Duality of thick subcategories in the bounded derived category over complete intersections via cohomological supports

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This is a joint work with
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2021.6.29

Goal : The containment of cohomological supports detects the containment of thick subcategories of bounded derived category over complete intersections.

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- Derived category
- Cohomological support
- Main result

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- The *derived category* $\mathsf{D}(R)$ is localization of $\mathsf{K}(R)$ by inverting quasi-isomorphisms.
- The *bounded derived category* $\mathsf{D}^f(R)$ is full subcategory of $\mathsf{D}(R)$ consisting of complexes with finitely generated bounded homology.

In general, K(R) and D(R) are NOT abelian categories. They are triangulated categories. They have

- shift [1]
- cone

For a complex X, $X[1]$: $(X[1])^i = X^{i+1}$ and $d_{X[1]} = -d_X$. For a chain map $f\colon X \to Y$,

$$\mathrm{Cone}(f)\colon \cdots \to X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \to \cdots .$$

Definition (Thick subcategory)

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- $\mathrm{thick}_{\mathrm{D}(R)}(X)$:= smallest thick subcategory of $\mathrm{D}(R)$ containing X.
- $\mathrm{perf}(R) := \{X \in \mathrm{D}(R) \mid X \text{ is quasi-isomorphic to } 0 \to P^n \to \cdots \to P^m \to 0, P^i \in R\text{−proj}\}$

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Example

$\mathrm{perf}(R) = \mathrm{thick}_{\mathrm{D}(R)}(R)$ is a thick subcategory of $\mathrm{D}^f(R)$.

For $X \in \mathrm{D}(R)$, $\operatorname{Supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \text{ is not acyclic}\}$.

Theorem (Hopkins,1987; Neeman,1992)

For perfect complexes M, N,

$$M \in \operatorname{thick}_{\mathrm{D}(R)}(N) \iff \operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N.$$

Complete intersection

Setting: $(Q, \mathfrak{m}, k)$ is a regular local ring. That is, $\mathrm{gldim}(Q) < \infty$ ($\iff$ $\mathsf{D}^f(Q) = \mathrm{thick}_{\mathsf{D}(Q)}(Q)$). e.g., $k[[x_1, \ldots, x_n]]$.

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- R= the complete intersection $Q/(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in \mathfrak{m}^2$ is Q-regular sequence ($f_{i+1} \colon Q/(f_1, \ldots, f_i) \hookrightarrow Q/(f_1, \ldots, f_i)$ for all $0 \leq i \leq n-1$).

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Example

$k[[x_1, \ldots, x_n]]/(x_1^{l_1}, \ldots, x_n^{l_n}), \quad l_i \geq 2.$

Choose a free resolution of k over R:

$$F\colon \cdots \to F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^{0} \to 0.$$

Lift F to a sequence over Q: $\tilde{F}\colon \cdots \to \tilde{F}^{-2} \xrightarrow{\tilde{d}^{-2}} \tilde{F}^{-1} \xrightarrow{\tilde{d}^{-1}} \tilde{F}^{0} \to 0.$

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- $\tilde{d}^2 \otimes_Q R = d^2 = 0 \implies \exists \tilde{t}_i\colon \tilde{F} \to \tilde{F}[2]$ $(1 \leq i \leq n)$ such that $\tilde{d}^2 = \sum_{i=1}^n \tilde{t}_i f_i$.

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- Each $t_i := \tilde{t}_i \otimes_Q R\colon F \to F[2]$ is a chain map!

$$(t_i)^* := \mathrm{H}^{j+2}(\mathrm{Hom}_R(t_i, k))\colon\ \mathrm{Ext}_R^j(k,k) \to \mathrm{Ext}_R^{j+2}(k,k).$$

$(t_i)^*$ is called the cohomological operator.

Cohomological operator II(Eisenbud)

Set $\mathcal{S} := k[\chi_1, \ldots, \chi_n]$, where $\deg(\chi_i) = 2$.

- There is an algebra homomorphism

$$\zeta_k \colon \mathcal{S} \to \mathrm{Ext}_R(k,k) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}_R^j(k,k), \ \ \chi_i \mapsto (t_i)^*(\mathrm{id}_k).$$

such that $(t_i)^*(\gamma) = \zeta_k(\chi_i) \circ \gamma = \gamma \circ \zeta_k(\chi_i)$ for all $\gamma \in \mathrm{Ext}_R^{|\gamma|}(k,k)$.

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- For $M \in \mathrm{D}(R)$, $\mathrm{Ext}_R(M, k) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^j_R(M, k)$ and $\mathrm{Ext}_R(k, M) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^j_R(k, M)$ are graded $\mathcal{S}$-modules.

Cohomological operator (Example)

Example

Let k be a field, $Q = k[[x]]$, $f = x^2$ and $R = Q/(f) = k[[x]]/(x^2)$. The free resolution of k is $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to 0$. Lift this over Q

$$\cdots \xrightarrow{x} Q \xrightarrow{x} Q \xrightarrow{x} Q \to 0.$$

Since $x^2 = 1 \cdot f$, the χ_f action on $\mathrm{Ext}^j_R(k,k) \cong k$ $(j \geq 0)$ is identity. Precisely, $\mathcal{S} = k[t^2] \hookrightarrow k[t] \cong \mathrm{Ext}_R(k,k)$, where $\deg(t) = 1$.

Cohomological support I

$\operatorname{Spec}\mathcal{S}$= the homogeneous spectrum of $\mathcal{S} = k[\chi_1, \ldots, \chi_n]$.

Definition (Cohomological support)

For $M \in \mathrm{D}(R)$. The *cohomological support* of M over R is

$$\mathrm{V}_R(M) := \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}_R(M, k) = \{\mathfrak{p} \in \operatorname{Spec}\mathcal{S} \mid \operatorname{Ext}_R(M, k)_{\mathfrak{p}} \neq 0\}.$$

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Example

- $\mathsf{V}_R(R) = \{(\chi_1, \ldots, \chi_n)\}$.
- $\mathsf{V}_R(k) = \operatorname{Spec}\mathcal{S}$.

Cohomological support II

Theorem (Gulliksen, 1974)

For $M \in \mathrm{D}^{\mathrm{f}}(R)$, $\operatorname{Ext}_R(M,k)$ and $\operatorname{Ext}_R(k,M)$ are finitely generated $\mathcal{S}$-modules.

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- For $M \in \mathrm{D}^f(R)$, $\mathrm{V}_R(M) = \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}_R(k,M)$.
- For $M \in \mathrm{D}^f(R)$,
$$\operatorname{Ext}_R^{\gg 0}(M,k) = 0 \iff \mathrm{V}_R(M) \subseteq \{(\chi_1, \ldots, \chi_n)\}.$$

Main result

Theorem (L-Pollitz, 2021)

For $M, N \in \mathsf{D}^{\mathsf{f}}(\mathsf{R})$,

$$M \in \operatorname{thick}_{\mathsf{D}(R)}(N) \iff \mathrm{V}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \subseteq \mathrm{V}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}), \textit{ for all } \mathfrak{p} \in \operatorname{Spec} R.$$

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Choose $N = R$, the theorem implies

- For $M \in \mathrm{D}^f(R)$,

$$\mathrm{pd}_R(M) < \infty \iff \mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty, \ \forall\, \mathfrak{p} \in \operatorname{Spec} R.$$

Application

Corollary (Stevenson, 2014)

For $M \in \mathrm{D}^{\mathrm{f}}(R)$, $\mathrm{RHom}_R(M, R) \in \mathrm{thick}_{\mathrm{D}(R)}(M)$.

Proof idea.

Set $(-)^{\vee} = \mathrm{RHom}(-, R)$ and $d = \dim(R)$. Then

$$\mathrm{Ext}_R(M, k) \cong \mathrm{Ext}_R(k^{\vee}, M^{\vee}) \cong \mathrm{Ext}_R(k, M^{\vee})[d].$$

Hence $\mathrm{V}_R(M) = \mathrm{Supp}_S \mathrm{Ext}_R(k, M^{\vee}) = \mathrm{V}_R(M^{\vee})$. Similar results hold when we localize at each $\mathfrak{p} \in \mathrm{Spec}\, R$. $\square$

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- Above result implies a result of Avramov-Buchweitz,
  For $M, N \in \mathrm{D}^f(R)$, $\mathrm{Ext}_R^{\gg 0}(M, N) = 0 \iff \mathrm{Ext}_R^{\gg 0}(N, M) = 0$.

Proof idea of main result

K^R(*resp.* K^Q) $=$ the Koszul algebra on a minimal set of generators of the maximal ideal of R(*resp.* Q).

$M \in \mathrm{thick}_{\mathsf{D}(R)}(N) \Longleftarrow \mathrm{V}_{R_\mathfrak{p}}(M_\mathfrak{p}) \subseteq \mathrm{V}_{R_\mathfrak{p}}(N_\mathfrak{p}), \forall\, \mathfrak{p} \in \mathrm{Spec}\, R.$.

- $R \to K^R = K^Q \otimes_Q R \xleftarrow{\simeq} K^Q \otimes_R \mathrm{Kos}^Q(\underline{f}) \xrightarrow{\simeq} k \otimes_Q \mathrm{Kos}^Q(\underline{f}) = \bigwedge(k[1])^c := \bigwedge$. Hence
$$\mathsf{D}^{\mathsf{f}}(R) \underset{\mathrm{t}}{\rightarrow} \mathsf{D}^{\mathsf{f}}(K^R) \underset{\mathrm{j}}{\overset{\cong}{\rightarrow}} \mathsf{D}^{\mathsf{f}}(\bigwedge) \underset{\mathrm{h}}{\overset{\cong}{\rightarrow}} \mathsf{D}^{\mathsf{f}}(\mathcal{S}).$$
- For $X \in \mathsf{D}^f(R)$, $\mathrm{V}_R(X) = \mathrm{Supp}_{\mathcal{S}}(\mathrm{hjt}(X))$. Hence if $X, Y \in \mathsf{D}^f(R)$ and $\mathrm{V}_R(X) \subseteq \mathrm{V}_R(Y)$, then $\mathrm{hjt}(X) \in \mathrm{thick}_{\mathsf{D}(\mathcal{S})}(\mathrm{hjt}(Y))$. Thus $\mathrm{t}(X) \in \mathrm{thick}_{\mathsf{D}(K^R)}(\mathrm{t}(Y))$.
- $K^{R_\mathfrak{p}} \otimes_R M \in \mathrm{thick}_{\mathsf{D}(K^{R_\mathfrak{p}})}(K^{R_\mathfrak{p}} \otimes_R N)$, $\forall\, \mathfrak{p} \in \mathrm{Spec}\, R$. Combine with BIK's local-global, $M \in \mathrm{thick}_{\mathsf{D}(R)}(N)$.

$\square$

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Thank you!