第2章 信息的表示与处理

100076202： 计算机系统导论

浮点数
Floating Point

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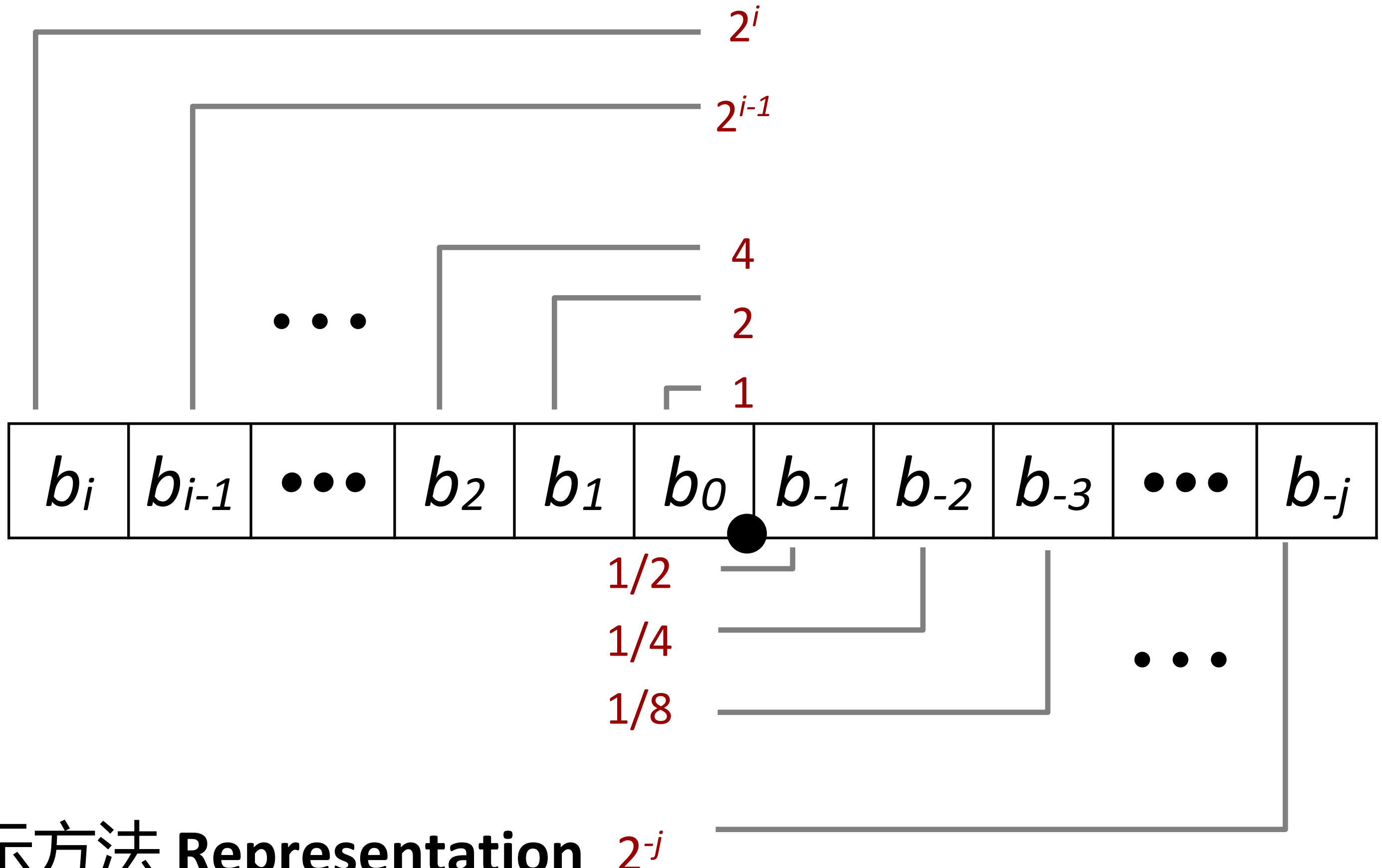
议题: 浮点数 Floating Point

- 背景：二进制小数 **Background: Fractional binary numbers**
- IEEE浮点标准：定义 **IEEE floating point standard: Definition**
- 示例和属性 **Example and properties**
- 舍入、加法和乘法 **Rounding, addition, multiplication**
- C语言中的浮点数 **Floating point in C**
- 小结 **Summary**

二进制小数
Fractional binary numbers

- What is 1011.101_2?

二进制小数 Fractional Binary Numbers



- 表示方法 **Representation**
  - "小数点"右边的位代表2的整数次幂分之一 Bits to right of "binary point" represent fractional powers of 2
  - 代表有理数 Represents rational number: $$\sum_{k=-j}^{i} b_k \times 2^k$$

二进制小数：示例
Fractional Binary Numbers: Examples

- 值 Value / 表示 Representation

值 Value	表示 Representation
5 3/4 =23/4	101.11_2
2 7/8=23/8	10.111_2
1 7/16=23/16	1.0111_2

- **观察 Observations**
  - 通过右移来除以2（无符号数）Divide by 2 by shifting right (unsigned)
  - 通过左移乘以2 Multiply by 2 by shifting left
  - 数字形式$0.111111\ldots_2$是刚好低于1.0的数 Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - 使用记法为1.0 - ε Use notation 1.0 – ε

可表示的数 Representable Numbers

- 限制#1 **Limitation #1**
  - 仅可以精确地表示x/2^k形式的数 Can only exactly represent numbers of the form $x/2^k$
    - 其它有理数有重复的比特位表示 Other rational numbers have repeating bit representations
  - 值 Value　　表示 Representation
    - 1/3　　`0.0101010101[01]`$\ldots_2$
    - 1/5　　`0.001100110011[0011]`$\ldots_2$
    - 1/10　　`0.0001100110011[0011]`$\ldots_2$

- 限制#2 **Limitation #2 ---定点数**
  - 在w比特位中仅有一个二进制小数点设置 Just one setting of binary point within the *w* bits
    - 有限的数值范围（非常小的值？非常大的值？） Limited range of numbers (very small values? very large?)

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IEEE浮点数 IEEE Floating Point

- **IEEE 754标准 IEEE Standard 754**
  - 1985年制定作为浮点运算的统一标准 Established in 1985 as uniform standard for floating point arithmetic
    - 在此之前，有很多异质的格式 Before that, many idiosyncratic formats
  - 得到所有主流CPU的支持 Supported by all major CPUs
  - 由Kahan为Intel处理器设计（获得1989年图灵奖） Designed by W. Kahan for Intel processors (Turing Award 1989)

- **由数值问题所驱动 Driven by numerical concerns**
  - 非常好的标准用于舍入、上溢和下溢 Nice standards for rounding, overflow, underflow
  - 在硬件上很难快速运算 Hard to make fast in hardware
    - 数值分析师在定义标准时比硬件设计师更占主导地位 Numerical analysts predominated over hardware designers in defining standard

浮点表示 Floating Point Representation

Example:
15213_{10} = $(-1)^0$ x 1.1101101101101_2 x 2^{13}

- 浮点数形式 **Numerical Form:**

  $$(-1)^s\ M\ 2^E$$

  - 符号位s确定数值是负还是正 **Sign bit *s*** determines whether number is negative or positive
  - 尾数M是范围在[1.0,2.0)之间的普通小数 **Significand *M*** normally a fractional value in range [1.0,2.0).
  - 阶码E是给浮点数指定2的E次幂权重 **Exponent *E*** weights value by power of two

- 编码 **Encoding**
  - 最高位是符号位s MSB s is sign bit ***s***
  - **exp**字段编码**E**（但不等于**E**） **exp** field encodes ***E*** (but is not equal to E)
  - **frac**字段编码**M**（但不等于**M**） **frac** field encodes ***M*** (but is not equal to M)

精度选项 Precision options

- 单精度浮点数 **Single precision: 32 bits**

s	exp	frac
1	8-bits	23-bits

- 双精度浮点数 **Double precision: 64 bits**

s	exp	frac
1	11-bits	52-bits

- 扩展精度（仅Intel） **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15-bits	63 or 64-bits

“规格化”值
“Normalized” Values

$v = (-1)^s M\, 2^E$

- 当阶码非全零和全一时 **When: exp ≠ 000…0 and exp ≠ 111…1**

- 阶码编码为一个有偏置值的有符号数： ***E* = *Exp* – *Bias***
  Exponent coded as a *biased* value: *E* = *Exp* – *Bias*
  - *Exp*： *exp*字段的无符号值 *Exp*: unsigned value of exp field
  - 偏置*Bias*= $2^{k-1} - 1$其中k是阶码位的位数 *Bias* = $2^{k-1} - 1$, where *k* is number of exponent bits
    - 单精度： 127 Single precision: 127 (Exp: 1…254, E: -126…127)
    - 双精度： 1023 Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- 尾数编码为带隐含的一个前导1 **Significand coded with implied leading 1: *M* =** $1.\text{xxx}\ldots\text{x}_2$
  - frac字段的比特位 xxx…x: bits of frac field
  - 当frac全零时值最小 Minimum when frac=000…0 (M = 1.0)
  - 当frac全一时值最大 Maximum when frac=111…1 (M = 2.0 – ε)

规格化编码示例
Normalized Encoding Example

$$v = (-1)^s \, M \, 2^E$$
$$E = Exp - Bias$$

- **值：单精度浮点数 Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **尾数 Significand**

  $M = 1.\underline{1101101101101}_2$

  $\texttt{frac} = \underline{1101101101101}0000000000_2$

- **阶码 Exponent**

  $E = 13$

  $Bias = 127$

  $Exp = 140 = 10001100_2$

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

非规格化值 Denormalized Values

$v = (-1)^s M\, 2^E$
$E = 1 - Bias$

- 条件：exp为全零 **Condition:** exp = 000…0
- 阶码值： **Exponent value: E = 1 – Bias (instead of E = 0 – *Bias*)**
- 尾数编码为隐含的一个前导零 **Significand coded with implied leading 0: M = 0.xxx…x$_2$**
  - `frac`字段比特位 **`xxx`…`x`**: bits of `frac`
- 情况 **Cases**
  - **`exp`** = 000…0, **`frac`** = 000…0
    - 代表0值 Represents zero value
    - 注意区别值：+0和-0（为何？） Note distinct values: +0 and –0 (why?)
  - **`exp`** = 000…0, **`frac`** ≠ 000…0
    - 最接近0.0的数值 Numbers closest to 0.0
    - 平均分布的 Equispaced

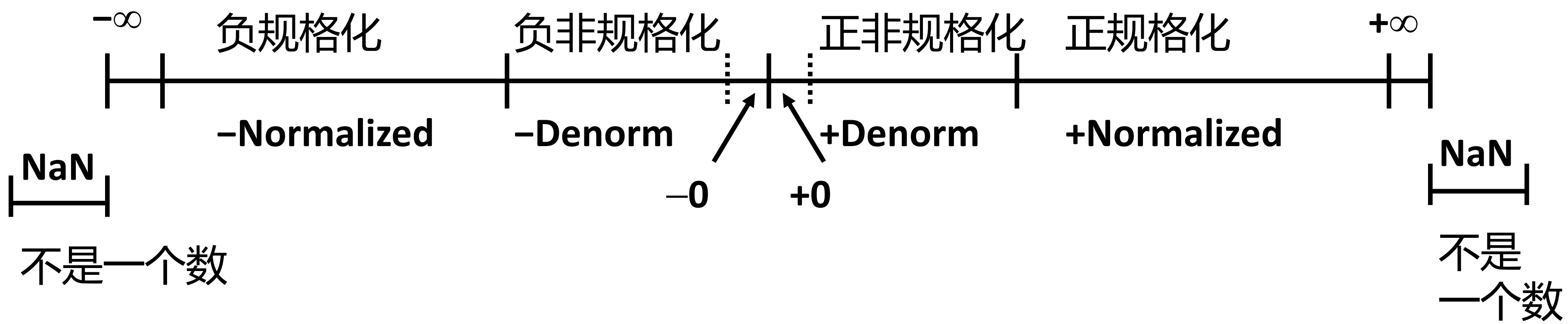
特殊值 Special Values

- 条件：exp为全一 **Condition: `exp` = `111`…`1`**

- 情况 **Case: `exp` = `111`…`1`, `frac` = `000`…`0`**
  - 代表值无穷大 Represents value ∞ (infinity)
  - 溢出运算 Operation that overflows
  - 正负均如此 Both positive and negative
  - 例如 E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- 情况 **Case: `exp` = `111`…`1`, `frac` ≠ `000`…`0`**
  - 不是一个数 Not-a-Number (NaN)
  - 代表无法确定数值的情况 Represents case when no numeric value can be determined
  - 例如 E.g., sqrt(–1), ∞ − ∞, ∞ × 0

可视化：浮点编码
Visualization: Floating Point Encodings



C语言中的浮点数示例

$v = (-1)^s M\, 2^E$

$E = \mathrm{exp} - \mathit{Bias}$

$\mathit{Bias} = 2^{k-1} - 1 = 127$

float: `0xC0A00000`

binary: ____ ____ ____ ____ ____ ____ ____ ____

1 | 8-bits | 23-bits

E =

S =

M =

$v = (-1)^s M\, 2^E =$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C语言中的浮点数示例#1

$v = (-1)^s M\, 2^E$

$E = \texttt{exp} - Bias$

float: `0xC0A00000`

$Bias = 2^{k-1} - 1 = 127$

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

E = `exp` – Bias = 129 – 127 = 2 (decimal)

S = 1 -> 负数

M = 1.010 0000 0000 0000 0000 0000

= 1 + 1/4 = 1.25

$v = (-1)^s M\, 2^E = (-1)^1 * 1.25 * 2^2 = -5$

C语言中的浮点数示例#2

$v = (-1)^s M\, 2^E$

$E = 1 - Bias$

float: 0x001C0000

$Bias = 2^{k-1} - 1 = 127$

binary: 0000 0000 0001 1100 0000 0000 0000 0000

0	0000 0000	001 1100 0000 0000 0000 0000
1	8-bits	23-bits

E = 1 – Bias = 1 – 127 = –126 (decimal)

S = 0 -> 正数

M = 0.001 1100 0000 0000 0000 0000

$= 1/8 + 1/16 + 1/32 = 7/32 = 7*2^{-5}$

$v = (-1)^s M\, 2^E = (-1)^0 * 7*2^{-5} * 2^{-126} = 7*2^{-131}$

$\approx 2.571393892 \times 10^{-39}$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

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微小的浮点数示例
Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- **8位浮点表示 8-bit Floating Point Representation**
  - 符号位是在最高有效位 the sign bit is in the most significant bit
  - 随后四位是阶码，偏置为7 the next four bits are the exponent, with a bias of 7
  - 最后的三位是尾数 the last three bits are the `frac`
- 与IEEE格式同样的通用形式 **Same general form as IEEE Format**
  - 规格化、非规格化 normalized, denormalized
  - 0、NaN和无穷大的表示 representation of 0, NaN, infinity

动态范围（仅正数）Dynamic Range (Positive Only)

$v = (-1)^s M\, 2^E$

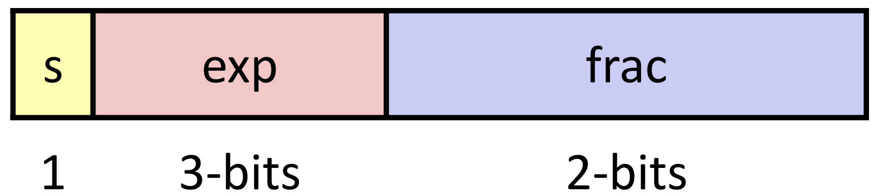
n: E = Exp – Bias

d: E = 1 – Bias

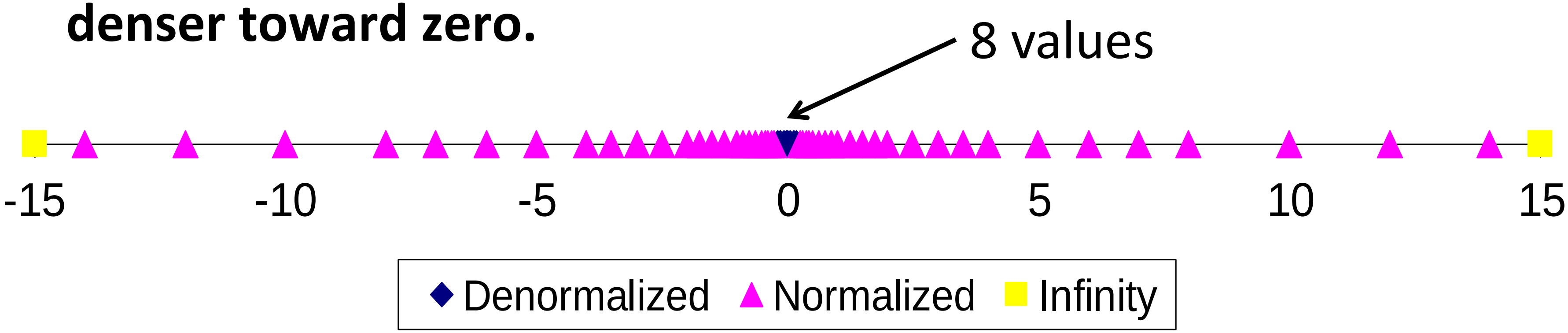
	s	exp	frac	E	Value	
非规格化数 Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	最接近0 closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	最大非规格化数 largest denorm
规格化数 Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	最小规格化数 smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	最接近1以下 closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	最接近1以上 closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	最大规格化数 largest norm
	0	1111	000	n/a	inf	无穷大 inf

值的分布 Distribution of Values

- 6位类IEEE格式 **6-bit IEEE-like format**
  - 3位阶码 e = 3 exponent bits
  - 2位尾数 f = 2 fraction bits
  - 偏置是3 Bias is $2^{3-1}-1 = 3$

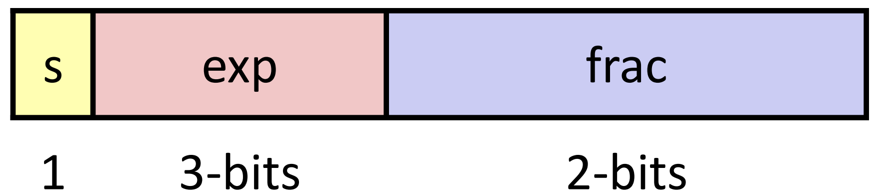


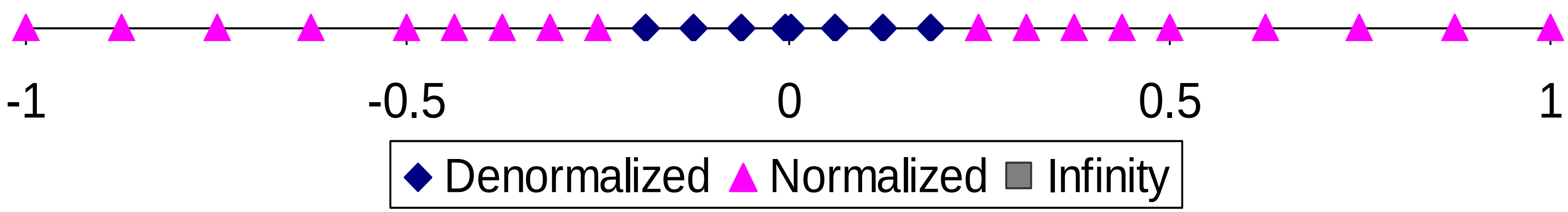
- 注意到越接近零分布越密集 **Notice how the distribution gets denser toward zero.**



值的分布（近处观察）Distribution of Values (close-up view)

- 6位类IEEE格式 **6-bit IEEE-like format**
  - 3位阶码 e = 3 exponent bits
  - 2位尾数 f = 2 fraction bits
  - 偏置量是3 Bias is 3





有趣的数值 Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	**00…00**	**00…00**	**0.0**
Smallest Pos. Denorm.	**00…00**	**00…01**	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.4×10^{-45}			
▪ Double ≈ 4.9×10^{-324}			
Largest Denormalized	**00…00**	**11…11**	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.18×10^{-38}			
▪ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	**00…01**	**00…00**	$1.0 \times 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
One	**01…11**	**00…00**	**1.0**
Largest Normalized	**11…10**	**11…11**	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
▪ Single ≈ 3.4×10^{38}			
▪ Double ≈ 1.8×10^{308}			

IEEE编码的特殊属性

Special Properties of the IEEE Encoding

- 浮点数的零和整数的零相同 **FP Zero Same as Integer Zero**
  - 所有比特位为0   All bits = 0
- 几乎可以使用无符号整数比较 **Can (Almost) Use Unsigned Integer Comparison**
  - 必须首先比较符号位 Must first compare sign bits
  - 必须考虑-0=0   Must consider −0 = 0
  - 不是一个数NaN的问题 NaNs problematic
    - 比任何其它值都大 Will be greater than any other values
    - 还应该比较什么？ What should comparison yield?
  - 否则都没有问题 Otherwise OK
    - 非规格化和规格化 Denorm vs. normalized
    - 规格化和无穷大 Normalized vs. infinity

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舍入 Rounding

- 舍入模式（用美元来说明） **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
向零 Towards zero	$1	$1	$1	$2	–$1
向下 Round down (–∞)	$1	$1	$1	$2	–$2
向上 Round up (+∞)	$2	$2	$2	$3	–$1
最近的偶数（默认） Nearest Even (default)	$1	$2	$2	$2	–$2

*将数字向上或者向下舍入，使得结果的最低有效位是偶数

近处观察向偶数舍入
Closer Look at Round-To-Even

- 默认的舍入模式 **Default Rounding Mode**
  - 没有深入汇编级很难理解任何其它类型的舍入 Hard to get any other kind without dropping into assembly
  - 所有其它的舍入都有统计偏差 All others are statistically biased
    - 一组正数的和将始终高估或低估 Sum of set of positive numbers will consistently be over- or under- estimated
- 适用于其它小数位/比特位位置 **Applying to Other Decimal Places / Bit Positions**
  - 当正好位于两个可能值中间时 When exactly halfway between two possible values
    - 舍入以便最低位是偶数 Round so that least significant digit is even
  - 例如舍入到最近的百分位 E.g., round to nearest hundredth

7.8949999	7.89	(低于中间值 Less than half way)
7.8950001	7.90	(高于中间值 Greater than half way)
7.8950000	7.90	(中间值-向上舍入 Half way—round up)
7.8850000	7.88	(中间值-向下舍入 Half way—round down)

舍入二进制数 Rounding Binary Numbers

- 二进制小数 **Binary Fractional Numbers**
  - 当最低位是0时为偶数 “Even” when least significant bit is 0
  - 当舍入位置右边=100…₂时为“中间值” “Half way” when bits to right of rounding position = $100\ldots_2$

- 举例 **Examples**
  - 舍入到最接近1/4（小数点右边2位）Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down向下)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up向上)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up向上)	3
2 5/8	10.10100_2	10.10_2	(1/2—down向下)	2 1/2

舍入 Rounding

1.BBGRXXX

监督位：结果的最低位

Guard bit: LSB of result

舍入位：删除的第一位

Round bit: 1st bit removed

固着位：剩余位的或

Sticky bit: OR of remaining bits

- 向上舍入条件 **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even向偶数舍入

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

浮点运算：基本思想
Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

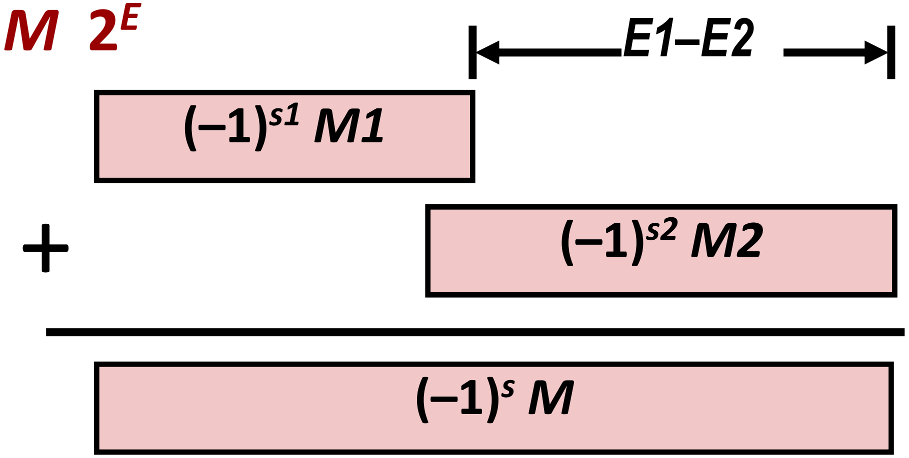
- $x \times_f y = \text{Round}(x \times y)$

- 基本思想 **Basic idea**
  - 首先计算精确的结果 First compute exact result
  - 使它适合需要的精度 Make it fit into desired precision
    - 如果阶码太大可能会溢出 Possibly overflow if exponent too large
    - 可能需要舍入才能适合尾数位数 Possibly round to fit into **`frac`**

浮点加法 Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - 假设 Assume $E1 > E2$ 小数点对齐 Get binary points lined up

- 精确的结果 **Exact Result:** $(-1)^s\ M\ \ 2^E$
  - 符号位s，尾数M：
  - Sign s, significand M:
    - 有符号数对齐并相加的结果
    - Result of signed align & add
  - 阶码E是E1 Exponent E: $E1$



- 修正 **Fixing**
  - 如果M大于等于2，M右移，E加一 If $M \geq 2$, shift M right, increment E
  - 如果M小于1，M左移k位，E减k if $M < 1$, shift M left k positions, decrement E by k
  - 如果E超过范围则溢出 Overflow if E out of range
  - 舍入M到适合frac的精度 Round M to fit `frac` precision

浮点加法 Floating Point Addition

$$1.010*2^2 + 1.110*2^3 = (0.1010 + 1.1100)*2^3$$
$$= 10.0110 * 2^3 = 1.00110 * 2^4 = 1.010 * 2^4$$

浮点加法的数学性质
Mathematical Properties of FP Add

- 相比于其它阿贝尔群 **Compare to those of Abelian Group**
  - 加法封闭吗？ Closed under addition? ***Yes***
    - 但可能产生无穷大或NaN But may generate infinity or NaN
  - 可交换吗？ Commutative? ***Yes***
  - 可结合吗？ Associative? ***No***
    - 溢出和舍入不精确 Overflow and inexactness of rounding
    - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
  - 0是加性恒等（单位元）的吗？ 0 is additive identity? ***Yes***
  - 每个元素都有加法逆元吗？ Every element has additive inverse?
    - 对，除了无穷大和NaNs Yes, except for infinities & NaNs ***Almost***
- 单调性 **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - 除了无穷大和NaNs Except for infinities & NaNs

浮点乘法 FP Multiplication

- $(-1)^{s1}\ M1\ \ 2^{E1}\quad \mathbf{x}\quad (-1)^{s2}\ M2\ \ 2^{E2}$
- 精确的结果 **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - 符号位s：异或 Sign *s*: $s1 \wedge s2$
  - 尾数M：相乘 Significand *M*: $M1 \times M2$
  - 阶码E：相加 Exponent *E*: $E1 + E2$
- 修正 **Fixing**
  - 如果M大于等于2，M右移，阶码E加一 If $M \geq 2$, shift *M* right, increment *E*
  - 如果E超过范围，溢出 If *E* out of range, overflow
  - 舍入M到适合frac的精度 Round *M* to fit `frac` precision
- 实现 **Implementation**
  - 最繁琐的工作是尾数相乘 Biggest chore is multiplying significands

4 位尾数: `1.010*2`^{2} `x 1.110*2`^{3} `= 10.0011*2`^{5}

`= 1.00011*2`^{6} `= 1.001*2`^{6}

浮点乘法的数学性质

Mathematical Properties of FP Mult

- 相比于交换环 **Compare to Commutative Ring**
  - 乘法封闭吗？ Closed under multiplication? ***Yes***
    - 但可能产生无穷大或NaN But may generate infinity or NaN
  - 乘法可交换吗？ Multiplication Commutative? ***Yes***
  - 乘法具有结合性吗？ Multiplication is Associative? ***No***
    - 可能溢出，舍入不精确 Possibility of overflow, inexactness of rounding
    - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
  - 1是乘法恒等（单位元）的吗？ 1 is multiplicative identity? ***Yes***
  - 乘法对加法是可分配的吗？ Multiplication distributes over addition? ***No***
    - 可能溢出，舍入不精确 Possibility of overflow, inexactness of rounding
    - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`
- 单调性 **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
    - 除了无穷大和NaNs Except for infinities & NaNs

议题: 浮点数 Floating Point

- 背景：二进制小数 Background: Fractional binary numbers
- IEEE浮点数标准：定义 IEEE floating point standard: Definition
- 示例和属性Example and properties
- 舍入、加法和乘法 Rounding, addition, multiplication
- C语言中的浮点数 Floating point in C
- 小结 Summary

C语言中的浮点数 Floating Point in C

- C语言确保两个级别的浮点数 **C Guarantees Two Levels**
  - **`float`** single precision 单精度
  - **`double`** double precision 双精度

- 转换/强制转换 **Conversions/Casting**
  - 在int，float和double之间强制转换改变比特位表示 Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float` → `int`**
    - 截断尾数部分 Truncates fractional part
    - 就像向零舍入 Like rounding toward zero
    - 当超过范围或NaN时没有定义：一般设置为TMin Not defined when out of range or NaN: Generally sets to TMin
  - **`int` → `double`**
    - 精确转换，只要int字长小于等于53位 Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int` → `float`**
    - 将按照舍入模式进行舍入 Will round according to rounding mode

创建浮点数 Creating Floating Point Number

- **步骤 Steps**
  - 用前导1规格化尾数 Normalize to have leading 1
  - 舍入以适合尾数位 Round to fit within fraction
  - 后规格化以处理舍入的影响 Postnormalize to deal with effects of rounding

s	exp	frac
1	4-bits	3-bits

- **案例研究 Case Study**
  - 转换8位无符号数成微小浮点数格式 Convert 8-bit unsigned numbers to tiny floating point format

示例数值 Example Numbers

128	10000000
13	00001101
17	00010001
19	00010011
138	10001010
63	00111111

规格化 Normalize

s	exp	frac
1	4-bits	3-bits

- 需求 **Requirement**
  - 设置小数点以便数值格式为1.xxxxx Set binary point so that numbers of form 1.xxxxx
  - 调整所有位得到前导1 Adjust all to have leading one
    - 随着尾数左移，阶码减一 Decrement exponent as shift left
    - **尾数右移，阶码+1**

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
13	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

后规格化 Postnormalize

- 问题 **Issue**
  - 舍入可能导致溢出 Rounding may have caused overflow
  - 通过右移一次进行处理同时阶码加一 Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
13	1.101	3		13
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

浮点数难题 Floating Point Puzzles

- 对于下面的每个C表达式 **For each of the following C expressions, either:** 完成其中一个工作
  - 对于所有的参数值解释其值为真 Argue that it is true for all argument values
  - 解释为何不为真 Explain why not true

```
int x = …;
float f = …;
double d = …;
```

假设d和f都不是NaN
Assume neither **d** nor **f** is NaN

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0` ⇒ `((d*2) < 0.0)` ✓
- `d > f` ⇒ `-f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

小结 Summary

- **IEEE浮点数有明确的数学性质 IEEE Floating Point has clear mathematical properties**
- 表示浮点数的形式为 **Represents numbers of form M x 2^E**
- 运算和实现是相互独立的 **One can reason about operations independent of implementation**
  - 好像采用完美的精度进行计算，然后进行舍入 As if computed with perfect precision and then rounded
- 与实数运算并不相同 **Not the same as real arithmetic**
  - 违背结合率和分配律 Violates associativity/distributivity
  - 对编译器和严谨数值应用程序员是一个挑战 Makes life difficult for compilers & serious numerical applications programmers

关于浮点数的灾难性影响（I）
Disastrous effects on floating Point (I)

- 浮点运算的不精确性 **The imprecision of floating-point arithmetic**
- 1991年2月25日，在第一次海湾战争期间，位于沙特阿拉伯Dharan 的美国爱国者导弹连未能拦截来袭的伊拉克飞毛腿导弹。飞毛腿袭击了美国陆军营房并杀死了28名士兵。**On February 25, 1991, during the first Gulf War, an American Patriot Missile battery in Dharan, Saudi Arabia, failed to intercept an incoming Iraqi Scud missile. The Scud struck an American Army barracks and killed 28 soldiers.**
- 爱国者系统包含一个内部时钟，以计数器的形式实现，每0.1 秒递增一次。为了确定以秒为单位的时间，程序会将此计数器的值乘以24比特位的小数，该小数是1/10的二进制近似值。**The Patriot system contains an internal clock, implemented as a counter that is incremented every 0.1 seconds. To determine the time in seconds, the program would multiply the value of this counter by a 24-bit quantity that was a fractional binary approximation to 1/10 .**

关于浮点数的灾难性影响（I）
Disastrous effects on floating Point (I)

- 程序采用0.1的近似值，其值为**x=0.00011001100110011001100**
  The program approximated 0.1, as a value x=0.00011001100110011001100
- **0.1-x = 0.00000000000000000000000[1100][1100]…2**
- **0.1-x=** $2^{-20} \times 0.1$ **=** 9.54×10^{-8}
- 开始100小时后，有0.343秒的差异 **After starting 100 hrs, there are 0.343s difference**
- 飞毛腿以2000米/秒左右的速度行驶，686米差距 **Scud travels at around 2000 m/s, 686 meters far off**
- 软件升级未完成，有时使用准确的定时，有时读取不准确的定时 **Software upgrade not completed, sometimes using accurate timing, reading inaccurate timing otherwise**

关于浮点数的灾难性影响（II）
Disastrous effects on floating Point (II)

- 1996年6月4日，阿丽亚娜Ariane 5号火箭的处女航行 **The maiden voyage of the Ariane 5 rocket, on June 4, 1996.**
- 发射后仅37秒，火箭就偏离了飞行路径，解体并爆炸 **Just 37 seconds after liftoff, the rocket veered off its flight path, broke up, and exploded.**
- 在将64位浮点数转换为16位有符号整数的过程中发生了溢出 **An overflow had occurred during the conversion of a 64-bit floating-point number to a 16-bit signed integer**
- 溢出的值测量了火箭的水平速度 **The value that overflowed measured the horizontal velocity of the rocket.**
- 在Ariane 4中，水平速度永远不会溢出16位数字。Ariane 5只是以5倍的速度重用相同的软件 **In the Ariane 4, the horizontal velocity would never overflow a 16-bit number. Ariane 5 just reuses the same software with 5 times higher velocity.**