ENPM667 Project 2 Report

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Part (A)

(A) Euler-Lagrange equation was used to derive the motion for the system.

The kinetic energy of the cart can be written as:

$$K_c = \frac{1}{2} M \dot{x}^2$$

To derive the kinetic energy of the loads, we first write the coordinates of the center of mass for each load. For load 1, the x, y coordinate of the mass is $(x + l_1 \sin\theta_1, -\cos\theta_1)$, assuming the y coordinate of where the cable connects with the cart is 0. Similarly, the x, y coordinate of load 2 is $(x + l_2 \sin\theta_2, -\cos\theta_2)$. Using the right-hand rule, θ_1 and θ_2 take on negative values when the loads are to the left of the center of mass of the cart while taking positive values when the loads are to the right.

Taking the derivative of the x, y coordinate of load 1:

$$(\dot{x}_1, \dot{y}_1) = (\dot{x} + l_1\dot{\theta}_1 \cos\theta_1, \dot{\theta}_1 \sin\theta_1)$$

Taking the derivative of the x, y coordinate of load 2:

$$(\dot{x}_2, \dot{y}_2) = (\dot{x} + l_2\dot{\theta}_2 \cos\theta_2, \dot{\theta}_2 \sin\theta_2)$$

The kinetic energy of load 1 is thus:

$$\begin{aligned} K_1 &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \\ &= \frac{1}{2} m_1 \left(\left(\dot{x} + l_1\dot{\theta}_1 \cos\theta_1\right)^2 + \left(l_1 \dot{\theta}_1 \sin\theta_1\right)^2 \right) \end{aligned}$$

And the kinetic energy of load 2 is thus:

$$\begin{aligned} K_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 \left(\left(\dot{x} + l_2\dot{\theta}_2 \cos\theta_2\right)^2 + \left(l_2 \dot{\theta}_2 \sin\theta_2\right)^2 \right) \end{aligned}$$

Now we calculate the potential energy of the system. The only potential energy is the gravitational potential energy. For the cart, the gravitational energy is constant and is thus ignored. The potential energy of load 1 is:

$$P_1 = -m_1 g l_1 \cos\theta_1$$

And the potential energy of load 2 is:

$$P_2 = -m_2 g l_2 \cos\theta_2$$

The general coordinates of the system are the horizontal position of the cart (x), the angles of load 1 and load 2 with respect to the vertical direction (θ_1 and θ_2). Thus, the Euler-Lagrange equation for the system is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

where $L = K - P$, and $K = K_c + K_1 + K_2$, $P = P_1 + P_2$. Expanding the equations, we get the following:

$$(M + m_1 + m_2)\ddot{x} + m_1 l_1\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1\right) + m_2 l_2\left(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right) = F$$

$$l_1\ddot{\theta}_1 + \ddot{x}\cos\theta_1 + g\sin\theta_1 = 0$$

$$l_2\ddot{\theta}_2 + \ddot{x}\cos\theta_2 + g\sin\theta_2 = 0$$

Solving for $\ddot{x}$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$, we get:

$$(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)\ddot{x} = F + m_1 g\sin\theta_1\cos\theta_1 + m_1 l_1\dot{\theta}_1^2\sin\theta_1 + m_2 g\sin\theta_2\cos\theta_2 + m_2 l_2\dot{\theta}_2^2\sin\theta_2$$

$$l_1\ddot{\theta}_1 = -\frac{F\cos\theta_1 + m_1 g\sin\theta_1 + m_2 g\sin\theta_2\cos(\theta_1 - \theta_2) + m_1 l_1\dot{\theta}_1^2\sin\theta_1\cos\theta_1 + m_2 l_2\dot{\theta}_2^2\sin\theta_2\cos\theta_1 + Mg\sin\theta_1}{M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2}$$

$$l_2\ddot{\theta}_2 = -\frac{F\cos\theta_2 + m_2 g\sin\theta_2 + m_1 g\sin\theta_1\cos(\theta_1-\theta_2) + m_1 l_1\dot{\theta}_1^2\sin\theta_1\cos\theta_2 + m_2 l_2\dot{\theta}_2^2\sin\theta_2\cos\theta_2 + Mg\sin\theta_2}{M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2}$$

Next, we choose the state variables to be $X = \left[x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2\right]^T = [x_1, x_2, x_3, x_4, x_5, x_6]^T$.

Thus, we can write the following equations for the state space model:

$$\begin{aligned}
\dot{x}_1 &= f_1(X,u) = x_2 \\
\dot{x}_2 &= f_2(X,u) \\
\dot{x}_3 &= f_3(X,u) = x_4 \\
\dot{x}_4 &= f_4(X,u) \\
\dot{x}_5 &= f_5(X,u) = x_6 \\
\dot{x}_6 &= f_6(X,u)
\end{aligned}$$

where the input $u = F$. The expressions of f_2, f_4, f_6 are the following:

$$f_2(X,u) = \frac{m_1 g\sin x_3\cos x_3 + m_1 l_1 x_4^2\sin x_3 + m_2 g\sin x_5\cos x_5 + m_2 l_2 x_6^2\sin x_5}{M + m_1\sin^2 x_3 + m_2\sin^2 x_5} + \frac{u}{M + m_1\sin^2 x_3 + m_2\sin^2 x_5}$$

$$f_4(X,u) = -\frac{m_1 g\sin x_3 + m_2 g\sin x_5\cos(x_3-x_5) + m_1 l_1 x_4^2\sin x_3\cos x_3 + m_2 l_2 x_6^2\sin x_5\cos x_3 + Mg\sin x_3}{l_1(M + m_1\sin^2 x_3 + m_2\sin^2 x_5)} - \frac{u\cos x_3}{l_1(M + m_1\sin^2 x_3 + m_2\sin^2 x_5)}$$

$$f_6(X,u) = -\frac{m_2 g\sin x_5 + m_1 g\sin x_3\cos(x_3-x_5) + m_1 l_1 x_4^2\sin x_3\cos x_5 + m_2 l_2 x_6^2\sin x_5\cos x_5 + Mg\sin x_5}{l_2(M + m_1\sin^2 x_3 + m_2\sin^2 x_5)} - \frac{u\cos x_5}{l_2(M + m_1\sin^2 x_3 + m_2\sin^2 x_5)}$$

Thus, the above is the nonlinear state space representation of the system.

Part (B)

At equilibrium point $x = 0, \theta_1 = 0, \theta_2 = 0$, all time derivative of all state variables need to be zero, which means x_2, x_4, x_6 all equal to 0. In addition, we also need f_2, f_4, f_6 to equal zero, which also requires that $u = 0$. To get the linearized system, we compute the Jacobian with respect to X and u, and evaluate at $X = \vec{0}$ and $u = 0$.

The expression of the Jacobians are as follows:

$$\nabla f_X = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}, \nabla f_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \vdots \\ \frac{\partial f_6}{\partial u} \end{bmatrix}$$

The linearized system around $X = \vec{0}$ and $u = 0$ can be written as the following:

$$\dot{X} = AX + Bu$$

where $A = \nabla f_X \big|_{X=\vec{0}}$ and $B = \nabla f_u \big|_{u=0}$, i.e., the Jacobians evaluated at $X = \vec{0}$ and $u = 0$.

We used the SymPy package to calculate the derivative and evaluate at $X = \vec{0}$ and $u = 0$, which gives the following result for A and B:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m_1 g}{M} & 0 & \frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{Mg + m_1 g}{M l_1} & 0 & -\frac{m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{M l_2} & 0 & -\frac{Mg + m_2 g}{M l_2} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{1}{M l_1} & 0 & -\frac{1}{M l_2} \end{bmatrix}^T$$

Part (C)

The linearized system is an LTI system, and thus to check the controllability, we calculated the controllability matrix:

$$[B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

which equals:

$$\begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{gm_2}{M^2l_2}-\frac{gm_1}{M^2l_1} & 0 & -\frac{-\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2}}{Ml_2}-\frac{-\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1}}{Ml_1} \\ \frac{1}{M} & 0 & -\frac{gm_2}{M^2l_2}-\frac{gm_1}{M^2l_1} & 0 & -\frac{-\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2}}{Ml_2}-\frac{-\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1}}{Ml_1} & 0 \\ 0 & -\frac{1}{Ml_1} & 0 & \frac{gm_2}{M^2l_1l_2}+\frac{Mg+gm_1}{M^2l_1^2} & 0 & -\frac{\frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2}}{Ml_2}-\frac{\frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2}}{Ml_1} \\ -\frac{1}{Ml_1} & 0 & \frac{gm_2}{M^2l_1l_2}+\frac{Mg+gm_1}{M^2l_1^2} & 0 & -\frac{\frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2}}{Ml_2}-\frac{\frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2}}{Ml_1} & 0 \\ 0 & -\frac{1}{Ml_2} & 0 & \frac{gm_1}{M^2l_1l_2}+\frac{Mg+gm_2}{M^2l_2^2} & 0 & -\frac{\frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2}}{Ml_2}-\frac{\frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2}}{Ml_1} \\ -\frac{1}{Ml_2} & 0 & \frac{gm_1}{M^2l_1l_2}+\frac{Mg+gm_2}{M^2l_2^2} & 0 & -\frac{\frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2}}{Ml_2}-\frac{\frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2}}{Ml_1} & 0 \end{bmatrix}$$

The determinant of the controllability matrix equals:

$$-\frac{g^6}{M^6l_1^6l_2^6}(l_1-l_2)^2$$

For the controllability matrix to be invertible, we need the above determinant to be non-zero, which leads to the condition $l_1 \neq l_2$ for the linearized system around $X = 0$ and $u = 0$ to be controllable.

Part (D)

Plugging in $M = 1000$kg, $m_1 = 100$kg, $m_2 = 100$kg, $l_1 = 20$m, $l_2 = 10$m and evaluating the controllability matrix yields:

$$\begin{bmatrix} 0 & 0.001 & 0 & -0.00014715 & 0 & 0.0001419482475 \\ 0.001 & 0 & -0.00014715 & 0 & 0.0001419482475 & 0 \\ 0 & -5.0\cdot10^{-5} & 0 & 3.18825\cdot10^{-5} & 0 & -2.2735778625\cdot10^{-5} \\ -5.0\cdot10^{-5} & 0 & 3.18825\cdot10^{-5} & 0 & -2.2735778625\cdot10^{-5} & 0 \\ 0 & -0.0001 & 0 & 0.000112815 & 0 & -0.00012486633975 \\ -0.0001 & 0 & 0.000112815 & 0 & -0.00012486633975 & 0 \end{bmatrix}$$

The rank of the above matrix equals 6 and thus the system is indeed controllable.

Plugging the values and evaluating A and B matrix:

$$A = \begin{bmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.981 & 0 & 0.981 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & -0.53955 & 0 & -0.04905 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0.001 \\ 0 \\ -5.0 \cdot 10^{-5} \\ 0 \\ -0.0001 \end{bmatrix}$$

To design the LQR controller for the linearized system, we first choose Q and R. We followed Bryson's rule where we chose the Q and R to be diagonal matrices (R is a scalar in our system) and each entry on the diagonal corresponds to the inverse of the maximum acceptable value squared for that state or input variable. We chose the maximum values as shown in the following table:

VARIABLE	MAXIMUM ACCEPTABLE VALUE
$\boldsymbol{x}$	10000 m
$\dot{\boldsymbol{x}}$	100 m/s
$\boldsymbol{\theta_1}$	5 °
$\dot{\boldsymbol{\theta}}_\mathbf{1}$	100 °/s
$\boldsymbol{\theta_2}$	5 °
$\dot{\boldsymbol{\theta}}_\mathbf{2}$	100 °/s
$\boldsymbol{u}$	500 N

We chose x's value to be large as we decided to place less emphasis on the absolute location of the cart as long as both θ_1 and θ_2 can be stabilized. With these values, we get the following Q and R (the angles are converted to rad):

$$Q = \begin{bmatrix} 10^{-8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 131.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3283 & 0 & 0 \\ 0 & 0 & 0 & 0 & 131.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3283 \end{bmatrix}$$

$$R = 4 \times 10^{-6}$$

Next, to find the coefficients of the linear feedback or K, we need to find the solution P to the following stationary Ricatti equation:

$$A^T P + PA - PBR^{-1}B^T P = -Q$$

and P is a symmetric positive definite matrix and the solution is $K = -R^{-1}B^T P$.

We used MATLAB built-in function "lqr" to solve this problem and found K, the value of which is shown below:

$$K = [-0.05, \quad -13.1, \quad -979.0, \quad 8304.9, \quad 3605.1, \quad 3617.6\,]$$

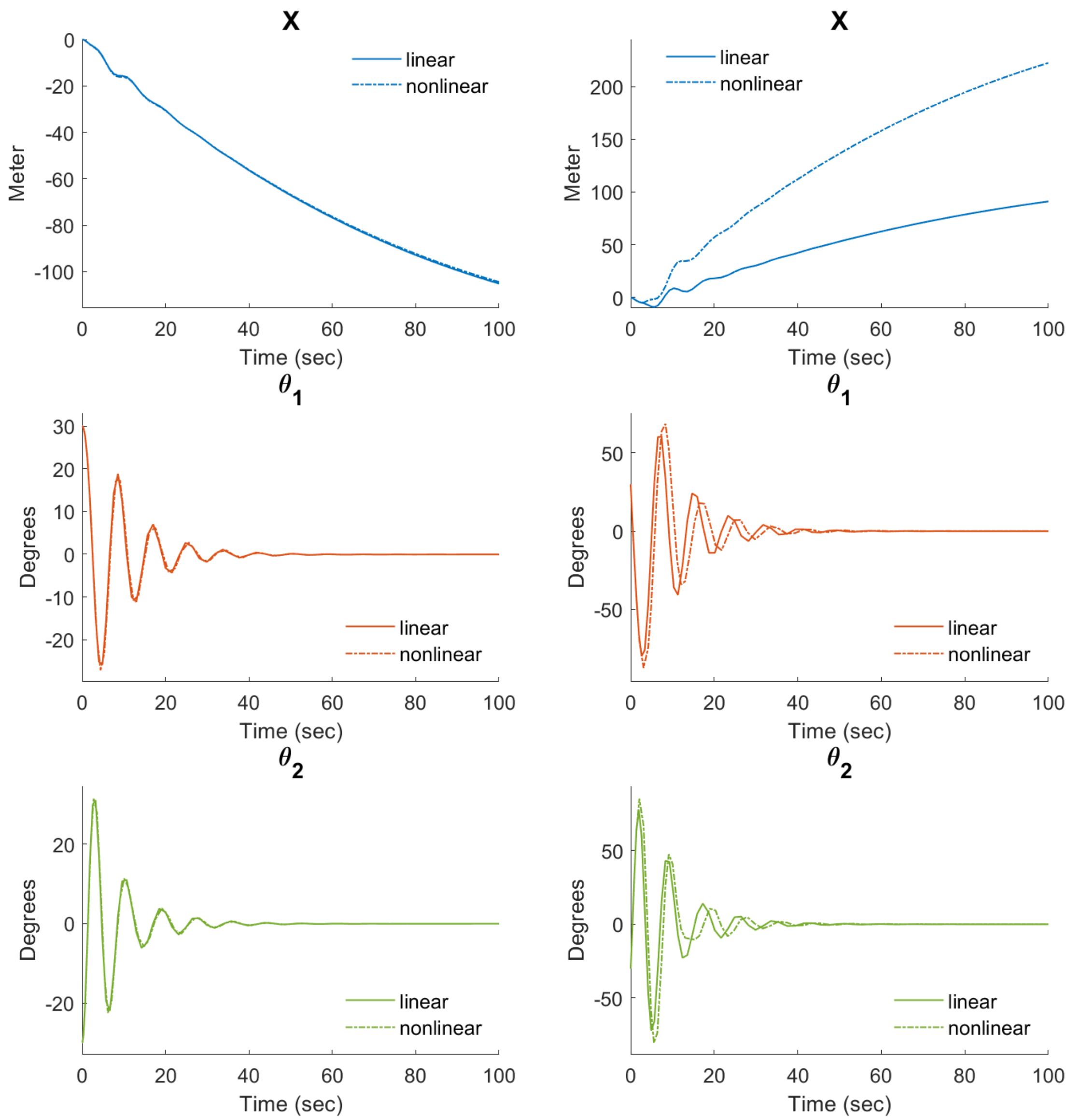

Figure 1 Simulation with linearized system (solid lines) and the original nonlinear system (dash-dot lines). Two initial conditions were chosen (left vs right column). In both conditions, $x(0) = 0$, $\dot{x}(0) = 0$, $\theta_1(0) = \frac{\pi}{6}$, $\theta_2(0) = -\frac{\pi}{6}$. For the left column, $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$. For the right column, $\dot{\theta}_1(0) = -1$ and $\dot{\theta}_2(0) = 1$.

With these feedback coefficients, we simulated the system dynamics with both the linearized system and the original nonlinear system (Figure 1). Two different initial conditions were chosen, which shared the following: $x(0) = 0$, $\dot{x}(0) = 0$, $\theta_1(0) = \frac{\pi}{6}$, $\theta_2(0) = -\frac{\pi}{6}$, but the first condition had $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$ (Figure 1, left column) while the second condition had $\dot{\theta}_1(0) = -1\ rad/s$ and $\dot{\theta}_2(0) = 1\ rad/s$ (Figure

1, right column). In both conditions, the controller could stabilize θ_1 and θ_2 asympotically at 0. As the panelty for $x^2(t)$ is neglectable in our cost function, $x(t)$ deviated significantly from the initial condition, which is expected.

Comparing the simulation with the linear and that with the nonlinear system, the trajectories from the linear and nonlinear system could be similar even with large initial angle displacement in θ_1 and θ_2, (Figure 1, left column). However, with a different choice of the initial condition (Figure 1, right column), the trajectories could differ significantly between the linear system and the nonlinear system, especially in terms of x. Nevertheless, the controller was able to drive both the linear and nonlinear systems back to the equilibrium point of $\dot{x} = \theta_1 = \dot{\theta}_1 = \theta_2 = \dot{\theta}_2 = 0$, suggesting that the closed-loop system is likely locally stable.

To check the stability of the system, we used Lyapunov's indirect method and we computed the eigenvalues of $A + BK$, the values of which are listed below:

-0.2832 + 1.0737i

-0.2832 - 1.0737i

-0.1074 + 0.7463i

-0.1074 - 0.7463i

-0.0045 + 0.0038i

-0.0045 - 0.0038i

As all of these eigenvalues have a negative real part, we conclude that the closed-loop system is stable around the equilibrium at least locally.

Another thing to notice is that $x(t)$, or the horizontal position of the cart has strayed significantly from the point we linearized the system, i.e., $x = 0$ and yet the dynamics of the linearized system still resembles the one from the nonlinear system (Figure 1, left column). This can be explained by the original nonlinear dynamics equation, as none of the equations depends on x_1 or the absolute position of the cart. This means that the linearized system is valid for the cart in equilibrium at any x, not just $x = 0$, which agrees with the actual physics of the system.

To see the different behavior of the controller we also tried lowering the maximum acceptable of x when designing the Q matrix. We ran simulations with the corresponding controller and we observed that $x(t)$

asymptotically returned to 0, which is expected as the modified controller also minimizes $x^2(t)$ in the long run (Figure 2). The MATLAB script for generating these figures is “sim1_PartD.m”.

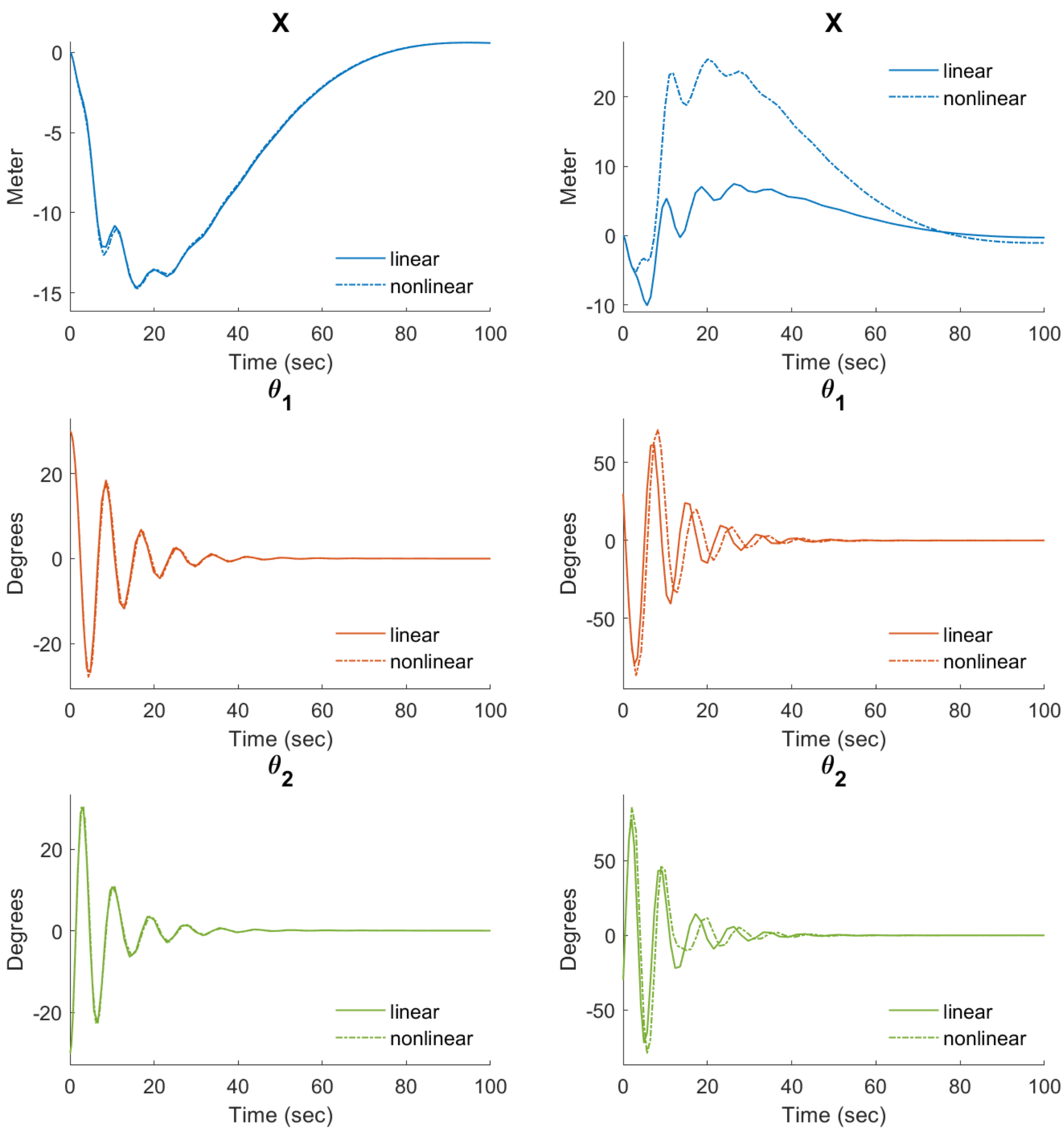

Figure 2 Simulation with the same parameters as in Figure 1 except for that in Q matrix the cost for $x^2(t)$ was increased (maximum acceptable value equals 100). The optimal controller accordingly minimizes $x^2(t)$ in the long run, i.e., the cart returns to the origin.

Part (E)

To test the observability of the system, we need to compute the rank of the observability matrix defined as:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The system is observable if and only if the rank of the observability matrix equals n, or in other words, if the observability matrix has full column rank.

(1) The output variable is $x(t)$. Thus we have $C = [1,0,0,0,0,0]$. The observability matrix is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} \\ 0 & 0 & -\frac{g^2m_1m_2}{M^2l_2} - \frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1} - \frac{gm_2(Mg+gm_2)}{M^2l_2} & 0 \\ 0 & 0 & 0 & -\frac{g^2m_1m_2}{M^2l_2} - \frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1} - \frac{gm_2(Mg+gm_2)}{M^2l_2} \end{bmatrix}$$

This is a 6 by 6 matrix and the determinant is as follows:

$$\frac{(-g^3l_1m_1m_2 + g^3l_2m_1m_2)^2}{M^4l_1^2l_2^2}$$

Thus, state variables are observable if and only if $l_1 \neq l_2$. Thus given the parameters specified in (D), where $l_1 \neq l_2$, we can confirm that the system is observable.

(2) The output variables are $\theta_1(t)$ and $\theta_2(t)$. The corresponding C matrix is:

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix is as follows:

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -\frac{Mg+gm_1}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\
0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} & 0 \\
0 & 0 & 0 & -\frac{Mg+gm_1}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} \\
0 & 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} \\
0 & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2} & 0 & \frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2} & 0 \\
0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2} & 0 \\
0 & 0 & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2} & 0 & \frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2} \\
0 & 0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2}
\end{bmatrix}
$$

It is obvious that the observability matrix does not have full column rank, and thus given $(\theta_1(t), \theta_2(t))$ pair as the system output, the system is not observable.

(3) The output variables are $x(t)$ and $\theta_2(t)$. The corresponding C matrix is:

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

The observability matrix is as follows:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} & 0 \\
0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} & 0 \\
0 & 0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} \\
0 & 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} \\
0 & 0 & -\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2} & 0 \\
0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2} & 0 \\
0 & 0 & 0 & -\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2} \\
0 & 0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2}
\end{bmatrix}
$$

Let the above matrix be M. It can be shown that M is full column rank if and only if M^TM is invertible. Plugging the parameter values specified in (D), the determinant of M^TM evaluates to be 17.24>0, thus confirming that given the $(x(t), \theta_2(t))$ pair as the output, the system is observable.

(4) The output variables are $x(t)$, $\theta_1(t)$ and $\theta_2(t)$. The corresponding C matrix is:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix is:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} & 0 \\
0 & 0 & -\frac{Mg+gm_1}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\
0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} & 0 \\
0 & 0 & 0 & \frac{gm_1}{M} & 0 & \frac{gm_2}{M} \\
0 & 0 & 0 & -\frac{Mg+gm_1}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} \\
0 & 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{Mg+gm_2}{Ml_2} \\
0 & 0 & -\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2} & 0 \\
0 & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2} & 0 & \frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2} & 0 \\
0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2} & 0 \\
0 & 0 & 0 & -\frac{g^2m_1m_2}{M^2l_2}-\frac{gm_1(Mg+gm_1)}{M^2l_1} & 0 & -\frac{g^2m_1m_2}{M^2l_1}-\frac{gm_2(Mg+gm_2)}{M^2l_2} \\
0 & 0 & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_1)^2}{M^2l_1^2} & 0 & \frac{gm_2(Mg+gm_2)}{M^2l_1l_2}+\frac{gm_2(Mg+gm_1)}{M^2l_1^2} \\
0 & 0 & 0 & \frac{gm_1(Mg+gm_2)}{M^2l_2^2}+\frac{gm_1(Mg+gm_1)}{M^2l_1l_2} & 0 & \frac{g^2m_1m_2}{M^2l_1l_2}+\frac{(Mg+gm_2)^2}{M^2l_2^2}
\end{bmatrix}$$

Again letting the above matrix be M and computing the determinant of M^TM by plugging the values specified in (D), the determinant evaluates to be 140.8>0. Therefore, given ($x(t)$, $\theta_1(t)$, $\theta_2(t)$) as the system output, the system is observable.

Part (F)

From part (E) we know that when the system outputs are the following 3 choices, the system is observable: $x(t)$, ($x(t)$,$\theta_2(t)$) and ($x(t)$, $\theta_1(t)$, $\theta_2(t)$). Below we design the "best" Luenberger observer for each case.

To design the "best" Luenberger observer L, we can find the LQR controller for the system (A^T, C^T) and take the transpose of the feedback coefficient. Given L, the dynamics of the Luenberger observer are described by:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(t) + L(y(t) - C\hat{X}(t))$$

where $\hat{X}$ is the estimation of the state variables and y is the output of the plant.

To design $\hat{Q}$ and $\hat{R}$ for the system (A^T, C^T), we start by choosing $\hat{Q}$ and $\hat{R}$ to have identical values along the diagonal, since the state variables of the transposed system do not necessarily have a one-to-one mapping with the original error state variables. Thus we chose both $\hat{Q}$ and $\hat{R}$ to be a diagonal matrix with diagonal entries equal to 100. Next, we solve for L using the MATLAB built-in function “lqr” by plugging in $A^T, C^T, \hat{Q}, \hat{R}$. The system’s response was simulated with the step input at $t = 0$, and the input amplitude is 100 N. The initial conditions for all following simulations are the same: $x(0) = 5$, $\dot{x}(0) = -0.5$, $\theta_1(0) = \frac{\pi}{9}$, $\theta_2(0) = -\frac{\pi}{9}$, $\dot{\theta}_1(0) = -0.5\, rad/s$, $\dot{\theta}_2(0) = 0.5\, rad/s$. We performed 3 simulations, taking the output variable to be $x(t)$, ($x(t)$ and $\theta_2(t)$), and $(x(t), \theta_1(t), \theta_2(t))$ respectively (Figure 3, Figure 4, and Figure 5). In all 3 simulations, the estimation $\hat{x}(t), \hat{\theta}_1(t), \hat{\theta}_2(t)$ could converge to the actual state variable values, even when applied to the nonlinear model, but the rate of convergence differed. Specifically, the convergence of the variables which consisted of the system output was faster than the convergence of the variables that did not appear in the system output. For example, in the first simulation (Figure 3), the convergence was fast for $x(t)$ but rather slow for $\theta_2(t)$ and $\theta_3(t)$. In contrast, when all 3 variables appear in the system output, the convergence was fast for all 3 variables (Figure 5). Thus, these observations point to the importance of the proper selection of the output variables for fast estimation convergence.

The MATLAB script for generating the figures in this part is “sim2_PartF.m”.

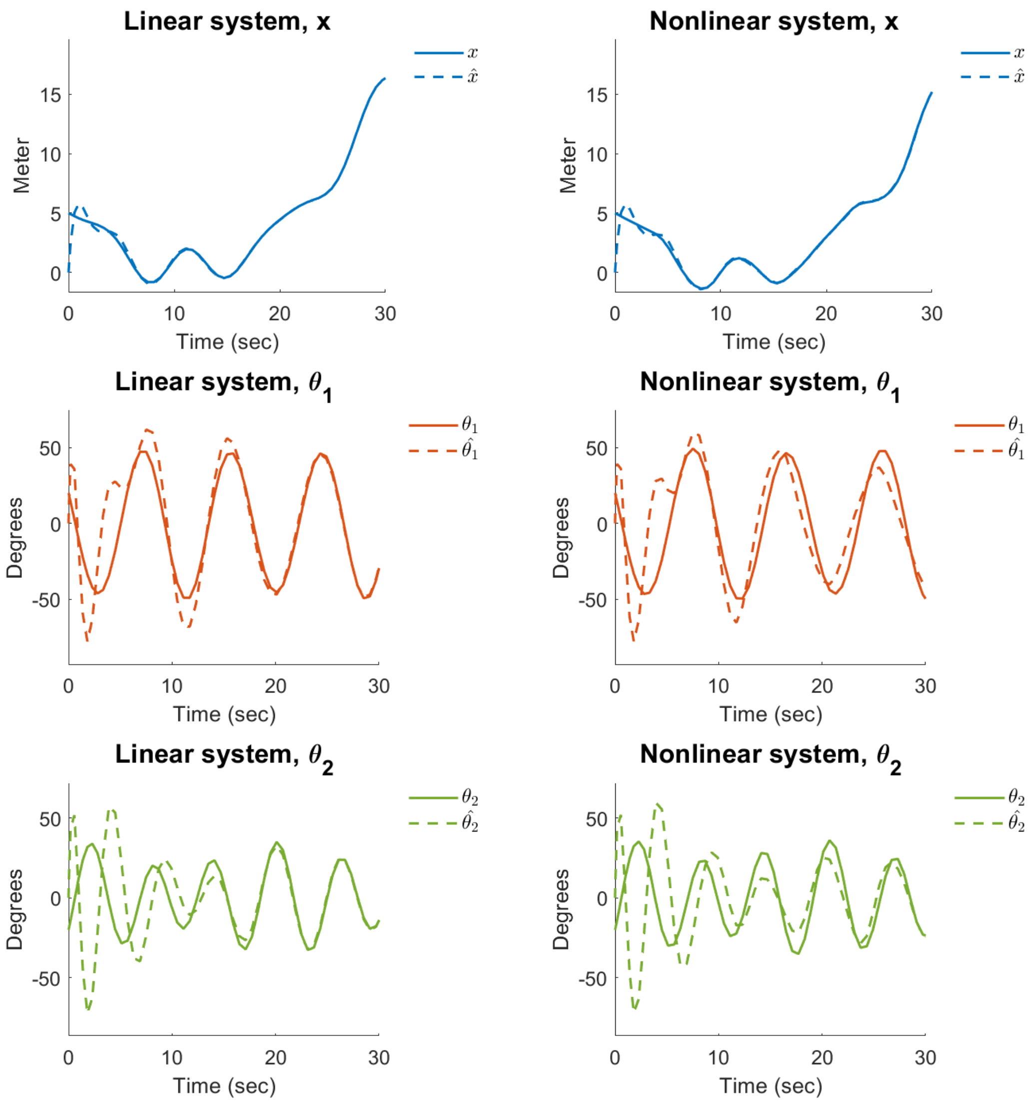

Figure 3 Simulation with the Luenberger observer. The system output is $x(t)$. $\hat{x}(t)$, $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are the estimation formed by the Luenberger observer. Left: simulation was performed with the linearized system. Right: the simulation was performed with the original nonlinear system.

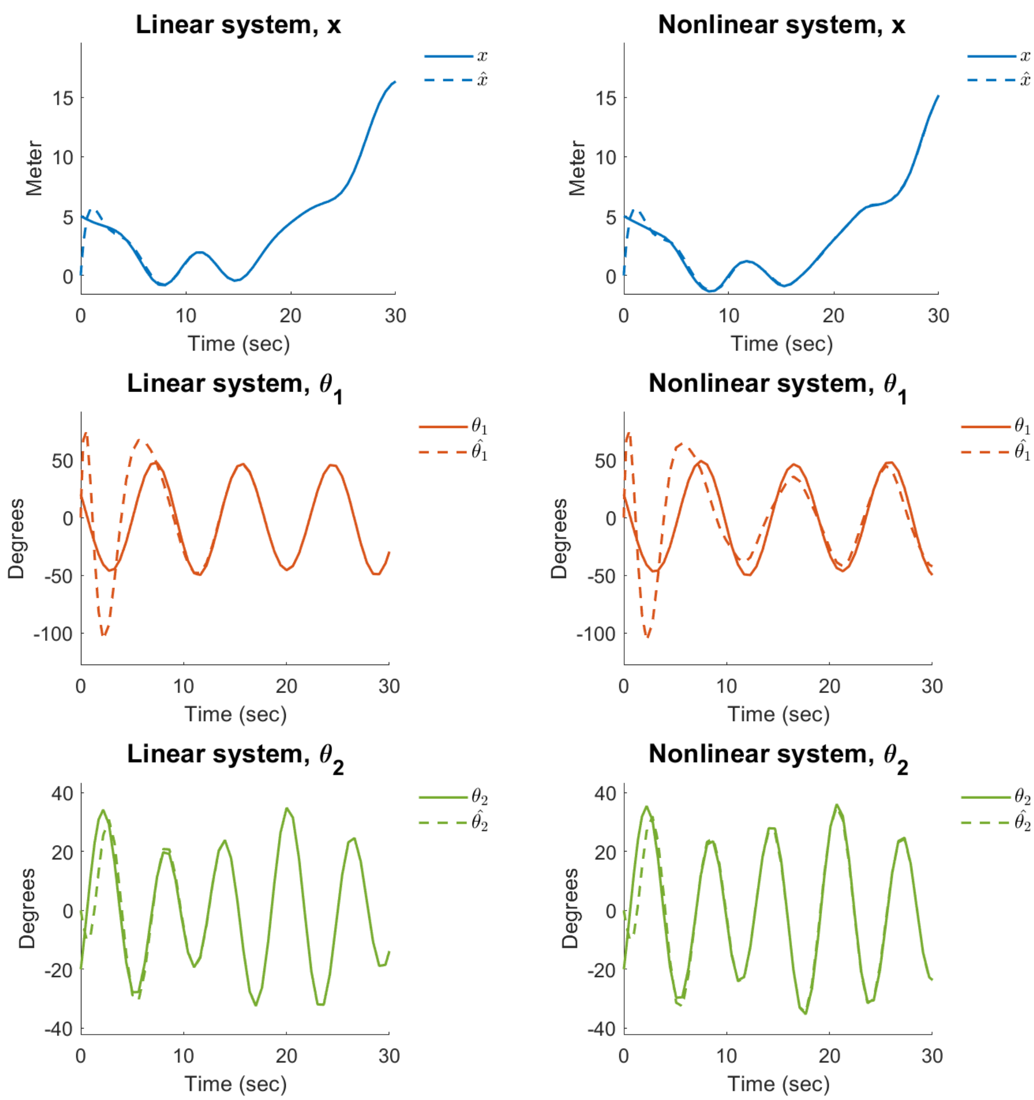

Figure 4 Simulation with the Luenberger observer. The system output is $(x(t), \theta_2(t))$. $\hat{x}(t)$, $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are the estimations formed by the Luenberger observer. Left: simulation was performed with the linearized system. Right: the simulation was performed with the original nonlinear system.

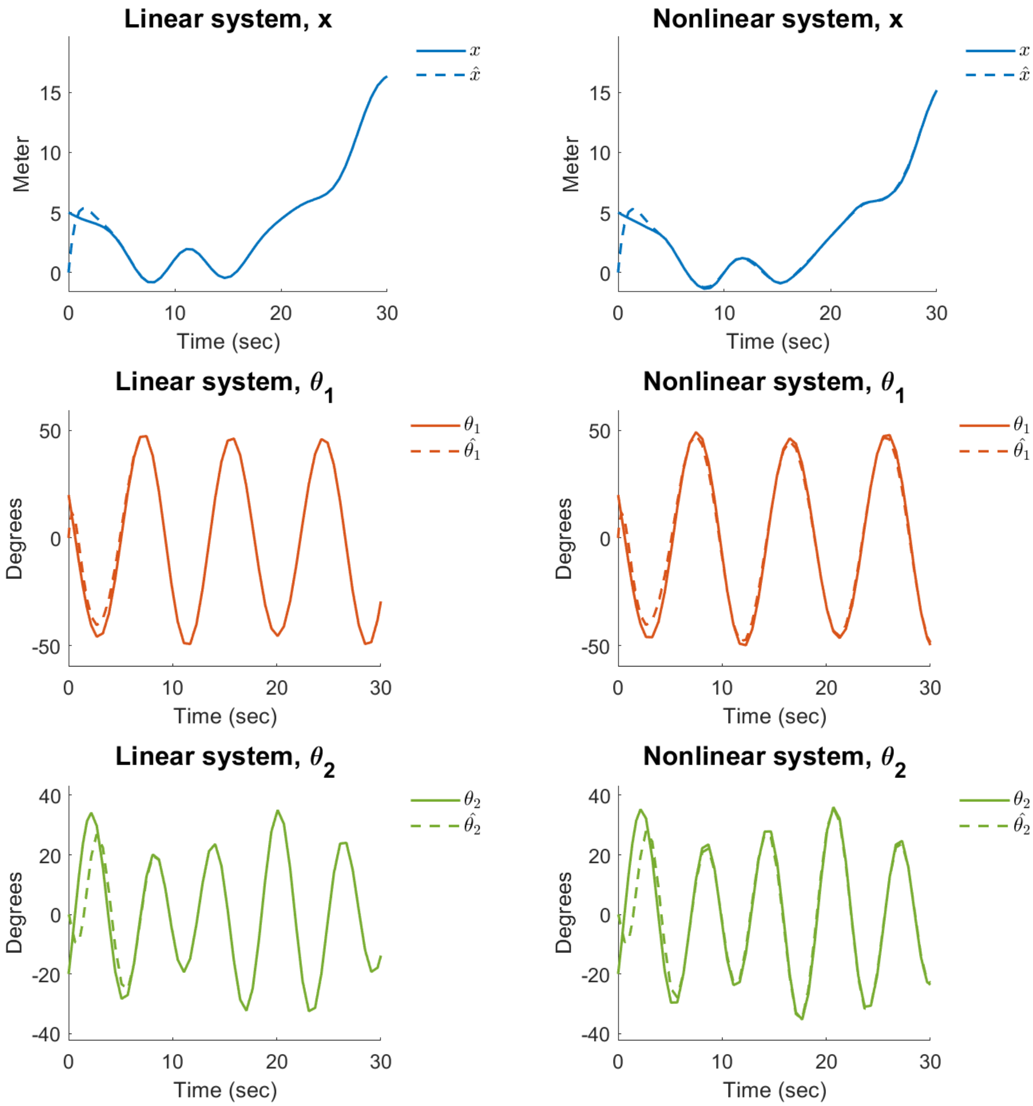

Figure 5 Simulation with the Luenberger observer. The system output is $(x(t), \theta_1(t), \theta_2(t))$. $\hat{x}(t)$, $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are the estimations formed by the Luenberger observer. Left: simulation was performed with the linearized system. Right: the simulation was performed with the original nonlinear system.

Part (G)

For the output feedback system, using the linearized model, we have the following dynamics equations:

$$\dot{X} = AX + B_K K \hat{X} + B_D u_D$$

$$\dot{\hat{X}} = A\hat{X} + B_K K\hat{X} + L(Y - C\hat{X})$$

$$Y = CX + v$$

where u_D and v represent process noise and measurement noise respectively.

Now express the above model with combined state variable $\left[X^T, \hat{X}^T\right]^T$:

$$\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \end{bmatrix} = \begin{bmatrix} A & B_K K \\ LC & A + B_K K - LC \end{bmatrix} \begin{bmatrix} X \\ \hat{X} \end{bmatrix} + \begin{bmatrix} B_D \\ 0 \end{bmatrix} u_D$$

which is equivalent to:

$$\begin{bmatrix} \dot{X} \\ \dot{X}_e \end{bmatrix} = \begin{bmatrix} A + B_K K & -B_K K \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} X \\ X_e \end{bmatrix} + \begin{bmatrix} B_D \\ B_D \end{bmatrix} u_D$$

Where $X_e = X - \hat{X}$. Thus, to design the output feedback controller, we apply the separation principle and design K with the LQR method while designing L using the Kalman-Bucy filter.

We used the same Q and R parameters as in part (D). To arrive at L, we first need to determine the smallest output vector for reasonable performance of the controller. From the result of part (E), we know that the smallest output is $x(t)$, for which we could find L such that (A^T, C^T) can be stabilized. However, the convergence of the estimate of θ_1 and θ_2 was poor, and we would like to have a more reliable estimate of all three variables. Thus, we chose the output variables to be $x(t), \theta_1(t), \theta_2(t)$. The corresponding C is:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For this part we simulated u_D and v as i.i.d Gaussian white noise. The covariance of the process noise and measurement noise for $x(t)$ is chosen to be 10 times larger than that of $\theta_1(t)$ and $\theta_2(t)$ due to the different scales of the signal. Let the covariance matrix of u_D be Σ_D and the covariance matrix of v be Σ_v. To find the optimal L, we first find P which is the solution to the following Ricatti equation:

$$AP + PA^T + B_D \Sigma_D B_D^T - PC^T \Sigma_v^{-1} CP = 0$$

We solved this problem with the MATLAB built-in function "icare". With P, we can find L:

$$L = PC^T \Sigma_v^{-1}$$

We performed simulations with the above design strategy and we systematically varied the noise variance to see the behavior of the system. For each choice of noise variance, the optimal L was re-computed. All simulations had the same initial condition: $x(0) = 5$, $\dot{x}(0) = -0.5$, $\theta_1(0) = \frac{\pi}{9}$, $\theta_2(0) = -\frac{\pi}{9}$, $\dot{\theta}_1(0) = -0.5\ rad/s$, $\dot{\theta}_2(0) = 0.5\ rad/s$. We performed two sets of simulations. In the first set, we kept the process noise variance at a neglectable level of 10^{-3} for x and 10^{-4} for θ_1 and θ_2, while varying the measurement noise variance between 0.01, 0.1, 1 for $x(t)$ and between 0.001, 0.01, 0.1 for $\theta_1(t)$ and $\theta_2(t)$.

Figure 6 shows the result of the first set of simulations, where across each column from left to right, the measurement noise covariance was increased by 10-fold. It can be observed that with minimum measurement noise (Figure 6, left column), the state estimations can quickly converge to the actual values, while with increasing measurement noise (mid and right columns), the estimations converged more slowly or in the extreme case failed to follow the actual values (right column). Due to the different rates of convergence of the observer, the controller drove $\theta_1(t)$ and $\theta_2(t)$ to 0 also at a different rate with different level of measurement noise, i.e., the larger the measurement noise, the slower $\theta_1(t)$ and $\theta_2(t)$ converged to 0. In addition, the control signal u also differed in amplitude across different measurement noise conditions (Figure 6, top row). With small measurement noise, the state estimations were reliable and the control effort was larger in amplitude, while with large measurement noise, the state estimations were unreliable and the control effort was accordingly smaller to reflect the uncertainty. These simulations demonstrated the effect of measurement noise on the performance of the output feedback controller.

In the second set of simulations, we held the measurement noise at a minimum while varying the process noise variance across 3 different levels. Figure 7 shows the results of these simulations, where across each column from left to right, the process noise covariance was increased by 10-fold. In all cases, the state variable estimations quickly converged to the actual values and the controller was able to drive $\theta_1(t)$ and $\theta_2(t)$ around 0. However, with increasing process noise, $\theta_1(t)$ and $\theta_2(t)$ would have a lot more oscillations around 0, which would be expected as the process noise started to match the signal in amplitude, causing more significant random drift in the states. Nevertheless, the controllers seemed robust against these random drifts and $\theta_1(t)$ and $\theta_2(t)$ remained around 0.

Given the above two sets of simulations, it seems that measurement noise has a bigger impact on the performance of the output feedback controller as the inaccurate state variable estimations prevented the controller from acting quickly to stabilize the system. Thus, the design of the LQG controller and its performance is heavily influenced by the noise structure.

The MATLAB script for generating Figure 6 and Figure 7 is “sim3_partG_part1.m”.

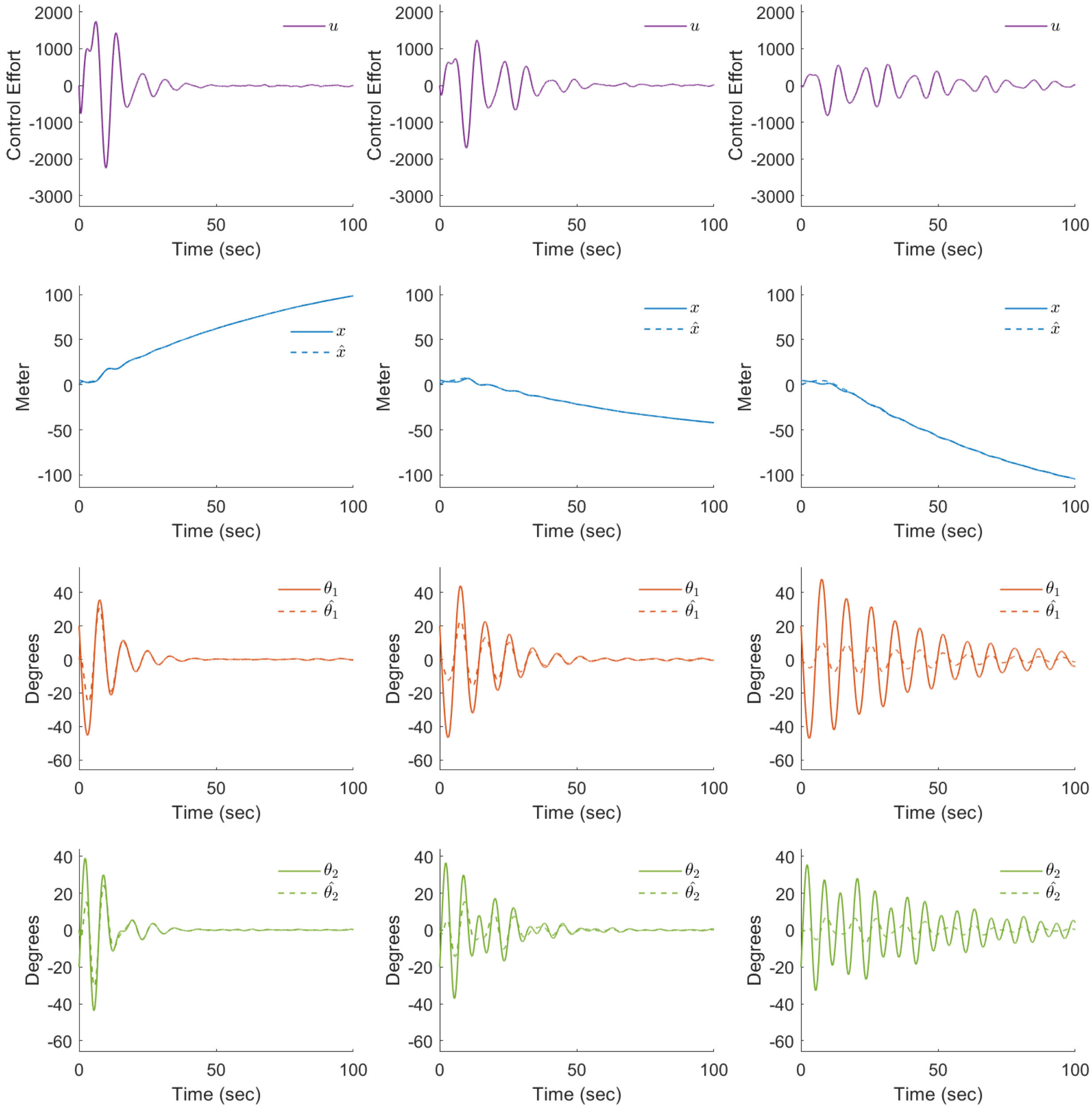

Figure 6 Simulations with output feedback control. Process noise was neglectable across all conditions. Left, mid and right columns correspond to 3 simulations with different levels of measurement noise covariance. Mid and right column has 10 and 100 times measurement noise covariance than the left column, respectively.

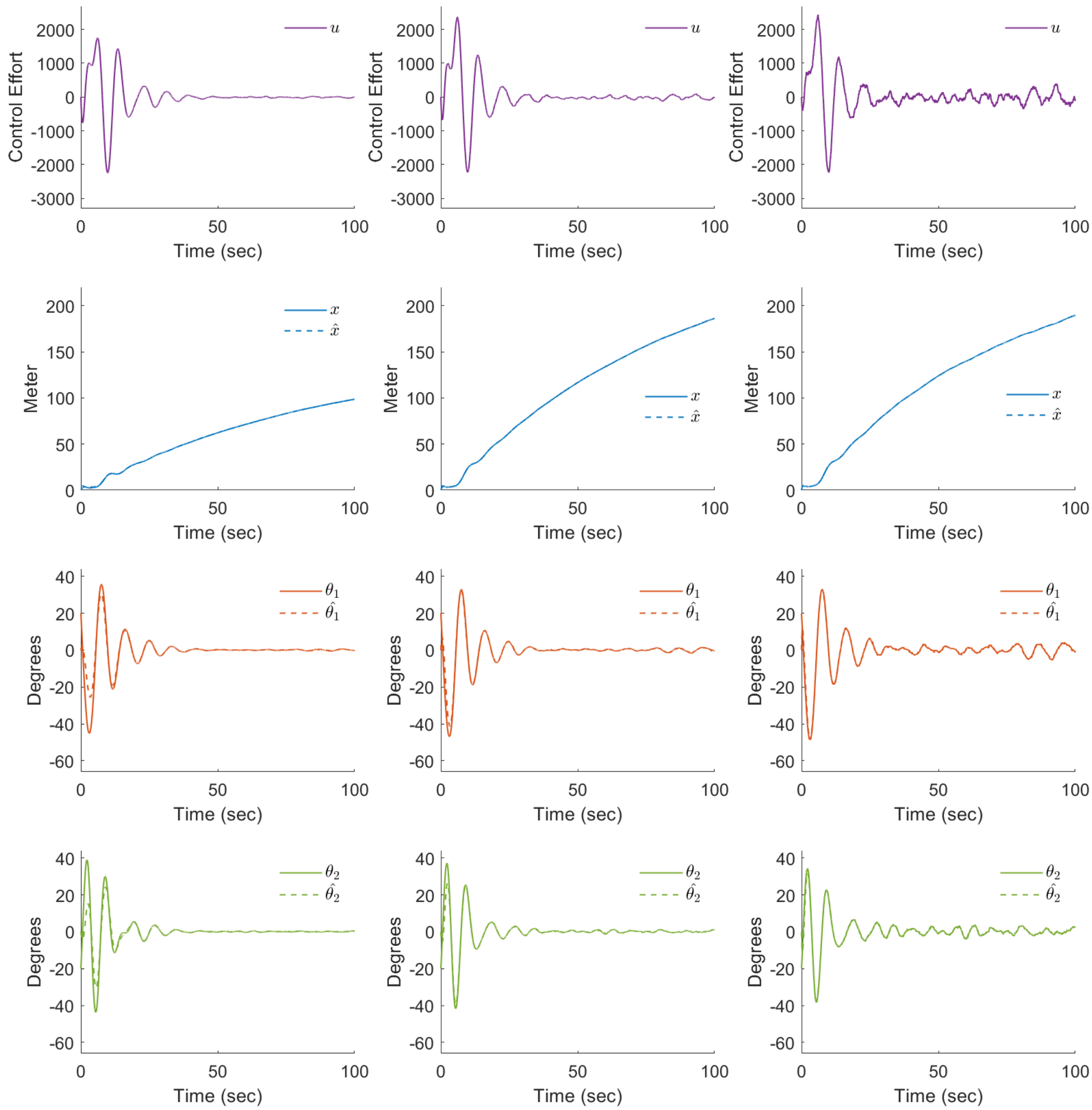

Figure 7 Simulations with output feedback control. Measurement noise was neglectable across all conditions. Left, mid and right columns correspond to 3 simulations with different levels of process noise covariance. Mid and right column has 10 and 100 times process noise covariance than the left column, respectively.

Finally, we consider tracking a constant reference X_d. At X_d, $\dot{X}$ equals zero. Consider the input u_∞ corresponding to X_d, we have the following using the linearized system:

$$\dot{X} = 0 = AX_d + B_K u_\infty$$

Let $\tilde{X}(t) = X(t) - X_d$ and $\tilde{u} = u - u_\infty$. We have the following representation of the system:

$$\begin{aligned} \dot{\tilde{X}} = \ \dot{X} &= A(\tilde{X} + X_d) + B_K(\tilde{u} + u_\infty) + B_D u_D \\ &= A\tilde{X} + B_K\tilde{u} + AX_d + B_K u_\infty + B_D u_D \\ &= \ A\tilde{X} + B_K\tilde{u} + B_D u_D \end{aligned}$$

Furthermore, we modified the Luenberger observer to track $\tilde{X}$ instead of X:

$$\begin{aligned} \dot{\hat{X}} &= A\hat{X} + B_K\tilde{u} + L(Y - CX_d - C\hat{X}) \\ &= A\hat{X} + B_K\tilde{u} + L(CX - CX_d - C\hat{X}) \\ &= A\hat{X} + B_K\tilde{u} + L(C\tilde{X} - C\hat{X}) \end{aligned}$$

To cope with constant disturbance, an integral variable is introduced, i.e.,

$$X_I(t) = \int_0^t S\hat{X}(\tau)d\tau$$

Note that it integrates $\hat{X}$ as we only have access to the state estimates. In addition, the matrix S is the selection matrix such that we can choose which variables to integrate. For example, if we integrate the $\hat{x}_1$ and $\hat{x}_3$, then S would be the following:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The time derivative of X_I is $\dot{X}_I = S\hat{X}$. Concatenating $\tilde{X}, \hat{X}, X_I$, we have:

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{\hat{X}} \\ \dot{X}_I \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ LC & A - LC & 0 \\ 0 & S & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \hat{X} \\ X_I \end{bmatrix} + \begin{bmatrix} B_K \\ 0 \\ 0 \end{bmatrix} \tilde{u} + \begin{bmatrix} B_D \\ 0 \\ 0 \end{bmatrix} u_D$$

Let $X_e = \tilde{X} - \hat{X}$, and further choosing the $\tilde{u} = K_1\hat{X} + K_2 X_I$, we could rewrite the above dynamics equation as:

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{X}_I \\ \dot{X}_e \end{bmatrix} = \begin{bmatrix} A + B_K K_1 & B_K K_2 & -B_K K_1 \\ S & 0 & -S \\ 0 & 0 & A - LC \end{bmatrix} \begin{bmatrix} \tilde{X} \\ X_I \\ X_e \end{bmatrix} + \begin{bmatrix} B_D \\ 0 \\ B_D \end{bmatrix} u_D$$

Let $\bar{X} = [\tilde{X}^T, X_I^T]^T$, $\bar{A} = \begin{bmatrix} A & 0 \\ S & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_K \\ 0 \end{bmatrix}$, $\bar{K} = [K_1, K_2]$, $\bar{B}_D = \begin{bmatrix} B_D \\ 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -B_K K_1 \\ -S \end{bmatrix}$. The above equation thus becomes:

$$\begin{bmatrix}\dot{\bar{X}}\\X_e\end{bmatrix}=\begin{bmatrix}\bar{A}+\bar{B}\bar{K} & A_{12}\\0 & A-LC\end{bmatrix}\begin{bmatrix}\bar{X}\\X_e\end{bmatrix}+\begin{bmatrix}\bar{B}_D\\B_D\end{bmatrix}u_D$$

Therefore to stabilize this system we could again apply the separation principle and we could design an LQR controller based on $\bar{A},\bar{B}$ and a chosen $\bar{Q},\bar{R}$, while L can be determined using the Kalman-Bucy filter. However, depending on the choice of the selection matrix S, a solution to the LQR controller might not be available, and thus the choice of S is determined from trial and error.

Given the system with the cart and two loads, the only constant reference the system is capable of tracking is in the form of $X_d=[c,0,0,0,0,0]^T$, meaning that the the system reaches equilibrium point at location c on the horizontal axis, which also suggests $u_\infty=0$. Thus, we choose to integrate only $\hat{x}_1$, and in fact, this is likely the only choice that will produce a stable LQR controller. Thus, the selection matrix S is simply:

$$S=[1\quad 0\quad 0\quad 0\quad 0\quad 0]$$

Next, we can choose $\bar{Q}$ with the following form:

$$\bar{Q}=\begin{bmatrix}Q_1 & 0\\0 & Q_2\end{bmatrix}$$

where Q_1 is the same as in part (D). Q_2 is a scalar constant and we choose a value of 0.1 to make the controller respond faster. $\bar{R}$ is the same as R in part (D). Given these choices, $\bar{K}$ is solved to be:

$$\bar{K}=[-753.1,-1807.8,4760.6,1327.8,4593.2,-1689.3,-158.1]$$

The input to the original plant is thus $u=\tilde{u}+u_\infty=\bar{K}\bar{X}+0=\bar{K}\bar{X}$. For the observer, L is computed similarly as before and we chose the process noise and the measurement noise to be the minimum among previously chosen levels of variance.

Finally, we performed simulations on the controller under two conditions (Figure 8). In the first condition, no external disturbance is applied (Figure 8, solid lines), while in the second condition, a force of -500 N is applied to the cart (Figure 8, dashed lines). In both conditions, the initial condition is $x(0)=\dot{x}(0)=\theta_1(0)=\dot{\theta}_1(0)=\theta_2(0)=\dot{\theta}_2(0)=0$, and the constant reference is $X_d=[10,0,0,0,0,0]^T$. When no disturbance was applied, the controller was able to drive the state to the constant reference, i.e., $x(t)$ stabilized at 10 while θ_1 and θ_2 stabilized at 0. With the constant -500 N disturbance to the system, the controller was able to generate +500 N in compensation (Figure 8, left upper panel). However, we get a tracking error when the disturbance was applied (Figure 8, right upper panel). The tracking error was due to the disturbance term in the observer error dynamics:

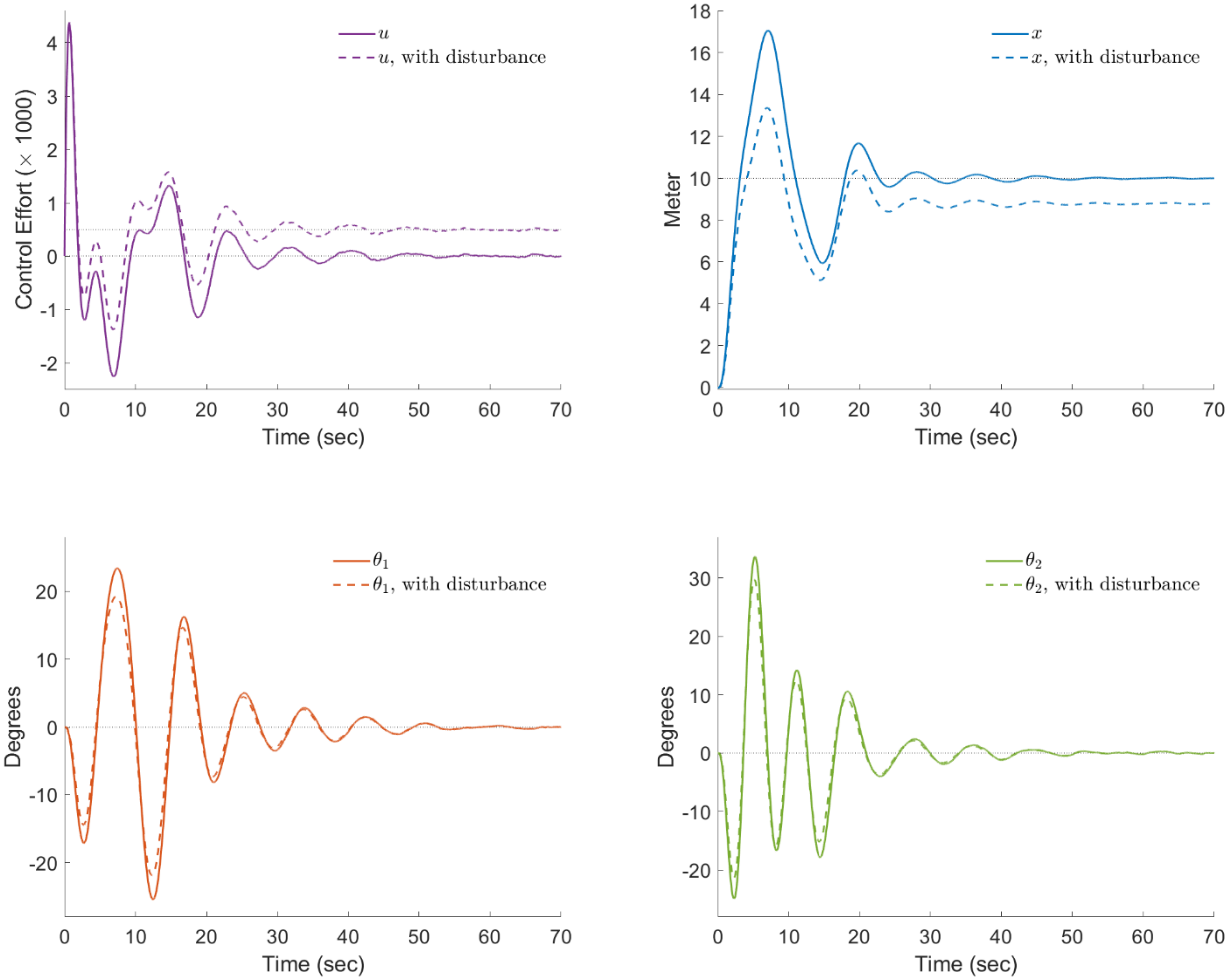

Figure 8 Simulation tracking constant reference with (dash lines) and without (solid lines) constant disturbance. The constant disturbance is -500 N. In the left upper panel, the two horizontal dotted lines correspond to 0 and 500 N respectively. The controller was able to compensate the disturbance.

$$\dot{X}_e = (A - LC)X_e + B_D u_D$$

Since $\lim_{t\to\infty} \dot{X}_e(t) = 0$, we have $\lim_{t\to\infty} X_e(t) = -(A - LC)^{-1} B_D u_D$, which means that given a constant disturbance, our state estimation will also have a constant error and will not converge to the actual value. In addition, since $\lim_{t\to\infty} \dot{X}_I(t) = 0$, we have $\lim_{t\to\infty} S\hat{X}(t) = 0$. And given that $\tilde{X}(t) = \hat{X}(t) + X_e(t)$, we have:

$$\lim_{t\to\infty} \tilde{X}(t) = \lim_{t\to\infty} \hat{X}(t) + X_e(t) = \lim_{t\to\infty} \hat{X}(t) - (A - LC)^{-1} B_D u_D$$

Premultiplying the selection matrix S:

$$\lim_{t\to\infty} S\tilde{X}(t) = \lim_{t\to\infty} S\hat{X}(t) - S(A - LC)^{-1} B_D u_D = -S(A - LC)^{-1} B_D u_D$$

which means:

$$\lim_{t\to\infty} SX(t) = \lim_{t\to\infty} S\tilde{X}(t) + SX_d = -S(A - LC)^{-1} B_D u_D + SX_d$$

Since $SX(t)$ in our design is simpily $x(t)$, we will have a tracking error in the actual state $x(t)$ which equals $-S(A - LC)^{-1} B_D u_D$, and this is due to the biased state estimation. By plugging in the values and performing the calculation, we could predict the value of the tracking error. To find the value of B_D, since the disturbance is also a force directly applied to the cart, we have $B_D = B_K$. Together, the value of the tracking error in $x(t)$ is calculated to be -1.2098, and that indeed agrees with our observation that $x(t = 70\ sec) = 8.7921$ in the second simulation with the constant disturbance, which is -1.2079 away from the desired value of 10.

The MATLAB script for generating Figure 8 and the above calculation is “sim4_partG_part2.m”.