Algorithmic Game Theory	**Fall 2019**

Lecture 1: An Introduction to (Algorithmic) Game Theory

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1.1 Introduction

Welcome to the reading group! The reading group's topics will tentatively cover three main topics:

1. Mechanism Design
2. Price of Anarchy
3. Equilibrium.

We'll be introducing each of these topics in a bit more detail today. To see a full topic list and sign up for presentations, you can visit `tinyurl.com/agttopics`. While the order presented here is the order that the topics will be presented in during the reading group, we'll be presenting them out of order today for clarity's sake.

1.2 Equilibrium

Before we can talk about what it means for a game to be in *equilibrium*, we need to talk about what games are and how we can even begin to analyze them.

1.2.1 The Prisoner's Dilemma

One of the most well-known games analyzed in Game Theory is the Prisoner's Dilemma. Partners-in-crime Jonathan and Debayan have been arrested for committing some crime, and are being held in separate rooms for interrogation. The detective offers each one of them a choice: testify that your partner masterminded the crime, and they'll get a maximum sentence for the crime while you'll be granted freedom. If neither prisoner testifies, both will be charged a light sentence based on limited evidence. On the other hand, if both prisoners testify, both will receive near-maximum sentences for the crime. We can represent this game as the following *payoff matrix*:

	Cooperate	Testify
Cooperate	$(3,3)$	$(0,5)$
Testify	$(5,0)$	$(1,1)$

The actions on the left represents Jonathan's *strategies*, while the actions on top represent Debayan's. Each ordered pair corresponds to a pair of strategies. The left value represents Jonathan's payoff, and the right value represents Debayan's.

1.2.2 Analysis of Prisoner's Dilemma

Now, what should each player do? In the real world, this question is very difficult; what's the value of partnership? What are the social and moral implications of betrayal?

However, in Game Theory, we instead turn to the mechanism I'm sure we're all familiar with: cold, rational logic. Before we do, let me introduce a few concepts to help us talk about games:

Definition 1.1 (Dominant Strategy)**.** *A player's strategy s is a **dominant strategy** if, for any other players' strategies, the player maximizes payoff by selecting strategy s.*

Looking at this situation, we can see with a little thought that there actually exists a dominant strategy for each player! If Jonathan cooperates, then Debayan has a better payoff for testifying and getting out with no punishment. On the other hand, if Jonathan tries to testify, then Debayan is better off testifying and bringing Jonathan down with him. In both cases, Debayan is better off testifying, so this is a dominant strategy. A symmetric argument can be made for Jonathan.

Definition 1.2 (Nash Equilibrium)**.** *A set of strategies $S = \{s_1 ... s_n\}$ is a **Nash Equilibrium** if there is no player i that can change their individual strategy and earn a better payoff.*

It's reasonable to think that a situation where no player can benefit from changing their move, as described by the Nash equilibrium definition above, is an equilibrium. In this game, we can see that if one player testifies, then the other player does better testifying as well, so this situation where both players testify is a *Nash equilibrium*. Morally, it seems an unfortunate consequence that the two never cooperate, but on the other hand perhaps it's for the betterment of society? Maybe?

Claim 1.3. *If each player in a game has a dominant strategy, then the game has exactly one Nash equilibrium, which occurs when every player selects their dominant strategy.*

Proof. Exercise. □

1.2.3 Prisoner's Dilemma, but forever this time

Here's a new question: what happens if Jonathan and Debayan are forced to repeatedly play the Prisoner's Dilemma game forever? This means that Jonathan and Debayan can change their strategy based on the outcomes of the previous rounds. Is it possible to find any kind of equilibrium if the two can't communicate?

Claim 1.4. *A **grim trigger** strategy is an equilibrium strategy in the Infinite Prisoner's Dilemma game.*

Informally, a *grim trigger* strategy is a strategy where a player begins by cooperating, and continues to do so until another player stops. In this game, Jonathan would begin the infinite prisoner's dilemma by cooperating, but if Debayan decides to testify then Jonathan will only testify from then on.

We can see that in this case, if both players adopt the grim trigger strategy, then either player will lose an infinite amount of value in total by choosing to testify. This is thus a Nash equilibrium in the repeated game.

There are many variants of this problem, all with interesting and sometimes nontrivial results! We can talk about what happens if a player doesn't believe in delayed gratification, or if they decide to only punish for a finite number of rounds, or if future utility gives diminishing returns. Hooray for cooperation and stuff!

1.2.4 Mixed Nash Equilibrium

With this, we cover one of the most revolutionary results in Game Theory:

Theorem 1.5 (Nash's Theorem (51))**.** *Every two-player game presentable in a matrix like this has a Nash equilibrium.*

Hold on, you might be thinking. What about games like Rock Paper Scissors? If you were thinking this, you'd be absolutely right; unless for some reason both players don't want the game to end, there's no pair of choices where both players would not benefit from diverging. For this reason, we introduce the following concept:

Definition 1.6 (Mixed strategy)**.** *For a player with strategy set $S = (s_1 \cdots s_n)$, a* ***Mixed strategy*** *is a strategy profile defined by a set $P = (p_1 \cdots p_n)$ where $\sum_{i=1}^{n} p_i = 1$ and where the player's choice of strategy in the game is selected at random from S with $P(s_i) = p_i$.*

Perhaps intuitively, we can see that the mixed strategy where a player selects each choice with equal probability is probably a good one. However, in introducing this concept, we see the concept of Nash equilibrium twisted in ways that we perhaps were not expecting. Let's say that player one decides to use the strategy profile $(R, P, S) = (1/3, 1/3, 1/3)$, and assume that this is part of a Nash equilibrium. In the Nash equilibria we've seen previously, we'd just take whatever strategy was best for Player 2, and these two strategies together would constitute a Nash equilibrium. But this is where the issue arises: what is Player 2's best strategy in this case?

Upon examination, we can see that in fact *any* strategy utilized by Player 2 is the "best" strategy, in the sense that they all give the exact same payoff. Player 1 has created an equilibrium state for Player 2 not by forcing them to take a bad compromise, as with the Prisoner's Dilemma, nor by giving them an agreeable outcome, but instead by simply rendering Player 2's decisions meaningless.

In this state, if both players select to choose the same strategy profile of selecting each choice uniformly at random, it's a Nash equilibrium because neither player derives any benefit from changing, but in reality neither player gets any punishment from changing either (for one game, at least).

Of course, Nash equilibrium is not the only form of equilibrium. In part 3 of this reading group, we will be looking at different types of equilibrium, how they arise naturally in games, and how(if) we can use algorithms to find them.

1.3 Mechanism Design

Mechanism design, in essence, is the study of how we can design games to make players do what we want them to, even if they are acting out of their own self-interest with no regard for our goals. Perhaps instead of talking about what this entails, we'll introduce it with a story about where poor mechanism design can cause serious issues with gameplay.

1.3.1 The 2012 Olympics Badminton Scandal

Watch this video for Professor Roughgarden's explanation: he does it more justice than I'd be able to.

1.3.2 Auctions

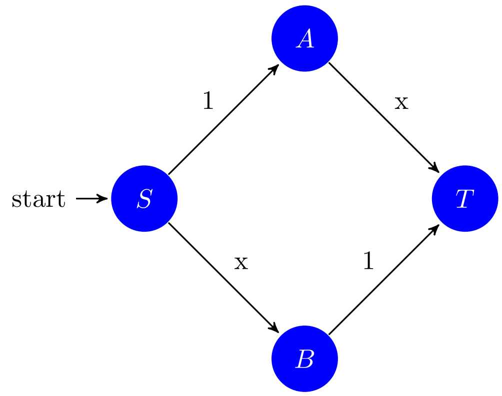
We covered auctions a bit in this lecture, but the next lecture will be all about auctions so you can just read about them next week!

1.4 Price of Anarchy

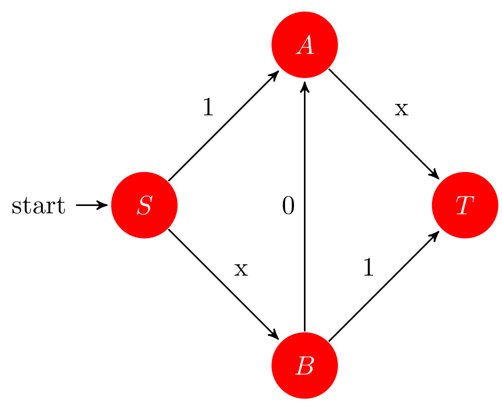
One question we ask in Algorithmic Game Theory is when selfish behavior is acceptable. We cannot always design systems to work well, so the best we can do is to study them as they are. What does this mean and why do we care? Braess' Paradox is a motivating example.

1.4.1 Braess' Paradox

In 1968, Dietrich Braess, a mathematician working on traffic modelling, thought about the fact that if every driver chose a route best for them via some shortcut, then this could slow down the overall traffic for all drivers on the road. He detailed this behavior by finding an example where this holds. Suppose we have a suburb S from which drivers are leaving, a train station T towards which drivers are driving, and two noninterfering routes through points A and B. The graph is drawn below. In general, the time that it takes to drive on a given road is a function of the number of cars on that road. Let's call x the fraction of all cars driving on any given road. Then, in general, the time that it takes to cross a road, t, will be some function of x, i.e. each road takes some time $t(x)$ to cross. In the network below, the road from S to A and from B to T are both slow, but independent of the fraction of cars traveling on them, $t(x) = 1$ (hour). On the other hand, the road from S to B and from A to T are both faster but dependent on the fraction of total drivers on the road, $t(x) = x$ (hours).



What will be the behavior of drivers on this road? Well, there are only two routes that drivers can take. In either route, the total time traveled will be $1 + x$ (hours). Thus, drivers should prefer either route equally, and the fraction going through each route will be half. Alternatively, we could think about what would happen if every driver was going down one route. If that were the case, then the other route would be much faster, so drivers would selfishly choose the other route until there is an even split of drivers on each route and there is no longer an advantage to choosing one route over the other. This means that the travel time through this network is 90 minutes, or $\frac{3}{2}$ hours. What if we added a shortcut from B to A which took virtually no time at all, i.e. $t(x) = 0$, as shown below? Intuitively, we might expect that the traffic flow cannot get worse, and that it might even get better!



Surprisingly, this is not what happens! If we follow the route from S to B to A to T, we see that the total travel time for that route is $x + x = 2x$, which is better than the $1 + x$ that we had for either of the other two routes (remember that x is a fraction less than 1). Thus, every driver would selfishly choose to take this route because it can't be any worse than traveling the other routes. But this means that the total fraction of drivers along this route will be 1, and the total travel time will be 2 hours. This is a greater overall travel time than when we didn't have this additional road/route! This is the essence of the paradox; that intuitively we add a noncongested route, but now we see that selfish behavior leads to worse behavior than if that route didn't exist.

Algorithmic Game Theory tries to measure how badly this system performs. It compares two different scenarios: if we were to allow people to act selfishly, how would the overall behavior of the system compare to if we played dictator? This comparison is called the Price of Anarchy, or POA, and it is the ratio between performance assuming strategic players, and best system performance. For example, in our roads example that we developed, selfish behavior leads to a travel time of 2, versus we can reach up to an efficient travel time of 3/2. This leads to a POA of $\frac{4}{3}$. This quantity was first defined in CS, even though economists and game theorists were aware of the inefficiencies that exist in equilibria. As computer scientists, we want to find situations in which the price of anarchy is close to unity; that is, selfish behavior is near-optimal. The applications in CS include network routing, resource allocation, and simple auction designs.

One additional note is that it may appear as if the system we defined is highly contrived. Surely, Braess' Paradox rarely occurs, right? In fact, in 1983, Steinberg and Zangwill found general conditions for the occurrence of Braess' paradox when adding a new route, and found that for random networks and routes, Braess' paradox occurs about as often as not. It even happens in real life. An example of this occurred in New York, 2009, when the city closed down Broadway near Times Square, and it led to better traffic flow overall. There are also several examples where states decide to add more roads to highways, only to have the situation result in greater congestion and longer travel times.

By the way, the idea that adding greater resources can lead to inefficiencies has many examples outside of traffic networks. For example, there exist analogies to Braess' Paradox in electrical power grids and in biology. An example of the latter is the Paradox of Enrichment, named by Michael Rosenzweig in 1971, where adding food and resources for a population can actually create great instability in the population. Another example in computer science is Belady's Anomaly, where increasing the number of page frames (chunks of data that we have loaded at any given time) leads to a greater number of page faults (having to load a new chunk of data from memory) because of a first-in, first-out implementation (wherein the chunk of data which has been there the longest is chosen to be cleared).