Algorithmic Game Theory **Fall 2019**

Lecture 4: Prophet Inequality

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4.1 Relevant Ideas from Previous Weeks

4.1.1 DSIC Auction

A **Dominant Strategy Incentive Compatible** auction is one in which every player has a dominant strategy which is to act according to solely their own preferences, and where this dominant strategy always results in nonnegative utility. It's a generally preferable auction to strive for because it means that strategies are really easy to players to determine and nobody leaves the auction unhappy.

4.1.2 Vickrey Auction

The **Vickrey Auction**, also knows as the **second-price auction**, is a single-item auction in which the top bidder receives the item and pays the second highest bid. The dominant strategy for each player is to simply bid their valuation of the item. Its claim to fame comes in being easy to implement, DSIC, and welfare-maximizing.

4.1.3 Revenue and Virtual Welfare

Last week, we learned about the difficulties in maximizing revenue, despite its obvious real-world applicability. In the setting where bidder i's valuation of an item is private but drawn from a distribution with CDF F_i and PDF f_i, we described the bidder's **virtual welfare** as a function of their private valuation:

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

While of course we can't know any bidder's virtual welfare because we don't know their valuation, we did find the surprising fact that for any auction, **the payment rule that maximizes expected revenue is exactly the payment rule that maximizes virtual welfare**. Of course, we'd like virtual welfare, whatever it is, to be monotone increasing as v_i increases in order to match our standard definitions of welfare and value. If F_i is a function that allows for this monotonicity, it is called **regular**.

4.2 What more is there?

At first glance, this should be a pretty satisfying answer to all of our problems. For any single-parameter auction with reasonably predictable valuation distributions F_i, we can just find the payment rule that best maximizes expected virtual welfare, and then we're done! If each distribution is regular (read: fairly well-behaved), this optimal auction can be made to be DSIC, which is great as well.

Unfortunately, life is never that easy. To see why, let's work on an exercise!

4.2.1 A Quick Exercise [Exercise Set 3, Exercise 26]

Consider a single-item auction where bidder i's valuation is drawn from its own regular distribution F_i (i.e., the F_i's can be different).

(a) Give a formula for the winner's payment in an optimal auction, in terms of the bidders' virtual valuation functions.

(b) Show by example that, in an optimal auction, the highest bidder need not win, even if it has a positive virtual valuation. [Hint: two bidders with valuations from different uniform distributions suffices.]

(c) Give an intuitive explanation of why the property in (b) might be beneficial to the revenue of an auction.

Unfortunately, as seen, optimal auctions can result in some pretty counterintuitive design, which may be theoretically revenue-maximizing but still awkward to implement. For this reason, we'll spend the rest of the lecture thinking about more practical alternatives that are also more robust to varied valuation distributions.

4.3 Prophet Inequality

Before we can introduce our next revenue-almost-maximizing auction, we'll take a detour into the intersection between Stopping Theory and auctions. Let's say we're trying to auction off our single item, and we have a list of n bidders. Instead of hearing all of the bids as in a classical auction, we hear each bid one at a time, and after hearing a bid we can choose to either accept that bid as payment or reject the offer and move on to the next bidder. We've done our research; while we don't know each bidder's bid π_i a priori, we do have a distribution for each bidder's potential bid, denotes as F_i for bidder i. Obviously, (unless of course we're prophets) no strategy can guarantee that we'll always accept the best bidder regardless of distributions, but there actually does exist a strategy that performs better than perhaps expected. This is outlined by the Prophet Inequality, introduced by Samuel-Cahn:

Theorem 4.1 (Prophet Inequality). *For the auction described earlier on independent distributions $F_1, \dots, F_n$, there is a strategy that guarantees*

$$\mathbb{E}[reward] \geq \frac{1}{2}\mathbb{E}[\max_i \pi_i].$$

In fact, there is a "threshold" strategy that guarantees this.

As a sanity check, it may be useful to think about this in comparison to the famous Secretary problem, where the setup is the same but we have no a priori distributions on any applicants. Even so, in that case there exists a strategy that accepts the best candidate with probability $1/e$. Given the amount of information we have in our setting, we'd hope to at least do comparably well, and the Prophet Inequality in fact does somewhat better than the $1/e$ guarantee.

Proof of Theorem 4.1. Let's say that we're using a strategy with threshold t, which means that we simply take any offer that is greater than or equal to t. To analyze the expected payoff from the auction, we'll split the result of the auction using our strategy into three cases: either no bid meets the threshold, more than one meets the threshold, or exactly one meets the threshold. While the first two cases are certainly reasonable cases to consider, the intuitive reason behind considering the middle case is that in this case, our strategy always yields the optimal payout.

- Case 1: No bid meets the threshold. Denote by $q(t)$ the probability that this occurs. In the real world, we'd just take the last bid regardless of what it is (why?), but in this case we'll assume we take nothing. Thus our expected value in this case is 0.
- Case 2: More than one bid is above the threshold. Frankly the analysis of this will take a lot of work and our bound is relatively relaxed, so we'll just assume in this case that we receive exactly t revenue from the auction to save us the trouble of doing other calculation.
- Case 3: Exactly one bid is above the threshold. If we were to be precise, the expectation from this is

$$\sum_{i=1}^{n} \mathbb{E}[\pi_i | \pi_i \geq t] \cdot \Pr[\pi_i \geq t] \cdot \Pr[(\forall j \neq i)\ \pi_j < t],$$

  but we'd much prefer simplicity, especially in coordination with the previous case. Note that in either Case 2 or Case 3, the minimum revenue is t, and these cases occur with probability $1 - q(t)$, so we'll instead look at the "excess" revenue generated in the case where exactly one bidder is above the threshold. We note that the expected excess revenue in this case is

$$\sum_{i=1}^{n} \mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t] \cdot \Pr[(\forall j \neq i)\ \pi_j < t].$$

  Now, we know that $\Pr[(\forall j \neq i)\ \pi_j < t] < q(t)$ (why?), so

$$\sum_{i=1}^{n} \mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t] \cdot \Pr[(\forall j \neq i)\ \pi_j < t] \geq q(t) \sum_{i=1}^{n} \mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t].$$

  Now, note that $\mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t]$ can be thought of as

$$\begin{aligned} \mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t] &= \mathbb{E}[\pi_i - t | \pi_i \geq t] \cdot \Pr[\pi_i \geq t] + 0 \cdot \Pr[\pi_i < t] \\ &= \mathbb{E}[\max(\pi_i - t, 0)], \end{aligned}$$

  so our expected bonus revenue can be expressed as

$$q(t) \sum_{i=1}^{n} \mathbb{E}[\max(\pi_i - t, 0)].$$

Combining the analysis from these cases gives us

$$\mathbb{E}[\text{Revenue}] \geq (1 - q(t)) \cdot t + q(t) \sum_{i=1}^{n} \mathbb{E}[\max(\pi_i - t, 0)].$$

With this in mind, our challenge is to somehow rewrite the prophet's outcome in terms of comparable items. Note that the prophet's expected outcome is exactly $\mathbb{E}[\max_i \pi_i]$, which doesn't include anything about t at all. Forcing it into the expression and using some manipulation towards our lower bound's expression gives us

$$\begin{aligned}
\mathbb{E}[\max_i \pi_i] &= \mathbb{E}[t + \max_i(\pi_i - t)] \\
&\leq \mathbb{E}[t + \max_i(\pi_i - t, 0)] \\
&= t + \mathbb{E}[\max_i(\pi_i - t, 0)] \\
&\leq t + \sum_{i=1}^{n} \mathbb{E}[\max(\pi_i - t, 0)].
\end{aligned}$$

Thus, selecting t such that $q(t) = 0.5$ and plugging this $q(t)$ into the lower bound immediately gives us the result we need. $\square$

One important note is that we actually deducted all of the benefit if multiple bidders are above the threshold. This means that *any* choice of bidder above the threshold, even always selecting the worst one for whatever reason, would still give us this guarantee!

4.4 Generalizing to a Single-Item Auction

In the context of a normal auction, we clearly aren't constrained to accept bids one at a time; we can receive them all at once! Even so, we do need to think about how we can maximize expected revenue. Remember that maximizing expected revenue is equivalent to maximizing virtual welfare, so we should look for a way to approximately find the maximal virtual welfare. For a single item, this tells us simply that in the optimal auction, the expected revenue is exactly $\mathbb{E}[\max_i \phi_i(v_i)]$, or the maximal virtual welfare (of course, on the condition that some bidder's virtual welfare is greater than zero). Notice that choosing the maximal value of something on expectation is something approximable using the Prophet Inequality! In particular, the Prophet Inequality tells us that, with the correct threshold strategy for selecting a bidder, we can guarantee always doing at least half as well as if we had been prophets all along.

In particular, in the proof of Prophet Inequality, we selected a threshold such that the probability that someone met the threshold was exactly $1/2$, which in this case means we must select some t such that

$$\Pr[\max_i \phi_i(v_i)] = \frac{1}{2}.$$

With this t, we concluded previously that selecting any bidder with virtual welfare greater than this value is automatically a good enough allocation mechanism, in particular that this allows us to do at least half as well as the best possible auction had we known each player's valuations.

One special case, in particular, is if we simply allocate the item to the highest bidder with virtual welfare at least t. Note that in this case we simply revert to the Vickrey Auction for payment, in the sense that the person who receives the item pays the least amount needed to beat every other bidder (as well as reach the t threshold). This auction is much simpler in terms of allocation, calculating payments, and general bidding, but it does have a lingering issue: it requires us to be fairly confident in our assessment of the bidder's valuation function. After all, a skewed valuation function changes the virtual welfare of any bidder.

4.5 Prior-Independent Auctions

This, of course, is a whole new can of worms; what if we don't know the bidder's valuation functions all too well? What if we don't know them at all? We'd like to be able to design an auction close to revenue-maximizing that doesn't rely on us having a great idea of each person's valuation. Along the same lines, we should be wondering how much benefit we actually derive from knowing the bidders' valuation distributions. Auctions like these that don't rely on prior information are called (for pretty clear reasons, I think) **prior-independent auctions**, and are an extremely valuable topic of research.

Of course, we've spent a decent portion of this group describing how the Vickrey Auction is an "awesome" auction in terms of welfare-maximization, efficiency, and DSIC, despite not requiring any knowledge of the bidders' valuations a priori, so the Vickrey auction should be considered a standard benchmark for prior-independent auctions. What's obvious, by definition, is that the optimal revenue-maximizing auction is at least as good as the Vickrey Auction. What should be less obvious, and probably quite surprising, is the following result:

Theorem 4.2 (Bulow-Klemperer Theorem)**.** *Let F be a regular distribution, and take an auction with every bidder following this distribution F. Then*

$$\mathbb{E}[\textit{Revenue(Vickrey)}\ (n+1\ \textit{bidders})] \geq \mathbb{E}[\textit{Revenue(Optimal Auction)}\ (n\ \textit{bidders})].$$

Huh. Let's just dive into the proof, I guess. We begin with a lemma about the strength of Vickrey Auctions.

Lemma 4.3. *In the space of single-item auctions where every bidder follows the same regular valuation distribution and the item is guaranteed to be sold to someone, the Vickrey Auction is the member that maximizes revenue.*

Proof of Lemma 4.3. Because each bidder follows the same valuation distribution, every bidder has the exact same formula for virtual welfare as well. As long as virtual welfare is monotone, which we know because F is given to be regular, the highest bidder will always have the highest virtual welfare, so it is always revenue-maximizing to allocate the good to the highest bidder. The only price structure that maintains DSIC is exactly the second-price auction. We note that the Vickrey Auction always sells the good even if the virtual valuation is negative. If the space did not require the item to be sold, then the Vickrey Auction can be suboptimal if every bidder's virtual welfare is negative. □

Proof of Theorem 4.2. To prove this Theorem, we construct an "intermediary" auction comparable to both sides of the inequality. Our auction is on $n+1$ bidders, and works as follows:

1. Run the optimal (not necessarily Vickrey) auction on the first n bidders, ignoring the $n+1$th bidder.
2. If the item was not sold, give it to bidder $n+1$ for free.

First, we note that this auction is on $n+1$ bidders, every bidder follows the same regular valuation, and the item is guaranteed to be sold. Thus, by Lemma 4.3, the Vickrey Auction guarantees at least as much expected revenue as this one. Next, note that the expected revenue of this auction is exactly equivalent to the expected revenue of the optimal auction on n bidders, because the last bidder can never contribute positively or negatively to the revenue. Thus, this bidder is never better than Vickrey on $n+1$ but always equal to the optimal auction on n bidders, so the Theorem follows. $\square$

To some extent, the fact that this Theorem holds is bittersweet. On one hand, it tells us that auction designers stand to gain more by simply increasing the number of participants than by trying to find the optimal auction on the bidders. This is great news for auctioneers who do not want to spend the time learning the preferences of their bidders. on the other hand, it to some extent discourages mechanism design; why bother perfecting auctions and learning preferences if the benefit is comparatively negligible? It's also clear through this and some simple analysis of the Vickrey Auction on different numbers of bidders that the expected revenue of the Vickrey Auction is at least $\frac{n}{n+1}$ times the expected revenue of the best possible auction on the bidders. That's a pretty sharp ratio, all things considered.

4.6 Virtual Valuations Practice

If there's time at the end, it might be good for us to work on a problem interpreting virtual valuations. Take a look at Problem 7 in Problem Set 2.