

Duality of thick subcategories in the bounded derived category over complete intersections via cohomological supports

Jian Liu

University of Science and Technology of China

This is a joint work with
Josh Pollitz (University of Utah)

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- Derived category
- Cohomological support
- Main result

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Derived category I

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- The *derived category* $D(R)$ is localization of $K(R)$ by inverting quasi-isomorphisms.
- The *bounded derived category* $D^f(R)$ is full subcategory of $D(R)$ consisting of complexes with finitely generated bounded homology.

In general, $K(R)$ and $D(R)$ are **NOT** abelian categories. They are triangulated categories. They have

- shift $[1]$
- cone

For a complex X , $X[1]$: $(X[1])^i = X^{i+1}$ and $d_{X[1]} = -d_X$. For a chain map $f: X \rightarrow Y$,

$$\text{Cone}(f): \dots \rightarrow X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \rightarrow \dots$$

Thick subcategory I

Definition (Thick subcategory)

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- $\text{thick}_{D(R)}(X) :=$ smallest thick subcategory of $D(R)$ containing X .
- $\text{perf}(R) := \{X \in D(R) \mid X \text{ is quasi-isomorphic to } 0 \rightarrow P^n \rightarrow \dots \rightarrow P^m \rightarrow 0, P^i \in R\text{-proj}\}$

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Example

$\text{perf}(R) = \text{thick}_{D(R)}(R)$ is a thick subcategory of $D^f(R)$.

Thick subcategory II

For $X \in D(R)$, $\text{Supp}_R X := \{\mathfrak{p} \in \text{Spec } R \mid X_{\mathfrak{p}} \text{ is not acyclic}\}.$

Theorem (Hopkins,1987; Neeman,1992)

For perfect complexes M, N ,

$$M \in \text{thick}_{D(R)}(N) \iff \text{Supp}_R M \subseteq \text{Supp}_R N.$$

Complete intersection

Setting: (Q, \mathfrak{m}, k) is a regular local ring. That is, $\text{gldim}(Q) < \infty$
($\iff D^f(Q) = \text{thick}_{D(Q)}(Q)$). e.g., $k[[x_1, \dots, x_n]]$.

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- $R =$ the **complete intersection** $Q/(f_1, \dots, f_n)$, where $f_1, \dots, f_n \in \mathfrak{m}^2$ is Q -regular sequence
($f_{i+1}: Q/(f_1, \dots, f_i) \hookrightarrow Q/(f_1, \dots, f_i)$ for all $0 \leq i \leq n-1$).

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Example

$$k[[x_1, \dots, x_n]]/(x_1^{l_1}, \dots, x_n^{l_n}), \quad l_i \geq 2.$$

Cohomological operator I (Eisenbud)

Choose a free resolution of k over R :

$$F: \dots \rightarrow F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \rightarrow 0.$$

Lift F to a sequence over Q : $\tilde{F}: \dots \rightarrow \tilde{F}^{-2} \xrightarrow{\tilde{d}^{-2}} \tilde{F}^{-1} \xrightarrow{\tilde{d}^{-1}} \tilde{F}^0 \rightarrow 0.$

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- $\tilde{d}^2 \otimes_Q R = d^2 = 0 \implies \exists \tilde{t}_i: \tilde{F} \rightarrow \tilde{F}[2] \ (1 \leq i \leq n)$ such that
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$$\tilde{d}^2 = \sum_{i=1}^n \tilde{t}_i f_i.$$

- Each $t_i := \tilde{t}_i \otimes_Q R: F \rightarrow F[2]$ is a chain map!

$$(t_i)^* := H^{j+2}(\text{Hom}_R(t_i, k)): \text{Ext}_R^j(k, k) \rightarrow \text{Ext}_R^{j+2}(k, k).$$

$(t_i)^*$ is called the **cohomological operator**.

Cohomological operator II(Eisenbud)

Set $\mathcal{S} := k[\chi_1, \dots, \chi_n]$, where $\deg(\chi_i) = 2$.

- There is an algebra homomorphism

$$\zeta_k: \mathcal{S} \rightarrow \operatorname{Ext}_R(k, k) := \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}_R^j(k, k), \quad \chi_i \mapsto (t_i)^*(\operatorname{id}_k).$$

such that $(t_i)^*(\gamma) = \zeta_k(\chi_i) \circ \gamma = \gamma \circ \zeta_k(\chi_i)$ for all $\gamma \in \operatorname{Ext}_R^{|\gamma|}(k, k)$.

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- For $M \in D(R)$, $\text{Ext}_R(M, k) := \bigoplus_{j \in \mathbb{Z}} \text{Ext}_R^j(M, k)$ and $\text{Ext}_R(k, M) := \bigoplus_{j \in \mathbb{Z}} \text{Ext}_R^j(k, M)$ are graded \mathcal{S} -module.

Cohomological operator (Example)

Example

Let k be a field, $Q = k[[x]]$, $f = x^2$ and $R = Q/(f) = k[[x]]/(x^2)$.
The free resolution of k is $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \rightarrow 0$. Lift this over Q

$$\cdots \xrightarrow{x} Q \xrightarrow{x} Q \xrightarrow{x} Q \rightarrow 0.$$

Since $x^2 = 1 \cdot f$, the χ_f action on $\text{Ext}_R^j(k, k) \cong k$ ($j \geq 0$) is identity.
Precisely, $S = k[t^2] \hookrightarrow k[t] \cong \text{Ext}_R(k, k)$, where $\deg(t) = 1$.

$\text{Spec } \mathcal{S} =$ the homogeneous spectrum of $\mathcal{S} = k[\chi_1, \dots, \chi_n]$.

Definition (Cohomological support)

For $M \in D(R)$. The *cohomological support* of M over R is

$$V_R(M) := \text{Supp}_{\mathcal{S}} \text{Ext}_R(M, k) = \{\mathfrak{p} \in \text{Spec } \mathcal{S} \mid \text{Ext}_R(M, k)_{\mathfrak{p}} \neq 0\}.$$

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Example

- $V_R(R) = \{(\chi_1, \dots, \chi_n)\}$.
- $V_R(k) = \text{Spec } \mathcal{S}$.

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For $M \in D^f(R)$, $\text{Ext}_R(M, k)$ and $\text{Ext}_R(k, M)$ are finitely generated S -modules.

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- For $M \in D^f(R)$, $V_R(M) = \text{Supp}_{\mathcal{S}} \text{Ext}_R(k, M)$.
- For $M \in D^f(R)$,

$$\text{Ext}_R^{\gg 0}(M, k) = 0 \iff V_R(M) \subseteq \{(\chi_1, \dots, \chi_n)\}.$$

Theorem (L-Pollitz, 2021)

For $M, N \in D^f(R)$,

$$M \in \text{thick}_{D(R)}(N) \iff V_{R_p}(M_p) \subseteq V_{R_p}(N_p), \text{ for all } p \in \text{Spec } R.$$

Theorem (L-Pollitz, 2021)

For $M, N \in D^f(R)$,

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- The case of $N = R$ is due to Bass: For $M \in D^f(R)$,
 $\text{pd}_R(M) < \infty \iff \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty, \forall \mathfrak{p} \in \text{Spec } R.$

Corollary (Stevenson, 2014)

For $M \in D^f(R)$, $\mathrm{RHom}_R(M, R) \in \mathrm{thick}_{D(R)}(M)$.

Proof idea.

Set $(-)^{\vee} = \mathrm{RHom}(-, R)$ and $d = \dim(R)$. Then

$$\mathrm{Ext}_R(M, k) \cong \mathrm{Ext}_R(k^{\vee}, M^{\vee}) \cong \mathrm{Ext}_R(k, M^{\vee})[d].$$

Hence $V_R(M) = \mathrm{Supp}_S \mathrm{Ext}_R(k, M^{\vee}) = V_R(M^{\vee})$. Similar results hold when we localize at each $\mathfrak{p} \in \mathrm{Spec} R$. □

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- Above result implies a result of Avramov-Buchweitz,

$$\text{For } M, N \in D^f(R), \quad \mathrm{Ext}_R^{\geq 0}(M, N) = 0 \iff \mathrm{Ext}_R^{\geq 0}(N, M) = 0.$$

Proof idea of main result

K^R (resp. K^Q) = the Koszul algebra on a minimal set of generators of the maximal ideal of R (resp. Q).

$$M \in \text{thick}_{D(R)}(N) \iff V_{R_p}(M_p) \subseteq V_{R_p}(N_p), \forall p \in \text{Spec } R..$$








- $R \rightarrow K^R = K^Q \otimes_Q R \xleftarrow{\simeq} K^Q \otimes_R \text{Kos}^Q(\underline{f}) \xrightarrow{\simeq} k \otimes_Q \text{Kos}^Q(\underline{f}) = \bigwedge(k[1])^c := \bigwedge$. Hence

$$D^f_t(R) \rightarrow D^f_t(K^R) \xrightarrow{\simeq} D^f_t(\bigwedge) \xrightarrow{\simeq} D^f_h(S).$$

- For $X \in D^f(R)$, $V_R(X) = \text{Supp}_S(\text{hjt}(X))$. Hence if $X, Y \in D^f(R)$ and $V_R(X) \subseteq V_R(Y)$, then $\text{hjt}(X) \in \text{thick}_{D(S)}(\text{hjt}(Y))$. Thus $t(X) \in \text{thick}_{D(K^R)}(t(Y))$.
- $K^{R_p} \otimes_R M \in \text{thick}_{D(K^{R_p})}(K^{R_p} \otimes_R N)$, $\forall p \in \text{Spec } R$. Combine with BIK's local-global, $M \in \text{thick}_{D(R)}(N)$.



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Thank you!