Noether different and Tate cohomology.

Notation: $f: T \to S$ be a ring hom, where T is a commutative Noether ring.

Under some mild assumptions,

⇝ The Noether different of f annihilates all the Tate cohomology group $\widehat{\mathrm{Ext}}_S(-,-)$.

Recall Noether different.

$$0 \to I_{S/T} \to S^e_T \xrightarrow{\mu} S \to 0$$

$S^e_T = S \otimes_T S^{op}$

S: regard as left S^e_T-module. $(b' \otimes b'') \cdot b := b'bb''$.

<u>Noether different</u> $\quad N_{S/T} := \mu(\mathrm{ann}((I_{S/T})_{S^e_T}))$

right annihilator.

Note that: $\mathrm{ann}_{S^e_T}(I_{S/T}) = \mathrm{Hom}_{S^e_T}(S, S^e_T)$. [In this case, $N_{S/T} \subseteq Z(S)$.]

$$\text{Observation:}\quad \begin{array}{ccc} \mathrm{Hom}_{S^e_T}(S, S^e_T) & \xrightarrow{\mathrm{Hom}_{S^e_T}(S,\mu)} & \mathrm{Hom}_{S^e_T}(S,S) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{ann}_{S^e_T}(I_{S/T}) & \xrightarrow{\mu} & Z(S) \end{array}$$

[identifying as abelian group!!]

Note that For an ideal α of a ring A: $\mathrm{Hom}_A(A/\alpha, A) \cong \mathrm{ann}_r(\alpha_A)$!! $= \{v \in A \mid \alpha \cdot v = 0\}$. 右零化子.

$$\rightsquigarrow \mu(\mathrm{ann}_{S^e_T}(I_{S/T})) = \mathrm{Im}\,\mathrm{Hom}_{S^e_T}(S, \mu).$$

Prop: $N_{S/T} = Z(S) \Leftrightarrow S$ is projective as a left module over S^e_T.

Pf: S is projective

$\Leftrightarrow$

$$\underbrace{S \overset{\exists \alpha}{\dashrightarrow} S^e_T \xrightarrow{\mu} S}_{\mathrm{id}_S}$$

above diagram

$\Leftrightarrow \mathrm{id}_S \in N_{S/T}$. i.e. $N_{S/T} = Z(S)$ $\quad$ Q.E.D.

Theorem (Buchweitz)

Let $f: T \longrightarrow S$ be a ring hom, where

① T is a comm. noetherian ring with finite Krull dim and $\mathrm{gl.dim}\, T < \infty$

② S is strongly Gorenstein and f_*S is f.g. projective over T.

Then the Noether different $N_{S/T}$ annihilates $\mathrm{Ext}^i_S(M,N) := \mathrm{Hom}_{D_{sg}(S)}(M, N[i])$. $\forall M, N \in D_{sg}(S)$, $\forall i \in \mathbb{Z}$.

[In other words, $N_{S/T}$ annihilates $D_{sg}(S)$].

Proof: Let $a \in N_{S/T} = Z(S)$.

ie $\exists \alpha$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[rr, bend right, "a"'] & S^e_T \arrow[r, "\mu"] & S. \end{tikzcd}$$

so

$$\begin{tikzcd}[column sep=large] - \overset{\mathbb{L}}{\underset{S}{\otimes}} S \arrow[r, "- \overset{\mathbb{L}}{\underset{S}{\otimes}} \alpha"] \arrow[d, equal] \arrow[rr, bend right=40, "- \overset{\mathbb{L}}{\underset{S}{\otimes}} a"'] & - \overset{\mathbb{L}}{\underset{S}{\otimes}} (S \overset{\mathbb{L}}{\underset{T}{\otimes}} S) \arrow[r, "- \overset{\mathbb{L}}{\underset{S}{\otimes}} \mu"] \arrow[d, equal] & - \overset{\mathbb{L}}{\underset{S}{\otimes}} S \arrow[d, equal] \\ \mathrm{id}_S & - \overset{\mathbb{L}}{\underset{T}{\otimes}} S. & \mathrm{id}_S. \end{tikzcd}$$

$- \overset{\mathbb{L}}{\underset{S}{\otimes}} (S \underset{T}{\otimes} S)$ $\overset{\shortparallel}{}$ $f_* T$ is proj

$\rightsquigarrow$ $\forall M \in D_{sg}(S)$.

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "a"'] & M \overset{\mathbb{L}}{\underset{T}{\otimes}} S \arrow[r] & M \end{tikzcd}$$

$M \overset{\mathbb{L}}{\underset{T}{\otimes}} S \in \mathrm{Perf}(S)$ as $\mathrm{pd}_T M < \infty$.

so $a: M \to M$ is zero in $D_{sg}(S)$.

any morphism $a \cdot \alpha = 0$. for $\alpha: M \to \Sigma^i N$ in $D_{sg}(S)$ $\square$

Example: A comm. ring. G a finite group

$\iota: A \longrightarrow A[G] =: B$

$I_{B/A} = \langle h \otimes 1 - 1 \otimes h \mid h \in G \rangle$

It is ~~not~~ ~~difficult to~~ see that $\sum_{g \in G} g \otimes g^{-1} \in \mathrm{ann}_{B^e_A}(I_{B/A})$ $\quad B^e_A = B \otimes_A B^{op}$.

$$\left(\sum_g g \otimes g^{-1}\right)(h \otimes 1 - 1 \otimes h) = \sum_g \left(gh \otimes g^{-1} - g \otimes hg^{-1}\right)$$
$$= 0$$

$$\Rightarrow \quad \sum_g g \otimes g^{-1} \in N_{A[G]/A}.$$ $\square$

Example (Simple extension).

A comm. ring. $B = \dfrac{A[x]}{(f(x))}$ be the simple extension defined by $f(x)$.

$$B^e_A \cong \frac{A[x,y]}{(f(x), f(y))} \xrightarrow{\mu} B.$$

$$y \longmapsto x$$

Note that: $I_{B/A} = \langle x-y \rangle$

As $f(x) - f(y) \in I_{B/A}$, $f(x) - f(y) = (x-y)\,g(x,y)$ for some $g \in A[x,y]$.

$\rightsquigarrow$ $g(x,y) \in \operatorname{ann}_{B^e_A}(I_{B/A})$

$\Rightarrow$ $\mu(g(x,y)) \in \mathcal{N}_{B/A}$. Note that $\mu(g(x,y)) = g(x,x)$

As g is continuous $= \lim\limits_{y \to x} g(x,y)$

$= \lim\limits_{y \to x} \dfrac{f(x) - f(y)}{x-y} = f'(x)$

$\boxed{//}$

Theorem (Buchweitz)

Let $R = \dfrac{k[[x_1, \cdots, x_n]]}{(f_1, \cdots, f_c)}$ be a complete int. (i.e. $f_1, \cdots, f_c$ is a $k[[x_1, \cdots, x_n]]$-regular sequence)

Then the jacobian ideal of R, denoted $J(R)$, annihilates $D_{sg}(R)$.

Recall: $J(R)$ is the ideal of R generated by all $c \times c$ minors of the jacobian matrix $\dfrac{\partial(f_1, \cdots, f_c)}{\partial(x_1, \cdots, x_n)}$.

Proof: It is well-known that:

$$J(R) = \sum_{A \subseteq R} J_{R/A}$$

Kahler different i.e. 0-th fitting ideal

noether normalization

$A \hookrightarrow R$ regular local ring

$\hookrightarrow$ Noether normalization.

Recall: Assume $R = \dfrac{A[Y_1, \cdots, Y_s]}{(g_1, \cdots, g_l)}$

then $J_{R/A} := I_s\left(\dfrac{\partial(g_1, \cdots, g_l)}{\partial(Y_1, \cdots, Y_s)}\right)$. This is denoted by $K(R/A)$ in Iyengar-Takahashi's paper.

Wang: On the fitting ideal in the free res. lemma 4.3.

As $A \hookrightarrow R$ is ~~normalization~~ noether normalization, $R \simeq \dfrac{A[y_1, \cdots, y_\ell]}{(g_1, \cdots, g_m)}$

claim: $\ell = m = n - c$.

Note that edim $R = n$. $\dim R = n - c = \dim A$.

The claim follows.

so $A \longrightarrow \dfrac{A[y_1, \cdots, y_m]}{(g_1, \cdots, g_m)} \simeq R$.

As R is CM and f.g. over A, gl.dim $A < \infty$ $\Rightarrow$ $\mathrm{pd}_A R = 0$ by Auslander-Buchsbaum formula.

$\mathrm{pd}_A R = \mathrm{depth} A - \mathrm{depth} R$.

$\boxed{\text{Fact: } J_{R/A} \subseteq N_{R/A}}$. [in this case $J_{R/A} \simeq N_{R/A}$

see Kunz, Kahler differentials 10.17]

If this is true, then by last theorem, we conclude that

$J_{R/A} \cdot D_{sg}(R) = 0$.

$\rightsquigarrow J(R) \cdot D_{sg}(R) = 0$

The proof of fact: we write a lemma.

Lemma: , then $J_{R/A} \subseteq N_{R/A}$. [see Wang, on the fitting ideal in free res. Lemma 5.8]

pf: Assume $R = \dfrac{A[X_1, \cdots, X_n]}{(f_1, \cdots, f_t)}$

then $R^t \xrightarrow{\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_t}{\partial X_1} \\ \frac{\partial f_1}{\partial X_n} & & \frac{\partial f_t}{\partial X_n} \end{pmatrix} = \left(\frac{\partial f_i}{\partial X_j}\right)} R^n \longrightarrow \Omega_{R/A} \to 0 \quad (1)$

Assume $I = \mathrm{Ker}(\mu: R_A^e \to R)$. then $\Omega_{R/A} \simeq I/I^2$.

Consider $(R_A^e)^s \xrightarrow{(g_{ij})} (R_A^e)^n \to I \to 0 \quad (*)$

$-\otimes_{R_A^e} R$ with (*), we have

$$R^s \xrightarrow{u(g_{ij})} R^n \longrightarrow \Omega_{R/A} \to 0 \qquad (2).$$

By (1)+(2), we have

$$I_n(u(g_{ij})) = I_n\left(\left(\frac{\partial f_i}{\partial x_j}\right)\right)$$
$$:= J_{R/A}$$

Note that

$$J_{R/A} := I_n\left(\frac{\partial(f_1,\dots,f_t)}{\partial(x_1,\dots,x_n)}\right)$$
$$= \mathrm{Fitt}^0(\Omega_{R/A})$$

$\hookrightarrow$ 这个一般叫 Kähler different.

Note that $I_n((g_{ij}))\cdot I = 0 \Longrightarrow I_n(u(g_{ij})) = u(I_n(g_{ij})) \subseteq \mathcal{N}_{R/A}$.

$$\left[\ \boxed{\ g_{ij}\ }\ \right]$$

M $n\times n$ submatrix

$$(R_A^e)^s \xrightarrow{(g_{ij})} (R_A^e)^n \longrightarrow I_{R/A} \to 0 \qquad (3)$$
$$e_i \longmapsto x_i\otimes 1 - 1\otimes x_i := \delta^i$$
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto (\delta^1,\dots,\delta^n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum \delta^i a_i.$$

We know

$$MM^* = \det(M)\cdot I_{n\times n}$$
$$= \begin{pmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{pmatrix}$$

Note that $I_{R/A}\cdot \det M = 0 \Longleftrightarrow (\delta^1,\dots,\delta^n)\begin{pmatrix} \det M & & \\ & \ddots & \\ & & \det M \end{pmatrix} = 0$

$$\Longleftrightarrow (\delta^1,\dots,\delta^n)MM^* = 0$$
$$\Longleftarrow (\delta^1,\dots,\delta^n)M = 0$$

This is clear by the choice of the exact seq (3)

$\boxed{////}$

总结: let A be an f.g. R-alg, say $R = \dfrac{A[x_1, \dots, x_n]}{(f_1, \dots, f_t)}$

then $I_n\left(\dfrac{\partial(f_1, \dots, f_t)}{\partial(x_1, \dots, x_n)}\right)$ is the 0-th fitting ideal of $\Omega_{R/A}$

This is ~~denote~~ by $J_{R/A}$ in Wang's paper!!

Iyengar–Takahashi ~~denote~~ this by Kähler different $K_{R/A}$

$\rightsquigarrow$ $K_{R/A} \subseteq N_{R/A}$. [it is equal up to radical],