Noether different and Tate cohomology.

Notation: $f: T \to S$ be a ring hom, where T is a commutative Noether ring.

Under some mild assumptions,

$\rightsquigarrow$ The Noether different of f annihilates all the Tate cohomology group $\widehat{\mathrm{Ext}}_S(-,-)$.

Recall Noether different.

$$0 \to I_{S/T} \to \underset{\underset{S \otimes_T S^{op}}{\shortparallel}}{S^e_T} \xrightarrow{\mu} S \to 0$$

S regard as left S^e_T-module. $(b' \otimes b'') \cdot b := b' b b''$.

Noether different $\quad N_{S/T} := \mu\left(\mathrm{ann}\left((I_{S/T})_{S^e_T}\right)\right)$ right annihilator.

Note that: $\quad \mathrm{ann}_{S^e_T}(I_{S/T}) = \mathrm{Hom}_{S^e_T}(S, S^e_T)$. $\quad$ [In this case, $N_{S/T} \subseteq Z(S)$.]

Observation:

$$\begin{array}{ccc} \mathrm{Hom}_{S^e_T}(S, S^e_T) & \xrightarrow{\mathrm{Hom}_{S^e_T}(S,\mu)} & \mathrm{Hom}_{S^e_T}(S, S) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{ann}_{S^e_T}(I_{S/T}) & \xrightarrow{\mu} & Z(S) \end{array}$$

[$\mapsto$ identifying as abelian group!!]

Note that
For an ideal α of a ring A
$\mathrm{Hom}_A(A/\alpha, A) \cong \mathrm{ann}(\alpha_A)$!!
$\mathrm{ann}(\alpha_A) = \{v \in A \mid \alpha \cdot v = 0\}$.
右零化子.

$\rightsquigarrow \mu\left(\mathrm{ann}_{S^e_T}(I_{S/T})\right) = \mathrm{Im}\,\mathrm{Hom}_{S^e_T}(S, \mu)$.

Prop: $N_{S/T} = Z(S) \iff S$ is projective as a left module over S^e_T.

Pf: $\quad S$ is projective

$\iff$

$$S \overset{\exists \alpha}{\dashrightarrow} S^e_T \xrightarrow{\mu} S, \quad \mu \circ \alpha = \mathrm{id}_S$$

above diagram
$\iff \mathrm{id}_S \in N_{S/T}$. i.e. $N_{S/T} = Z(S)$ $\quad \boxed{\text{Q.E.D}}$

Theorem 1 (Buchweitz)

Let $f: T \longrightarrow S$ be a ring hom, where

① T is a comm. noetherian ring with finite Krull dim and $\mathrm{gl.dim}\, T < \infty$

② S is strongly Gorenstein and $f_* S$ is f.g. projective over T.

Then the Noether different $N_{S/T}$ annihilates $\mathrm{Ext}^i_S(M,N) := \mathrm{Hom}_{D_{sg}(S)}(M, N[i])$.
$\forall M, N \in D_{sg}(S)$, $\forall i \in \mathbb{Z}$.

[In other words, $N_{S/T}$ annihilates $D_{sg}(S)$].

Proof: Let $a \in N_{S/T} = Z(S)$.

i.e. $\exists\, \alpha$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[rr, bend right, "a"'] & S^e_T \arrow[r, "u"] & S. \end{tikzcd}$$

so

$$\begin{tikzcd} -\overset{L}{\underset{S}{\otimes}} S \arrow[r, "-\overset{L}{\underset{S}{\otimes}}\alpha"] \arrow[d, equal] \arrow[rr, bend right=60, "-\overset{L}{\underset{S}{\otimes}} a"'] & -\overset{L}{\underset{S}{\otimes}}(S \underset{T}{\otimes} S) \arrow[r, "-\overset{L}{\underset{S}{\otimes}} u"] \arrow[d, equal] & -\overset{L}{\underset{S}{\otimes}} S \arrow[d, equal] \\ \mathrm{id}_S & -\overset{L}{\underset{T}{\otimes}} S. & \mathrm{id}_S. \end{tikzcd}$$

$-\overset{L}{\underset{S}{\otimes}}(S \overset{L}{\underset{T}{\otimes}} S) = -\overset{L}{\underset{S}{\otimes}}(S \underset{T}{\otimes} S)$ " $f_* T$ is proj

$\rightsquigarrow$ $\forall M \in D_{sg}(S)$.

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "a"'] & M \overset{L}{\underset{T}{\otimes}} S \arrow[r] & M \end{tikzcd}$$

$M \overset{L}{\underset{T}{\otimes}} S \in \mathrm{Perf}(S)$ as $\mathrm{pd}_T M < \infty$.

so $a: M \to M$ is zero in $D_{sg}(S)$.

any morphism $a \cdot \alpha = 0$. for $\alpha: M \to \Sigma^i N$ in $D_{sg}(S)$ $\boxed{\text{///}}$

Example: A comm. ring. G a finite group

$\iota: A \to A[G] =: B$

$I_{B/A} = \langle h\otimes 1 - 1 \otimes h \mid h \in G \rangle$

It is ~~not~~ difficult to see that $\sum_{g\in G} g \otimes g^{-1} \in \mathrm{ann}_{B^e_A}(I_{B/A})$ $\quad B^e_A = B \underset{A}{\otimes} B^{op}$.

$$\Big(\sum_g g\otimes g^{-1}\Big)(h\otimes 1 - 1\otimes h) = \sum_g \big(gh\otimes g^{-1} - g\otimes hg^{-1}\big)$$
$$= 0$$

$$\Rightarrow \quad \sum_g g\otimes g^{-1} \in N_{A[G]/A}. \quad \boxed{\text{///}}$$

Example (Simple extension).

A comm. ring. $B = \frac{A[x]}{(f(x))}$ be the simple extension defined by $f(x)$.

$$B_A^e \cong \frac{A[x,y]}{(f(x), f(y))} \xrightarrow{\mu} B.$$

$$y \longmapsto x$$

Note that: $I_{B/A} = \langle x-y \rangle$

As $f(x)-f(y) \in I_{B/A}$, $f(x)-f(y) = (x-y)g(x,y)$ for some $g \in A[x,y]$.

$\rightsquigarrow$ $g(x,y) \in \mathrm{ann}_{B_A^e}(I_{B/A})$

$\Rightarrow$ $\mu(g(x,y)) \in \mathcal{N}_{B/A}$. Note that $\mu(g(x,y)) = g(x,x)$

As g is continuous $= \lim_{y\to x} g(x,y)$

$= \lim_{y \to x} \frac{f(x)-f(y)}{x-y} = f'(x)$

$\boxed{//}$

$I_{B/A}$ has no relation with f !

assume $f(x) = g(x)h(x)$,

then $g(x)h(y) \in \mathrm{Ker}(\mu)$, ?

check $g(x)h(y) \in I = \langle x-y\rangle$

$g(x)h(y)$

$= g(x)(h(y) - h(x) + h(x))$

$= \underbrace{g(x)h(x)}_{=0\ ?\ \text{in}\ B_A^e} + \underbrace{g(x)(h(y)-h(x))}_{\in\langle x-y\rangle}$.

Theorem 2 (Buchweitz)

Let $R = \frac{k[[x_1,\cdots,x_n]]}{(f_1,\cdots,f_c)}$ be a complete int. (i.e. $f_1,\cdots,f_c$ is a $k[[x_1,\cdots,x_n]]$-regular sequence)

Then the jacobian ideal of R, denoted $J(R)$, annihilates $\mathrm{Dsg}(R)$.

Recall: $J(R)$ is the ideal of R generated by all $c\times c$ minors of the jacobian matrix $\frac{\partial(f_1,\cdots,f_c)}{\partial(x_1,\cdots,x_n)}$.

Proof: It is well-known that:

$$J(R) = \sum_{A \subseteq R} J_{R/A}$$

$A \subseteq R$: noether normalization

$J_{R/A}$: Kähler different, i.e. 0-th fitting ideal

Wang: On the fitting ideal in the free res. Lemma 4.3.

$A \hookrightarrow R$ — regular local ring

$\hookrightarrow$ Noether normalization.

Recall: Assume $R = \frac{A[Y_1,\cdots,Y_s]}{(g_1,\cdots,g_c)}$

then $J_{R/A} := I_s\left(\frac{\partial(g_1,\cdots,g_c)}{\partial(Y_1,\cdots,Y_s)}\right)$. This is denoted by $K(R/A)$ in Iyengar–Takahashi's paper.

As $A \hookrightarrow R$ is ~~normalization~~ noether normalization, $R \simeq \dfrac{A[y_1, \dots, y_\ell]}{(g_1, \dots, g_m)}$

claim: $\ell = m = n-c$.

Note that edim $R = n$. $\dim R = n - c = \dim A$.

The claim follows.

So $A \longrightarrow \dfrac{A[y_1, \dots, y_m]}{(g_1, \dots, g_m)} \simeq R$.

As R is CM and f.g. over A, gl.dim $A < \infty$ $\Rightarrow$ $\mathrm{pd}_A R < \infty$ by Auslander-Buchsbaum formula.
$$\mathrm{pd}_A R = \operatorname{depth} A - \operatorname{depth} R.$$

$\boxed{\text{Fact: } J_{R/A} \subseteq N_{R/A}}$. [in this case $J_{R/A} = N_{R/A}$

see Kunz, Kähler differentials 10.17]

If this is true, then by last theorem, we conclude that
$$J_{R/A} \cdot D_{sg}(R) = 0.$$
$$\rightsquigarrow J(R) \cdot D_{sg}(R) = 0$$

The proof of fact: we write a lemma.

Lemma: , then $J_{R/A} \subseteq N_{R/A}$. [see Wang, on the fitting ideal in free res. Lemma 5.8]

pf: Assume $R = \dfrac{A[X_1, \dots, X_n]}{(f_1, \dots, f_t)}$

then
$$R^t \xrightarrow{\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_t}{\partial X_1} \\ \frac{\partial f_1}{\partial X_n} & \cdots & \frac{\partial f_t}{\partial X_n} \end{pmatrix} = \left(\frac{\partial f_i}{\partial X_j}\right)} R^n \longrightarrow \Omega_{R/A} \to 0 \qquad (1)$$

Assume $I = \mathrm{Ker}(\mu: R^e_A \to R)$. then $\Omega_{R/A} \simeq I/I^2$.

Consider
$$(R^e_A)^s \xrightarrow{(g_{ij})} (R^e_A)^n \to I \to\!\!\!\!\to \quad (*)$$

$- \otimes_{R_A^e} R$ with $(*)$, we have

$$R^S \xrightarrow{u(g_{ij})} R^n \longrightarrow \Omega_{R/A} \to 0 \qquad (2).$$

By (1)+(2), we have

$$I_n(u(g_{ij})) = I_n\left(\left(\frac{\partial f_i}{\partial x_j}\right)\right)$$

$$:= J_{R/A}$$

Note that

$$J_{R/A} := I_n\left(\frac{\partial(f_1, \dots, f_t)}{\partial(x_1, \dots, x_n)}\right)$$

$$= \mathrm{Fitt}^0(\Omega_{R/A})$$

$\hookrightarrow$ 这个一般叫 Kähler different.

Note that $I_n((g_{ij})) \cdot I = 0 \Rightarrow I_n(u(g_{ij})) = u(I_n(g_{ij})) \subseteq N_{R/A}$.

$$\begin{bmatrix} \boxed{g_{ij}} & \end{bmatrix}$$

M $n \times n$ submatrix

$$(R_A^e)^S \xrightarrow{(g_{ij})} (R_A^e)^n \longrightarrow I_{R/A} \to 0 \qquad (3)$$

$$e_i \longmapsto x_i \otimes 1 - 1 \otimes x_i := \delta^i$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto (\delta^1, \dots, \delta^n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum \delta^i a_i.$$

we know

$$MM^* = \det(M) \cdot I_{n \times n}$$

$$= \begin{pmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{pmatrix}$$

Note that

$$I_{R/A} \cdot \det M = 0 \Leftrightarrow (\delta^1, \dots, \delta^n) \begin{pmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{pmatrix} = 0$$

$$\Leftrightarrow (\delta^1, \dots, \delta^n) M M^* = 0$$

$$\Leftarrow (\delta^1, \dots, \delta^n) M = 0$$

This is clear by the choice of the exact seq (3)

$\boxed{///}$

总结: let A be an fg k-alg, say $R = \dfrac{A[x_1, \dots, x_n]}{(f_1, \dots, f_t)}$

then $I_n\left(\dfrac{\partial(f_1, \dots, f_t)}{\partial(x_1, \dots, x_n)}\right)$ is the 0-th fitting ideal of $\Omega_{R/A}$

This is denote by $J_{R/A}$ in Wang's paper!!

Iyengar-Takahashi denote this by Kähler different $K_{R/A}$

$\rightsquigarrow$ $K_{R/A} \subseteq N_{R/A}$. [it is equal up to radical],

Rmk: Indeed, in Theorem 1 of Buchweitz, we don't need S is Gorenstein.

$\Rightarrow$ Theorem 1'

let $f: T \to S$ be a ring hom. s.t

① T is a finite Krull dim. and gl.dim $T < \infty$

② $f_* S$ is f.g. projective over T.

Then $N_{S/T}$ annihilates the singularity cat. of S.

Note that in Theorem 2, we also don't need R is complete ring.

Combine with Theorem 1'. we have

Theorem 2': Let R be a Cohen-Macaulay local ring.

Set $\operatorname{jac}(R) := \sum\limits_{\substack{A \subseteq R \\ \text{noether normalization}}} K(R/A)$. $\qquad$ $\big|\; K(R/A) := J(R/A)$

Then $\operatorname{jac}(R)$ annihilates $D_{sg}(R)$.

See a more general version in Iyengar-Takahashi's paper:

Let R be a comm. noether ring, then $\operatorname{jac}(R)^s \cdot D_{sg}(R) = 0$ for some $s \in \mathbb{N}$.