

# Duality of thick subcategories in the bounded derived category over complete intersections via cohomological supports

Jian Liu

University of Science and Technology of China

This is a joint work with  
Josh Pollitz (University of Utah)

2021.6.29

**Goal :** The containment of cohomological supports detects the containment of thick subcategories of bounded derived category over complete intersections.

**Goal :** The containment of cohomological supports detects the containment of thick subcategories of bounded derived category over complete intersections.

- Derived category
- Cohomological support
- Main result

Let  $R$  be a commutative noetherian ring. Let  $C(R)$  denote the category of complexes of  $R$ -modules.

Let  $R$  be a commutative noetherian ring. Let  $C(R)$  denote the category of complexes of  $R$ -modules.

- The *homotopy category*  $K(R)$  is defined by identifying homotopy in  $C(R)$ .

Let  $R$  be a commutative noetherian ring. Let  $C(R)$  denote the category of complexes of  $R$ -modules.

- The *homotopy category*  $K(R)$  is defined by identifying homotopy in  $C(R)$ .
- The *derived category*  $D(R)$  is localization of  $K(R)$  by inverting quasi-isomorphisms.

# Derived category I

Let  $R$  be a commutative noetherian ring. Let  $C(R)$  denote the category of complexes of  $R$ -modules.

- The *homotopy category*  $K(R)$  is defined by identifying homotopy in  $C(R)$ .
- The *derived category*  $D(R)$  is localization of  $K(R)$  by inverting quasi-isomorphisms.
- The *bounded derived category*  $D^f(R)$  is full subcategory of  $D(R)$  consisting of complexes with finitely generated bounded homology.

In general,  $K(R)$  and  $D(R)$  are **NOT** abelian categories. They are triangulated categories. They have

- shift  $[1]$
- cone

For a complex  $X$ ,  $X[1]$ :  $(X[1])^i = X^{i+1}$  and  $d_{X[1]} = -d_X$ . For a chain map  $f: X \rightarrow Y$ ,

$$\text{Cone}(f): \dots \rightarrow X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \rightarrow \dots$$



# Thick subcategory I

## Definition (Thick subcategory)

A full subcategory  $\mathcal{T}$  of  $D(R)$  (or  $D^f(R)$ ) is called *thick* if it is closed under taking  $[1]$ , cone and direct summands.

## Definition (Thick subcategory)

A full subcategory  $\mathcal{T}$  of  $D(R)$  (or  $D^f(R)$ ) is called *thick* if it is closed under taking  $[1]$ , cone and direct summands.

- $\text{thick}_{D(R)}(X) :=$  smallest thick subcategory of  $D(R)$  containing  $X$ .
- $\text{perf}(R) := \{X \in D(R) \mid X \text{ is quasi-isomorphic to } 0 \rightarrow P^n \rightarrow \dots \rightarrow P^m \rightarrow 0, P^i \in R\text{-proj}\}$

# Thick subcategory I

## Definition (Thick subcategory)

A full subcategory  $\mathcal{T}$  of  $D(R)$  (or  $D^f(R)$ ) is called *thick* if it is closed under taking  $[1]$ , cone and direct summands.

- $\text{thick}_{D(R)}(X) :=$  smallest thick subcategory of  $D(R)$  containing  $X$ .
- $\text{perf}(R) := \{X \in D(R) \mid X \text{ is quasi-isomorphic to } 0 \rightarrow P^n \rightarrow \dots \rightarrow P^m \rightarrow 0, P^i \in R\text{-proj}\}$

## Example

$\text{perf}(R) = \text{thick}_{D(R)}(R)$  is a thick subcategory of  $D^f(R)$ .

# Thick subcategory II

For  $X \in D(R)$ ,  $\text{Supp}_R X := \{\mathfrak{p} \in \text{Spec } R \mid X_{\mathfrak{p}} \text{ is not acyclic}\}.$

Theorem (Hopkins,1987; Neeman,1992)

*For perfect complexes  $M, N$ ,*

$$M \in \text{thick}_{D(R)}(N) \iff \text{Supp}_R M \subseteq \text{Supp}_R N.$$

# Complete intersection

Setting:  $(Q, \mathfrak{m}, k)$  is a regular local ring. That is,  $\text{gldim}(Q) < \infty$   
( $\iff D^f(Q) = \text{thick}_{D(Q)}(Q)$ ). e.g.,  $k[[x_1, \dots, x_n]]$ .

# Complete intersection

Setting:  $(Q, \mathfrak{m}, k)$  is a regular local ring. That is,  $\text{gldim}(Q) < \infty$   
( $\iff D^f(Q) = \text{thick}_{D(Q)}(Q)$ ). e.g.,  $k[[x_1, \dots, x_n]]$ .

- $R =$  the **complete intersection**  $Q/(f_1, \dots, f_n)$ , where  $f_1, \dots, f_n \in \mathfrak{m}^2$  is  $Q$ -regular sequence  
( $f_{i+1}: Q/(f_1, \dots, f_i) \hookrightarrow Q/(f_1, \dots, f_i)$  for all  $0 \leq i \leq n-1$ ).

# Complete intersection

Setting:  $(Q, \mathfrak{m}, k)$  is a regular local ring. That is,  $\text{gldim}(Q) < \infty$   
( $\iff D^f(Q) = \text{thick}_{D(Q)}(Q)$ ). e.g.,  $k[[x_1, \dots, x_n]]$ .

- $R =$  the **complete intersection**  $Q/(f_1, \dots, f_n)$ , where  
 $f_1, \dots, f_n \in \mathfrak{m}^2$  is  $Q$ -regular sequence  
( $f_{i+1}: Q/(f_1, \dots, f_i) \hookrightarrow Q/(f_1, \dots, f_i)$  for all  $0 \leq i \leq n-1$ ).

## Example

$$k[[x_1, \dots, x_n]]/(x_1^{l_1}, \dots, x_n^{l_n}), \quad l_i \geq 2.$$

# Cohomological operator I (Eisenbud)

Choose a free resolution of  $k$  over  $R$ :

$$F: \dots \rightarrow F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \rightarrow 0.$$

Lift  $F$  to a sequence over  $Q$ :  $\tilde{F}: \dots \rightarrow \tilde{F}^{-2} \xrightarrow{\tilde{d}^{-2}} \tilde{F}^{-1} \xrightarrow{\tilde{d}^{-1}} \tilde{F}^0 \rightarrow 0.$



# Cohomological operator I (Eisenbud)

Choose a free resolution of  $k$  over  $R$ :

$$F: \dots \rightarrow F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \rightarrow 0.$$

Lift  $F$  to a sequence over  $Q$ :  $\tilde{F}: \dots \rightarrow \tilde{F}^{-2} \xrightarrow{\tilde{d}^{-2}} \tilde{F}^{-1} \xrightarrow{\tilde{d}^{-1}} \tilde{F}^0 \rightarrow 0.$

- $\tilde{d}^2 \otimes_Q R = d^2 = 0 \implies \exists \tilde{t}_i: \tilde{F} \rightarrow \tilde{F}[2] \ (1 \leq i \leq n)$  such that  
$$\tilde{d}^2 = \sum_{i=1}^n \tilde{t}_i f_i.$$

# Cohomological operator I (Eisenbud)

Choose a free resolution of  $k$  over  $R$ :

$$F: \dots \rightarrow F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \rightarrow 0.$$

Lift  $F$  to a sequence over  $Q$ :  $\tilde{F}: \dots \rightarrow \tilde{F}^{-2} \xrightarrow{\tilde{d}^{-2}} \tilde{F}^{-1} \xrightarrow{\tilde{d}^{-1}} \tilde{F}^0 \rightarrow 0$ .

- $\tilde{d}^2 \otimes_Q R = d^2 = 0 \implies \exists \tilde{t}_i: \tilde{F} \rightarrow \tilde{F}[2] \ (1 \leq i \leq n)$  such that
$$\tilde{d}^2 = \sum_{i=1}^n \tilde{t}_i f_i.$$

- Each  $t_i := \tilde{t}_i \otimes_Q R: F \rightarrow F[2]$  is a chain map!

$$(t_i)^* := H^{j+2}(\text{Hom}_R(t_i, k)): \text{Ext}_R^j(k, k) \rightarrow \text{Ext}_R^{j+2}(k, k).$$

$(t_i)^*$  is called the **cohomological operator**.

# Cohomological operator II(Eisenbud)

Set  $\mathcal{S} := k[\chi_1, \dots, \chi_n]$ , where  $\deg(\chi_i) = 2$ .

- There is an algebra homomorphism

$$\zeta_k: \mathcal{S} \rightarrow \operatorname{Ext}_R(k, k) := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_R^j(k, k), \quad \chi_i \mapsto (t_i)^*(\operatorname{id}_k).$$

such that  $(t_i)^*(\gamma) = \zeta_k(\chi_i) \circ \gamma = \gamma \circ \zeta_k(\chi_i)$  for all  $\gamma \in \operatorname{Ext}_R^{|\gamma|}(k, k)$ .

# Cohomological operator II(Eisenbud)

Set  $\mathcal{S} := k[\chi_1, \dots, \chi_n]$ , where  $\deg(\chi_i) = 2$ .

- There is an algebra homomorphism

$$\zeta_k: \mathcal{S} \rightarrow \operatorname{Ext}_R(k, k) := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_R^j(k, k), \quad \chi_i \mapsto (t_i)^*(\operatorname{id}_k).$$

such that  $(t_i)^*(\gamma) = \zeta_k(\chi_i) \circ \gamma = \gamma \circ \zeta_k(\chi_i)$  for all  $\gamma \in \operatorname{Ext}_R^{|\gamma|}(k, k)$ .

- For  $M \in D(R)$ ,  $\operatorname{Ext}_R(M, k) := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_R^j(M, k)$  and  $\operatorname{Ext}_R(k, M) := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_R^j(k, M)$  are graded  $\mathcal{S}$ -module.

# Cohomological operator (Example)

## Example

Let  $k$  be a field,  $Q = k[[x]]$ ,  $f = x^2$  and  $R = Q/(f) = k[[x]]/(x^2)$ .  
The free resolution of  $k$  is  $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \rightarrow 0$ . Lift this over  $Q$

$$\cdots \xrightarrow{x} Q \xrightarrow{x} Q \xrightarrow{x} Q \rightarrow 0.$$

Since  $x^2 = 1 \cdot f$ , the  $\chi_f$  action on  $\text{Ext}_R^j(k, k) \cong k$  ( $j \geq 0$ ) is identity.  
Precisely,  $S = k[t^2] \hookrightarrow k[t] \cong \text{Ext}_R(k, k)$ , where  $\deg(t) = 1$ .

$\text{Spec } \mathcal{S} =$  the homogeneous spectrum of  $\mathcal{S} = k[\chi_1, \dots, \chi_n]$ .

## Definition (Cohomological support)

For  $M \in D(R)$ . The *cohomological support* of  $M$  over  $R$  is

$$V_R(M) := \text{Supp}_{\mathcal{S}} \text{Ext}_R(M, k) = \{\mathfrak{p} \in \text{Spec } \mathcal{S} \mid \text{Ext}_R(M, k)_{\mathfrak{p}} \neq 0\}.$$

$\text{Spec } \mathcal{S} =$  the homogeneous spectrum of  $\mathcal{S} = k[\chi_1, \dots, \chi_n]$ .

## Definition (Cohomological support)

For  $M \in D(R)$ . The *cohomological support* of  $M$  over  $R$  is

$$V_R(M) := \text{Supp}_{\mathcal{S}} \text{Ext}_R(M, k) = \{\mathfrak{p} \in \text{Spec } \mathcal{S} \mid \text{Ext}_R(M, k)_{\mathfrak{p}} \neq 0\}.$$

## Example

- $V_R(R) = \{(\chi_1, \dots, \chi_n)\}$ .
- $V_R(k) = \text{Spec } \mathcal{S}$ .

## Theorem (Gulliksen, 1974)

*For  $M \in D^f(R)$ ,  $\text{Ext}_R(M, k)$  and  $\text{Ext}_R(k, M)$  are finitely generated  $S$ -modules.*



## Theorem (Gulliksen, 1974)

*For  $M \in D^f(R)$ ,  $\text{Ext}_R(M, k)$  and  $\text{Ext}_R(k, M)$  are finitely generated  $S$ -modules.*

- For  $M \in D^f(R)$ ,  $V_R(M)$  is Zariski closed subset in  $\text{Spec } S$ .

## Theorem (Gulliksen, 1974)

*For  $M \in D^f(R)$ ,  $\text{Ext}_R(M, k)$  and  $\text{Ext}_R(k, M)$  are finitely generated  $\mathcal{S}$ -modules.*

- For  $M \in D^f(R)$ ,  $V_R(M)$  is Zariski closed subset in  $\text{Spec } \mathcal{S}$ .
- For  $M \in D^f(R)$ ,  $V_R(M) = \text{Supp}_{\mathcal{S}} \text{Ext}_R(k, M)$ .

## Theorem (Gulliksen, 1974)

*For  $M \in D^f(R)$ ,  $\text{Ext}_R(M, k)$  and  $\text{Ext}_R(k, M)$  are finitely generated  $\mathcal{S}$ -modules.*

- For  $M \in D^f(R)$ ,  $V_R(M)$  is Zariski closed subset in  $\text{Spec } \mathcal{S}$ .
- For  $M \in D^f(R)$ ,  $V_R(M) = \text{Supp}_{\mathcal{S}} \text{Ext}_R(k, M)$ .
- For  $M \in D^f(R)$ ,

$$\text{Ext}_R^{\gg 0}(M, k) = 0 \iff V_R(M) \subseteq \{(\chi_1, \dots, \chi_n)\}.$$

## Theorem (L-Pollitz, 2021)

For  $M, N \in D^f(R)$ ,

$$M \in \text{thick}_{D(R)}(N) \iff V_{R_p}(M_p) \subseteq V_{R_p}(N_p), \text{ for all } p \in \text{Spec } R.$$

## Theorem (L-Pollitz, 2021)

For  $M, N \in D^f(R)$ ,

$$M \in \text{thick}_{D(R)}(N) \iff V_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq V_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}), \text{ for all } \mathfrak{p} \in \text{Spec } R.$$

Choose  $N = R$ , the theorem implies

- For  $M \in D^f(R)$ ,

$$\text{pd}_R(M) < \infty \iff \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty, \forall \mathfrak{p} \in \text{Spec } R.$$

## Corollary (Stevenson, 2014)

For  $M \in D^f(R)$ ,  $\mathrm{RHom}_R(M, R) \in \mathrm{thick}_{D(R)}(M)$ .

## Proof idea.

Set  $(-)^{\vee} = \mathrm{RHom}(-, R)$  and  $d = \dim(R)$ . Then

$$\mathrm{Ext}_R(M, k) \cong \mathrm{Ext}_R(k^{\vee}, M^{\vee}) \cong \mathrm{Ext}_R(k, M^{\vee})[d].$$

Hence  $V_R(M) = \mathrm{Supp}_S \mathrm{Ext}_R(k, M^{\vee}) = V_R(M^{\vee})$ . Similar results hold when we localize at each  $\mathfrak{p} \in \mathrm{Spec} R$ . □

## Corollary (Stevenson, 2014)

For  $M \in D^f(R)$ ,  $\mathrm{RHom}_R(M, R) \in \mathrm{thick}_{D(R)}(M)$ .

## Proof idea.

Set  $(-)^{\vee} = \mathrm{RHom}(-, R)$  and  $d = \dim(R)$ . Then

$$\mathrm{Ext}_R(M, k) \cong \mathrm{Ext}_R(k^{\vee}, M^{\vee}) \cong \mathrm{Ext}_R(k, M^{\vee})[d].$$

Hence  $V_R(M) = \mathrm{Supp}_S \mathrm{Ext}_R(k, M^{\vee}) = V_R(M^{\vee})$ . Similar results hold when we localize at each  $\mathfrak{p} \in \mathrm{Spec} R$ . □

- Above result implies a result of Avramov-Buchweitz,

$$\text{For } M, N \in D^f(R), \quad \mathrm{Ext}_R^{\geq 0}(M, N) = 0 \iff \mathrm{Ext}_R^{\geq 0}(N, M) = 0.$$

# Proof idea of main result

$K^R$  (resp.  $K^Q$ ) = the Koszul algebra on a minimal set of generators of the maximal ideal of  $R$  (resp.  $Q$ ).

$$M \in \text{thick}_{D(R)}(N) \iff V_{R_p}(M_p) \subseteq V_{R_p}(N_p), \forall p \in \text{Spec } R..$$

- $R \rightarrow K^R = K^Q \otimes_Q R \xleftarrow{\simeq} K^Q \otimes_R \text{Kos}^Q(\underline{f}) \xrightarrow{\simeq} k \otimes_Q \text{Kos}^Q(\underline{f}) = \bigwedge(k[1])^c := \bigwedge$ . Hence








$$D^f_t(R) \rightarrow D^f_t(K^R) \xrightarrow{\simeq} D^f_t(\bigwedge) \xrightarrow{\simeq} D^f_h(S).$$

- For  $X \in D^f(R)$ ,  $V_R(X) = \text{Supp}_S(\text{hjt}(X))$ . Hence if  $X, Y \in D^f(R)$  and  $V_R(X) \subseteq V_R(Y)$ , then  $\text{hjt}(X) \in \text{thick}_{D(S)}(\text{hjt}(Y))$ . Thus  $t(X) \in \text{thick}_{D(K^R)}(t(Y))$ .
- $K^{R_p} \otimes_R M \in \text{thick}_{D(K^{R_p})}(K^{R_p} \otimes_R N)$ ,  $\forall p \in \text{Spec } R$ . Combine with BIK's local-global,  $M \in \text{thick}_{D(R)}(N)$ .





# References

-  L. L. Avramov, R. O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142**, 2000, 285–318.
-  D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260**, 1980, 35–64.
-  T. H. Gulliksen, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. **34**, 1974, 167–183.
-  M. J. Hopkins, *Global methods in homotopy theory*, Lond. Math. Soc. Lecture Note Ser. **117**, Cambridge University Press, 1987, 73–96.
-  J. Liu, J. Pollitz, *Duality and symmetry of complexity over complete intersections via exterior homology*, Proc. Amer. Math. Soc. **149**, 2021, 619–631.
-  A. Neeman, *The chromatic tower for  $D(R)$* , Topology, **31**, 1992, 519–532, With an appendix by Marcel Bökstedt.
-  G. Stevenson, *Duality for bounded derived categories of complete intersections*, Bull. Lond. Math. Soc. **46**, 2014, 245–257.

Thank you!