

Noether different and Tate cohomology.

Notation: $f: T \rightarrow S$ be a map, where T is a commutative Noether ring.
Under some mild assumptions,

\Rightarrow The Noether different of f annihilates all the Tate cohomology group $\overline{\text{Ext}}_S^i(\cdot, \cdot)$.

Recall Noether different.

$$0 \rightarrow I_{S/T} \rightarrow S_T^e \xrightarrow{\mu} S \rightarrow 0$$

$S_T^e \cong S^{\otimes 2}$

\downarrow
regard as left S_T^e -module. $(b' \otimes b'') \cdot b := b' b b''$.

Noether different $N_{S/T} := \mu(\text{ann}((I_{S/T})_{S_T^e}))$ right annihilator.

Note that: $\text{ann}_{S_T^e}(I_{S/T}) = \text{Hom}_{S_T^e}(S, S_T^e)$. [In this case, $N_{S/T} \subseteq Z(S)$.]

Observation: $\text{Hom}_{S_T^e}(S, S_T^e) \xrightarrow{\text{Hom}_{S_T^e}(S, \mu)} \text{Hom}_{S_T^e}(S, S)$

Note that $\text{Hom}_A(A/\alpha, A) \cong \text{ann}_A(\alpha_A)$!! For an ideal α of a ring A

$\text{ann}_{S_T^e}(I_{S/T}) \xrightarrow{\mu} Z(S)$

$\downarrow \cong \left[\text{identifying as abelian group!!} \right]$

$\Rightarrow \mu(\text{ann}_{S_T^e}(I_{S/T})) = \text{Im Hom}_{S_T^e}(S, \mu)$.

Prop: $N_{S/T} = Z(S) \Leftrightarrow S$ is projective as a left module over S_T^e .

Pf: S is projective

\Leftrightarrow

$$S \xrightarrow{\exists \alpha} S_T^e \xrightarrow{\mu} S$$

above diagram \Leftrightarrow $\text{id}_S \in N_{S/T}$. i.e $N_{S/T} = Z(S)$ $\forall \alpha$

Theorem (Buchsatz)

Let $f: T \rightarrow S$ be a ring hom. where

① T is a comm. noetherian ring with finite Krull dim and $\text{gl. dim } T < \infty$

② S is strongly Gorenstein and $f_* S$ is f.g. projective over T .

Then the Noether different $N_{S/T}$ annihilates $\text{Ext}_S^i(M, N) := \text{Hom}_{D_{\text{sgl}(S)}}(M, N[i])$.
 $\forall M, N \in D_{\text{sgl}(S)}, \forall i \in \mathbb{Z}$.

[In other words, $N_{S/T}$ annihilates $D_{\text{sgl}(S)}$].

Proof: Let $a \in N_{S/T} = Z(S)$.

i.e. $\exists a$

$$S \xrightarrow{a} S \xrightarrow{a} S \xrightarrow{a} \dots$$

$$\begin{array}{ccccc} \bigoplus_S S & \xrightarrow{\sim \otimes_S a} & \bigoplus_S (S \otimes_S S) & \xrightarrow{\sim \otimes_S a} & \bigoplus_S S \\ \parallel & & \parallel & & \parallel \\ \bigoplus_S S & & \bigoplus_S S & & \bigoplus_S S \\ \parallel & & \parallel & & \parallel \\ \bigoplus_S S & & \bigoplus_S S & & \bigoplus_S S \end{array}$$

$\Rightarrow \forall M \in D_{\text{sgl}(S)}$.

$$M \xrightarrow{\alpha \in \text{Def}(S)} M \otimes_S S \xrightarrow{\alpha} M$$

so $a: M \rightarrow M$ is zero in $D_{\text{sgl}(S)}$.

any morphism $a \cdot a = 0$. for $a: M \rightarrow \Sigma N$ in $D_{\text{sgl}(S)}$ □

Example: A comm. ring. G a finite group

$$l: A \rightarrow A[G] := B$$

$$I_{B/A} = \langle h \otimes 1 - 1 \otimes h \mid h \in G \rangle$$

It is not difficult to see that $\sum_{g \in G} g \otimes g^{-1} \in \text{ann}_B(I_{B/A})$ $B_A^e = B \otimes_A B^{op}$

$$\left(\sum_g g \otimes g^{-1} \right) (h \otimes 1 - 1 \otimes h) = \sum_g (gh \otimes g^{-1} - g \otimes hg^{-1}) = 0$$

$$\Rightarrow \sum_g g \otimes g^{-1} \in N_{B(A[G])/A} \quad \square$$

Example (Simple extension).

A comm. ring. $B = \frac{A[x]}{(f(x))}$ be the simple extension defined by $f(x)$.

$$B_A^e \cong \frac{A[x, y]}{(f(x), f(y))} \xrightarrow{u} B.$$

$$y \longmapsto x$$

Note that: $I_{B/A} = \langle x - y \rangle$

As $f(x) - f(y) \in I_{B/A}$, $f(x) - f(y) = (x - y)g(x, y)$ for some $g \in A[x, y]$.

$$\Rightarrow g(x, y) \in \text{ann}_R(I_{B/A})$$

$$\Rightarrow u(g(x, y)) \in I_{B/A}. \quad \text{Note that } u(g(x, y)) = g(x, x)$$

As g is continuous $= \lim_{y \rightarrow x} g(x, y)$

$$= \lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y} = f'(x) \quad \square$$

$I_{B/A}$ has no relation with f .
assume $f(x) = g(x)h(x)$. / check $g(x)h(y) \in I_{B/A}$

$$\begin{aligned} \text{then } g(x)h(y) &\in \text{Ker}(u), & g(x)h(y) \\ &= g(x)(h(y) - h(x) + h(x)) \\ &= g(x)(h(y) - h(x)) + g(x)h(x) \\ &\in \langle x - y \rangle + \langle f(x) \rangle \\ &\in \langle x - y \rangle. \end{aligned}$$

Theorem (Buchweitz)

Let $R = \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_c)}$ be a complete int. (i.e. f_1, \dots, f_c is a $k[x_1, \dots, x_n]$ -regular sequence)

Then the jacobian ideal of R , denoted $J(R)$, annihilates $\text{Deg}(R)$.

Recall: $J(R)$ is the ideal of R generated by all $c \times c$ minors of the jacobian matrix $\frac{\partial(f_1, \dots, f_c)}{\partial(x_1, \dots, x_n)}$.

Proof: It is well-known that:

$$J(R) = \sum_{A \in R} J_{R/A}$$

noether normalization
Kähler different
ie. 0-th fitting
ideal

regular local ring
 $A \hookrightarrow R$

Noether normalization

Recall: Assume $R = \frac{A[x_1, \dots, x_s]}{(g_1, \dots, g_t)}$

then $J_{R/A} = I_s(\frac{\partial(g_1, \dots, g_t)}{\partial(x_1, \dots, x_s)})$. This is denoted by

Wang: On the fitting ideal in the free res. lemma 4.3, $K(R/A)$ in Iyengar-Takahashi's paper.

As $A \hookrightarrow R$ is ^{nachher} ~~normalization~~, $R \cong \frac{A[y_1, \dots, y_e]}{(g_1, \dots, g_m)}$

Claim: $\ell = m = n - c$.

Note that $\dim R = n$. $\dim R = n - c = \dim A$.

The claim follows.

$$\text{So } A \longrightarrow \frac{A[y_1, \dots, y_m]}{(g_1, \dots, g_m)} \cong R.$$

As R is CM and f.g. over A , $\Rightarrow \text{pd}_A R = 0$ by Auslander-Buchsbaum formula.
 $\text{gl. dim } A < \infty$ $\quad \quad \quad \text{depth } A = \text{depth } R$.

Fact: $J_{R/A} \subseteq N_{R/A}$. [in this case $J_{R/A} = N_{R/A}$
 see Kunz, Kohler differentials 10.17]

If this is true, then by last theorem, we conclude that

$$J_{R/A} \cdot \text{D}_{\text{sg}}(R) = 0.$$

$$\leadsto J(R) \cdot \text{D}_{\text{sg}}(R) = 0$$

The proof of fact: we write a lemma.

Lemma: , then $J_{R/A} \subseteq N_{R/A}$. [see Wang, on the
 fitting ideal in free res.]
 Lemma 5.8

PS: Assume $R = \frac{A[x_1, \dots, x_n]}{(f_1, \dots, f_t)}$

then $R \xrightarrow{\left(\frac{\partial f_i}{\partial x_j} \right)} R^n \rightarrow \Omega_{R/A} \rightarrow 0 \quad (*)$

Assume $I = \ker(d: A^c \rightarrow R)$.
 then $\Omega_{R/A} \cong I/I^2$.
[Consider $(R^c)^s \xrightarrow{(g_{ij})} (R^c)^n \rightarrow I \rightarrow 0 \quad (*)$

— $\otimes_{\mathbb{R}} R$ with $(*)$, we have

$$R^S \xrightarrow{u(\partial^j)} R^n \longrightarrow S_2 R/A \rightarrow 0 \quad (2).$$

By (1) + (2), we have

$$I_n(u|g_{ij}) = I_n\left(\frac{\partial f_i}{\partial x_j}\right)$$

$$:= \int_{R_A}$$

Note that

$$J_{R/A} := \text{Im} \left(\frac{\partial f, \dots, \partial t}{\partial x, \dots, \partial y} \right)$$

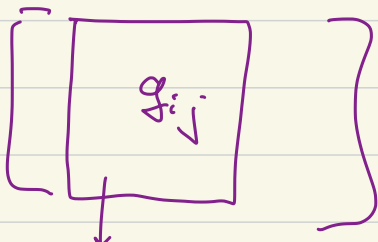
$$= \mathbb{F}_q^{\times}(\Sigma_{4/A})$$

→ 这个一般叫

Kahket different.

Note that

$$I_n(g_{ij}) \cdot I = 0. \Rightarrow I_n(u(g_{ij})) = u(I_n(g_{ij})) \subseteq N_{K/A}$$



Max n submatrix

$$(R_A)^S \xrightarrow{(g;j)} (R_A)^\eta \rightarrow I_{R_A}^{\sim \circ} \quad (3)$$

$$e_i \mapsto x_{i-1} \otimes x_i := \delta^i$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto (s^1, \dots, s^n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum s^i a_i.$$

we know

$$MM^* = \det(M) \cdot I_{n \times n}$$

$$= \begin{pmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{pmatrix}$$

Note flat

Note that $I_{F/A} \cdot \det M = 0 \iff (s', \dots, s^n) \begin{pmatrix} \det M \\ \vdots \\ \det M \end{pmatrix} = 0$

$$\Leftrightarrow (s' \dots, s^n) M M^* = 0$$

$$\leftarrow (s^1, \dots, s^n) \models \phi = 0$$

This is clear by the choice of the exact seq (3)



Ex: let A be an k -alg, say $R = \frac{A[x_1, \dots, x_n]}{(f_1, \dots, f_t)}$

then $I_n\left(\frac{\partial(f_1, \dots, f_t)}{\partial(x_1, \dots, x_n)}\right)$ is the 0-th fitting ideal of $J_{R/A}$

This is denoted by $J_{R/A}$ in Wang's paper!!

Iyengar - Takahashi denote this by Kohler different $K_{R/A}$

$\leadsto K_{R/A} \subseteq J_{R/A}$. [it is equal up to radical],