Triangulated cat. of periodic complexes and orbit categories.

Def: Let $\mathcal{A}$ be an additive cat. $T: \mathcal{A} \to \mathcal{A}$ be an autoequivalence.
The orbit cat. $\mathcal{A}/T$ is defined to be
objects: same with $\mathcal{A}$.
Morphisms: $\mathrm{Hom}_{\mathcal{A}/T}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, T^iY)$.

Fact: ① $X \cong T^iX$ in $\mathcal{A}/T$.
②
$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[rr, bend right, "\pi"'] & \mathcal{A} \arrow[r, "\pi"] & \mathcal{A}/T \end{tikzcd}$$
i.e. $\pi \circ T \cong \pi$.

Moreover, for additive functor $F: \mathcal{A} \to \mathcal{B}$.
if $F \circ T \cong F$, then $\exists \bar{F}$ s.t.
$$\begin{tikzcd} \mathcal{A} \arrow[d, "\pi"'] \arrow[r, "F"] & \mathcal{B} \\ \mathcal{A}/T \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$
.

Question: If $\mathcal{A}$ is a Δ-cat, is $\mathcal{A}/T$ a Δ-cat such that $\pi: \mathcal{A} \to \mathcal{A}/T$ is an exact functor?

Example (Peng-Xiao 1997)

Let R be an finite dimensional hereditary k-alg $\boxed{k \text{ field}}$
Then $D^b(R\text{mod})/[2] \simeq K_2(R\text{proj})$ is a Δ-cat.

Happel called this "root cat."

Fact: Neeman observed that the above question is negative.
$R = \frac{k[x]}{(x^2)}$, then $D^b(R\text{mod})/[2]$ is NOT a Δ-cat.

Keller (2005) constructed the triangulated hull of certain orbit cat.

idea : For a Δcat $(\mathcal{A}, T)$ with a dg enhancement $\mathcal{B}$.

if the equivalence $T: \mathcal{A} \longrightarrow \mathcal{A}$ can lift to a dg equivalence $T: \mathcal{B} \to \mathcal{B}$.

then

$$\mathcal{A}/T \simeq H^0(\mathcal{B}/T) \xhookrightarrow{Y} D(\mathcal{B}/T).$$

We call $D(\mathcal{B}/T)^c$ is <u>the triangulated hull</u> of $\mathcal{A}/T$.

- Here $\mathcal{B}$ is a pretriangulated cat.
  i.e $Y: H^0(\mathcal{B}) \xhookrightarrow{\sim} D(\mathcal{B})^c$ is equivalence up to direct summands.
  ($\Rightarrow$ This gives the Δ-structure of $H^0\mathcal{B}$)
- $\mathcal{A} \simeq H^0\mathcal{B}$ Δ-equivalence.

<u>Theorem</u> (Zhao (2014), Stai (2015))

Let R be a finite dimensional alg. over a field and $\mathrm{gl.dim} R < \infty$.
Then $D^b(R\text{-mod})/[n] \hookrightarrow D_n(R\text{-mod})$ is the embedding of the orbit cat. into its Δ-hull.

This generalizes Peng-Xiao's result.

<u>Question</u> : What's the triangulated hull of $D^b(R\text{-mod})/[n]$ for a general R.

<u>Theorem</u> (L)

Let R be a left Noetherian ring R. Then
$D^b(R\text{-mod})/[n] \hookrightarrow K_n(R\text{-Inj})^c$, $\mathrm{perf}(R)/[n] \hookrightarrow D_n(R\text{-Mod})^c$.
are the embeddings of the orbit. cat. into their Δ-hull.

<u>Rmk</u> : ① The result is inspired by

$$D^b(R\text{-mod}) \simeq K(R\text{-Inj})^c$$
$$\mathrm{perf}(R) \simeq D(R\text{-Mod})^c.$$

② The above result generalizes Zhao and Stai's result.
If gl.dim $R < \infty$. Then $K_n(R\text{-Inj}) \simeq D_n(R\text{Mod})$.
and $D_n(R\text{Mod})^c \simeq D_n(R\text{-mod})$.

Now we study Δ-cat. of periodic complexes.

Let $\mathcal{A}$ be an additive cat. Let $C_n(\mathcal{A})$ denote the cat. of n-periodic complexes.

$C_n(\mathcal{A})$ $\begin{cases} \text{Object}: n\text{-periodic complex } (\text{a complex } X \text{ is } \underline{n\text{-periodic}} \text{ if } X^i = X^{i+n},\ d^i = d^{i+n}) \\ \text{Morphism}: f: X \to Y. (f \text{ chain map}, f^i = f^{i+n}) \end{cases}$

<u>Fact</u>: $C_n(\mathcal{A})$ is abelian cat.

Similarly, one can define homotopy and cone
$\rightsquigarrow$ Thus we can get homotopy cat $K_n(\mathcal{A})$ (This is Δ-cat.).

If $\mathcal{A}$ is abelian cat. $D_n(\mathcal{A}) := K_n(\mathcal{A}) / \text{acyclic complexes}$.

<u>Fact</u>: if $\mathcal{A}$ has $\oplus$, then $C(\mathcal{A})$, $C_n(\mathcal{A})$, $K(\mathcal{A})$, $K_n(\mathcal{A})$ have $\oplus$.
If $\mathcal{A}$ is an AB4 cat. (has $\oplus$ and $\oplus$ is exact), then $D(\mathcal{A})$ and $D_n(\mathcal{A})$ have $\oplus$.

If $\mathcal{A}$ has $\oplus$. given a complex over $\mathcal{A}$, one can construct

$$\Delta(X) := \cdots \to \coprod_{i \equiv 0 \bmod n} X^i \longrightarrow \coprod_{i \equiv 1 \bmod n} X^i \to \cdots \quad C_n(\mathcal{A}).$$

$\Longrightarrow$ $C(\mathcal{A}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} C_n(\mathcal{A})$ adjoint pair. (Δ, ∇ are exact functors).

This induces adjoint $K(\mathcal{A}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} K_n(\mathcal{A})$ between Δ-cat.

<u>unit</u>: $\eta_X : X \longrightarrow \nabla\Delta X = \coprod_{i \in \mathbb{Z}} X[ni]$ is a split injection.

If $\mathcal{A}$ is AB4, $\exists$ $D(\mathcal{A}) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} D_n(\mathcal{A})$ between Δ-cat.

Prop: ① Let $\mathcal{A}$ be an additive cat with $\oplus$. Then $\Delta: K(\mathcal{A}) \to K_n(\mathcal{A})$ preserves compact objects. If $K(\mathcal{A})$ is compactly gen, then $K_n(\mathcal{A})$ is compactly gen. by $\Delta(K(\mathcal{A})^c)$ and Δ induces a fully faithful functor

$$\bar{\Delta}: K(\mathcal{A})^c/[n] \hookrightarrow K_n(\mathcal{A})^c.$$

② let $\mathcal{A}$ be an AB4 cat. Then $\Delta: D(\mathcal{A}) \to D_n(\mathcal{A})$ preserves compact obj. If $D(\mathcal{A})$ is compactly gen. then $D_n(\mathcal{A})$ is also compactly gen. and Δ induces a fully faithful functor $\bar{\Delta}: D(\mathcal{A})^c/[n] \hookrightarrow D_n(\mathcal{A})^c$.

proof: We prove ① for example.

1. Let $X \in K(\mathcal{A})^c$. want: $\Delta(X)$ is compact.

$$\mathrm{Hom}_{K_n(\mathcal{A})}(\Delta X, \coprod_i Y_i) \cong \mathrm{Hom}_{K(\mathcal{A})}(X, \nabla(\coprod_i Y_i))$$

$\nabla(\coprod_i Y_i) \cong \coprod_i (\nabla(Y_i))$ $\leftarrow$ ∇ preserves $\oplus$.

X compact $\downarrow$

$$\cong \coprod_i \mathrm{Hom}_{K(\mathcal{A})}(X, \nabla(Y_i))$$

$$\cong \coprod_i \mathrm{Hom}_{K_n(\mathcal{A})}(\Delta X, Y_i).$$

$\rightsquigarrow$ ΔX is compact.

2. If $K(\mathcal{A})$ is compactly gen, we want to show $K_n(\mathcal{A})$ is compactly gen. by $\Delta(K(\mathcal{A})^c)$

Assume $\mathrm{Hom}_{K_n(\mathcal{A})}(\Delta(K(\mathcal{A})^c), X) = 0 \overset{?}{\Rightarrow} X = 0$

$$\cong \qquad\qquad \Updownarrow \text{exercise}$$

$\mathrm{Hom}_{K(\mathcal{A})}(K(\mathcal{A})^c, \nabla X) = 0 \xRightarrow{\text{by assumption}} \nabla X = 0$

3.
$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "\Delta"] \arrow[d, "\pi"'] & K_n(\mathcal{A}) \\ K(\mathcal{A})/[n] \arrow[ur, dashed, "\exists \bar{\Delta}"'] & \end{tikzcd}$$

Note that $X[n] \cong X(n)$, $x \in X^i \mapsto (-1)^{ni} x$.

$\Delta \circ [n] \cong \Delta \circ (n) = \Delta$.

$\bar{\Delta}$ restricts $\bar{\Delta} : K(\mathcal{A})^c/[n] \longrightarrow K_n(\mathcal{A})^c$.

Want: this is f.f.

Indeed, if $X, Y \in K(\mathcal{A})^c$

$$\begin{aligned} \mathrm{Hom}_{K_n(\mathcal{A})}(\Delta X, \Delta Y) &\cong \mathrm{Hom}_{K(\mathcal{A})}(X, \nabla\Delta Y) \\ & \qquad\qquad \nabla\Delta Y \cong \coprod_{i \in \mathbb{Z}} Y[ni] \\ &\cong \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{K(\mathcal{A})}(X, Y[n]^i) \\ &:= \mathrm{Hom}_{K(\mathcal{A})/[n]}(X, Y) \xrightarrow{\bar{\Delta}} \mathrm{Hom}_{K_n(\mathcal{A})}(\Delta X, \Delta Y) \end{aligned}$$

Thus $\bar{\Delta} : K(\mathcal{A})^c/[n] \to K_n(\mathcal{A})^c$ is f.f.

$\boxed{\checkmark}$

Corollary: There are fully faithful embeddings:

$$\bar{\Delta}\circ i : D^b(R\text{mod})/[n] \hookrightarrow K_n(R\text{-Inj})^c$$

$$\bar{\Delta} : \mathrm{perf}(R)/[n] \hookrightarrow D_n(R\text{-Mod})^c.$$

$K_n(R\text{-Inj})^c$ is compactly gen. by image of $\bar{\Delta}\circ i$. $D_n(R\text{-Mod})$ is compactly gen. by image of $\bar{\Delta}$.

Proof: $i : D^b(R\text{-mod}) \xrightarrow[\text{Krause}]{\simeq} K(R\text{-Inj})^c \xrightarrow{\Delta} K_n(R\text{-Inj})^c$.

$\mathrm{perf}(R) = D(R\text{-Mod})^c \xrightarrow{\Delta} D_n(R\text{-Mod})^c$

The desired result follows from last proposition.

$\boxed{\checkmark}$

Note that: $i : D^b(R\text{-mod}) \xrightarrow{\sim} K(R\text{-Inj})^c$.

$\operatorname{Im} i = \{X \in K^+(R\text{-Inj}) \mid H(X) := \bigoplus_{i \in \mathbb{Z}} H^i(X) \text{ is f.g. } R\text{-mod}\}$

$:= K^{+,f}(R\text{-Inj})$

$= H^0(C^{+,f}_{dg}(R\text{-Inj}))$

with same obj. in $K^{+,f}(R\text{-Inj})$

Morphism space is hom complex.

This is a dg enhancement of $D^b(R\text{-mod})$

$\operatorname{perf}_{dg}(R)$: $\begin{cases} \text{same objs in } \operatorname{perf}(R) \\ \text{Morphism space is hom space} \end{cases}$ $\longrightarrow$ dg enhancement of $\operatorname{perf}(R)$.

Theorem (L)

$$D(C^{+,f}_{dg}(R\text{-Inj})/[n]) \simeq K_n(R\text{-Inj})$$

$$D(\operatorname{perf}_{dg}(R)/[n]) \simeq D_n(R\text{-Mod}).$$

Rmk: This will imply the embedding of the orbit cat into its triangulated hull.

Sketch proof: the first one.

$$\Phi : K_n(R\text{-Inj}) \longrightarrow D(C^{+,f}_{dg}(R\text{-Inj})/[n])$$
$$X \longmapsto \operatorname{Hom}_{\mathbb{P}}(-, X)$$

$X^\wedge : C^{+,f}_{dg}(R\text{-Inj}) \to C_{dg}(R)$

$$\begin{tikzcd} C^{+,f}_{dg}(R\text{-Inj}) \arrow[r] \arrow[d] & C_{dg}(R) \\ C^{+,f}_{dg}(R\text{-Inj})/[n] \arrow[ur, dashed] & \end{tikzcd}$$

Fact: Φ preserves $\bigoplus$.

$\exists$ comm. diagram:

$$\begin{tikzcd} K_n(R\text{-Inj}) \arrow[r, "\Phi"] & D(C^{+,f}_{dg}(R\text{-Inj})/[n]) \\ K_n(R\text{-Inj})^c \arrow[u, "\text{inc}"] \arrow[ur, phantom, "2"] & H^0(C^{+,f}_{dg}(R\text{-Inj})/[n]) \arrow[l, "\overline{\Delta}"'] \arrow[u, "Y"'] \end{tikzcd}$$

$\Rightarrow \Phi$ is equivalence $\boxed{//}$