{14}\, a_{23}\, a_{24} +$
$a_{13}\, a_{24}^2 + a_{12}^2\, a_{34} - a_{12}\, a_{13}\, a_{34} - a_{12}\, a_{14}\, a_{34} + a_{13}\, a_{14}\, a_{34} - a_{12}\, a_{23}\, a_{34} - a_{14}\, a_{23}\, a_{34} - a_{12}\, a_{24}\, a_{34} - a_{13}\, a_{24}\, a_{34} + a_{23}\, a_{24}\, a_{34} + a_{12}\, a_{34}^2\Big\}$

$\Big\{$A New Component:4, $u_{14}^5\Big\}$

{A New Component:1, a_{13}}

$\Big\{$A New Component:2,
$a_{12}\, a_{13}\, a_{23} - a_{12}\, a_{14}\, a_{23} - a_{13}\, a_{14}\, a_{23} + a_{14}^2\, a_{23} + a_{14}\, a_{23}^2 - a_{12}\, a_{13}\, a_{24} + a_{13}^2\, a_{24} + a_{12}\, a_{14}\, a_{24} - a_{13}\, a_{14}\, a_{24} - a_{13}\, a_{23}\, a_{24} - a_{14}\, a_{23}\, a_{24} +$
$a_{13}\, a_{24}^2 + a_{12}^2\, a_{34} - a_{12}\, a_{13}\, a_{34} - a_{12}\, a_{14}\, a_{34} + a_{13}\, a_{14}\, a_{34} - a_{12}\, a_{23}\, a_{34} - a_{14}\, a_{23}\, a_{34} - a_{12}\, a_{24}\, a_{34} - a_{13}\, a_{24}\, a_{34} + a_{23}\, a_{24}\, a_{34} + a_{12}\, a_{34}^2\Big\}$

$\Big\{$A New Component:3, $u_{14}^3\Big\}$

{A New Component:4, $a_{13}\, u_{13} + a_{14}\, u_{13} - a_{34}\, u_{13} - a_{13}\, u_{14}$}

{Total 3 Branch(s) of New Component(s) Discovered}

```
{a12 a14 u12 u13 + a13 a14 u12 u13 − a14 a23 u12 u13 − a12 a13 u12 u14 − a13 a14 u12 u14 +
   a13 a24 u12 u14 − a12 a13 u13 u14 − a12 a14 u13 u14 + a12 a34 u13 u14 + 2 a12 a13 u14^2,
 a12^2 a13^3 a23 u13 u14^3 − a12 a13^3 a14 a23 u13 u14^3 − a12^2 a13 a14^2 a23 u13 u14^3 + a12 a13 a14^3 a23 u13 u14^3 +
  a12 a13^2 a14 a23^2 u13 u14^3 + a12 a13 a14^2 a23^2 u13 u14^3 − a12^2 a13^3 a24 u13 u14^3 + a12 a13^4 a24 u13 u14^3 +
  a12^2 a13 a14^2 a24 u13 u14^3 − a12 a13^2 a14^2 a24 u13 u14^3 − a12 a13^3 a23 a24 u13 u14^3 − 2 a12 a13^2 a14 a23 a24 u13 u14^3 −
  a12 a13 a14^2 a23 a24 u13 u14^3 + a12 a13^3 a24^2 u13 u14^3 + a12 a13^2 a14 a24^2 u13 u14^3 + a12^2 a13^2 a34 u13 u14^3 −
  a12^2 a13^3 a34 u13 u14^3 + a12^2 a13 a14 a34 u13 u14^3 − 2 a12^2 a13^2 a14 a34 u13 u14^3 + a12 a13^3 a14 a34 u13 u14^3 −
  a12^2 a13 a14^2 a34 u13 u14^3 + a12 a13^2 a14^2 a34 u13 u14^3 − 2 a12^2 a13^2 a23 a34 u13 u14^3 − 2 a12 a13 a14^2 a23 a34 u13 u14^3 −
  a12 a13 a14 a23^2 a34 u13 u14^3 − 2 a12 a13^3 a24 a34 u13 u14^3 − 2 a12^2 a13 a14 a24 a34 u13 u14^3 +
  2 a12 a13^2 a23 a24 a34 u13 u14^3 + 2 a12 a13 a14 a23 a24 a34 u13 u14^3 − a12 a13^2 a24^2 a34 u13 u14^3 −
  a12^2 a13^2 a34^2 u13 u14^3 + 2 a12^2 a13 a14 a34^2 u13 u14^3 + 2 a12^2 a13 a14 a34^2 u13 u14^3 − a12 a13^2 a14 a34^2 u13 u14^3 +
  a12^2 a13 a23 a34^2 u13 u14^3 + a12 a13 a14 a23 a34^2 u13 u14^3 + a12^2 a13 a24 a34^2 u13 u14^3 + a12 a13^2 a24 a34^2 u13 u14^3 −
  a12 a13 a23 a24 a34^2 u13 u14^3 − a12^2 a13 a34^3 u13 u14^3 − a12^2 a13^2 a23 u14^4 + a12^2 a13^2 a14 a23 u14^4 +
  a12 a13^3 a14 a23 u14^4 − a12 a13^2 a14^2 a23 u14^4 − a12 a13^2 a14 a23^2 u14^4 + a12^2 a13^3 a24 u14^4 − a12 a13^4 a24 u14^4 −
  a12^2 a13^2 a14 a24 u14^4 + a12 a13^3 a14 a24 u14^4 + a12 a13^3 a23 a24 u14^4 + a12 a13^2 a14 a23 a24 u14^4 − a12 a13^3 a24^2 u14^4 −
  a12^3 a13^2 a34 u14^4 + a12^2 a13^3 a34 u14^4 + a12^2 a13^2 a14 a34 u14^4 − a12 a13^3 a14 a34 u14^4 + a12^2 a13^2 a23 a34 u14^4 +
  a12 a13^2 a14 a23 a34 u14^4 + a12^2 a13^2 a24 a34 u14^4 + a12 a13^3 a24 a34 u14^4 − a12 a13^2 a23 a24 a34 u14^4 − a12^2 a13^2 a34^2 u14^4,
 a12^2 a13^4 a14 a23 u14^5 − a12^2 a13^3 a14^2 a23 u14^5 − a12 a13^4 a14^2 a23 u14^5 + a12 a13^3 a14^3 a23 u14^5 +
  a12 a13^3 a14^2 a23^2 u14^5 − a12^2 a13^4 a14 a24 u14^5 + a12 a13^5 a14 a24 u14^5 + a12^2 a13^3 a14^2 a24 u14^5 −
  a12 a13^4 a14^2 a24 u14^5 − a12 a13^4 a14 a23 a24 u14^5 − a12 a13^3 a14^2 a23 a24 u14^5 + a12 a13^4 a14 a24^2 u14^5 +
  a12^3 a13^3 a14 a34 u14^5 − a12^2 a13^4 a14 a34 u14^5 − a12^2 a13^3 a14^2 a34 u14^5 + a12 a13^4 a14^2 a34 u14^5 −
  a12^2 a13^3 a14 a23 a34 u14^5 − a12 a13^3 a14^2 a23 a34 u14^5 − a12^2 a13^3 a14 a24 a34 u14^5 − a12 a13^4 a14 a24 a34 u14^5 +
  a12 a13^3 a14 a23 a24 a34 u14^5 + a12^2 a13^3 a14 a34^2 u14^5, a12 − u12 v12, a13 − u13 v13, a14 − u14 v14}
```

```
CharacteristicForm[defcs, ord, Padding → "0"]
```

```
a12^2 a13^3 a23 u13 u14^3 − a12 a13^3 a14 a23 u13 u14^3 − a12^2 a13 a14^2 a23 u13 u14^3 + a12 a13 a14^3 a23 u13 u14^3 + a12 a13^2 a14 a23^2 u13
```

CTP_III: Testing Problem on Symmetric Polynomial Systems

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[#, {x1, x2, x3, x4}] &@
 CharacteristicSet[{x1 x2 x3 + x4, x1 x3 x4 + x2, x1 x2 x4 + x3, x2 x3 x4 + x1}, {x1, x2, x3, x4},
  TracePrintOn → True]
```

{CS_STEP:1, $\{x_1 x_2 x_3 + x_4\}$}

$\{$CS_STEP:2, $\{x_3 - x_1^2 x_2^2 x_3,\ x_1 x_2 x_3 + x_4\}\}$

{A New Component:1, $1 + x_1 x_2$}

{A New Component:2, x_3}

$\{$CS_STEP:3, $\{x_1 (-1 + x_1^2 x_2^2),\ x_3 - x_1^2 x_2^2 x_3,\ x_1 x_2 x_3 + x_4\}\}$

{A New Component:1, $-1 + x_1 x_2$}

{A New Component:2, $1 + x_1 x_2$}

{A New Component:1, $1 + x_1 x_2$}

{A New Component:2, x_3}

{Total 3 Branch(s) of New Component(s) Discovered}

$$\left\{\left\{x_2 \to -\frac{1}{x_1},\ x_3 \to 0,\ x_4 \to 0\right\},\ \left\{x_2 \to \frac{1}{x_1},\ x_3 \to 0,\ x_4 \to 0\right\}\right\}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[#, {x1, x2, x3, x4}] &@
 CharacteristicSet[{x1 x3 + x4 x2, x1 x4 + x2 x3, x1 x2 + x3 x4, x1 x3 + x2 x4}, {x1, x2, x3, x4},
  TracePrintOn → True]
```

{CS_STEP:1, $\{x_2 x_3 + x_1 x_4\}$}

$\{$CS_STEP:2, $\{(x_1^2 - x_2^2) x_3,\ x_2 x_3 + x_1 x_4\}\}$

{A New Component:1, $x_1 + x_2$}

{A New Component:2, x_3}

$\{$CS_STEP:3, $\{-x_1^4 x_2 + x_1^2 x_2^3,\ (x_1^2 - x_2^2) x_3,\ x_2 x_3 + x_1 x_4\}\}$

{A New Component:1, $x_1 - x_2$}

{A New Component:2, x_2}

{A New Component:3, $x_1 + x_2$}

{A New Component:1, $x_1 + x_2$}

{A New Component:2, x_3}

{Total 3 Branch(s) of New Component(s) Discovered}

$$\{\{x_2 \to 0,\ x_3 \to 0,\ x_4 \to 0\},\ \{x_2 \to -x_1,\ x_3 \to 0,\ x_4 \to 0\},\ \{x_2 \to x_1,\ x_3 \to 0,\ x_4 \to 0\}\}$$

CTP_IV: Some Testing Problem from [5]

Problem 1. $PS = \{x_4^2+x_1x_4^2-x_2x_4-x_1x_2x_4+x_1x_2+3x_2, x_1x_4+x_3-x_1x_2, x_3x_4-2x_2^2-$
with variable ordering $x_1 \prec \cdots \prec x_4$.

Problem 2. $PS = \{p_1, \ldots, p_8\}$ with variable ordering $b \prec c_2 \prec c_3 \prec a \prec b_3 \prec b_2 \prec$
where $p_1 = b_1 + b_2 + b_3 - a - b$, $p_2 = 2b_2c_2 + 2b_3c_3 - 1 - b - 2b^2 + 2ab$,
$p_3 = 3b_2c_2^2 + 3b_3c_3^2 - a - 3ab^2 + 4b + 3b^2 + 3b^3$,
$p_4 = 6b_3a_{32}c_2 - a - 3ab - 6ab^2 + 4b + 6b^2 + 6b^3$,
$p_5 = 4b_2c_2^3 + 4b_3c_3^3 - 1 - b - 10b^2 - 6b^3 - 4b^4 + 4ab + 4ab^3$,
$p_6 = 8b_3c_3a_{32}c_2 - 1 - 3b - 14b^2 - 12b^3 - 8b^4 + 4ab + 4ab^2 + 8ab^3$,
$p_7 = 12b_3a_{32}c_2^2 - 1 - b - 14b^2 - 18b^3 - 12b^4 + 8ab + 12ab^2 + 12ab^3$,
$p_8 = 1 + 7b + 26b^2 + 36b^3 + 24b^4 - 8ab - 24ab^2 - 24ab^3$.

Problem 3. $PS = \{y^2 - p_1, \frac{\partial p_2}{\partial x_1}, \frac{\partial p_2}{\partial x_2}, \frac{\partial p_2}{\partial x_3}, \frac{\partial p_2}{\partial x_4}, \frac{\partial p_2}{\partial x_5}, \frac{\partial p_2}{\partial x_6}, \frac{\partial p_2}{\partial \lambda_1}, \frac{\partial p_2}{\partial \lambda_2}, \frac{\partial p_2}{\partial \lambda_3}\}$ with variab
ing $x_1 \prec \cdots \prec x_6 \prec \lambda_1 \prec \lambda_2 \prec \lambda_3 \prec y$, where $p_1 = (x_4 + x_5)(x_5 + x_6)(x_6 + x_4$
$p_2 = p_1 + \lambda_1(x_2^2x_6 - 1) + \lambda_2(x_1^2x_4 - 1) + \lambda_3(x_3^2x_5 - 1)$.

Problem 4. $PS = \{x^2 + y^2 + z^2 - r^2, xy + z^2 - 1, xyz - x^2 - y^2 - z + 1\}$ with variable
$r \prec x \prec y \prec z$.

Problem 5. $AS = \{a^4 + a^3 + a^2 + a + 1\}$ and $F = 16x^4 + 8x^3 + 4x^2 + 2x + 1$.

Problem 6. $AS = \{-1 + b + 6b^2 + 12b^3\}$ and $F = 745092b - 252156 + 540900c + 210326$
$2010720b^2+7117713c^2b-132367c^2+3076830c^3-7843500c^3b^2+2792322c^3b-3779244bc-107$
$21225240bc^5 + 26306208b^2c^5 + 8257464c^5 - 436536c^4 + 6094008b^2c^4 + 594432bc^4$.

Problem 7. $AS = \{r^2 - 2 + z^2, -rz + y + 4y^2\}$ and $F = -370x^2y - 10x^3 + 60x^2z + 4xy$
$74rzy + 2rzx + 37rz - 37y + 12r^3 - 24r$ with variable ordering $z \prec y \prec x$.

Figure 2

■ Problem 1

```
ps = {x4^2 + x1 x4^2 - x2 x4 - x1 x2 x4 + x1 x2 + 3 x2, x1 x4 + x3 - x1 x2, x3 x4 - 2 x2^2 - x1 x2 - 1};
ord = {x1, x2, x3, x4};
defcs = CharacteristicSet[ps, ord, TracePrintOn → True];
CharacteristicForm[defcs, ord]
WuRittEqnsSolve[defcs, ord]
```

{CS_STEP:1, $\{x_3 + x_1 (-x_2 + x_4)\}$}

$\{$CS_STEP:2, $\{-x_1 - x_1^2 x_2 - 2 x_1 x_2^2 + x_1 x_2 x_3 - x_3^2,\ x_3 + x_1 (-x_2 + x_4)\}\}$

$\{$CS_STEP:3, $\{-x_1 (1 + x_1 - 2 x_1 x_2 + 2 x_2^2 + 2 x_1 x_2^2),\ -x_1 - x_1^2 x_2 - 2 x_1 x_2^2 + x_1 x_2 x_3 - x_3^2,\ x_3 + x_1 (-x_2 + x_4)\}\}$

$\{$A New Component:1, $1 + x_1 - 2 x_1 x_2 + 2 x_2^2 + 2 x_1 x_2^2\}$

{Total 1 Branch(s) of New Component(s) Discovered}

$$\begin{pmatrix} -x_1 - x_1^2 + 2 x_1^2 x_2 - 2 x_1 x_2^2 - 2 x_1^2 x_2^2 & \{x_1, x_2, 00, 00\} \\ -x_1 - x_1^2 x_2 - 2 x_1 x_2^2 + x_1 x_2 x_3 - x_3^2 & \{x_1, x_2, x_3, 00\} \\ -x_1 x_2 + x_3 + x_1 x_4 & \{x_1, x_2, x_3, x_4\} \end{pmatrix}$$

$$\left\{\left\{x_2 \to \frac{x_1 - \sqrt{-2 - 4 x_1 - x_1^2}}{2 (1 + x_1)},\ x_3 \to \frac{x_1^2}{4 + 4 x_1} - \frac{x_1 \sqrt{-2 - 4 x_1 - x_1^2}}{4 (1 + x_1)} - \frac{1}{2 \sqrt{2}}\right.\right.$$
$$\sqrt{x_1}\ \sqrt{\left(\left(x_1 \left(-1 - 4 x_1^2 + 12 \sqrt{-2 - 4 x_1 - x_1^2}\right) + x_1 \left(-14 + 3 \sqrt{-2 - 4 x_1 - x_1^2}\right)\right)\right) / (1 + x_1)^2},$$
$$x_4 \to \frac{1}{4} \left(\frac{x_1}{1 + x_1} - \frac{\sqrt{-2 - 4 x_1 - x_1^2}}{1 + x_1} + \frac{1}{\sqrt{x_1}} \right.$$
$$\left.\left. \sqrt{2}\ \sqrt{-\left(\left(x_1 \left(1 + 4 x_1^2 - 12 \sqrt{-2 - 4 x_1 - x_1^2}\right) + x_1 \left(14 - 3 \sqrt{-2 - 4 x_1 - x_1^2}\right)\right)\right) / (1 + x_1)^2} \right) \right\},$$

$$\left\{x_2 \to \frac{x_1 + \sqrt{-2 - 4 x_1 - x_1^2}}{2 (1 + x_1)},\ x_3 \to \frac{x_1^2}{4 + 4 x_1} - \frac{x_1 \sqrt{-2 - 4 x_1 - x_1^2}}{4 (1 + x_1)} + \frac{1}{2 \sqrt{2}}\right.$$
$$\sqrt{x_1}\ \sqrt{\left(\left(x_1 \left(-1 - 4 x_1^2 + 12 \sqrt{-2 - 4 x_1 - x_1^2}\right) + x_1 \left(-14 + 3 \sqrt{-2 - 4 x_1 - x_1^2}\right)\right)\right) / (1 + x_1)^2},$$
$$x_4 \to \frac{1}{4} \left(\frac{x_1}{1 + x_1} + \frac{3 \sqrt{-2 - 4 x_1 - x_1^2}}{1 + x_1} - \frac{1}{\sqrt{x_1}} \right.$$
$$\left.\left. \sqrt{2}\ \sqrt{-\left(\left(x_1 \left(1 + 4 x_1^2 - 12 \sqrt{-2 - 4 x_1 - x_1^2}\right) + x_1 \left(14 - 3 \sqrt{-2 - 4 x_1 - x_1^2}\right)\right)\right) / (1 + x_1)^2} \right) \right\}\right\}$$

■ Problem 2

```
p1 = b1 + b2 + b3 - a - b;
p2 = 2 b2 c2 + 2 b3 c3 - 1 - b - 2 b^3 + 2 a b;
p3 = 3 b2 c2^2 + 3 b3 c3^2 - a - 3 a b^2 + 4 b + 3 b^2 + 3 b^3;
p4 = 6 b3 a32 c2 - a - 3 a b - 6 a b^2 + 4 b + 6 b^2 + 6 b^3;
p5 = 4 b2 c2^3 + 4 b3 c3^3 - 1 - b - 10 b^2 - 6 b^3 - 4 b^4 + 4 a b + 4 a b^3;
p6 = 8 b3 c3 a32 c2 - 1 - 3 b - 14 b^2 - 12 b^3 - 8 b^4 + 4 a b + 4 a b^2 + 8 a b^3;
p7 = 12 b3 a32 c2^2 - 1 - b - 14 b^2 - 18 b^3 - 12 b^4 + 8 a b + 12 a b^2 + 12 a b^3;
p8 = 1 + 7 b + 26 b^2 + 36 b^3 + 24 b^4 - 8 a b - 24 a b^2 - 24 a b^3;
ps = {p1, p2, p3, p4, p5, p6, p7, p8};
ord = {b, c2, c3, a, b3, b2, a32, b1};
```

```
defcs = CharacteristicSet[ps, ord, TracePrintOn → True]
```

$$\{\text{CS\_STEP:1},\ \{1+(7-8\,a)\,b+(26-24\,a)\,b^2+(36-24\,a)\,b^3+24\,b^4,$$
$$-1+(-1+2\,a)\,b-2\,b^3+2\,b_2\,c_2+2\,b_3\,c_3,\ -a+(4-3\,a)\,b-6\,(-1+a)\,b^2+6\,b^3+6\,a_{32}\,b_3\,c_2,\ -a-b+b_1+b_2+b_3\}\}$$

$$\{\text{CS\_STEP:2},$$
$$\{(1+b)\,(54\,b^3+36\,b^4+c_2+9\,b\,c_2+6\,b^2\,(3+2\,c_2)),\ -(1+b)\,(42\,b^3+108\,b^4+72\,b^5+b^2\,(9-12\,c_3)+b\,(3-9\,c_3)-c_3),$$
$$1+(7-8\,a)\,b+(26-24\,a)\,b^2+(36-24\,a)\,b^3+24\,b^4,\ -1+36\,b^5+b^4\,(90-48\,c_2)+72\,b^6\,c_2+$$
$$b^3\,(63-6\,c_2-72\,b_3\,c_2\,c_3+72\,b_3\,c_3^2)+b\,(-7+24\,b_3\,c_3^2+c_2\,(9-24\,b_3\,c_3))+3\,b^2\,(1+24\,b_3\,c_3^2+c_2\,(9-24\,b_3\,c_3)),$$
$$-1+(-1+2\,a)\,b-2\,b^3+2\,b_2\,c_2+2\,b_3\,c_3,\ -a+(4-3\,a)\,b-6\,(-1+a)\,b^2+6\,b^3+6\,a_{32}\,b_3\,c_2,\ -a-b+b_1+b_2+b_3\}\}$$

$$\{\text{A New Component:1},\ 18\,b^2+54\,b^3+36\,b^4+c_2+9\,b\,c_2+12\,b^2\,c_2\}$$

$$\{\text{A New Component:1},\ 3\,b+9\,b^2+42\,b^3+108\,b^4+72\,b^5-c_3-9\,b\,c_3-12\,b^2\,c_3\}$$

$$\{\text{CS\_STEP:3},$$
$$\{18\,b^2\,(1+b)^3\,(-1-16\,b-58\,b^2+21\,b^3+612\,b^4+1692\,b^5+1512\,b^6-2952\,b^7-8640\,b^8-3456\,b^9+10\,368\,b^{10}+10\,368\,b^{11}),$$
$$(1+b)\,(54\,b^3+36\,b^4+c_2+9\,b\,c_2+6\,b^2\,(3+2\,c_2)),\ -(1+b)\,(42\,b^3+108\,b^4+72\,b^5+b^2\,(9-12\,c_3)+b\,(3-9\,c_3)-c_3),$$
$$1+(7-8\,a)\,b+(26-24\,a)\,b^2+(36-24\,a)\,b^3+24\,b^4,\ -1+36\,b^5+b^4\,(90-48\,c_2)+72\,b^6\,c_2+$$
$$b^3\,(63-6\,c_2-72\,b_3\,c_2\,c_3+72\,b_3\,c_3^2)+b\,(-7+24\,b_3\,c_3^2+c_2\,(9-24\,b_3\,c_3))+3\,b^2\,(1+24\,b_3\,c_3^2+c_2\,(9-24\,b_3\,c_3)),$$
$$-1+(-1+2\,a)\,b-2\,b^3+2\,b_2\,c_2+2\,b_3\,c_3,\ -a+(4-3\,a)\,b-6\,(-1+a)\,b^2+6\,b^3+6\,a_{32}\,b_3\,c_2,\ -a-b+b_1+b_2+b_3\}\}$$

$$\{\text{A New Component:1},\ (1+b)^3\}$$

$$\{\text{A New Component:2},\ 1+2\,b\}$$

$$\{\text{A New Component:3},\ 1+3\,b+3\,b^2\}$$

$$\{\text{A New Component:4},\ -1-11\,b+6\,b^2+96\,b^3+144\,b^4+72\,b^5-720\,b^6-864\,b^7+1728\,b^8\}$$

$$\{\text{A New Component:1},\ 18\,b^2+54\,b^3+36\,b^4+c_2+9\,b\,c_2+12\,b^2\,c_2\}$$

$$\{\text{A New Component:1},\ 3\,b+9\,b^2+42\,b^3+108\,b^4+72\,b^5-c_3-9\,b\,c_3-12\,b^2\,c_3\}$$

$$\{\text{Total 5 Branch(s) of New Component(s) Discovered}\}$$

$$\{1+7\,b-8\,a\,b+26\,b^2-24\,a\,b^2+36\,b^3-24\,a\,b^3+24\,b^4,$$
$$-18\,b^2-342\,b^3-1962\,b^4-3636\,b^5+8730\,b^6+63\,594\,b^7+152\,010\,b^8+130\,896\,b^9-$$
$$202\,824\,b^{10}-660\,960\,b^{11}-519\,696\,b^{12}+404\,352\,b^{13}+1\,057\,536\,b^{14}+746\,496\,b^{15}+186\,624\,b^{16},$$
$$-a-b+b_1+b_2+b_3,\ 18\,b^2+72\,b^3+90\,b^4+36\,b^5+c_2+10\,b\,c_2+21\,b^2\,c_2+12\,b^3\,c_2,$$
$$-a+4\,b-3\,a\,b+6\,b^2-6\,a\,b^2+6\,b^3+6\,a_{32}\,b_3\,c_2,$$
$$-3\,b-12\,b^2-51\,b^3-150\,b^4-180\,b^5-72\,b^6+c_3+10\,b\,c_3+21\,b^2\,c_3+12\,b^3\,c_3,$$
$$-1-b+2\,a\,b-2\,b^3+2\,b_2\,c_2+2\,b_3\,c_3,\ -1-7\,b+3\,b^2+63\,b^3+90\,b^4+36\,b^5+9\,b\,c_2+27\,b^2\,c_2-6\,b^3\,c_2-$$
$$48\,b^4\,c_2+72\,b^6\,c_2-24\,b\,b_3\,c_2\,c_3-72\,b^2\,b_3\,c_2\,c_3-72\,b^3\,b_3\,c_2\,c_3+24\,b\,b_3\,c_3^2+72\,b^2\,b_3\,c_3^2+72\,b^3\,b_3\,c_3^2\}$$

```
WuRittEqnsSolve[defcs, ord]
```

```
$Aborted
```

■ Problem 3

$$p1 = (x_4+x_5)\,(x_5+x_6)\,(x_6+x_4)\,x_2^2\,x_1^2\,x_3^2;$$
$$p2 = p1+\lambda_1\,(x_2^2\,x_6-1)+\lambda_2\,(x_1^2\,x_4-1)+\lambda_3\,(x_3^2\,x_5-1);$$
$$ps = \{y^2-p1,\ \partial_{x_1}p2,\ \partial_{x_2}p2,\ \partial_{x_3}p2,\ \partial_{x_4}p2,\ \partial_{x_5}p2,\ \partial_{x_6}p2,\ \partial_{\lambda_1}p2,\ \partial_{\lambda_2}p2,\ \partial_{\lambda_3}p2\};$$
$$ord = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ y\};$$

```
CharacteristicSet[ps, ord, TracePrintOn → True, MaxSteps → 10]
```

$$\{\text{CS\_STEP:1}, \{y^2 - x_1^2 x_2^2 x_3^2 (x_4 + x_5) (x_4 + x_6) (x_5 + x_6), -1 + x_1^2 x_4, -1 + x_3^2 x_5, -1 + x_2^2 x_6, x_2^2 (x_1^2 x_3^2 (x_4 + x_5) (x_4 + x_5 + 2 x_6) + \lambda_1), x_1^2 (x_2^2 x_3^2 (x_5 + x_6) (2 x_4 + x_5 + x_6) + \lambda_2), x_3^2 (x_1^2 x_2^2 (x_4 + x_6) (x_4 + 2 x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1^2 x_3^2 x_4^2 + 2 x_1^2 x_3^2 x_4 x_5 + x_1^2 x_3^2 x_5^2 + 2 x_1^2 x_3^2 x_4 x_6 + 2 x_1^2 x_3^2 x_5 x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2 x_2^2 x_3^2 x_4 x_5 + x_2^2 x_3^2 x_5^2 + 2 x_2^2 x_3^2 x_4 x_6 + 2 x_2^2 x_3^2 x_5 x_6 + x_2^2 x_3^2 x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2 x_2^2 x_4^2 + 2 x_1^2 x_2^2 x_4 x_5 + 2 x_1^2 x_2^2 x_4 x_6 + 2 x_1^2 x_2^2 x_5 x_6 + x_1^2 x_2^2 x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:2}, \{y^2 - x_1^2 x_2^2 x_3^2 (x_4 + x_5) (x_4 + x_6) (x_5 + x_6), -2 x_1^4 x_3^2 + 2 x_2^4 x_3^2 + 2 x_1^2 (x_2^4 - x_3^4), -1 + x_1^2 x_4, -1 + x_3^2 x_5, -1 + x_2^2 x_6, x_2^2 (x_1^2 x_3^2 (x_4 + x_5) (x_4 + x_5 + 2 x_6) + \lambda_1), x_1^2 (x_2^2 x_3^2 (x_5 + x_6) (2 x_4 + x_5 + x_6) + \lambda_2), x_3^2 (x_1^2 x_2^2 (x_4 + x_6) (x_4 + 2 x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_2^2 + x_1 x_3\}$$

$$\{\text{A New Component:2}, x_1^2 + x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2 x_3^2 x_4^2 + 2 x_1^2 x_3^2 x_4 x_5 + x_1^2 x_3^2 x_5^2 + 2 x_1^2 x_3^2 x_4 x_6 + 2 x_1^2 x_3^2 x_5 x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2 x_2^2 x_3^2 x_4 x_5 + x_2^2 x_3^2 x_5^2 + 2 x_2^2 x_3^2 x_4 x_6 + 2 x_2^2 x_3^2 x_5 x_6 + x_2^2 x_3^2 x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2 x_2^2 x_4^2 + 2 x_1^2 x_2^2 x_4 x_5 + 2 x_1^2 x_2^2 x_4 x_6 + 2 x_1^2 x_2^2 x_5 x_6 + x_1^2 x_2^2 x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:3}, \{y^2 - x_1^2 x_2^2 x_3^2 (x_4 + x_5) (x_4 + x_6) (x_5 + x_6), 2 (x_1^4 - x_2^4) (x_2^2 x_3^2 + x_1^2 (x_2^2 + x_3^2)), -1 + x_1^2 x_4, -1 + x_3^2 x_5, -1 + x_2^2 x_6, x_2^2 (x_1^2 x_3^2 (x_4 + x_5) (x_4 + x_5 + 2 x_6) + \lambda_1), x_1^2 (x_2^2 x_3^2 (x_5 + x_6) (2 x_4 + x_5 + x_6) + \lambda_2), x_3^2 (x_1^2 x_2^2 (x_4 + x_6) (x_4 + 2 x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 + x_2\}$$

$$\{\text{A New Component:2}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:3}, x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2 x_3^2 x_4^2 + 2 x_1^2 x_3^2 x_4 x_5 + x_1^2 x_3^2 x_5^2 + 2 x_1^2 x_3^2 x_4 x_6 + 2 x_1^2 x_3^2 x_5 x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2 x_2^2 x_3^2 x_4 x_5 + x_2^2 x_3^2 x_5^2 + 2 x_2^2 x_3^2 x_4 x_6 + 2 x_2^2 x_3^2 x_5 x_6 + x_2^2 x_3^2 x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2 x_2^2 x_4^2 + 2 x_1^2 x_2^2 x_4 x_5 + 2 x_1^2 x_2^2 x_4 x_6 + 2 x_1^2 x_2^2 x_5 x_6 + x_1^2 x_2^2 x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:4}, \{y^2 - x_1^2 x_2^2 x_3^2 (x_4 + x_5) (x_4 + x_6) (x_5 + x_6), 2 x_1^4 x_2^2 (x_1^8 + x_1^6 x_2^2 - x_1^2 x_2^6 - x_2^8), 2 (x_1^4 - x_2^4) (x_2^2 x_3^2 + x_1^2 (x_2^2 + x_3^2)), -1 + x_1^2 x_4, -1 + x_3^2 x_5, -1 + x_2^2 x_6, x_2^2 (x_1^2 x_3^2 (x_4 + x_5) (x_4 + x_5 + 2 x_6) + \lambda_1), x_1^2 (x_2^2 x_3^2 (x_5 + x_6) (2 x_4 + x_5 + x_6) + \lambda_2), x_3^2 (x_1^2 x_2^2 (x_4 + x_6) (x_4 + 2 x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 - x_2\}$$

$$\{\text{A New Component:2}, x_2^2\}$$

$$\{\text{A New Component:3}, x_1 + x_2\}$$

$$\{\text{A New Component:4}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:5}, x_1^2 - x_1 x_2 + x_2^2\}$$

$$\{\text{A New Component:6}, x_1^2 + x_1 x_2 + x_2^2\}$$

$$\{\text{A New Component:1}, x_1 + x_2\}$$

$$\{\text{A New Component:2}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:3}, x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2 x_3^2 x_4^2 + 2 x_1^2 x_3^2 x_4 x_5 + x_1^2 x_3^2 x_5^2 + 2 x_1^2 x_3^2 x_4 x_6 + 2 x_1^2 x_3^2 x_5 x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2 x_2^2 x_3^2 x_4 x_5 + x_2^2 x_3^2 x_5^2 + 2 x_2^2 x_3^2 x_4 x_6 + 2 x_2^2 x_3^2 x_5 x_6 + x_2^2 x_3^2 x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2 x_2^2 x_4^2 + 2 x_1^2 x_2^2 x_4 x_5 + 2 x_1^2 x_2^2 x_4 x_6 + 2 x_1^2 x_2^2 x_5 x_6 + x_1^2 x_2^2 x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:5}, \{y^2 - x_1^2 x_2^2 x_3^2 (x_4 + x_5) (x_4 + x_6) (x_5 + x_6), 2 x_1^4 x_2^2 (x_1^8 + x_1^6 x_2^2 - x_1^2 x_2^6 - x_2^8), 2 (x_1^4 - x_2^4) (x_2^2 x_3^2 + x_1^2 (x_2^2 + x_3^2)), -1 + x_1^2 x_4, -1 + x_3^2 x_5, -1 + x_2^2 x_6, x_2^2 (x_1^2 x_3^2 (x_4 + x_5) (x_4 + x_5 + 2 x_6) + \lambda_1), x_1^2 (x_2^2 x_3^2 (x_5 + x_6) (2 x_4 + x_5 + x_6) + \lambda_2), x_3^2 (x_1^2 x_2^2 (x_4 + x_6) (x_4 + 2 x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 - x_2\}$$

$$\{\text{A New Component:2}, x_2^2\}$$

$$\{\text{A New Component:3}, x_1 + x_2\}$$

$$\{\text{A New Component:4}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:5}, x_1^2 - x_1 x_2 + x_2^2\}$$

$$\{\text{A New Component:6}, x_1^2 + x_1\, x_2 + x_2^2\}$$

$$\{\text{A New Component:1}, x_1 + x_2\}$$

$$\{\text{A New Component:2}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:3}, x_1^2\, x_2^2 + x_1^2\, x_3^2 + x_2^2\, x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_3^2\, x_4^2 + 2\, x_1^2\, x_3^2\, x_4\, x_5 + x_1^2\, x_3^2\, x_5^2 + 2\, x_1^2\, x_3^2\, x_4\, x_6 + 2\, x_1^2\, x_3^2\, x_5\, x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2\, x_2^2\, x_3^2\, x_4\, x_5 + x_2^2\, x_3^2\, x_5^2 + 2\, x_2^2\, x_3^2\, x_4\, x_6 + 2\, x_2^2\, x_3^2\, x_5\, x_6 + x_2^2\, x_3^2\, x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_2^2\, x_4^2 + 2\, x_1^2\, x_2^2\, x_4\, x_5 + 2\, x_1^2\, x_2^2\, x_4\, x_6 + 2\, x_1^2\, x_2^2\, x_5\, x_6 + x_1^2\, x_2^2\, x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:6}, \{y^2 - x_1^2\, x_2^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_6)\, (x_5 + x_6), 2\, x_1^4\, x_2^2\, (x_1^8 + x_1^6\, x_2^2 - x_1^2\, x_2^6 - x_2^8),$$
$$2\, (x_1^4 - x_2^4)\, (x_2^2\, x_3^2 + x_1^2\, (x_2^2 + x_3^2)), -1 + x_1^2\, x_4, -1 + x_3^2\, x_5, -1 + x_2^2\, x_6, x_2^2\, (x_1^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_5 + 2\, x_6) + \lambda_1),$$
$$x_1^2\, (x_2^2\, x_3^2\, (x_5 + x_6)\, (2\, x_4 + x_5 + x_6) + \lambda_2), x_3^2\, (x_1^2\, x_2^2\, (x_4 + x_6)\, (x_4 + 2\, x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 - x_2\}$$

$$\{\text{A New Component:2}, x_2^2\}$$

$$\{\text{A New Component:3}, x_1 + x_2\}$$

$$\{\text{A New Component:4}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:5}, x_1^2 - x_1\, x_2 + x_2^2\}$$

$$\{\text{A New Component:6}, x_1^2 + x_1\, x_2 + x_2^2\}$$

$$\{\text{A New Component:1}, x_1 + x_2\}$$

$$\{\text{A New Component:2}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:3}, x_1^2\, x_2^2 + x_1^2\, x_3^2 + x_2^2\, x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_3^2\, x_4^2 + 2\, x_1^2\, x_3^2\, x_4\, x_5 + x_1^2\, x_3^2\, x_5^2 + 2\, x_1^2\, x_3^2\, x_4\, x_6 + 2\, x_1^2\, x_3^2\, x_5\, x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2\, x_2^2\, x_3^2\, x_4\, x_5 + x_2^2\, x_3^2\, x_5^2 + 2\, x_2^2\, x_3^2\, x_4\, x_6 + 2\, x_2^2\, x_3^2\, x_5\, x_6 + x_2^2\, x_3^2\, x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_2^2\, x_4^2 + 2\, x_1^2\, x_2^2\, x_4\, x_5 + 2\, x_1^2\, x_2^2\, x_4\, x_6 + 2\, x_1^2\, x_2^2\, x_5\, x_6 + x_1^2\, x_2^2\, x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:7}, \{y^2 - x_1^2\, x_2^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_6)\, (x_5 + x_6), 2\, x_1^4\, x_2^2\, (x_1^8 + x_1^6\, x_2^2 - x_1^2\, x_2^6 - x_2^8),$$
$$2\, (x_1^4 - x_2^4)\, (x_2^2\, x_3^2 + x_1^2\, (x_2^2 + x_3^2)), -1 + x_1^2\, x_4, -1 + x_3^2\, x_5, -1 + x_2^2\, x_6, x_2^2\, (x_1^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_5 + 2\, x_6) + \lambda_1),$$
$$x_1^2\, (x_2^2\, x_3^2\, (x_5 + x_6)\, (2\, x_4 + x_5 + x_6) + \lambda_2), x_3^2\, (x_1^2\, x_2^2\, (x_4 + x_6)\, (x_4 + 2\, x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 - x_2\}$$

$$\{\text{A New Component:2}, x_2^2\}$$

$$\{\text{A New Component:3}, x_1 + x_2\}$$

$$\{\text{A New Component:4}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:5}, x_1^2 - x_1\, x_2 + x_2^2\}$$

$$\{\text{A New Component:6}, x_1^2 + x_1\, x_2 + x_2^2\}$$

$$\{\text{A New Component:1}, x_1 + x_2\}$$

$$\{\text{A New Component:2}, x_1^2 + x_2^2\}$$

$$\{\text{A New Component:3}, x_1^2\, x_2^2 + x_1^2\, x_3^2 + x_2^2\, x_3^2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_3^2\, x_4^2 + 2\, x_1^2\, x_3^2\, x_4\, x_5 + x_1^2\, x_3^2\, x_5^2 + 2\, x_1^2\, x_3^2\, x_4\, x_6 + 2\, x_1^2\, x_3^2\, x_5\, x_6 + \lambda_1\}$$

$$\{\text{A New Component:1}, 2\, x_2^2\, x_3^2\, x_4\, x_5 + x_2^2\, x_3^2\, x_5^2 + 2\, x_2^2\, x_3^2\, x_4\, x_6 + 2\, x_2^2\, x_3^2\, x_5\, x_6 + x_2^2\, x_3^2\, x_6^2 + \lambda_2\}$$

$$\{\text{A New Component:1}, x_1^2\, x_2^2\, x_4^2 + 2\, x_1^2\, x_2^2\, x_4\, x_5 + 2\, x_1^2\, x_2^2\, x_4\, x_6 + 2\, x_1^2\, x_2^2\, x_5\, x_6 + x_1^2\, x_2^2\, x_6^2 + \lambda_3\}$$

$$\{\text{CS\_STEP:8}, \{y^2 - x_1^2\, x_2^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_6)\, (x_5 + x_6), 2\, x_1^4\, x_2^2\, (x_1^8 + x_1^6\, x_2^2 - x_1^2\, x_2^6 - x_2^8),$$
$$2\, (x_1^4 - x_2^4)\, (x_2^2\, x_3^2 + x_1^2\, (x_2^2 + x_3^2)), -1 + x_1^2\, x_4, -1 + x_3^2\, x_5, -1 + x_2^2\, x_6, x_2^2\, (x_1^2\, x_3^2\, (x_4 + x_5)\, (x_4 + x_5 + 2\, x_6) + \lambda_1),$$
$$x_1^2\, (x_2^2\, x_3^2\, (x_5 + x_6)\, (2\, x_4 + x_5 + x_6) + \lambda_2), x_3^2\, (x_1^2\, x_2^2\, (x_4 + x_6)\, (x_4 + 2\, x_5 + x_6) + \lambda_3)\}\}$$

$$\{\text{A New Component:1}, x_1 - x_2\}$$

$$\{\text{A New Component:2}, x_2^2\}$$

$$\{\text{A New Component:3}, x_1 + x_2\}$$

{A New Component:4, $x_1^2 + x_2^2$}

{A New Component:5, $x_1^2 - x_1\,x_2 + x_2^2$}

{A New Component:6, $x_1^2 + x_1\,x_2 + x_2^2$}

{A New Component:1, $x_1 + x_2$}

{A New Component:2, $x_1^2 + x_2^2$}

{A New Component:3, $x_1^2\,x_2^2 + x_1^2\,x_3^2 + x_2^2\,x_3^2$}

{A New Component:1, $x_1^2\,x_3^2\,x_4^2 + 2\,x_1^2\,x_3^2\,x_4\,x_5 + x_1^2\,x_3^2\,x_5^2 + 2\,x_1^2\,x_3^2\,x_4\,x_6 + 2\,x_1^2\,x_3^2\,x_5\,x_6 + \lambda_1$}

{A New Component:1, $2\,x_2^2\,x_3^2\,x_4\,x_5 + x_2^2\,x_3^2\,x_5^2 + 2\,x_2^2\,x_3^2\,x_4\,x_6 + 2\,x_2^2\,x_3^2\,x_5\,x_6 + x_2^2\,x_3^2\,x_6^2 + \lambda_2$}

{A New Component:1, $x_1^2\,x_2^2\,x_4^2 + 2\,x_1^2\,x_2^2\,x_4\,x_5 + 2\,x_1^2\,x_2^2\,x_4\,x_6 + 2\,x_1^2\,x_2^2\,x_5\,x_6 + x_1^2\,x_2^2\,x_6^2 + \lambda_3$}

{CS_STEP:9, $\{y^2 - x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_4 + x_6)\,(x_5 + x_6),\ 2\,x_1^4\,x_2^2\,(x_1^8 + x_1^6\,x_2^2 - x_1^2\,x_2^6 - x_2^8),$
$2\,(x_1^4 - x_2^4)\,(x_2^2\,x_3^2 + x_1^2\,(x_2^2 + x_3^2)),\ -1 + x_1^2\,x_4,\ -1 + x_3^2\,x_5,\ -1 + x_2^2\,x_6,\ x_2^2\,(x_1^2\,x_3^2\,(x_4 + x_5)\,(x_4 + x_5 + 2\,x_6) + \lambda_1),$
$x_1^2\,(x_2^2\,x_3^2\,(x_5 + x_6)\,(2\,x_4 + x_5 + x_6) + \lambda_2),\ x_3^2\,(x_1^2\,x_2^2\,(x_4 + x_6)\,(x_4 + 2\,x_5 + x_6) + \lambda_3)\}$}

{A New Component:1, $x_1 - x_2$}

{A New Component:2, x_2^2}

{A New Component:3, $x_1 + x_2$}

{A New Component:4, $x_1^2 + x_2^2$}

{A New Component:5, $x_1^2 - x_1\,x_2 + x_2^2$}

{A New Component:6, $x_1^2 + x_1\,x_2 + x_2^2$}

{A New Component:1, $x_1 + x_2$}

{A New Component:2, $x_1^2 + x_2^2$}

{A New Component:3, $x_1^2\,x_2^2 + x_1^2\,x_3^2 + x_2^2\,x_3^2$}

{A New Component:1, $x_1^2\,x_3^2\,x_4^2 + 2\,x_1^2\,x_3^2\,x_4\,x_5 + x_1^2\,x_3^2\,x_5^2 + 2\,x_1^2\,x_3^2\,x_4\,x_6 + 2\,x_1^2\,x_3^2\,x_5\,x_6 + \lambda_1$}

{A New Component:1, $2\,x_2^2\,x_3^2\,x_4\,x_5 + x_2^2\,x_3^2\,x_5^2 + 2\,x_2^2\,x_3^2\,x_4\,x_6 + 2\,x_2^2\,x_3^2\,x_5\,x_6 + x_2^2\,x_3^2\,x_6^2 + \lambda_2$}

{A New Component:1, $x_1^2\,x_2^2\,x_4^2 + 2\,x_1^2\,x_2^2\,x_4\,x_5 + 2\,x_1^2\,x_2^2\,x_4\,x_6 + 2\,x_1^2\,x_2^2\,x_5\,x_6 + x_1^2\,x_2^2\,x_6^2 + \lambda_3$}

{Total 41 Branch(s) of New Component(s) Discovered}

$\{2\,x_1^{12}\,x_2^2 + 2\,x_1^{10}\,x_2^4 - 2\,x_1^6\,x_2^8 - 2\,x_1^4\,x_2^{10},\ 2\,x_1^6\,x_2^2 - 2\,x_1^2\,x_2^6 + 2\,x_1^6\,x_3^2 + 2\,x_1^4\,x_2^2\,x_3^2 - 2\,x_1^2\,x_2^4\,x_3^2 - 2\,x_2^6\,x_3^2,$
$-1 + x_1^2\,x_4,\ -1 + x_3^2\,x_5,\ -1 + x_2^2\,x_6,\ y^2 - x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_4 + x_6)\,(x_5 + x_6),$
$x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_4 + x_6) + x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_5 + x_6) + x_2^2\,\lambda_1,$
$x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_5 + x_6) + x_1^2\,x_2^2\,x_3^2\,(x_4 + x_6)\,(x_5 + x_6) + x_1^2\,\lambda_2,$
$x_1^2\,x_2^2\,x_3^2\,(x_4 + x_5)\,(x_4 + x_6) + x_1^2\,x_2^2\,x_3^2\,(x_4 + x_6)\,(x_5 + x_6) + x_3^2\,\lambda_3\}$

■ Problem 4

```
ps = {x^2 + y^2 + z^2 - r^2, x y + z^2 - 1, x y z - x^2 - y^2 - z + 1};
ord = {x, z, y};
```

```
defcs = CharacteristicSet[ps, ord, TracePrintOn → True]
```

$$\{\text{CS\_STEP:1}, \{-1+x\,y+z^2\}\}$$

$$\{\text{CS\_STEP:2}, \{-x^4-(-1+z^2)^2-x^2(-1+z^3), -1+x\,y+z^2\}\}$$

$$\{\text{CS\_STEP:3}, \{-x^2(-1+r^2-z^2+z^3), -1+x\,y+z^2\}\}$$

$$\{\text{A New Component:1}, -1+r^2-z^2+z^3\}$$

$$\{\text{CS\_STEP:4}, \{x^2(-2-x^4-z+z^2-x^2z^2+r^2(1+x^2+z)), -1+x\,y+z^2\}\}$$

$$\{\text{A New Component:1}, 2-r^2-r^2x^2+x^4+z-r^2z-z^2+x^2z^2\}$$

$$\{\text{CS\_STEP:5},\ \{x^2(1+x^2(-4+z)-x^4(-1+z)+x^6(-1+z)-2z-r^4(1+x^2+z)+r^2(1-x^4(-1+z)+2z+x^2(3+z))), -1+x\,y+z^2\}\}$$

$$\{\text{A New Component:1},\ 1+r^2-r^4-4x^2+3r^2x^2-r^4x^2+x^4+r^2x^4-x^6-2z+2r^2z-r^4z+x^2z+r^2x^2z-x^4z-r^2x^4z+x^6z\}$$

$$\{\text{CS\_STEP:6},\ \{-x^2(-1+x^2)^2(9+r^8+10x^2+6x^4+6x^6+4x^8+x^{10}+x^{12}-r^6(8+6x^2+x^4+x^6)+r^4(22+19x^2+9x^4+5x^6+3x^8)-r^2(24+23x^2+13x^4+12x^6+4x^8+3x^{10})),\ x^2(1+x^2(-4+z)-x^4(-1+z)+x^6(-1+z)-2z-r^4(1+x^2+z)+r^2(1-x^4(-1+z)+2z+x^2(3+z))), -1+x\,y+z^2\}\}$$

$$\{\text{A New Component:1}, x^2\}$$

$$\{\text{A New Component:2}, (1+x)^2\}$$

$$\{\text{A New Component:3}, 9-24r^2+22r^4-8r^6+r^8+10x^2-23r^2x^2+19r^4x^2-6r^6x^2+6x^4-13r^2x^4+9r^4x^4-r^6x^4+6x^6-12r^2x^6+5r^4x^6-r^6x^6+4x^8-4r^2x^8+3r^4x^8+x^{10}-3r^2x^{10}+x^{12}\}$$

$$\{\text{A New Component:1},\ 1+r^2-r^4-4x^2+3r^2x^2-r^4x^2+x^4+r^2x^4-x^6-2z+2r^2z-r^4z+x^2z+r^2x^2z-x^4z-r^2x^4z+x^6z\}$$

$$\{\text{Total 5 Branch(s) of New Component(s) Discovered}\}$$

$$\{-9x^2+24r^2x^2-22r^4x^2+8r^6x^2-r^8x^2+8x^4-25r^2x^4+25r^4x^4-10r^6x^4+2r^8x^4+5x^6-9r^2x^6+7r^4x^6-3r^6x^6-r^8x^6-4x^8+9r^2x^8-6r^4x^8+5r^6x^8+2x^{10}-7r^2x^{10}-2r^4x^{10}-r^6x^{10}+x^{12}+7r^2x^{12}+r^4x^{12}+r^6x^{12}-3x^{14}-2r^2x^{14}-3r^4x^{14}+x^{16}+3r^2x^{16}-x^{18},\ x^2+r^2x^2-r^4x^2-4x^4+3r^2x^4-r^4x^4+x^6+r^2x^6-x^8-2x^2z+2r^2x^2z-r^4x^2z+x^4z+r^2x^4z-x^6z-r^2x^6z+x^8z, -1+x\,y+z^2\}$$

```
WuRittEqnsSolve[defcs, ord]
```

$$\left\{\left\{x\to -1,\ z\to \frac{3-2r^2}{-1+r^2},\ \frac{3-2r^2}{-1+r^2}\to -\sqrt{1+y}\right\},\right.$$

$$\left\{x\to -1,\ z\to \frac{3-2r^2}{-1+r^2},\ \frac{3-2r^2}{-1+r^2}\to \sqrt{1+y}\right\},\ \left\{x\to 0,\ z\to \frac{1+r^2-r^4}{2-2r^2+r^4},\ \frac{3-2r^2}{-1+r^2}\to -\sqrt{1+y}\right\},$$

$$\left\{x\to 0,\ z\to \frac{1+r^2-r^4}{2-2r^2+r^4},\ \frac{3-2r^2}{-1+r^2}\to \sqrt{1+y}\right\},\ \left\{x\to 1,\ z\to \frac{3-2r^2}{-1+r^2},\ \frac{1+r^2-r^4}{2-2r^2+r^4}\to -1\right\},$$

$$\left\{x\to 1,\ z\to \frac{3-2r^2}{-1+r^2},\ \frac{1+r^2-r^4}{2-2r^2+r^4}\to 1\right\},\ \left\{x\to -\sqrt{\text{Root}\left[9-24r^2+22r^4-8r^6+r^8+10\#1-23r^2\#1+19r^4\#1-6r^6\#1+6\#1^2-13r^2\#1^2+9r^4\#1^2-r^6\#1^2+6\#1^3-12r^2\#1^3+5r^4\#1^3-r^6\#1^3+4\#1^4-4r^2\#1^4+3r^4\#1^4+\#1^5-3r^2\#1^5+\#1^6\ \&,\ 1\right]},\right.$$

$$z\to \left(1+r^2-r^4-\left(4-3r^2+r^4\right)\text{Root}\left[9-24r^2+22r^4-8r^6+r^8+10\#1-23r^2\#1+19r^4\#1-6r^6\#1+6\#1^2-13r^2\#1^2+9r^4\#1^2-r^6\#1^2+6\#1^3-12r^2\#1^3+5r^4\#1^3-r^6\#1^3+4\#1^4-4r^2\#1^4+3r^4\#1^4+\#1^5-3r^2\#1^5+\#1^6\ \&,\ 1\right]+\left(1+r^2\right)\text{Root}\left[9-24r^2+22r^4-8r^6+r^8+10\#1-23r^2\#1+19r^4\#1-6r^6\#1+6\#1^2-13r^2\#1^2+9r^4\#1^2-r^6\#1^2+6\#1^3-12r^2\#1^3+5r^4\#1^3-r^6\#1^3+4\#1^4-4r^2\#1^4+3r^4\#1^4+\#1^5-3r^2\#1^5+\#1^6\ \&,\ 1\right]^2-\text{Root}\left[9-24r^2+22r^4-8r^6+r^8+10\#1-23r^2\#1+19r^4\#1-6r^6\#1+6\#1^2-13r^2\#1^2+9r^4\#1^2-r^6\#1^2+6\#1^3-\right.$$

```
        12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
   (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
          4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3), (1 + r^2 - r^4)/(2 - 2 r^2 + r^4) -> -1},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
          19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
   (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
          4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3), (1 + r^2 - r^4)/(2 - 2 r^2 + r^4) -> 1},
 {x -> -Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
          19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) /
   (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
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        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3), (3 - 2 r^2)/(-1 + r^2) -> -Sqrt[1 - y]},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3), (3 - 2 r^2)/(-1 + r^2) -> Sqrt[1 - y]},
 {x -> -Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3), (3 - 2 r^2)/(-1 + r^2) -> -Sqrt[1 - y]},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3), (3 - 2 r^2)/(-1 + r^2) -> Sqrt[1 - y]},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3), (3 - 2 r^2)/(-1 + r^2) -> -Sqrt[1 - y]},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
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        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 -
        4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 3]^3), (3 - 2 r^2)/(-1 + r^2) -> Sqrt[1 - y]},
 {x -> -Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^3),
  (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
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        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) →
  -Sqrt[1 + y Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]])},
 {x → Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
      13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
      r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]],
  z → (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^3) /
   (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 4]^3)},
 {x → Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
      13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
      r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]],
  z → (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
   (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
     Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) →
  Sqrt[1 + y Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]])},
 {x → -Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
      13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
      r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]],
  z → (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5] +
     (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
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        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^3),
 (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) ->
  -Sqrt[(1 - y Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]])},
 {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]],
  z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
        19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^2 -
    Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^3) /
  (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
        6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
        5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5] +
    (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
        6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^2 -
```

```
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 5]^3),
  (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]^3) ->
    Sqrt(1 - y Sqrt(Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 1]))},
  {x -> -Sqrt(Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]),
   z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
          19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^3),
  (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
```

```
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) ->
   -Sqrt[(1 + y Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]])},
  {x -> Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
        13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
        r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]],
   z -> (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 +
          19 r^4 #1 - 6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 6]^3),
   (1 + r^2 - r^4 - (4 - 3 r^2 + r^4) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) /
    (2 - 2 r^2 + r^4 - (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 -
          6 r^6 #1 + 6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 +
          5 r^4 #1^3 - r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2] +
      (1 + r^2) Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
          6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^2 -
      Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 + 6 #1^2 -
          13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
          r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]^3) ->
   Sqrt[(1 + y Sqrt[Root[9 - 24 r^2 + 22 r^4 - 8 r^6 + r^8 + 10 #1 - 23 r^2 #1 + 19 r^4 #1 - 6 r^6 #1 +
```

```
6 #1^2 - 13 r^2 #1^2 + 9 r^4 #1^2 - r^6 #1^2 + 6 #1^3 - 12 r^2 #1^3 + 5 r^4 #1^3 -
r^6 #1^3 + 4 #1^4 - 4 r^2 #1^4 + 3 r^4 #1^4 + #1^5 - 3 r^2 #1^5 + #1^6 &, 2]) }}
```

CTP_V: Some Testing Problem from [6]

■ XGao Problem

Characteristic Set and Characteristic Form

```
xgao = CharacteristicSet[{2 * (x5 + x6) + x1, (x5^2 + 4 * x5 x6 + x6^2) - x2, 2 x5 x6 (x5 + x6) + x3,
    x5^2 x6^2 - x4}, {x1, x2, x3, x4, x5, x6}, TracePrintOn → True]
```

$$\{\text{CS\_STEP:1},\ \{x_1 + 2\,(x_5 + x_6)\}\}$$

$$\{\text{CS\_STEP:2},\ \{x_1^2 - 4\,x_1\,x_5 - 4\,(x_2 + 2\,x_5^2),\ x_1 + 2\,(x_5 + x_6)\}\}$$

$$\{\text{CS\_STEP:3},\ \{x_1^3 - 4\,x_1\,x_2 + 8\,x_3,\ x_1^4 - 8\,x_1^2\,x_2 + 16\,(x_2^2 - 4\,x_4),\ x_1^2 - 4\,x_1\,x_5 - 4\,(x_2 + 2\,x_5^2),\ x_1 + 2\,(x_5 + x_6)\}\}$$

```
{x1^3 - 4 x1 x2 + 8 x3, x1^4 - 8 x1^2 x2 + 16 x2^2 - 64 x4, x1^2 - 4 x2 - 4 x1 x5 - 8 x5^2, x1 + 2 (x5 + x6)}
```

```
Initial[#, {x1, x2, x3, x4, x5, x6}] & /@ xgao
```

```
{8, -64, -8, 2}
```

```
CharacteristicForm[xgao, {x1, x2, x3, x4, x5, x6}]
```

$$\begin{pmatrix} x_1^3 - 4\,x_1\,x_2 + 8\,x_3 & \{x_1, x_2, x_3, 00, 00, 00\} \\ x_1^4 - 8\,x_1^2\,x_2 + 16\,x_2^2 - 64\,x_4 & \{x_1, x_2, 00, x_4, 00, 00\} \\ x_1^2 - 4\,x_2 - 4\,x_1\,x_5 - 8\,x_5^2 & \{x_1, x_2, 00, 00, x_5, 00\} \\ x_1 + 2\,(x_5 + x_6) & \{x_1, 00, 00, 00, x_5, x_6\} \end{pmatrix}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[xgao, {x1, x2, x3, x4, x5, x6}]
```

$$\left\{\left\{x_3 \to \frac{1}{8}\left(-x_1^3 + 4\,x_1\,x_2\right),\ x_4 \to \frac{1}{64}\left(x_1^2 - 4\,x_2\right)^2,\right.\right.$$
$$\left.\left.x_5 \to \frac{1}{4}\left(-x_1 - \sqrt{3\,x_1^2 - 8\,x_2}\right),\ x_6 \to \frac{1}{4}\left(-x_1 + \sqrt{3\,x_1^2 - 8\,x_2}\right)\right\}\right\}$$

■ Pyramid Problem

Characteristic Set and Characteristic Form

```
wdk = CharacteristicSet[{x2 - x1 + 1, x9 (x3 - 1) + x5, x9 (x6 - 1) + x7, x7 x2 + x5 (x6 - x2),
    x3^2 + x6^2 + (x7 - x5)^2 - 4 x1^2, (x3 - x1)^2 + x2^2 + x5^2 - 4 x1^2, 4 x3^2 - x1^2 - (x6 - x2)^2 - x7^2,
    x3^2 - x1 (x3 + x1)}, Reverse@{x1, x2, x3, x4, x5, x6, x7, x8, x9}, TracePrintOn → True]
```

$$\{\text{CS\_STEP:1},\ \{x_7 + (-1 + x_6)\,x_9,\ x_5 + (-1 + x_3)\,x_9,\ x_5\,x_6 + x_2\,(-x_5 + x_7),\ 1 - x_1 + x_2\}\}$$

$$\{\text{CS\_STEP:2},\ \{x_7 + (-1 + x_6)\,x_9,$$
$$x_5^2\left(x_5^2 - 3\,x_5\,x_7 + x_7^2\right) + 2\,x_5\,(3\,x_5 - 2\,x_7)\,x_7\,x_9 - \left(5\,x_5^2 - 5\,x_5\,x_7 + x_7^2\right)x_9^2,\ x_5 + (-1 + x_3)\,x_9,\ x_5\,x_6 + x_2\,(-x_5 + x_7),\ 1 - x_1 + x_2\}\}$$

$$\{\text{CS\_STEP:3}, \{x_7+(-1+x_6)\,x_9,\ -x_7\left(-4\,x_5^3+x_5^2\,x_7+x_7^3\right)+2\left(-4\,x_5^3-2\,x_5^2\,x_7+3\,x_5\,x_7^2+x_7^3\right)x_9-\left(-20\,x_5^2+24\,x_5\,x_7+\left(-6+x_5^2\right)x_7^2-2\,x_5\,x_7^3+x_7^4\right)x_9^2,\ x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{CS\_STEP:4}, \{x_7+(-1+x_6)\,x_9,\ x_7^3\left(x_5^2+x_7^2\right)-2\,x_7^2\left(7\,x_5^2-5\,x_5\,x_7+2\,x_7^2\right)x_9-2\,x_7\left(-20\,x_5^2+20\,x_5\,x_7-5\,x_7^2+x_5\,x_7^3\right)x_9^2-2\left(16\,x_5^2-20\,x_5\,x_7+\left(6-9\,x_5^2\right)x_7^2+3\,x_5\,x_7^3\right)x_9^3-x_7\left(48\,x_5^2-28\,x_5\,x_7+\left(-2+x_5^2\right)x_7^2-2\,x_5\,x_7^3+x_7^4\right)x_9^4+8\left(5\,x_5^2-5\,x_5\,x_7+x_7^2\right)x_9^5,\ x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{CS\_STEP:5}, \{x_7+(-1+x_6)\,x_9,\ -8\,x_7\left(x_7^2-3\,x_7\,x_9+2\,x_9^2\right)\left((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\left(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3\right)x_9^2+x_7\left(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3\right)x_9^3+\left(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5\right)x_9^4+x_7\left(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3\right)x_9^5+\left(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5\right)x_9^6+x_7\left(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3\right)x_9^7\right), x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, x_7-x_9\}$$

$$\{\text{A New Component:3}, x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7\}$$

$$\{\text{CS\_STEP:6}, \{16\,x_7^2\,(-x_7+x_9)^3\left(x_7^8-36\,x_7^7\,x_9+x_7^6\left(518+x_7^2\right)x_9^2+2\,x_7^5\left(-1858+7\,x_7^2\right)x_9^3-2\,x_7^4\left(-6908+342\,x_7^2+x_7^4\right)x_9^4+8\,x_7^3\left(-3470+851\,x_7^2+11\,x_7^4\right)x_9^5-2\,x_7^2\left(-14\,976+13\,656\,x_7^2+328\,x_7^4+x_7^6\right)x_9^6-4\,x_7\left(3968-13\,484\,x_7^2-90\,x_7^4+3\,x_7^6\right)x_9^7+\left(3072-54\,464\,x_7^2+6344\,x_7^4+732\,x_7^6+x_7^8\right)x_9^8-4\,x_7\left(-6384+4516\,x_7^2+930\,x_7^4+13\,x_7^6\right)x_9^9+\left(-3840+19\,360\,x_7^2+7744\,x_7^4+186\,x_7^6+x_7^8\right)x_9^{10}-2\,x_7\left(3680+3864\,x_7^2+106\,x_7^4+x_7^6\right)x_9^{11}+80\,x_7^2\left(38+x_7^2\right)x_9^{12}), x_7+(-1+x_6)\,x_9,\ -8\,x_7\left(x_7^2-3\,x_7\,x_9+2\,x_9^2\right)\left((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\left(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3\right)x_9^2+x_7\left(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3\right)x_9^3+\left(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5\right)x_9^4+x_7\left(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3\right)x_9^5+\left(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5\right)x_9^6+x_7\left(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3\right)x_9^7\right), x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, (x_7-x_9)^3\}$$

$$\{\text{A New Component:3}, -x_7^3+14\,x_7^2\,x_9-40\,x_7\,x_9^2+32\,x_9^3-18\,x_7^2\,x_9^3+48\,x_7\,x_9^4+x_7^3\,x_9^4-40\,x_9^5\}$$

$$\{\text{A New Component:4}, -x_7^4+20\,x_7^3\,x_9-130\,x_7^2\,x_9^2-x_7^4\,x_9^2+164\,x_7\,x_9^3-12\,x_7^3\,x_9^3-48\,x_9^4+116\,x_7^2\,x_9^4+x_7^4\,x_9^4-92\,x_7\,x_9^5-32\,x_7^3\,x_9^5+38\,x_7^2\,x_9^6+x_7^4\,x_9^6\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, x_7-x_9\}$$

$$\{\text{A New Component:3}, x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7\}$$

$$\{\text{CS\_STEP:7}, \{64\,x_7^2\,x_9^2\,(-x_7+x_9)^3\left(11\,x_7^7-200\,x_7^6\,x_9+2\,x_7^5\left(566+23\,x_7^2\right)x_9^2-12\,x_7^4\left(238+81\,x_7^2\right)x_9^3+\left(3264\,x_7^3+9296\,x_7^5-79\,x_7^7\right)x_9^4+4\,x_7^2\left(-248-10\,152\,x_7^2+615\,x_7^4\right)x_9^5-4\,x_7\left(224-22\,452\,x_7^2+6177\,x_7^4+33\,x_7^6\right)x_9^6+8\left(64-12\,932\,x_7^2+12\,613\,x_7^4+92\,x_7^6\right)x_9^7+x_7\left(57\,920-201\,488\,x_7^2+6504\,x_7^4+85\,x_7^6\right)x_9^8-8\left(1488-25\,588\,x_7^2+5507\,x_7^4+309\,x_7^6\right)x_9^9+2\,x_7\left(-48\,160+48\,712\,x_7^2+5550\,x_7^4+43\,x_7^6\right)x_9^{10}+4\left(3648-23\,984\,x_7^2-4888\,x_7^4+19\,x_7^6\right)x_9^{11}+\left(35\,776\,x_7+13\,520\,x_7^3-1688\,x_7^5-17\,x_7^7\right)x_9^{12}+4\left(-160+320\,x_7^2+982\,x_7^4+13\,x_7^6\right)x_9^{13}-4\,x_7\left(1040+952\,x_7^2+9\,x_7^4\right)x_9^{14}+1440\,x_7^2\,x_9^{15}), x_7+(-1+x_6)\,x_9,\ -8\,x_7\left(x_7^2-3\,x_7\,x_9+2\,x_9^2\right)\left((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\left(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3\right)x_9^2+x_7\left(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3\right)x_9^3+\left(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5\right)x_9^4+x_7\left(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3\right)x_9^5+\left(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5\right)x_9^6+x_7\left(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3\right)x_9^7\right), x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, (x_7-x_9)^3\}$$

$$\{\text{A New Component:3},\ x_9^2\}$$

$$\{\text{A New Component:4},\ -x_7^3+14\,x_7^2\,x_9-40\,x_7\,x_9^2+32\,x_9^3-18\,x_7^2\,x_9^3+48\,x_7\,x_9^4+x_7^3\,x_9^4-40\,x_9^5\}$$

$$\{\text{A New Component:5},\ 11\,x_7^3-24\,x_7^2\,x_9+46\,x_7^3\,x_9^2+8\,x_9^3-434\,x_7^2\,x_9^3+580\,x_7\,x_9^4-68\,x_7^3\,x_9^4-176\,x_9^5+498\,x_7^2\,x_9^5-392\,x_7\,x_9^6-86\,x_7^3\,x_9^6+8\,x_9^7+58\,x_7^2\,x_9^7+52\,x_7\,x_9^8+17\,x_7^3\,x_9^8-18\,x_7^2\,x_9^9\}$$

$$\{\text{A New Component:1},\ x_7-2\,x_9\}$$

$$\{\text{A New Component:2},\ x_7-x_9\}$$

$$\{\text{A New Component:3},\ x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7\}$$

$$\{\text{CS\_STEP:8},\ \{128\,x_7^2\,x_9^4\,(-x_7+x_9)^3\,(-1+x_9^2)\,(-5513\,x_7^6+98\,174\,x_7^5\,x_9-x_7^4\,(538\,028+3567\,x_7^2)\,x_9^2+(1\,354\,152\,x_7^3-32\,290\,x_7^5)\,x_9^3+2\,x_7^2\,(-859\,904+118\,650\,x_7^2+17\,665\,x_7^4)\,x_9^4-4\,x_7\,(-262\,720+126\,936\,x_7^2+155\,069\,x_7^4)\,x_9^5-4\,(59\,008-113\,640\,x_7^2-869\,915\,x_7^4+11\,842\,x_7^6)\,x_9^6+8\,x_7\,(-17\,972-1\,132\,346\,x_7^2+176\,095\,x_7^4)\,x_9^7+2\,(-1984+5\,996\,856\,x_7^2-4\,098\,366\,x_7^4+1743\,x_7^6)\,x_9^8-4\,x_7\,(1\,930\,176-5\,112\,164\,x_7^2+335\,065\,x_7^4)\,x_9^9+2\,(924\,928-12\,675\,184\,x_7^2+3\,927\,866\,x_7^4+15\,633\,x_7^6)\,x_9^{10}+4\,x_7\,(3\,780\,392-4\,398\,408\,x_7^2+188\,877\,x_7^4)\,x_9^{11}-4\,(832\,992-4\,556\,340\,x_7^2+1\,096\,421\,x_7^4+7727\,x_7^6)\,x_9^{12}-8\,x_7\,(1\,010\,088-1\,032\,574\,x_7^2+40\,419\,x_7^4)\,x_9^{13}+4\,(245\,632-1\,407\,776\,x_7^2+554\,305\,x_7^4+5140\,x_7^6)\,x_9^{14}-8\,x_7\,(53\,132+596\,158\,x_7^2+3417\,x_7^4)\,x_9^{15}+(1\,283\,456+4\,491\,536\,x_7^2-221\,108\,x_7^4-2557\,x_7^6)\,x_9^{16}+2\,x_7\,(-682\,816+376\,824\,x_7^2+14\,619\,x_7^4)\,x_9^{17}-(230\,912+984\,288\,x_7^2+89\,144\,x_7^4+891\,x_7^6)\,x_9^{18}+2\,x_7\,(252\,592+50\,076\,x_7^2-801\,x_7^4)\,x_9^{19}+18\,(-320-1000\,x_7^2+808\,x_7^4+9\,x_7^6)\,x_9^{20}-36\,x_7\,(1040+612\,x_7^2+9\,x_7^4)\,x_9^{21}+12\,960\,x_7^2\,x_9^{22}),\ x_7+(-1+x_6)\,x_9,\ -8\,x_7\,(x_7^2-3\,x_7\,x_9+2\,x_9^2)\,((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\,(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3)\,x_9^2+x_7\,(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3)\,x_9^3+(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5)\,x_9^4+x_7\,(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3)\,x_9^5+(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5)\,x_9^6+x_7\,(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3)\,x_9^7),\ x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1},\ x_7-2\,x_9\}$$

$$\{\text{A New Component:2},\ (x_7-x_9)^3\}$$

$$\{\text{A New Component:3},\ -1+x_9\}$$

$$\{\text{A New Component:4},\ x_9^4\}$$

$$\{\text{A New Component:5},\ 1+x_9\}$$

$$\{\text{A New Component:6},\ -x_7^3+14\,x_7^2\,x_9-40\,x_7\,x_9^2+32\,x_9^3-18\,x_7^2\,x_9^3+48\,x_7\,x_9^4+x_7^3\,x_9^4-40\,x_9^5\}$$

$$\{\text{A New Component:7},\ 5513\,x_7^2-9966\,x_7\,x_9+3688\,x_9^2+3567\,x_7^2\,x_9^2-9872\,x_7\,x_9^3+4672\,x_9^4-29\,817\,x_7^2\,x_9^4+58\,006\,x_7\,x_9^5-23\,064\,x_9^6+50\,935\,x_7^2\,x_9^6-74\,100\,x_7\,x_9^7+23\,232\,x_9^8-33\,303\,x_7^2\,x_9^8+8222\,x_7\,x_9^9+13\,688\,x_9^{10}+19\,669\,x_7^2\,x_9^{10}-17\,320\,x_7\,x_9^{11}-2944\,x_9^{12}-2395\,x_7^2\,x_9^{12}+5818\,x_7\,x_9^{13}-72\,x_9^{14}-891\,x_7^2\,x_9^{14}-468\,x_7\,x_9^{15}+162\,x_7^2\,x_9^{16}\}$$

$$\{\text{A New Component:1},\ x_7-2\,x_9\}$$

$$\{\text{A New Component:2},\ x_7-x_9\}$$

$$\{\text{A New Component:3},\ x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7\}$$

$$\{\text{CS\_STEP:9}, \{1280\,x_7^2\,x_9^6\,(-x_7+x_9)^3\,(-1+x_9^2)^2\,(11+46\,x_9^2-68\,x_9^4-86\,x_9^6+17\,x_9^8)$$
$$(85\,845\,x_7^5-1\,415\,610\,x_7^4\,x_9+(6\,510\,900\,x_7^3-1\,989\,971\,x_7^5)\,x_9^2+24\,x_7^2\,(-519\,865+1\,450\,623\,x_7^2)\,x_9^3+$$
$$4\,x_7\,(2\,552\,040-41\,542\,481\,x_7^2+2\,306\,568\,x_7^4)\,x_9^4-8\,(336\,720-41\,917\,655\,x_7^2+23\,971\,063\,x_7^4)\,x_9^5+$$
$$(-300\,133\,056\,x_7+940\,743\,992\,x_7^3-25\,661\,218\,x_7^5)\,x_9^6+8\,(11\,862\,548-240\,152\,247\,x_7^2+78\,280\,315\,x_7^4)\,x_9^7+$$
$$x_7\,(1\,752\,601\,792-3\,134\,527\,688\,x_7^2+39\,602\,851\,x_7^4)\,x_9^8-2\,(286\,614\,128-3\,217\,714\,060\,x_7^2+647\,046\,919\,x_7^4)\,x_9^9+$$
$$(-5\,910\,560\,736\,x_7+6\,656\,077\,372\,x_7^3-21\,026\,249\,x_7^5)\,x_9^{10}+32\,(61\,153\,411-429\,453\,725\,x_7^2+50\,795\,377\,x_7^4)\,x_9^{11}-$$
$$2\,x_7\,(-6\,354\,645\,616+4\,388\,073\,438\,x_7^2+14\,102\,767\,x_7^4)\,x_9^{12}-4\,(1\,064\,728\,616-4\,588\,171\,928\,x_7^2+273\,848\,717\,x_7^4)\,x_9^{13}+$$
$$4\,x_7\,(-4\,306\,749\,656+1\,649\,124\,852\,x_7^2+12\,618\,333\,x_7^4)\,x_9^{14}+16\,(369\,876\,772-891\,923\,279\,x_7^2+13\,173\,999\,x_7^4)\,x_9^{15}+$$
$$(13\,899\,051\,648\,x_7-2\,099\,696\,624\,x_7^3-27\,353\,733\,x_7^5)\,x_9^{16}+(-5\,058\,113\,728+5\,219\,867\,824\,x_7^2+205\,294\,634\,x_7^4)\,x_9^{17}+$$
$$x_7\,(-5\,772\,314\,816-524\,668\,180\,x_7^2+1\,489\,035\,x_7^4)\,x_9^{18}-8\,(-309\,380\,120-48\,614\,717\,x_7^2+22\,028\,371\,x_7^4)\,x_9^{19}+$$
$$(401\,622\,368\,x_7+804\,414\,212\,x_7^3+5\,867\,916\,x_7^5)\,x_9^{20}+8\,(-71\,370\,616-163\,537\,153\,x_7^2+7\,280\,618\,x_7^4)\,x_9^{21}+$$
$$(802\,244\,224\,x_7-337\,895\,304\,x_7^3-3\,172\,018\,x_7^5)\,x_9^{22}-40\,(865\,772-14\,417\,345\,x_7^2+216\,901\,x_7^4)\,x_9^{23}+$$
$$x_7\,(-382\,116\,672+71\,606\,328\,x_7^2+753\,325\,x_7^4)\,x_9^{24}+2\,(20\,464\,080-58\,560\,932\,x_7^2+415\,535\,x_7^4)\,x_9^{25}+$$
$$(67\,463\,840\,x_7-10\,334\,172\,x_7^3-112\,911\,x_7^5)\,x_9^{26}-48\,(-21\,466-307\,283\,x_7^2+4733\,x_7^4)\,x_9^{27}+$$
$$6\,x_7\,(-890\,960+335\,466\,x_7^2+3843\,x_7^4)\,x_9^{28}-108\,(28\,120+29\,036\,x_7^2+427\,x_7^4)\,x_9^{29}+1\,844\,640\,x_7\,x_9^{30}),\ x_7+(-1+x_6)\,x_9,$$
$$-8\,x_7\,(x_7^2-3\,x_7\,x_9+2\,x_9^2)\,((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\,(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3)\,x_9^2+$$
$$x_7\,(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3)\,x_9^3+(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5)\,x_9^4+$$
$$x_7\,(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3)\,x_9^5+(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5)\,x_9^6+$$
$$x_7\,(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3)\,x_9^7),\ x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, (x_7-x_9)^3\}$$

$$\{\text{A New Component:3}, (-1+x_9)^2\}$$

$$\{\text{A New Component:4}, x_9^6\}$$

$$\{\text{A New Component:5}, (1+x_9)^2\}$$

$$\{\text{A New Component:6}, 1+x_9^2\}$$

$$\{\text{A New Component:7}, -x_7^3+14\,x_7^2\,x_9-40\,x_7\,x_9^2+32\,x_9^3-18\,x_7^2\,x_9^3+48\,x_7\,x_9^4+x_7^3\,x_9^4-40\,x_9^5\}$$

$$\{\text{A New Component:8}, 11+35\,x_9^2-103\,x_9^4+17\,x_9^6\}$$

$$\{\text{A New Component:9}, -85\,845\,x_7+42\,090\,x_9+1\,989\,971\,x_7\,x_9^2-1\,430\,206\,x_9^3-9\,312\,117\,x_7\,x_9^4+7\,168\,934\,x_9^5+$$
$$27\,651\,189\,x_7\,x_9^6-21\,615\,538\,x_9^7-48\,914\,968\,x_7\,x_9^8+39\,526\,116\,x_9^9+48\,677\,438\,x_7\,x_9^{10}-43\,061\,548\,x_9^{11}-$$
$$20\,709\,434\,x_7\,x_9^{12}+25\,206\,092\,x_9^{13}-1\,795\,894\,x_7\,x_9^{14}-7\,164\,900\,x_9^{15}+6\,644\,299\,x_7\,x_9^{16}-34\,798\,x_9^{17}-$$
$$3\,284\,929\,x_7\,x_9^{18}+497\,610\,x_9^{19}+776\,383\,x_7\,x_9^{20}-17\,490\,x_9^{21}-112\,911\,x_7\,x_9^{22}-37\,962\,x_9^{23}+23\,058\,x_7\,x_9^{24}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, x_7-x_9\}$$

$$\{\text{A New Component:3},$$
$$x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-$$
$$16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-$$
$$16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7\}$$

$$\{\text{CS\_STEP:10}, \{10\,240\,x_7^2\,x_9^7\,(-x_7+x_9)^3\,(1+x_9^2)^4$$
$$(-x_7^4+16\,x_7^3\,x_9-68\,x_7^2\,x_9^2-2\,x_7\,(-56+9\,x_7^2)\,x_9^3+(-64+84\,x_7^2+x_7^4)\,x_9^4-2\,x_7\,(68+x_7^2)\,x_9^5+80\,x_9^6)$$
$$(11+24\,x_9^2-138\,x_9^4+120\,x_9^6-17\,x_9^8)^2\,(942\,723\,000-4\,777\,589\,172\,x_9^2+1\,569\,852\,839\,x_9^4+44\,530\,684\,300\,x_9^6-$$
$$141\,668\,869\,110\,x_9^8+166\,759\,379\,630\,x_9^{10}+72\,708\,925\,022\,x_9^{12}-569\,168\,038\,618\,x_9^{14}+1\,024\,216\,992\,962\,x_9^{16}-$$
$$1\,139\,804\,332\,306\,x_9^{18}+902\,064\,646\,684\,x_9^{20}-525\,847\,440\,730\,x_9^{22}+223\,361\,740\,894\,x_9^{24}-65\,548\,437\,046\,x_9^{26}+$$
$$11\,181\,312\,562\,x_9^{28}-41\,074\,190\,x_9^{30}-526\,676\,706\,x_9^{32}+137\,364\,174\,x_9^{34}-16\,244\,307\,x_9^{36}+774\,198\,x_9^{38}),\ x_7+(-1+x_6)\,x_9,$$
$$-8\,x_7\,(x_7^2-3\,x_7\,x_9+2\,x_9^2)\,((x_5-2\,x_7)\,x_7^4+5\,x_7^3\,(-5\,x_5+4\,x_7)\,x_9+x_7^2\,(104\,x_5-68\,x_7+5\,x_5\,x_7^2+2\,x_7^3)\,x_9^2+$$
$$x_7\,(-152\,x_5+94\,x_7+9\,x_5\,x_7^2-16\,x_7^3)\,x_9^3+(72\,x_5-44\,x_7-76\,x_5\,x_7^2+56\,x_7^3-9\,x_5\,x_7^4+2\,x_7^5)\,x_9^4+$$
$$x_7\,(92\,x_5-52\,x_7+37\,x_5\,x_7^2-16\,x_7^3)\,x_9^5+(-40\,x_5+20\,x_7-44\,x_5\,x_7^2+20\,x_7^3+3\,x_5\,x_7^4-2\,x_7^5)\,x_9^6+$$
$$x_7\,(20\,x_5-10\,x_7-5\,x_5\,x_7^2+4\,x_7^3)\,x_9^7),\ x_5+(-1+x_3)\,x_9,\ x_5\,x_6+x_2\,(-x_5+x_7),\ 1-x_1+x_2\}\}$$

$$\{\text{A New Component:1}, x_7-2\,x_9\}$$

$$\{\text{A New Component:2}, (x_7-x_9)^3\}$$

{A New Component:3, $(-1+x_9)^2$}

{A New Component:4, x_9^7}

{A New Component:5, $(1+x_9)^2$}

{A New Component:6, $-3+x_9^2$}

{A New Component:7, $(1+x_9^2)^4$}

{A New Component:8, $-x_7^3+14\,x_7^2\,x_9-40\,x_7\,x_9^2+32\,x_9^3-18\,x_7^2\,x_9^3+48\,x_7\,x_9^4+x_7^3\,x_9^4-40\,x_9^5$}

{A New Component:9, $-19+51\,x_9^2-49\,x_9^4+9\,x_9^6$}

{A New Component:10, $(11+35\,x_9^2-103\,x_9^4+17\,x_9^6)^2$}

{A New Component:11, $3000-8092\,x_9^2-2525\,x_9^4+24\,847\,x_9^6-27\,282\,x_9^8+13\,610\,x_9^{10}-3737\,x_9^{12}+531\,x_9^{14}$}

{A New Component:12, $5513+3567\,x_9^2-29\,817\,x_9^4+50\,935\,x_9^6-33\,303\,x_9^8+19\,669\,x_9^{10}-2395\,x_9^{12}-891\,x_9^{14}+162\,x_9^{16}$}

{A New Component:1, $x_7-2\,x_9$}

{A New Component:2, x_7-x_9}

{A New Component:3,
$x_5\,x_7^4-2\,x_7^5-25\,x_5\,x_7^3\,x_9+20\,x_7^4\,x_9+104\,x_5\,x_7^2\,x_9^2-68\,x_7^3\,x_9^2+5\,x_5\,x_7^4\,x_9^2+2\,x_7^5\,x_9^2-152\,x_5\,x_7\,x_9^3+94\,x_7^2\,x_9^3+9\,x_5\,x_7^3\,x_9^3-$
$16\,x_7^4\,x_9^3+72\,x_5\,x_9^4-44\,x_7\,x_9^4-76\,x_5\,x_7^2\,x_9^4+56\,x_7^3\,x_9^4-9\,x_5\,x_7^4\,x_9^4+2\,x_7^5\,x_9^4+92\,x_5\,x_7\,x_9^5-52\,x_7^2\,x_9^5+37\,x_5\,x_7^3\,x_9^5-$
$16\,x_7^4\,x_9^5-40\,x_5\,x_9^6+20\,x_7\,x_9^6-44\,x_5\,x_7^2\,x_9^6+20\,x_7^3\,x_9^6+3\,x_5\,x_7^4\,x_9^6-2\,x_7^5\,x_9^6+20\,x_5\,x_7\,x_9^7-10\,x_7^2\,x_9^7-5\,x_5\,x_7^3\,x_9^7+4\,x_7^4\,x_9^7$}

{Total 11 Branch(s) of New Component(s) Discovered}

$\{1-x_1+x_2,\ x_5\,(-x_2+x_6)+x_2\,x_7,\ x_5+(-1+x_3)\,x_9,\ x_7+(-1+x_6)\,x_9,$
$-8\,x_5\,x_7^7+16\,x_7^8+224\,x_5\,x_7^6\,x_9-208\,x_7^7\,x_9-1448\,x_5\,x_7^5\,x_9^2+1056\,x_7^6\,x_9^2-40\,x_5\,x_7^7\,x_9^2-16\,x_7^8\,x_9^2+$
$4112\,x_5\,x_7^4\,x_9^3-2704\,x_7^5\,x_9^3+48\,x_5\,x_7^6\,x_9^3+176\,x_7^7\,x_9^3-5888\,x_5\,x_7^3\,x_9^4+3696\,x_7^4\,x_9^4+744\,x_5\,x_7^5\,x_9^4-864\,x_7^6\,x_9^4+$
$72\,x_5\,x_7^7\,x_9^4-16\,x_7^8\,x_9^4+4160\,x_5\,x_7^2\,x_9^5-2560\,x_7^3\,x_9^5-2704\,x_5\,x_7^4\,x_9^5+2016\,x_7^5\,x_9^5-512\,x_5\,x_7^6\,x_9^5+176\,x_7^7\,x_9^5-$
$1152\,x_5\,x_7\,x_9^6+704\,x_7^2\,x_9^6+3744\,x_5\,x_7^3\,x_9^6-2304\,x_7^4\,x_9^6+1384\,x_5\,x_7^5\,x_9^6-576\,x_7^6\,x_9^6-24\,x_5\,x_7^7\,x_9^6+16\,x_7^8\,x_9^6-$
$2432\,x_5\,x_7^2\,x_9^7+1312\,x_7^3\,x_9^7-1808\,x_5\,x_7^4\,x_9^7+816\,x_7^5\,x_9^7+112\,x_5\,x_7^6\,x_9^7-80\,x_7^7\,x_9^7+640\,x_5\,x_7\,x_9^8-320\,x_7^2\,x_9^8+$
$1184\,x_5\,x_7^3\,x_9^8-560\,x_7^4\,x_9^8-168\,x_5\,x_7^5\,x_9^8+128\,x_7^6\,x_9^8-320\,x_5\,x_7^2\,x_9^9+160\,x_7^3\,x_9^9+80\,x_5\,x_7^4\,x_9^9-64\,x_7^5\,x_9^9,$
$1\,168\,071\,505\,920\,000\,x_7^9\,x_9^7-22\,193\,358\,612\,480\,000\,x_7^8\,x_9^8+139\,000\,509\,204\,480\,000\,x_7^7\,x_9^9+$
$3\,849\,701\,234\,565\,120\,x_7^9\,x_9^9-426\,346\,099\,660\,800\,000\,x_7^6\,x_9^{10}-52\,119\,036\,350\,177\,280\,x_7^8\,x_9^{10}+$
$724\,204\,333\,670\,400\,000\,x_7^5\,x_9^{11}+296\,920\,579\,096\,289\,280\,x_7^7\,x_9^{11}-45\,083\,527\,552\,460\,800\,x_7^9\,x_9^{11}-$
$696\,170\,617\,528\,320\,000\,x_7^4\,x_9^{12}-888\,853\,344\,999\,628\,800\,x_7^6\,x_9^{12}+909\,528\,644\,636\,047\,360\,x_7^8\,x_9^{12}+$
$355\,093\,737\,799\,680\,000\,x_7^3\,x_9^{13}+1\,501\,416\,563\,943\,014\,400\,x_7^5\,x_9^{13}-5\,767\,710\,683\,461\,621\,760\,x_7^7\,x_9^{13}-$
$49\,094\,149\,896\,560\,640\,x_7^9\,x_9^{13}-74\,756\,576\,378\,880\,000\,x_7^2\,x_9^{14}-1\,439\,393\,593\,467\,371\,520\,x_7^4\,x_9^{14}+$
$17\,738\,885\,903\,206\,615\,040\,x_7^6\,x_9^{14}+88\,414\,821\,913\,006\,080\,x_7^8\,x_9^{14}+731\,114\,289\,081\,876\,480\,x_7^3\,x_9^{15}-$
$30\,147\,992\,427\,667\,496\,960\,x_7^5\,x_9^{15}+641\,596\,232\,534\,077\,440\,x_7^7\,x_9^{15}+794\,631\,963\,397\,744\,640\,x_7^9\,x_9^{15}-$
$152\,935\,158\,538\,567\,680\,x_7^2\,x_9^{16}+28\,991\,593\,107\,439\,984\,640\,x_7^4\,x_9^{16}-2\,869\,459\,902\,300\,712\,960\,x_7^6\,x_9^{16}-$
$15\,297\,590\,995\,821\,496\,320\,x_7^8\,x_9^{16}-14\,797\,786\,302\,344\,888\,320\,x_7^3\,x_9^{17}+5\,228\,658\,110\,371\,573\,760\,x_7^5\,x_9^{17}+$
$95\,974\,237\,394\,567\,505\,920\,x_7^7\,x_9^{17}-714\,991\,391\,054\,581\,760\,x_7^9\,x_9^{17}+3\,118\,565\,285\,743\,820\,800\,x_7^2\,x_9^{18}-$
$5\,180\,422\,423\,518\,535\,680\,x_7^4\,x_9^{18}-294\,316\,979\,684\,117\,207\,040\,x_7^6\,x_9^{18}+27\,731\,155\,843\,626\,659\,840\,x_7^8\,x_9^{18}+$
$2\,757\,899\,080\,252\,702\,720\,x_7^3\,x_9^{19}+499\,827\,357\,555\,632\,066\,560\,x_7^5\,x_9^{19}-193\,659\,742\,902\,780\,866\,560\,x_7^7\,x_9^{19}-$
$5\,568\,224\,468\,012\,615\,680\,x_7^9\,x_9^{19}-617\,591\,769\,355\,550\,720\,x_7^2\,x_9^{20}-480\,545\,974\,801\,898\,209\,280\,x_7^4\,x_9^{20}+$
$608\,986\,066\,605\,148\,835\,840\,x_7^6\,x_9^{20}+81\,601\,988\,674\,424\,453\,120\,x_7^8\,x_9^{20}+$
$245\,229\,730\,931\,676\,282\,880\,x_7^3\,x_9^{21}-1\,040\,298\,844\,299\,166\,433\,280\,x_7^5\,x_9^{21}-$
$475\,127\,350\,003\,981\,178\,880\,x_7^7\,x_9^{21}+13\,935\,674\,093\,691\,043\,840\,x_7^9\,x_9^{21}-$
$51\,665\,412\,363\,436\,687\,360\,x_7^2\,x_9^{22}+1\,002\,625\,043\,852\,161\,269\,760\,x_7^4\,x_9^{22}+$
$1\,427\,621\,180\,038\,146\,590\,720\,x_7^6\,x_9^{22}-340\,849\,649\,316\,003\,164\,160\,x_7^8\,x_9^{22}-$
$513\,381\,102\,037\,892\,136\,960\,x_7^3\,x_9^{23}-2\,412\,097\,602\,119\,612\,170\,240\,x_7^5\,x_9^{23}+$
$2\,239\,535\,373\,351\,685\,550\,080\,x_7^7\,x_9^{23}+4\,790\,338\,083\,527\,526\,400\,x_7^9\,x_9^{23}+$
$108\,712\,414\,329\,957\,253\,120\,x_7^2\,x_9^{24}+2\,314\,742\,109\,881\,666\,887\,680\,x_7^4\,x_9^{24}-$
$6\,943\,695\,132\,191\,040\,368\,640\,x_7^6\,x_9^{24}+213\,586\,576\,433\,777\,162\,240\,x_7^8\,x_9^{24}-$
$1\,178\,673\,744\,307\,636\,142\,080\,x_7^3\,x_9^{25}+11\,821\,963\,223\,324\,952\,596\,480\,x_7^5\,x_9^{25}-$
$1\,778\,086\,122\,781\,456\,046\,080\,x_7^7\,x_9^{25}-59\,517\,344\,762\,465\,546\,240\,x_7^9\,x_9^{25}+$
$247\,501\,642\,305\,004\,175\,360\,x_7^2\,x_9^{26}-11\,378\,977\,026\,572\,935\,536\,640\,x_7^4\,x_9^{26}+$

$$
\begin{aligned}
&5\,799\,738\,157\,685\,952\,522\,240\,x_7^6\,x_9^{26} + 945\,884\,357\,142\,792\,908\,800\,x_7^8\,x_9^{26} + \\
&5\,816\,716\,222\,416\,928\,686\,080\,x_7^3\,x_9^{27} - 9\,994\,192\,504\,414\,331\,801\,600\,x_7^5\,x_9^{27} - \\
&5\,629\,516\,375\,751\,732\,490\,240\,x_7^7\,x_9^{27} + 66\,810\,570\,814\,822\,103\,040\,x_7^9\,x_9^{27} - \\
&1\,228\,628\,685\,107\,278\,807\,040\,x_7^2\,x_9^{28} + 9\,660\,614\,048\,681\,105\,244\,160\,x_7^4\,x_9^{28} + \\
&16\,995\,014\,857\,683\,887\,984\,640\,x_7^6\,x_9^{28} - 2\,103\,174\,784\,354\,584\,995\,840\,x_7^8\,x_9^{28} - \\
&4\,962\,224\,426\,143\,100\,600\,320\,x_7^3\,x_9^{29} - 28\,737\,738\,144\,904\,933\,171\,200\,x_7^5\,x_9^{29} + \\
&14\,342\,652\,338\,650\,871\,859\,200\,x_7^7\,x_9^{29} + 58\,658\,766\,620\,635\,422\,720\,x_7^9\,x_9^{29} + \\
&1\,055\,773\,932\,454\,525\,992\,960\,x_7^2\,x_9^{30} + 27\,599\,887\,928\,998\,679\,347\,200\,x_7^4\,x_9^{30} - \\
&44\,859\,254\,208\,149\,193\,840\,640\,x_7^6\,x_9^{30} + 736\,371\,342\,205\,190\,000\,640\,x_7^8\,x_9^{30} - \\
&14\,077\,723\,704\,778\,947\,297\,280\,x_7^3\,x_9^{31} + 76\,532\,928\,054\,557\,201\,080\,320\,x_7^5\,x_9^{31} - \\
&7\,333\,325\,817\,479\,377\,489\,920\,x_7^7\,x_9^{31} - 198\,314\,334\,337\,120\,778\,240\,x_7^9\,x_9^{31} + \\
&2\,963\,708\,426\,372\,718\,264\,320\,x_7^2\,x_9^{32} - 73\,725\,182\,523\,077\,648\,015\,360\,x_7^4\,x_9^{32} + \\
&24\,698\,215\,927\,966\,069\,411\,840\,x_7^6\,x_9^{32} + 3\,054\,708\,221\,296\,257\,914\,880\,x_7^8\,x_9^{32} + \\
&37\,726\,710\,732\,249\,864\,110\,080\,x_7^3\,x_9^{33} - 42\,892\,680\,828\,221\,194\,096\,640\,x_7^5\,x_9^{33} - \\
&17\,952\,958\,378\,447\,588\,014\,080\,x_7^7\,x_9^{33} + 118\,816\,966\,759\,588\,638\,720\,x_7^9\,x_9^{33} - \\
&7\,981\,490\,180\,691\,332\,300\,800\,x_7^2\,x_9^{34} + 41\,544\,600\,859\,549\,746\,831\,360\,x_7^4\,x_9^{34} + \\
&53\,920\,326\,666\,924\,501\,780\,480\,x_7^6\,x_9^{34} - 4\,797\,515\,211\,191\,991\,029\,760\,x_7^8\,x_9^{34} - \\
&21\,366\,233\,278\,478\,889\,533\,440\,x_7^3\,x_9^{35} - 91\,018\,925\,474\,528\,976\,056\,320\,x_7^5\,x_9^{35} + \\
&33\,645\,342\,465\,848\,619\,110\,400\,x_7^7\,x_9^{35} + 147\,755\,164\,176\,094\,648\,320\,x_7^9\,x_9^{35} + \\
&4\,554\,393\,027\,837\,819\,453\,440\,x_7^2\,x_9^{36} + 87\,408\,377\,908\,151\,465\,246\,720\,x_7^4\,x_9^{36} - \\
&105\,914\,301\,349\,946\,719\,744\,000\,x_7^6\,x_9^{36} + 1\,394\,493\,831\,937\,451\,202\,560\,x_7^8\,x_9^{36} - \\
&44\,616\,543\,155\,593\,771\,417\,600\,x_7^3\,x_9^{37} + 180\,978\,786\,457\,020\,040\,970\,240\,x_7^5\,x_9^{37} - \\
&14\,982\,006\,245\,281\,470\,003\,200\,x_7^7\,x_9^{37} - 278\,465\,109\,483\,010\,048\,000\,x_7^9\,x_9^{37} + \\
&9\,403\,328\,546\,535\,231\,324\,160\,x_7^2\,x_9^{38} - 174\,436\,404\,063\,937\,491\,517\,440\,x_7^4\,x_9^{38} + \\
&51\,118\,590\,710\,028\,288\,010\,240\,x_7^6\,x_9^{38} + 3\,746\,999\,657\,952\,846\,745\,600\,x_7^8\,x_9^{38} + \\
&89\,320\,314\,327\,193\,469\,255\,680\,x_7^3\,x_9^{39} - 89\,077\,355\,820\,444\,240\,691\,200\,x_7^5\,x_9^{39} - \\
&20\,951\,570\,919\,766\,935\,818\,240\,x_7^7\,x_9^{39} + 114\,506\,779\,816\,394\,188\,800\,x_7^9\,x_9^{39} - \\
&18\,915\,039\,591\,745\,515\,683\,840\,x_7^2\,x_9^{40} + 86\,320\,319\,727\,217\,938\,391\,040\,x_7^4\,x_9^{40} + \\
&61\,864\,964\,944\,502\,902\,804\,480\,x_7^6\,x_9^{40} - 5\,054\,731\,134\,387\,746\,007\,040\,x_7^8\,x_9^{40} - \\
&44\,374\,152\,747\,666\,326\,487\,040\,x_7^3\,x_9^{41} - 103\,905\,629\,801\,445\,109\,493\,760\,x_7^5\,x_9^{41} + \\
&35\,742\,689\,183\,939\,973\,253\,120\,x_7^7\,x_9^{41} + 131\,218\,642\,110\,230\,876\,160\,x_7^9\,x_9^{41} + \\
&9\,452\,355\,380\,032\,264\,929\,280\,x_7^2\,x_9^{42} + 99\,685\,581\,364\,837\,779\,660\,800\,x_7^4\,x_9^{42} - \\
&112\,723\,676\,165\,213\,369\,579\,520\,x_7^6\,x_9^{42} + 1\,922\,641\,947\,138\,505\,256\,960\,x_7^8\,x_9^{42} - \\
&50\,889\,020\,685\,853\,173\,104\,640\,x_7^3\,x_9^{43} + 192\,697\,890\,325\,489\,583\,452\,160\,x_7^5\,x_9^{43} - \\
&18\,632\,302\,137\,964\,769\,177\,600\,x_7^7\,x_9^{43} - 199\,585\,494\,680\,029\,214\,720\,x_7^9\,x_9^{43} + \\
&10\,727\,140\,549\,254\,699\,253\,760\,x_7^2\,x_9^{44} - 185\,762\,181\,184\,916\,353\,269\,760\,x_7^4\,x_9^{44} + \\
&62\,632\,901\,486\,733\,794\,119\,680\,x_7^6\,x_9^{44} + 1\,671\,862\,721\,598\,935\,787\,520\,x_7^8\,x_9^{44} + \\
&95\,138\,789\,482\,129\,284\,956\,160\,x_7^3\,x_9^{45} - 108\,789\,280\,574\,962\,741\,288\,960\,x_7^5\,x_9^{45} - \\
&7\,146\,718\,312\,542\,911\,907\,840\,x_7^7\,x_9^{45} + 86\,393\,744\,010\,516\,193\,280\,x_7^9\,x_9^{45} - \\
&20\,153\,287\,286\,857\,766\,993\,920\,x_7^2\,x_9^{46} + 105\,298\,233\,492\,740\,723\,671\,040\,x_7^4\,x_9^{46} + \\
&18\,925\,216\,664\,434\,100\,377\,600\,x_7^6\,x_9^{46} - 2\,655\,284\,882\,750\,673\,899\,520\,x_7^8\,x_9^{46} - \\
&54\,053\,102\,695\,307\,201\,740\,800\,x_7^3\,x_9^{47} - 30\,737\,860\,655\,316\,322\,181\,120\,x_7^5\,x_9^{47} + \\
&18\,010\,383\,886\,138\,611\,568\,640\,x_7^7\,x_9^{47} + 42\,032\,195\,455\,544\,637\,440\,x_7^9\,x_9^{47} + \\
&11\,489\,689\,839\,511\,458\,283\,520\,x_7^2\,x_9^{48} + 29\,242\,819\,669\,070\,060\,052\,480\,x_7^4\,x_9^{48} - \\
&56\,199\,073\,230\,905\,913\,200\,640\,x_7^6\,x_9^{48} + 1\,430\,410\,926\,732\,733\,081\,600\,x_7^8\,x_9^{48} - \\
&14\,873\,410\,334\,046\,405\,754\,880\,x_7^3\,x_9^{49} + 95\,798\,208\,394\,798\,981\,550\,080\,x_7^5\,x_9^{49} - \\
&12\,270\,954\,274\,664\,071\,792\,640\,x_7^7\,x_9^{49} - 77\,113\,653\,701\,113\,507\,840\,x_7^9\,x_9^{49} + \\
&3\,117\,675\,741\,482\,572\,840\,960\,x_7^2\,x_9^{50} - 92\,289\,020\,043\,308\,312\,698\,880\,x_7^4\,x_9^{50} + \\
&40\,372\,401\,713\,048\,260\,638\,720\,x_7^6\,x_9^{50} - 1\,577\,224\,177\,132\,810\,240\,x_7^8\,x_9^{50} + \\
&47\,254\,741\,483\,580\,705\,751\,040\,x_7^3\,x_9^{51} - 69\,765\,186\,443\,489\,863\,045\,120\,x_7^5\,x_9^{51} + \\
&2\,255\,872\,426\,746\,298\,429\,440\,x_7^7\,x_9^{51} + 41\,249\,706\,730\,830\,243\,840\,x_7^9\,x_9^{51} - \\
&10\,006\,349\,351\,563\,915\,264\,000\,x_7^2\,x_9^{52} + 67\,431\,851\,793\,263\,309\,619\,200\,x_7^4\,x_9^{52} - \\
&8\,869\,603\,030\,575\,139\,737\,600\,x_7^6\,x_9^{52} - 531\,218\,290\,495\,361\,505\,280\,x_7^8\,x_9^{52} - \\
&34\,579\,670\,663\,514\,924\,646\,400\,x_7^3\,x_9^{53} + 15\,954\,551\,456\,784\,739\,123\,200\,x_7^5\,x_9^{53} + \\
&2\,901\,155\,277\,958\,489\,569\,280\,x_7^7\,x_9^{53} - 2\,028\,937\,725\,999\,513\,600\,x_7^9\,x_9^{53} +
\end{aligned}
$$

$$7\,339\,114\,753\,169\,011\,507\,200\,x_7^2\,x_9^{54} - 15\,561\,715\,363\,985\,987\,420\,160\,x_7^4\,x_9^{54} -$$
$$8\,473\,750\,811\,408\,136\,366\,080\,x_7^6\,x_9^{54} + 393\,899\,045\,266\,234\,798\,080\,x_7^8\,x_9^{54} +$$
$$8\,008\,085\,267\,157\,415\,690\,240\,x_7^3\,x_9^{55} + 14\,177\,716\,844\,157\,837\,803\,520\,x_7^5\,x_9^{55} -$$
$$2\,984\,007\,808\,192\,195\,768\,320\,x_7^7\,x_9^{55} - 11\,281\,448\,033\,689\,692\,160\,x_7^9\,x_9^{55} -$$
$$1\,708\,499\,878\,249\,079\,767\,040\,x_7^2\,x_9^{56} - 13\,600\,167\,927\,526\,603\,448\,320\,x_7^4\,x_9^{56} +$$
$$9\,563\,534\,038\,533\,855\,948\,800\,x_7^6\,x_9^{56} - 147\,143\,123\,269\,097\,379\,840\,x_7^8\,x_9^{56} +$$
$$6\,954\,973\,974\,276\,148\,428\,800\,x_7^3\,x_9^{57} - 16\,416\,440\,327\,902\,057\,123\,840\,x_7^5\,x_9^{57} +$$
$$1\,467\,409\,007\,559\,876\,884\,480\,x_7^7\,x_9^{57} + 8\,216\,120\,203\,094\,568\,960\,x_7^9\,x_9^{57} -$$
$$1\,469\,958\,773\,693\,204\,725\,760\,x_7^2\,x_9^{58} + 15\,842\,316\,847\,677\,461\,176\,320\,x_7^4\,x_9^{58} -$$
$$4\,964\,064\,806\,887\,372\,032\,000\,x_7^6\,x_9^{58} + 24\,582\,008\,171\,025\,960\,960\,x_7^8\,x_9^{58} -$$
$$8\,118\,601\,532\,338\,608\,128\,000\,x_7^3\,x_9^{59} + 8\,638\,654\,446\,116\,199\,485\,440\,x_7^5\,x_9^{59} -$$
$$431\,189\,235\,840\,768\,921\,600\,x_7^7\,x_9^{59} - 3\,107\,244\,849\,151\,416\,320\,x_7^9\,x_9^{59} +$$
$$1\,721\,328\,674\,681\,308\,610\,560\,x_7^2\,x_9^{60} - 8\,361\,607\,314\,879\,190\,876\,160\,x_7^4\,x_9^{60} +$$
$$1\,564\,047\,569\,431\,497\,748\,480\,x_7^6\,x_9^{60} + 3\,377\,452\,352\,033\,310\,720\,x_7^8\,x_9^{60} +$$
$$4\,288\,294\,357\,010\,299\,125\,760\,x_7^3\,x_9^{61} - 2\,767\,050\,316\,994\,638\,684\,160\,x_7^5\,x_9^{61} +$$
$$65\,355\,314\,787\,015\,055\,360\,x_7^7\,x_9^{61} + 664\,074\,092\,095\,283\,200\,x_7^9\,x_9^{61} - 910\,261\,117\,617\,025\,515\,520\,x_7^2\,x_9^{62} +$$
$$2\,687\,489\,245\,972\,243\,415\,040\,x_7^4\,x_9^{62} - 277\,365\,443\,415\,133\,440\,000\,x_7^6\,x_9^{62} -$$
$$2\,885\,205\,848\,037\,068\,800\,x_7^8\,x_9^{62} - 1\,379\,076\,529\,930\,534\,912\,000\,x_7^3\,x_9^{63} +$$
$$506\,906\,421\,764\,017\,479\,680\,x_7^5\,x_9^{63} + 2\,690\,288\,473\,614\,110\,720\,x_7^7\,x_9^{63} - 44\,994\,819\,220\,858\,880\,x_7^9\,x_9^{63} +$$
$$292\,981\,138\,988\,080\,824\,320\,x_7^2\,x_9^{64} - 495\,489\,396\,119\,722\,270\,720\,x_7^4\,x_9^{64} +$$
$$5\,771\,143\,913\,455\,349\,760\,x_7^6\,x_9^{64} + 649\,463\,329\,187\,645\,440\,x_7^8\,x_9^{64} + 254\,430\,726\,504\,710\,635\,520\,x_7^3\,x_9^{65} -$$
$$16\,465\,024\,644\,978\,667\,520\,x_7^5\,x_9^{65} - 3\,676\,089\,767\,053\,486\,080\,x_7^7\,x_9^{65} - 17\,777\,761\,752\,739\,840\,x_7^9\,x_9^{65} -$$
$$54\,107\,831\,023\,769\,354\,240\,x_7^2\,x_9^{66} + 17\,197\,857\,504\,617\,881\,600\,x_7^4\,x_9^{66} + 10\,827\,700\,362\,535\,311\,360\,x_7^6\,x_9^{66} -$$
$$36\,964\,660\,234\,055\,680\,x_7^8\,x_9^{66} - 8\,873\,942\,652\,636\,282\,880\,x_7^3\,x_9^{67} - 18\,144\,038\,262\,836\,858\,880\,x_7^5\,x_9^{67} +$$
$$807\,353\,048\,659\,046\,400\,x_7^7\,x_9^{67} + 6\,198\,349\,056\,952\,320\,x_7^9\,x_9^{67} + 1\,900\,809\,161\,869\,393\,920\,x_7^2\,x_9^{68} +$$
$$17\,429\,751\,549\,656\,924\,160\,x_7^4\,x_9^{68} - 2\,979\,164\,497\,747\,251\,200\,x_7^6\,x_9^{68} - 13\,279\,110\,548\,060\,160\,x_7^8\,x_9^{68} -$$
$$8\,939\,557\,166\,256\,947\,200\,x_7^3\,x_9^{69} + 5\,290\,687\,853\,543\,096\,320\,x_7^5\,x_9^{69} - 80\,201\,734\,410\,332\,160\,x_7^7\,x_9^{69} -$$
$$926\,324\,663\,500\,800\,x_7^9\,x_9^{69} + 1\,897\,764\,773\,988\,270\,080\,x_7^2\,x_9^{70} - 5\,142\,804\,160\,417\,873\,920\,x_7^4\,x_9^{70} +$$
$$392\,568\,527\,490\,693\,120\,x_7^6\,x_9^{70} + 3\,316\,982\,633\,902\,080\,x_7^8\,x_9^{70} + 2\,639\,669\,801\,573\,744\,640\,x_7^3\,x_9^{71} -$$
$$735\,528\,800\,507\,904\,000\,x_7^5\,x_9^{71} + 1\,748\,099\,617\,689\,600\,x_7^7\,x_9^{71} + 71\,253\,603\,072\,000\,x_7^9\,x_9^{71} -$$
$$560\,999\,623\,623\,966\,720\,x_7^2\,x_9^{72} + 722\,213\,908\,707\,778\,560\,x_7^4\,x_9^{72} - 25\,830\,044\,665\,712\,640\,x_7^6\,x_9^{72} -$$
$$315\,027\,664\,680\,960\,x_7^8\,x_9^{72} - 370\,820\,633\,913\,262\,080\,x_7^3\,x_9^{73} + 53\,573\,500\,508\,221\,440\,x_7^5\,x_9^{73} +$$
$$325\,106\,405\,775\,360\,x_7^7\,x_9^{73} - 2\,291\,130\,593\,280\,x_7^9\,x_9^{73} + 78\,849\,494\,124\,134\,400\,x_7^2\,x_9^{74} -$$
$$53\,523\,151\,282\,298\,880\,x_7^4\,x_9^{74} + 513\,904\,500\,725\,760\,x_7^6\,x_9^{74} + 11\,455\,652\,966\,400\,x_7^8\,x_9^{74} +$$
$$27\,487\,858\,455\,429\,120\,x_7^3\,x_9^{75} - 1\,594\,169\,783\,562\,240\,x_7^5\,x_9^{75} - 20\,620\,175\,339\,520\,x_7^7\,x_9^{75} -$$
$$5\,846\,920\,603\,729\,920\,x_7^2\,x_9^{76} + 1\,677\,107\,594\,280\,960\,x_7^4\,x_9^{76} + 16\,037\,914\,152\,960\,x_7^6\,x_9^{76} -$$
$$861\,465\,103\,073\,280\,x_7^3\,x_9^{77} - 4\,582\,261\,186\,560\,x_7^5\,x_9^{77} + 183\,290\,447\,462\,400\,x_7^2\,x_9^{78}\}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[wdk, {x1, x2, x3, x4, x5, x6, x7, x8, x9}]
```

$$\left\{\left\{x_2 \to -1 + x_1,\ x_7 \to \frac{x_5\,(-1 + x_1 - x_6)}{-1 + x_1},\ x_9 \to -\frac{x_5}{-1 + x_3},\right.\right.$$
$$\left.\left. -\frac{x_5}{-1 + x_3} \to \frac{x_5\,(1 - x_1 + x_6)}{(-1 + x_1)\,(-1 + x_6)},\ -\frac{x_5}{-1 + x_3} \to \frac{x_5\,(1 - x_1 + x_6)}{2 - 2\,x_1},\ -\frac{x_5}{-1 + x_3} \to -1\right\}\right\}$$

■ WWT Problem

Characteristic Set and Characteristic Form

```
wwt = CharacteristicSet[{x1^2 + x2^2 - 1, y1^2 + y2^2 - 1, z1^2 + z2^2 - 1, x1 y2 + x2 y1 - z1,
    x2 y2 - x1 y1 + z2, x0 - (x1 + y1 + z1)}, {x0, x1, x2, y1, y2, z1, z2}, TracePrintOn → True]
```

$$\{\text{CS\_STEP:1}, \{-1+x_1^2+x_2^2, -1+y_1^2+y_2^2, x_0-x_1-y_1-z_1, -x_1 y_1+x_2 y_2+z_2\}\}$$

$$\{\text{CS\_STEP:2}, \{-1+x_1^2+x_2^2, -x_0+x_1+y_1+x_2 y_1+x_1 y_2, x_0-x_1-y_1-z_1, -x_1 y_1+x_2 y_2+z_2\}\}$$

$$\{\text{CS\_STEP:3}, \{-1+x_1^2+x_2^2, x_0^2+2(1+x_2) y_1 (x_1+y_1)-2 x_0 (x_1+y_1+x_2 y_1), -x_0+x_1+y_1+x_2 y_1+x_1 y_2, x_0-x_1-y_1-z_1, -x_1 y_1+x_2 y_2+z_2\}\}$$

$$\{-1+x_1^2+x_2^2, x_0^2-2 x_0 x_1-2 x_0 y_1+2 x_1 y_1-2 x_0 x_2 y_1+2 x_1 x_2 y_1+2 y_1^2+2 x_2 y_1^2, -x_0+x_1+y_1+x_2 y_1+x_1 y_2, x_0-x_1-y_1-z_1, -x_1 y_1+x_2 y_2+z_2\}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[wwt, {x0, x1, x2, y1, y2, z1, z2}]
```

$$\left\{\left\{x_2 \to -\sqrt{1-x_1^2},\right.\right.$$
$$y_1 \to \left(x_1 - x_1\sqrt{1-x_1^2} + x_0\left(-1+\sqrt{1-x_1^2}\right) + \sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right) \Big/ \left(2\left(-1+\sqrt{1-x_1^2}\right)\right),$$
$$y_2 \to \frac{1}{2x_1}\left(x_0\left(1+\sqrt{1-x_1^2}\right) - x_1\left(1+\sqrt{1-x_1^2}\right) + \sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right),$$
$$z_1 \to \left(x_1 - x_1\sqrt{1-x_1^2} + x_0\left(-1+\sqrt{1-x_1^2}\right) - \sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right) \Big/ \left(2\left(-1+\sqrt{1-x_1^2}\right)\right), z_2 \to$$
$$\left(-x_0x_1^2+x_1^3-\left(-1+\sqrt{1-x_1^2}\right)\sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right) \Big/ \left(2x_1\left(-1+\sqrt{1-x_1^2}\right)\right)\Big\},$$
$$\left\{x_2 \to \sqrt{1-x_1^2}, y_1 \to \left(x_1 - x_1\sqrt{1-x_1^2} + x_0\left(-1+\sqrt{1-x_1^2}\right) - \sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right) \Big/ \left(2\left(-1+\sqrt{1-x_1^2}\right)\right), y_2 \to \frac{1}{2x_1\left(-1+\sqrt{1-x_1^2}\right)}\left(-x_1^3-2x_1\left(-1+\sqrt{1-x_1^2}\right)+\right.\right.$$
$$\left.x_0\left(-2+x_1^2+2\sqrt{1-x_1^2}\right)+\left(1+\sqrt{1-x_1^2}\right)\sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right),$$
$$z_1 \to \left(x_1 - x_1\sqrt{1-x_1^2} + x_0\left(-1+\sqrt{1-x_1^2}\right) + \sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right) \Big/ \left(2\left(-1+\sqrt{1-x_1^2}\right)\right), z_2 \to -\frac{1}{2x_1\left(-1+\sqrt{1-x_1^2}\right)}\left(x_1^3+2x_1\left(-1+\sqrt{1-x_1^2}\right)-\right.$$
$$\left.\left.x_0\left(-2+x_1^2+2\sqrt{1-x_1^2}\right)+\left(1+\sqrt{1-x_1^2}\right)\sqrt{\left(-x_1^2\left(-2+x_0^2-2x_0x_1+x_1^2+2\sqrt{1-x_1^2}\right)\right)}\right)\right\}\right\}$$

CTP_VI: Some Testing Problems From [1]

■ Original Case

Polynomial System with ord and const

```
p1 = x^2 - 2 x z + 1;
p2 = x y + z^2;
p3 = 3 y^2 - 2 z^2;
ps = {p1, p2, p3};
ord = {x, y, z};
```

Characteristic Set and Characteristic For

```
tcs = CharacteristicSet[ps, ord, TracePrintOn → True]
```

$\{CS\_STEP:1, \{1 + x^2 - 2\,x\,z\}\}$

$\{CS\_STEP:2, \{1 + 2\,x^2 + x^4 + 4\,x^3\,y,\ 1 + x^2 - 2\,x\,z\}\}$

$\{CS\_STEP:3, \{-(1 + x^2)^2\,(-3 - 6\,x^2 + 5\,x^4),\ 1 + 2\,x^2 + x^4 + 4\,x^3\,y,\ 1 + x^2 - 2\,x\,z\}\}$

$\{A\ New\ Component:1, -3 - 6\,x^2 + 5\,x^4\}$

{Total 1 Branch(s) of New Component(s) Discovered}

$$\{3 + 12\,x^2 + 10\,x^4 - 4\,x^6 - 5\,x^8,\ 1 + 2\,x^2 + x^4 + 4\,x^3\,y,\ 1 + x^2 - 2\,x\,z\}$$

```
CharacteristicForm[tcs, ord, Padding → 0]
```

$$\begin{pmatrix} 3 + 12\,x^2 + 10\,x^4 - 4\,x^6 - 5\,x^8 & \{x, 0, 0\} \\ 1 + 2\,x^2 + x^4 + 4\,x^3\,y & \{x, y, 0\} \\ 1 + x^2 - 2\,x\,z & \{x, 0, z\} \end{pmatrix}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[tcs, ord]
```

$$\left\{\{x \to -i,\ y \to 0,\ z \to 0\},\ \{x \to -i,\ y \to 0,\ z \to 0\},\ \{x \to i,\ y \to 0,\ z \to 0\},\right.$$

$$\{x \to i,\ y \to 0,\ z \to 0\},\ \left\{x \to -\sqrt{\frac{3}{5}+\frac{2\sqrt{6}}{5}},\ y \to \frac{22+8\sqrt{6}}{\sqrt{5}\left(3+2\sqrt{6}\right)^{3/2}},\ z \to -\frac{4+\sqrt{6}}{\sqrt{5\left(3+2\sqrt{6}\right)}}\right\},$$

$$\left\{x \to \sqrt{\frac{3}{5}+\frac{2\sqrt{6}}{5}},\ y \to -\frac{2\left(11+4\sqrt{6}\right)}{\sqrt{5}\left(3+2\sqrt{6}\right)^{3/2}},\ z \to \frac{4+\sqrt{6}}{\sqrt{5\left(3+2\sqrt{6}\right)}}\right\},$$

$$\left\{x \to -i\sqrt{\frac{1}{5}\left(-3+2\sqrt{6}\right)},\ y \to -\frac{2i\left(-11+4\sqrt{6}\right)}{\sqrt{5}\left(-3+2\sqrt{6}\right)^{3/2}},\ z \to -\frac{i\left(-4+\sqrt{6}\right)}{\sqrt{5\left(-3+2\sqrt{6}\right)}}\right\},$$

$$\left.\left\{x \to i\sqrt{\frac{1}{5}\left(-3+2\sqrt{6}\right)},\ y \to \frac{2i\left(-11+4\sqrt{6}\right)}{\sqrt{5}\left(-3+2\sqrt{6}\right)^{3/2}},\ z \to \frac{i\left(-4+\sqrt{6}\right)}{\sqrt{5\left(-3+2\sqrt{6}\right)}}\right\}\right\}$$

■ **Similary Case**

Characteristic Set and Characteristic For

```
defcs = CharacteristicSet[{x1^2 - 2 x1 x3 + 5, x1 x2^2 + x2 x3^2, 3 x2^2 - 8 x3^2}, {x1, x2, x3},
  TracePrintOn → True]
```

$\{\text{CS\_STEP:1},\ \{5 + x_1^2 - 2\,x_1\,x_3\}\}$

$\{\text{CS\_STEP:2},\ \{-50 - 2\,x_1^4 + x_1^2\,(-20 + 3\,x_2^2),\ 5 + x_1^2 - 2\,x_1\,x_3\}\}$

$\{\text{CS\_STEP:3},\ \{(5 + x_1^2)^2\,(8\,x_1 + 3\,x_2),\ 5 + x_1^2 - 2\,x_1\,x_3\}\}$

$\{\text{A New Component:1},\ 8\,x_1 + 3\,x_2\}$

$\{\text{CS\_STEP:4},\ \{2\,(5 + x_1^2)^2\,(-75 - 30\,x_1^2 + 29\,x_1^4),\ (5 + x_1^2)^2\,(8\,x_1 + 3\,x_2),\ 5 + x_1^2 - 2\,x_1\,x_3\}\}$

$\{\text{A New Component:1},\ -75 - 30\,x_1^2 + 29\,x_1^4\}$

$\{\text{A New Component:1},\ 8\,x_1 + 3\,x_2\}$

$\{\text{Total 3 Branch(s) of New Component(s) Discovered}\}$

$$\left\{-3750 - 3000\,x_1^2 + 700\,x_1^4 + 520\,x_1^6 + 58\,x_1^8,\ 200\,x_1 + 80\,x_1^3 + 8\,x_1^5 + 75\,x_2 + 30\,x_1^2\,x_2 + 3\,x_1^4\,x_2,\ 5 + x_1^2 - 2\,x_1\,x_3\right\}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[defcs, {x1, x2, x3}]
```

$$\left\{\left\{x_1 \to -i\sqrt{5},\ x_2 \to \frac{8\,i\sqrt{5}}{3},\ x_3 \to 0\right\},\ \left\{x_1 \to -i\sqrt{5},\ x_2 \to \frac{8\,i\sqrt{5}}{3},\ x_3 \to 0\right\},\right.$$
$$\left\{x_1 \to i\sqrt{5},\ x_2 \to -\frac{8\,i\sqrt{5}}{3},\ x_3 \to 0\right\},\ \left\{x_1 \to i\sqrt{5},\ x_2 \to -\frac{8\,i\sqrt{5}}{3},\ x_3 \to 0\right\},$$
$$\left\{x_1 \to -\sqrt{\frac{15}{29}+\frac{20\sqrt{6}}{29}},\ x_2 \to \frac{8}{3}\sqrt{\frac{5}{29}\left(3+4\sqrt{6}\right)},\ x_3 \to -2\left(8+\sqrt{6}\right)\sqrt{\frac{5}{29\left(3+4\sqrt{6}\right)}}\right\},$$
$$\left\{x_1 \to \sqrt{\frac{15}{29}+\frac{20\sqrt{6}}{29}},\ x_2 \to -\frac{8}{3}\sqrt{\frac{5}{29}\left(3+4\sqrt{6}\right)},\ x_3 \to 2\left(8+\sqrt{6}\right)\sqrt{\frac{5}{29\left(3+4\sqrt{6}\right)}}\right\},$$
$$\left\{x_1 \to -i\sqrt{\frac{5}{29}\left(-3+4\sqrt{6}\right)},\ x_2 \to \frac{8}{3}\,i\sqrt{\frac{5}{29}\left(-3+4\sqrt{6}\right)},\ x_3 \to -2\,i\left(-8+\sqrt{6}\right)\sqrt{\frac{5}{29\left(-3+4\sqrt{6}\right)}}\right\},$$
$$\left.\left\{x_1 \to i\sqrt{\frac{5}{29}\left(-3+4\sqrt{6}\right)},\ x_2 \to -\frac{8}{3}\,i\sqrt{\frac{5}{29}\left(-3+4\sqrt{6}\right)},\ x_3 \to 2\,i\left(-8+\sqrt{6}\right)\sqrt{\frac{5}{29\left(-3+4\sqrt{6}\right)}}\right\}\right\}$$

CTP_VII: TRUSS Testing Problem

Polynomial System with ord and const

```
f1 = u1^2 + x1^2 - 1;
f2 = u2^2 + x2^2 - 1;
f3 = AB x2^2 + AD x1;
f4 = AB u2 + AD u1 + Ay;
f5 = 2 AD u1 - BD;
f6 = 2 AB u2 + BD + Fy;
ps = {f1, f2, f3, f5, f6};
ord = {u2, u1, x2, x1, BD, AB, AD, Ay, Fy};
```

Characteristic Set and Characteristic For

```
defCS = CharacteristicSet[ps, ord, TracePrintOn → True]
```

$$\left\{\text{CS\_STEP:1},\ \left\{-1+u_2^2+x_2^2,\ -1+u_1^2+x_1^2,\ -BD+2\,AD\,u_1,\ BD+F_y+2\,AB\,u_2\right\}\right\}$$
$$\left\{\text{CS\_STEP:2},\ \left\{-1+u_2^2+x_2^2,\ -1+u_1^2+x_1^2,\ -2\,AB\,u_1\left(-1+u_2^2\right)+BD\,x_1,\ -BD+2\,AD\,u_1,\ BD+F_y+2\,AB\,u_2\right\}\right\}$$

$$\left\{-BD+2\,AD\,u_1,\ BD+F_y+2\,AB\,u_2,\ 2\,AB\,u_1-2\,AB\,u_1\,u_2^2+BD\,x_1,\ -1+u_1^2+x_1^2,\ -1+u_2^2+x_2^2\right\}$$

The Relations Between Dependent and Independent Variables

```
WuRittEqnsSolve[defCS, ord]
```

$$\left\{\left\{u_1 \to \frac{BD}{2\,AD},\ u_2 \to -\frac{BD + F_y}{2\,AB},\ x_1 \to \left(-4\,AB^2 + BD^2 + 2\,BD\,F_y + F_y^2\right) \Big/ (4\,AB\,AD),\right.\right.$$

$$\left.\left.\left(-4\,AB^2 + BD^2 + 2\,BD\,F_y + F_y^2\right) \Big/ (4\,AB\,AD) \to -\sqrt{1 - \frac{BD^2}{4\,AD^2}},\ x_2 \to -\sqrt{1 - \frac{(BD + F_y)^2}{4\,AB^2}}\right\}\right\}$$

CTP_VIII: Zeros Decompositon Problem Testing

Not Available Yet

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