Notes on Functional Local Projections

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1 Motivation

In heterogeneous agent macro, we typically assume that law of motion for aggregate output Y is

$$Y' = h(Y, \lambda, Z)$$

where λ is the wealth distribution, and Z is some aggregate shock. Then h refers to the law of motion for output. In cases where we have near-aggregation like in Krussel-Smith, then we may use parametric methods to solve for h. Our approach is to estimate h using nonparametric methods in causal inference.

Question: Is FLP robust to the Lucas Critique? No. Because for causal inference we will rely on exogeneous shocks, that is, shocks that policymakers are unable to forecast for. Still, FLP will be able to estimate $\frac{\partial Y'}{\partial Z}$ and $\frac{\partial Y'}{\partial \lambda}\frac{\partial \lambda}{\partial Z}$. So to summarize, we cannot estimate h, but we can estimate some partial derivatives that are informative of h.

What is the issue of using summary statistics in our regressions? Again, failure of Krussel-Smith near-aggregation is important. When our policy functions cannot be described by finite moments of the wealth distribution, we are introducing a lot of inaccuracy in our regression estimates.

2 Functional local projections, a primer

Consider the VAR process indexed by coefficient A and kernel $B(x)$:

$$Y_t = AY_{t-1} + \int B(x) l_{t-1}(x)\, dx + \epsilon_t$$

$Y_t \in \mathbb{R}$ macro variable, $l_{t-1}(x) \in \mathcal{C}(\mathbb{R})$ cross-sectional density. Iteration on the VAR gives the functional local projection

$$Y_{t+h} = A_h Y_{t-1} + \int B_h(x) l_{t-1}(x)\, dx + u_{t+h}.$$

We will separately consider each of the following assumptions: $E[u_{t+h}|l_{t-1}(x), Y_{t-1}] = 0$, $E[u_{t+h}|Y_{t-1}, Z] = 0,\ Z \in \mathbb{R}^{dz}$

Approximate cross-sectional density $l_t(x)$ and kernel $B_h(x)$ with K basis functions ζ_k and ξ_k:

$$l_t^K(x) = \sum_{k=1}^{K} \alpha_{k,t} \zeta_k(x) = \zeta'(x)\alpha_t$$

$$B_h^K(x) = B_h \xi(x)$$

$\alpha_t, B_h \in \mathbb{R}^K$ sufficient statistics for $l_t(x)$ and $B_h(x)$ respectively. Then the functional local projection can be written as

$$Y_{t+h} = A_h Y_{t-1} + \Phi_h \alpha_{t-1} + u_{t+h}$$

where $\Phi_h = B_h \int \xi(x)\zeta'(x)\, dx$

Under Assumption 1, we have the OLS model

$$Y_{t+h} = A_h Y_{t-1} + \Phi_h \alpha_{t-1} + u_{t+h}, \quad E[u_{t+h}|Y_{t-1}, \alpha_t] = 0$$

We can rewrite the equation as

$$Y_{t+h} = \gamma_h X_{t-1} + u_{t+h}, \quad E[u_{t+h}|X_{t-1}] = 0$$

where $X_{t-1} = (Y_{t-1}, \alpha'_{t-1})'$ Then we have the closed form OLS solution

$$\gamma_h = E[Y_{t+h}X'_{t-1}]E[X_{t-1}X'_{t-1}]^{-1}$$

For estimation, replace α_t with the nonparametric density estimates $\hat{\alpha}_t$ and expectations with sample averages.

Under Assumption 2, we have the moment condition

$$m(Z, \gamma_h) = E[Y_{t+h} - A_h Y_{t-1} - \Phi_h \alpha_{t-1}|Z] = 0$$

There is generally no closed form solution for ϕ_h, as $\dim(\alpha) > \dim(Z)$ Instead, we follow a similar procedure to Ai+Chen (2003). The moment condition implies $\gamma_h = (A_h, \Phi_h)$ solves

$$\inf_{\gamma_h} E[m(Z, \gamma_h)'\Sigma(Z)^{-1}m(Z, \gamma_h)]$$

Approximate the moment condition $m(Z, \gamma_h)$ using sieves with

$$\hat{m}(Z, \gamma_h) = \sum_{t=1}^{T}(Y_{t+h} - A_h Y_{t-1} - \Phi_h \alpha_{t-1})p^{k_T}(Z_t)'(P'P)^{-1}p^{k_T}(Z)$$

where $p^{k_T} \in \mathbb{R}^{k_T}$ and $P = (p^{k_T}(Z_1), ..., p^{k_T}(Z_T))'$; k_T smoothing parameter

But actually, there is an easier way to estimate FLP-IV. We will use a 2SLS approach akin to Chen and Christensen (2018). First run the regression

$$\alpha_t = \psi b(Z_t) + v_t$$

Then use fitted $\hat{\alpha}$ in FLP equation. We can also write the estimates for Φ as the projection

$$\Phi_h = [\boldsymbol{\alpha}' B(B'B)^{-} B'\boldsymbol{\alpha}]^{-} \boldsymbol{\alpha}' B(B'B)^{-} B'Y$$

where $B = (b^K(Z_1), ..., b^K(Z_n))$ and $b^K(z) = (b_{K1}(z), ..., b_{KK}(z))$.

3 Estimating the Propagation of Symmetric Microeconomic Shocks

In this section, we consider the functional local projection

$$Y_{t+h} = A_h Y_{t-1} + \int B_h(x) l_t(x)\, dx + u_{t+h}$$

where Y is log TFP growth and $l(x)$ is the log density of earnings. Suppose the government gives a one-time stimulus check of x^* dollars at time t to every person. Then the log density of earnings at time t has the form $l_t^*(x) = l_t(x + x^*)$. We may write the counterfactual Y^* where the fiscal transfer occured as

$$Y_{t+h}^* = A_h Y_{t-1} + \int B_h(x) l_{t-1}^*(x)\, dx + u_{t+h}.$$

Approximating the functional local projection gives

$$Y_{t+h}^* = A_h Y_{t-1} + \Phi_h^* \alpha_t + u_{t+h}^*.$$

where

$$\Phi_h^* = B_h \int \xi(x) \zeta'(x + x^*)\, dx.$$

The impulse response may be written as

$$IR_h = E[Y^*_{t+h}] - E[Y_{t+h}] = (\Phi^*_h - \Phi_h)\alpha_t$$

Therefore, the impulse response of stimulus checks depends on the distribution of earnings. In the paper, we can graph the impulse response for relevant values of α_t in the same plot. For instance, we may graph impulse responses functions at the estimated steady state distribution α^*_t or at the most recent time period of data available. Another parameter of interest that we can identify is the effect of heterogeneity on the propagation of shocks. In particular, let

$$\delta_h = IR_h(\alpha^1) - IR_h(\alpha^2)$$

where α^1 and α^2 refer to two different distributions. One interpretation of δ_h is that it measures how sensitive the impulse response function is to the distribution of agents in the economy.

4 Constructing Aysmmetric Shocks

Way 1: Let $l^*_t(x) = \zeta'(x)\alpha^*_t$ be the counterfactual distribution. Then

$$Y_{t+h} = A_h Y_{t-1} + \int B_h(x) l_t(x)\, dx + u_{t+h}$$

$$Y^*_{t+h} = A_h Y_{t-1} + \int B_h(x) l^*_t(x)\, dx + u_{t+h}.$$

$$IR_{h,t} = \Phi_h(\alpha^*_t - \alpha_t)$$

Question: How to construct $l^*_t(x)$? There is a simple and intuitive way to do this! Suppose we observe draws of the micro variable x_{it} from the distribution $l_t(x)$ for $i = 1, ..., N$. Then consider a policy f such that $f(x_{it})$ is the counterfactual micro variable. Note that if $f(x_{it}) = x_{it} + x^*$ then we are in the case of symmetric microeconomic shocks. There are other functions we may consider. First, let $f = (1+r)x_{it}$ represent a function that increases the value of the micro variable x_{it} by a factor of $(1+r)$. This function could represent the macro effects of increased savings from a contractionary monetary policy shock, which is known as the indirect effect of monetary policy in the HANK literature. On the other hand, we can study the effects of targeted fiscal transfers with the piecewise function

$$f = \begin{cases} x_{it} + w_1, & x_{it} \leq m, \\ x_{it}, & \text{otherwise.} \end{cases}$$

This fiscal transfer function f gives a stimulus check w_1 to every person making less than m dollars.

Way 2 (I think we abandon this approach but have left it here for documentation):
Let $\Delta l_t(x)$ be a distributional shock to the cross-sectional density $l_t(x)$. Consider the case $\Delta l_t(x) = g_t(x)$ where $g_t(x)$ is a known continuous function. By the properties of sieves, there exists a α_t^* such that we have

$$g_t(x) = \zeta'(x)\alpha_t^* + o_p(1)$$

Hence, we have that α_t^* is a solution to the minimization problem

$$\min_{\alpha_t} |g_t(x) - \zeta'(x)\alpha_t|$$

We are interested in estimating the impulse response of a distributional shock $\Delta l_t(x)$ on a macro aggregate Y_t. In particular, define the impulse response as

$$IR_h = E[Y_{t+h}|\Delta l_t(x) = g_t(x)] - E[Y_{t+h}|\Delta l_t(x) = 0]$$

Recall the functional local projection has the form

$$Y_{t+h} = A_h Y_{t-1} + \Phi_h \alpha_t + u_{t+h}$$

Then substitution gives

$$IR_h = \Phi_h \alpha_t^*$$

Question: How do you construct the function $g_t(x)$? Unclear with this method.